The phase-plane picture for a class of fourth-order conservative differential equations

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Abstract

We study the bounded solutions of a class of fourth-order equations

$$-\gamma u'''' + u'' + f(u) = 0, \qquad \gamma > 0.$$

We show that when γ is not too large then the paths in the (u, u')-plane of two bounded solutions do not cross. Moreover, the conserved quantity associated with the equation puts an ordering on the bounded solutions in the phase-plane and a continuation theorem shows that they fill up part of the phase-plane. We apply these results to the Extended Fisher-Kolmogorov (EFK) equation, a fourth-order model equation for bi-stable systems. The uniqueness and ordering results imply that as long as the stable equilibrium points are real saddles the bounded solutions of the stationary EFK equation correspond exactly to those of the classical second-order Fisher-Kolmogorov equation. Besides, we establish the asymptotic stability of the heteroclinic solution of the EFK equation.

1 Introduction

In this paper we study the bounded solutions of fourth-order differential equations of the form

$$-\gamma u'''' + u'' + f(u) = 0, \qquad \gamma > 0, \tag{1}$$

where $f(u) = \frac{dF(u)}{du}$, and $F(u)$ is called the *potential*. By a bounded solution we mean a function $u(x) \in C^4(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which satisfies (1) for all $x \in \mathbb{R}$. For small positive γ Equation (1) is a singular perturbation of the (mechanical) equation

$$u'' + f(u) = 0. \tag{2}$$

We investigate the correspondence between bounded solutions of (1) and (2).

Note that (1) is both translation invariant and reversible (invariant under the transformation $x \to -x$). Besides, there is a constant of integration. When we multiply (1) by u' and integrate, we obtain the *energy* or Hamiltonian

$$\mathcal{E}[u] \stackrel{\text{def}}{=} -\gamma u''' u' + \frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 + F(u) = E, \tag{3}$$

where E is constant along solutions.

In recent years fourth order equations of the form (1) have attracted a wide interest, and two special cases have been thoroughly studied. Firstly, when the potential is

$$F(u) = -\frac{1}{4}(u^2 - 1)^2, \tag{4}$$

Equation (1) is the stationary version of the Extended Fisher-Kolmogorov (EFK) equation, which has been studied by shooting methods [PT1–4] and through variational approaches [PTV, KV, KKV]. Generalisations of the EFK potential have been studied in [PRT], including potentials with maxima of unequal height. Secondly, in the study of a strut on a nonlinear elastic foundation and in the study of shallow water waves, Equation (1) arises with the potential

$$F(u) = -\frac{1}{2}u^2 + \frac{1}{3}u^3. \tag{5}$$

The homoclinic orbits of this equation have been studied both analytically [AT, CT, BS, Bu] and numerically [BCT, CS]. In these studies a striking feature is that the behaviour of solutions changes dramatically when the parameter γ reaches the lowest value for which one of the equilibrium point becomes a saddle-focus. Below this critical value the solutions that have been found, are as tame as for the second order equation. When one of the equilibrium points becomes a saddle-focus, an outburst of new solutions appears.

The situation for $\gamma < 0$ seems to be much less understood. We refer to [Ch] for an overview of equations of the form

$$u'''' - Au'' + Bu = f(u, u', u'', u''') \qquad A, B \in \mathbb{R}.$$

As remarked, the character of the equilibrium point plays a dominating role. If an equilibrium point is a center for the second order equation, then it is a saddle-center for all $\gamma > 0$. On the other hand, if an equilibrium point is a saddle for the second order equation, then it is a real saddle for small (positive) values of γ. The character of such a point changes to saddle-focus as γ increases beyond some critical value.

Since (1) is a singular perturbation of the equation for $\gamma = 0$, it is natural to ask when it inherits solutions from the second order equation. For *small* γ this question can be answered using singular perturbation theory [AH, F, J]. Here we follow an approach that leads to uniqueness results for a wider range of γ-values. The method is based on repeated application of the maximum principle. In [BCT] this idea has been used to prove the uniqueness of the homoclinic orbit for the potential in (5).

We shall first state two general theorems and subsequently draw detailed conclusions for the case of the EFK equation. In fact, the general theorems presented here, are a natural extension of the result for the EFK equation, of which a short summary has been published in [Be].

We consider functions $f(u) \in C^1(\mathbb{R})$ and define, for $-\infty \leq a < b \leq \infty$,

$$\omega(a,b) \stackrel{\text{def}}{=} \max\bigl\{0, \max_{u\in[a,b]} -f'(u)\bigr\}.$$

We are only interested in cases where $\omega(a,b) < \infty$. We will often drop the dependence of ω on a and b, when it clear which constants a and b are meant. Also, we introduce sets of bounded functions

$$\mathcal{B}(a,b) \stackrel{\text{def}}{=} \big\{u \in C^4(\mathbb{R}) \bigm| u(x) \in [a,b] \text{ for all } x \in \mathbb{R}\big\}.$$

In the following we often have an a priori bound on the set of all bounded solutions, i.e., for some $-\infty \le a < b \le \infty$ all bounded solutions of (1) are in $\mathcal{B}(a,b)$. It is important to keep in mind that these a priori bounds are usually valid for a range of values of γ.

As will be clear from the statement of the theorems below, a better bound leads to a lower value of ω, which in turn leads to a stronger result.

The bounded solutions of the second order equation ($\gamma = 0$) are found directly from the phase-plane. Our first theorem states that the (u,u')-plane preserves the uniqueness property for the fourth-order equation as long as γ is not too large.

Theorem 1. *Let u_1 and u_2 be bounded solutions of (1), i.e., u_1 and u_2 are in $\mathcal{B}(a,b)$ for some $-\infty < a < b < \infty$. Suppose that $\gamma \in \big(0, \frac{1}{4\omega(a,b)}\big]$. Then the paths of u_1 and u_2 in the (u,u')-plane do not cross.*

Remark 1. It turns out that we need to give a meaning to the case $\gamma = \infty$. A scaling in x, which is discussed later on, shows that the natural extension of (1) for $\gamma = \infty$ is

$$-u'''' + f(u) = 0. \qquad \bullet$$

The following theorem shows that the energy $\mathcal{E}[u]$ (see (3)) is a parameter that orders the bounded solutions in the phase-plane.

Theorem 2. *Let $u_1, u_2 \in \mathcal{B}(a,b)$ be bounded solutions of (1) for some $\gamma \in \big(0, \frac{1}{4\omega(a,b)}\big]$. Suppose that (after translation) $u_1(0) = u_2(0)$ and either $u_1'(0) > u_2'(0) \ge 0$ or $u_1'(0) < u_2'(0) \le 0$. Then $\mathcal{E}[u_1] > \mathcal{E}[u_2]$.*

We now give some examples. For the double well potential $F(u) = \frac{1}{4}(u^2-1)^2$ (note that this is not the EFK potential in (4)) we have that $\omega(-\infty,\infty) = 1$ and thus any two bounded solutions do not cross in the (u,u')-plane for $\gamma \in (0, \frac{1}{4}]$. Besides, in this parameter range the energy ordering of Theorem 2 holds for all bounded solutions of (1).

In the case of the periodic potential $F(u) = \cos u$, we again have $\omega(-\infty,\infty) = 1$. In this case Theorem 1 combined with the periodicity of the potential, shows that for $\gamma \in (0, \frac{1}{4}]$ every bounded solution has its range in an interval of length at most 2π. We note that in both cases $\gamma = \frac{1}{4}$ is exactly the value where the character of some of the equilibrium points changes from real-saddle to saddle-focus.

In the previous two examples we did not need an a priori bound. However, for the EFK potential (4), the existence of a uniform bound on the bounded solutions is needed to obtain a finite ω. The results for the EFK equation are discussed in detail in Section 1.1. For the potential (5) only a lower bound is needed.

Let us now assume that for some $\gamma > 0$ we have an a priori bound on the set of bounded solutions, i.e., all bounded solutions of (1) are in $\mathcal{B}(a,b)$ for some $-\infty \le a < b \le \infty$, and let us assume that $\omega = \omega(a,b) < \infty$. Then if $\gamma \in [0, \frac{1}{4\omega}]$, bounded solutions of (1) do not cross by Theorem 1, and Theorem 2 gives an ordering of the bounded solutions in the (u,u')-plane in term of the energy. An immediate consequence of Theorem 1 and the reversibility of (1), is that when $\gamma \in [0, \frac{1}{4\omega}]$, any bounded solution of (1) is symmetric with respect to its extrema (therefore the analysis in Theorem 2 is restricted to the upper half-plane). This implies that the only possible bounded solutions are

- equilibrium points,
- homoclinic solutions with one extremum,
- monotone heteroclinic solutions,
- periodic solutions with a unique maximum and minimum value.

Another implication is that there are at most two bounded solutions in the stable and unstable manifolds of the equilibrium points.

We will use the following formulation. If $\tilde{u}(x)$ is a solution of (1), then by the transformation

$$u(x) = \tilde{u}(\sqrt[4]{\gamma}\, x) \qquad \text{and} \qquad q = -\frac{1}{\sqrt{\gamma}}\,,$$

it is transformed to a solution of

$$u'''' + qu'' - f(u) = 0, \qquad q < 0. \tag{6}$$

We examine the case where $q \le -2\sqrt{\omega}$, corresponding to $\gamma \in (0, \frac{1}{4\omega}]$. It should be clear that solutions of (6) with $q \le -2\sqrt{\omega}$ correspond to solutions of (1) with $0 < \gamma \le \frac{1}{4\omega}$ and vice versa. The energy in the new setting is

$$\mathcal{E}[u] \stackrel{\text{def}}{=} -u'''u' + \frac{1}{2}(u'')^2 - \frac{q}{2}(u')^2 + F(u). \tag{7}$$

For $q \le -2\sqrt{\omega}$ we define λ and μ such that

$$\lambda\mu = \omega \qquad \text{and} \qquad \lambda + \mu = -q,$$

or explicitly,

$$\lambda = -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \omega} \qquad \text{and} \qquad \mu = -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \omega}\,.$$

It is easily seen that λ and μ are positive real number if and only if $q \le -2\sqrt{\omega}$. In that case we have

$$0 \le \lambda \le \sqrt{\omega} \le \mu.$$

Equation (6) can be factorised as

$$\begin{cases} u'' - \lambda u = w \\ w'' - \mu w = f(u) + \omega u, \end{cases} \tag{8}$$

and the definition of ω ensures that $f(u) + \omega u$ is a non-decreasing function of u for $u \in [a, b]$.

The central tool in this paper is a comparison lemma which shows that if the initial data of two solutions obey certain inequalities, then at most one of the solutions can be bounded.

Lemma 1 (Comparison Lemma). *Let u and v be solutions of (6) such that, for some $-\infty < a < b < \infty$,*

$$a \le u(x), v(x) \le b \qquad \text{for all } x \in [0, \infty).$$

Suppose that $q \le -2\sqrt{\omega(a,b)}$ and

$$\begin{array}{cc} u(0) \ge v(0), & u'(0) \ge v'(0), \\ u''(0) - \lambda u(0) \ge v''(0) - \lambda v(0), & u'''(0) - \lambda u'(0) \ge v'''(0) - \lambda v'(0). \end{array}$$

Then $u(x) - v(x) \equiv C$ on $[0, \infty)$ for some constant $C \in \mathbb{R}$, and $C = 0$ if $\omega(a, b) \neq 0$.

Note that when the bounds a and b are sharper, then ω and λ are smaller, hence the conditions in the statement of the lemma are weaker. The proof of this lemma relies on the factorisation (8) of Equation (6).

We remark that both the splitting (8) of the differential operator and the Comparison Lemma can be extended to sixth and higher order equations. However, the increasing dimension of the phase-space and the lack of additional conserved quantities (like the energy) make it a difficult task to extend the uniqueness results to such higher order equations.

This paper mainly deals with uniqueness of solutions, but the information we obtain about the shape of solutions of (1) for γ not too large also allows us to conclude that any periodic solution belongs to a continuous family of solutions.

Theorem 3. *Let u be a periodic solution of (1) and let $a \equiv \min u(x)$ and $b \equiv \max u(x)$. Suppose that $\gamma \in \left(0, \frac{1}{4\omega(a,b)}\right]$. Then u belongs to a continuous one-parameter family of periodic solutions, parametrised by the energy E.*

1.1 An example: the EFK equation

The stationary version of the Extended Fisher-Kolmogorov (EFK) equation is given by

$$-\gamma u'''' + u'' + u - u^3 = 0, \qquad \gamma > 0. \tag{9}$$

The EFK equation is a generalisation (see [CER, DS]) of the classical Fisher-Kolmogorov (FK) equation ($\gamma = 0$). Clearly (9) is a special case of (1) with the potential $F(u) = -\frac{1}{4}(u^2 - 1)^2$ (we note that in some literature about the EFK equation the function $F(u) = \frac{1}{4}(u^2-1)^2$ is called the potential). In the form of (6) the EFK equation becomes

$$u'''' + qu'' - u + u^3 = 0. \tag{10}$$

Linearisation around $u = -1$ and $u = +1$ shows that the character of these equilibrium points depends crucially on the value of γ. For $0 < \gamma \leq \frac{1}{8}$ they are real saddles (real eigenvalues), whereas for $\gamma > \frac{1}{8}$ they are saddle-foci (complex eigenvalues). The behaviour of solutions of (9) is dramatically different in these two parameter regions.

For $\gamma \in (0, \frac{1}{8}]$ the solutions are calm. It was proved in [PT1] that there exists a monotonically increasing heteroclinic solution (or *kink*) connecting -1 with $+1$ (by symmetry there is also a monotonically decreasing kink connecting $+1$ with -1). This solution is antisymmetric with respect to its (unique) zero. Moreover, it is unique in the class of monotone antisymmetric functions. In [PT4] it was shown that in every energy level $E \in (-\frac{1}{4}, 0)$ there exists a periodic solution, which is symmetric with respect to its extrema and antisymmetric with respect to its zeros. Remark that these solutions correspond exactly to the solutions of the FK equation ($\gamma = 0$).

In contrast, for $\gamma > \frac{1}{8}$ families of complicated heteroclinic solutions [KKV, KV, PT2] and chaotic solutions [PT3] have been found. The outburst of solutions for $\gamma > \frac{1}{8}$ is due to the saddle-focus character of the equilibrium points ± 1.

We will prove that as long as the equilibrium points are real-saddles, i.e. $\gamma \leq \frac{1}{8}$, or, $q \leq -\sqrt{8}$, bounded solutions are uniformly bounded above by 1 and below by -1. To prove this, we first recall a bound already proved in [PT3, PRT], stating that any bounded solution of (9) for $\gamma > 0$ ($q \leq 0$) obeys

$$|u(x)| < \sqrt{2} \qquad \text{for all } x \in \mathbb{R}. \tag{11}$$

This bound is deduced from the shape of the potential and the energy identity. It already shows that Theorems 1 and 2 hold for $\omega = 5$, i.e., for any pair of bounded solutions of (9) with $\gamma \in (0, \frac{1}{20}]$. The method used to obtain this a priori estimate on all bounded solutions is applicable to a class of potentials which strictly decrease to $-\infty$ as $|u| \to \infty$.

The a priori bound can be sharpened in the case of the EFK equation.

Theorem 4. *For any $\gamma \leq \frac{1}{8}$, let u be a bounded solution of (9) on $\mathbb{R}$. Then $|u(x)| \leq 1$ for all $x \in \mathbb{R}$.*

Using the a priori bound (11), the sharper bound is obtained by applying the maximum principle twice to the factorisation of (10). Remark that a sharper bound is not possible since $u = \pm 1$ are equilibrium points of (9).

This theorem implies that we can sharpen the results of Theorems 1 and 2 to $\gamma \in (0, \frac{1}{8}]$, i.e., for all values γ for which the equilibrium points ± 1 are real saddles. It follows that for $\gamma \in (0, \frac{1}{8}]$ bounded solutions do not cross in the (u, u')-plane and they are ordered by their energies.

Remark 2. For the potential in Equation (5) an upper bound is not needed since $f'(u) = -1 + 2u > 0$ for $u > \frac{1}{2}$. An a priori lower bound of $a = 0$ and $\gamma_0 = \frac{1}{4}$ can be found in the same way as in the proof of Theorem 4. Therefore Theorems 1 and 2 hold for $\gamma \in (0, \frac{1}{4}]$ or $q \leq -2$ (see also [BCT]). •

We want to emphasise that the methods used in this paper to obtain a priori bounds on bounded solutions are by no means exhaustive. They are sufficient for the EFK equation but for other potentials different methods may be more suitable. For example, the techniques from this paper can be combined with geometric reasoning in the (u, u'')-plane to obtain a priori bounds on the bounded solutions in fixed energy levels, as is done in [Pe] for potentials that are polynomials of degree four. This allows an extension of the results on uniqueness to values of γ for which some of the equilibrium points are real saddles whereas other equilibrium points are saddle-foci.

The existence of bounded solutions corresponding to the solutions of the FK equation has been proved in [PT1, PT4, PTV]. From Theorems 1 and 2 it can be deduced that there is a complete correspondence between the bounded stationary solutions of the EFK equation and those of the FK equation ($\gamma = 0$).

Theorem 5. *The only bounded solutions of* (9) *for* $\gamma \in (0, \frac{1}{8}]$ *are the three equilibrium points, the two monotone antisymmetric kinks and a one-parameter family of periodic solutions, parametrised by the energy* $E \in (-\frac{1}{4}, 0)$.

The multitude of solutions which exist for $\gamma > \frac{1}{8}$, shows that this bound is sharp. Among other things, Theorem 5 proves the conjecture in [PT1] that the kink for $\gamma \in (0, \frac{1}{8}]$ is unique. We mention that the uniqueness of the kink for $\gamma \in (0, \frac{1}{8}]$ can also be proved using so-called twist maps [Kw].

In the proof of Theorem 5 we do not use the symmetry of the potential F in an essential manner (it merely reduces the length of the proofs). Using the symmetry F we obtain some additional results. Firstly, for $\gamma \leq \frac{1}{8}$ any bounded solution of (9) is antisymmetric with respect to its zeros. Secondly, the periodic solutions can also be parametrised by the period

$$L \in \left(2\pi\sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}, \infty\right).$$

Thirdly, we prove that the heteroclinic orbit is a transversal intersection of the stable and unstable manifold.

Theorem 6. *For* $\gamma \in (0, \frac{1}{8}]$ *the unique monotonically increasing heteroclinic solution of* (9) *is the transverse intersection of the unstable manifold of* -1 *and the stable manifold of* $+1$ *in the zero energy set.*

Finally, the monotonically increasing kink $\tilde{u}(x)$ with $\tilde{u}(0) = 0$ and its translates are asymptotically stable for the full time-dependent EFK equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3. \tag{12}$$

Theorem 7. *Let $u(x,t)$ be a solution of (12). For any $\gamma \in (0, \frac{1}{8}]$ there exists an $\varepsilon > 0$ such that if $\|u(x,0) - \tilde{u}(x + x_0)\|_{H^1} < \varepsilon$ for some $x_0 \in \mathbb{R}$, then there exists a $\delta \in \mathbb{R}$, depending on $u(x,0)$ (and small when ε is small), such that*

$$\lim_{t\to\infty} \|u(x,t) - \tilde{u}(x + x_0 + \delta)\|_{H^1} = 0.$$

We remark that the kink is also asymptotically stable in the space of bounded uniformly continuous functions.

The outline of the paper is the following. In Section 2 we prove the Comparison Lemma, and it then follows that bounded solutions do not cross each other in the (u, u')-plane, as formulated in Theorem 1. Section 3 is devoted to the proof of Theorem 2. In Section 4 we prove the a priori bound from Theorem 4, and in Section 5 the proof of Theorem 5 is completed. Besides, we prove the antisymmetry of bounded solutions, and we show that the periodic solutions can be parametrised by their period. In Section 6, we prove that the unstable manifold of -1 intersects the stable manifold of $+1$ transversally as stated in Theorem 6. Theorem 7 on the asymptotic stability of the kink for the EFK equation is proved in Section 7. Finally, in Section 8 we deal with the continuation and existence of solutions of (1) and in particular we prove Theorem 3.

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2 Uniqueness property

In this section we prove the Comparison Lemma and Theorem 1, which states that for $q \leq -2\sqrt{\omega}$ bounded solutions of (6) are unique in the (u, u')-plane.

Remark 3. For the results in this section, the condition that $f(u)$ is continuously differentiable can be weakened. When $f(u)$ is in $C^0(\mathbb{R})$, then $\omega(a,b)$ is defined as the lowest non-negative number such that $f(u) + \omega(a,b)u$ is non-decreasing as a function of u on $[a,b]$. •

We start with the proof of the Comparison Lemma, which is at the heart of most of the results in this paper. The proof proceeds along the same lines as in [BCT].
Proof. Let $u(x)$ and $v(x)$ satisfy the assumptions in Lemma 1. If $u(x) - v(x) \equiv C$, then by the assumptions $C \geq 0$ and $\lambda C \leq 0$, thus $C = 0$ if $\lambda \neq 0$, i.e., if $\omega \neq 0$.

Suppose now that $u(x) - v(x) \not\equiv C$. Let k be the smallest integer for which $u^{(k)}(0) \neq v^{(k)}(0)$. Then, by uniqueness of solutions, $k \in \{0, 1, 2, 3\}$ and $u^{(k)}(0) > v^{(k)}(0)$ by the hypotheses.

Hence there exists a $\sigma > 0$ such that

$$u(x) > v(x) \qquad \text{on } (0, \sigma).$$

Now let

$$\phi(x) \equiv u''(x) - \lambda u(x), \qquad \text{and} \qquad \psi(x) \equiv v''(x) - \lambda v(x).$$

Then, by the hypotheses,

$$\phi(0) - \psi(0) \geq 0, \qquad \text{and} \qquad \phi'(0) - \psi'(0) \geq 0. \tag{13}$$

Besides, writing $h(u) = f(u) + \omega(a, b)u$,

$$(\phi - \psi)''(x) - \mu(\phi - \psi)(x) = h(u(x)) - h(v(x)) \qquad \text{on } (0, \sigma). \tag{14}$$

Since $a \leq u(x) < v(x) \leq b$ on $(0, \sigma)$ and since $h(u)$ is a non-decreasing function on $[a, b]$ by the definition of $\omega(a, b)$, we have that

$$h(u(x)) - h(v(x)) \geq 0 \qquad \text{on } (0, \sigma). \tag{15}$$

It is immediate from (13), (14) and (15) that

$$\phi(x) - \psi(x) \geq 0 \qquad \text{on } (0, \sigma).$$

This is to say

$$(u - v)''(x) - \lambda(u - v)(x) \geq 0 \qquad \text{on } (0, \sigma).$$

Besides, by the hypotheses of the lemma we have that

$$(u - v)(0) \geq 0, \qquad \text{and} \qquad (u - v)'(0) \geq 0. \tag{16}$$

Therefore

$$u(x) > v(x) \qquad \text{on } (0, \sigma],$$

and since $a \leq u(x), v(x) \leq b$,

$$h(u(x)) \geq h(v(x)) \qquad \text{on } (0, \sigma].$$

Thus we obtain that

$$\sup\{\sigma \mid u(x) > v(x) \text{ for all } x \in (0, \sigma)\} = \infty,$$

and

$$(\phi - \psi)''(x) - \mu(\phi - \psi)(x) \geq 0 \qquad \text{on } (0, \infty),$$

$$(u - v)''(x) - \lambda(u - v)(x) \geq 0 \qquad \text{on } (0, \infty).$$

It follows from (16) and the assumption that $u(x) - v(x) \not\equiv C$, that

$$(u - v)(x) \to \infty \qquad \text{as } x \to \infty.$$

Clearly, if $u(x)$ and $v(x)$ are bounded this is not possible. This concludes the proof of the Comparison Lemma. □

Theorem 1 is a consequence of the Comparison Lemma. Let u_1 and u_2 be bounded solutions of (6) for $q \le -2\sqrt{\omega}$ (corresponding to bounded solutions of (1) for $0 < \gamma \le \frac{1}{4\omega}$). Suppose by contradiction that the paths of u_1 and u_2 cross in the (u, u')-plane. Then, after translation, we have that $u_1(0) = u_2(0)$ and $u_1'(0) = u_2'(0)$. Without loss of generality we may assume that $u_1''(0) \ge u_2''(0)$. Now if $u_1'''(0) \ge u_2'''(0)$, then by the Comparison Lemma we conclude that $u_1(x) - u_2(x) = C$ for some $C \in \mathbb{R}$. Since $u_1(0) = u_2(0)$ this implies that $u_1(x) \equiv u_2(x)$.

On the other hand, if $u_1'''(0) < u_2'''(0)$, then we define $\tilde{u}_1(x) = u_1(-x)$ and $\tilde{u}_2(x) = u_2(-x)$. Clearly $\tilde{u}_1$ and $\tilde{u}_2$ are also bounded solutions of (10). We now apply the Comparison Lemma to $\tilde{u}_1$ and $\tilde{u}_2$, and find as before that $\tilde{u}_1(x) \equiv \tilde{u}_2(x)$, which concludes the proof of Theorem 1.

We now touch upon a lemma that gives a lot of information about the shape of bounded solutions. It states that every bounded solution is symmetric with respect to its extrema.

Lemma 2. *Let $u \in \mathcal{B}(a, b)$ be a bounded solution of (1) for some $\gamma \in \left(0, \frac{1}{4\omega(a,b)}\right]$. Suppose that $u'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then $u(x_0 + x) = u(x_0 - x)$ for all $x \in \mathbb{R}$.*

Proof. After translation we may take $x_0 = 0$. Now we define $v(x) = u(-x)$. By reversibility $v(x)$ is also a bounded solution of (1). Clearly $u(0) = v(0)$ and $u'(0) = v'(0)$. From Theorem 1 we conclude that $u(x) \equiv v(x)$. □

Remark 4. It should be clear that when a solution is bounded for $x > 0$, then it either has an infinite number of extrema or it tends to a limit monotonically. We will show in Lemma 3 that such a limit can only be an equilibrium point. It therefore follows from Lemma 2 that if all bounded solutions of (1) are in $\mathcal{B}(a, b)$, then for $\gamma \in (0, \frac{1}{4\omega}]$ the only possible bounded solutions are equilibrium points, homoclinic solutions with one extremum, monotone kinks and periodic orbits with a unique maximum and minimum. •

3 Energy ordering

To fill in the remaining details of the phase-plane picture we use Theorem 2, which establishes an ordering in terms of the energy $\mathcal{E}$ of the paths in the (u, u')-plane. In this section we will use the notation of Equation (6). Before we start with the proof of Theorem 2, we obtain some preliminary results.

The following lemma shows that when a solution tends to a limit monotonically, then this limit has to be an equilibrium point. We denote the set of zeros of $f(u)$ by $\mathcal{A}$.

Lemma 3. *Let $u(x)$ be a solution of (6) for $q < 0$, which is bounded on $[x_0, \infty)$ for some $x_0 \in \mathbb{R}$. Suppose that $u'(x) \geq 0$ for all $x > x_0$ or $u'(x) \leq 0$ for all $x > x_0$. Then*

$$\lim_{x\to\infty} u(x) \in \mathcal{A} \qquad \textit{and} \qquad \lim_{x\to\infty} u^{(i)}(x) = 0 \quad \textit{for } i = 1, 2, 3.$$

Proof. We may assume that $u'(x) \geq 0$ for $x \geq x_0$ (the other case is completely analogous). It is then clear that

$$\lim_{x\to\infty} u(x) \stackrel{\text{def}}{=} L_0$$

exists and $u(x)$ increases towards L_0 as $x \to \infty$. Since $u(x)$ is bounded for $x > x_0$, L_0 is finite.

We now consider the function $w = u''' + qu'$. Then $w(x)$ satisfies

$$w'' = u' f'(u).$$

We first show that $u''(x)$ tends to zero as $x \to \infty$. If $f'(L_0) \neq 0$ (the other case will be dealt with later), then $f'(u)$ has a sign for x large enough, by which we mean that either $f'(u) \geq 0$ for large x, or $f'(u) \leq 0$ for large x. Since $u'(x) \geq 0$, it follows that $w''(x)$ has a sign for x large enough, hence so does $w(x)$. The fact that $w(x) = u'''(x) + qu'(x)$ has a sign for x large enough implies that

$$\lim_{x\to\infty} u''(x) + qu(x) \stackrel{\text{def}}{=} L_1$$

exists and $u''(x) \to L_1 - qL_0$ as $x \to \infty$. Moreover, since $u(x)$ is bounded, we must have

$$\lim_{x\to\infty} u''(x) = 0.$$

If $f'(L_0) = 0$, then we consider $w = u''' + \frac{q}{2}u'$. We now have

$$w'' + \frac{q}{2}w = u'\left(f'(u) + \frac{q^2}{4}\right).$$

Since $f'(L_0) + \frac{q^2}{4}$ is positive for x large enough, we conclude from the maximum principle that $w(x) = u'''(x) - qu'(x)$ has a sign for x large enough. As before we see that

$$\lim_{x\to\infty} u''(x) = 0.$$

The fact that $u(x) \to L_0$ and $u''(x) \to 0$, implies that $u'(x) \to 0$ as $x \to \infty$. Because $u^{(iv)} = -qu'' + f(u)$, we see that

$$\lim_{x\to\infty} u^{(iv)}(x) \stackrel{\text{def}}{=} L_2 = f(L_0),$$

and, since $u(x)$ is bounded, $L_2 = 0$ and thus $L_0 \in \mathcal{A}$. Finally, the fact that $u''(x) \to 0$ and $u^{(iv)}(x) \to 0$, implies that $u'''(x) \to 0$ as $x \to \infty$. □

Remark 5. For $q = 0$ the situation is slightly more subtle, but when $f'(u)$ has a sign as u tends to L_0 monotonically, then the proof still holds. Since we consider bounded solutions of (6) for $q \leq -2\sqrt{\omega}$, this difficulty only arises when $\omega = 0$, which (by the definition of ω) implies that $f'(u) \geq 0$ for all values of u involved, hence the lemma holds for this case. •

We prove that $u''' - \lambda u'(x)$ is negative if and only if $u'(x)$ is positive.

Lemma 4. *Let $u \in \mathcal{B}(a,b)$ be a bounded solution of (6) for some $q \leq -2\sqrt{\omega(a,b)}$. Then (with $\mathrm{sign}(0) \stackrel{\mathrm{def}}{=} 0$)*

$$\mathrm{sign}\big(u'''(x) - \lambda u'(x)\big) = -\mathrm{sign}\big(u'(x)\big) \qquad \textit{for all } x \in \mathbb{R}. \tag{17}$$

Proof. Let $x_0 \in \mathbb{R}$ be arbitrary. We may assume that $u'(x_0) \geq 0$ (for $u'(x_0) < 0$ the proof is analogous). We see from Lemma 2 that (17) holds if $u'(x_0) = 0$. We thus assume that $u'(x_0) > 0$. Since $u(x)$ is bounded there exist $-\infty \leq x_a < x_0 < x_b \leq \infty$, such that $u'(x_a) = u'(x_b) = 0$ and $u'(x) > 0$ on (x_a, x_b). By Lemmas 2 and 3 we have that $u'''(x_a) = u'''(x_b) = 0$. Let $w \equiv u''' - \lambda u'$. Then $w(x)$ satisfies the system

$$\begin{cases} w'' - \mu w = u'(f'(u) + \omega), \\ w(x_a) = u'''(x_a) - \lambda u'(x_a) = 0, \\ w(x_b) = u'''(x_b) - \lambda u'(x_b) = 0. \end{cases}$$

Since $u'(x) > 0$ on (x_a, x_b), we have by the definition of ω that $u'(f'(u) + \omega) \geq 0$. By the strong maximum principle we obtain that $w(x) < 0$ for all $x \in (x_a, x_b)$, and especially, $w(x_0) < 0$. This completes the proof. □

Remark 6. It follows from the boundary point lemma and the preceding proof that if a bounded solutions of (1) for $q \leq -2\sqrt{\omega}$ attains a maximum at some point x_0, then

$$u''(x_0) < 0 \qquad \text{and} \qquad u^{(iv)}(x_0) - \lambda u''(x_0) > 0.$$

Moreover, it is seen from the differential equation that

$$f(u(x_0)) = u^{(iv)}(x_0) + q u''(x_0) > -\mu u''(x_0) > 0,$$

i.e., maxima only occur at positive values of $f(u)$. •

We immediately obtain the following consequence.

Lemma 5. *Let $u \in \mathcal{B}(a,b)$ be a bounded solution of (6) for some $q \leq -2\sqrt{\omega(a,b)}$. Then*

$$H(x) \stackrel{\mathrm{def}}{=} -\mathcal{E}[u] + F(u(x)) + \tfrac{1}{2}(u''(x))^2 \leq 0 \qquad \textit{for all } x \in \mathbb{R}.$$

Proof. By the energy identity we have

$$H = u' \left\{ u''' + \frac{q}{2} u' \right\} = u'(u''' - \lambda u') - C(u')^2,$$

where $C = \sqrt{\left(\frac{q}{2}\right)^2 - \omega} \geq 0$. It is easily seen from Lemma 4 that the assertion holds. □

We will now start the proof of Theorem 2. Let u_1 and u_2 satisfy the assumptions in Theorem 2. We only consider the case where $u_1'(0) > u_2'(0) \geq 0$. The other case follows by symmetry. For contradiction we assume that $\mathcal{E}[u_1] \leq \mathcal{E}[u_2]$. It will be proved in Lemma 7 that we can then find points x_1 and x_2 such that $u_1(x_1) = u_2(x_2)$ and $u_1''(x_1) = u_2''(x_2)$. Subsequently, in Lemma 8, we show that the energy identity (7) ensures that we can apply the Comparison Lemma to u_1 and u_2 resulting in a contradiction.

It should be clear that u_1 and u_2 are not translates of one another, because this would contradict the result on symmetry with respect to extrema, obtained in Lemma 2.

We make the following change of variables on intervals $[x_a, x_b]$ where the function $u(x)$ is strictly monotone on the interior (see [PT1]). Denoting the inverse of $u(x)$ by $x(u)$, we set

$$t = u \qquad \text{and} \qquad z(t) = [u'(x(t))]^2 .$$

We now get for $t \in [t_a, t_b] = [u(x_a), u(x_b)]$

$$z'(t) = 2u''(x(t)).$$

If $x_a = -\infty$, then we write $z'(t_a) = \lim_{t \to t_a} z'(t)$ (the limit exists by Lemma 3).

Before we proceed with the general case, we first consider the special case where two different solutions tend to the same equilibrium point as $x \to -\infty$. The next lemma in fact shows that there are at most two bounded solution in the unstable manifold of each equilibrium point.

Lemma 6. *Let u_1, $u_2 \in \mathcal{B}(a, b)$ be two different non-constant bounded solutions of (6) for some $q \leq -2\sqrt{\omega(a, b)}$. Suppose there exists an $\tilde{u} \in \mathcal{A}$ such that*

$$\lim_{x \to -\infty} u_1(x) = \lim_{x \to -\infty} u_2(x) = \tilde{u}.$$

Then $u_1(x)$ decreases to $\tilde{u}$ and $u_2(x)$ increases to $\tilde{u}$ as $x \to -\infty$, or vice versa.

Proof. By Remark 4, u_1 and u_2 can only tend to $\tilde{u}$ monotonically. Suppose u_1 and u_2 both decrease towards $\tilde{u}$ as $x \to -\infty$ (the case where they both increase towards $\tilde{u}$ is analogous). It follows from Remark 4 that u_1 and u_2 can only tend to $\tilde{u}$ monotonically as $x \to \infty$. We may thus assume that $u_1'(x) > 0$ and $u_2'(x) > 0$ for $x \in (-\infty, x_0)$.

For $t \in (\tilde{u}, \tilde{u} + \varepsilon_0)$, where $\varepsilon > 0$ is sufficiently small, let z_1 and z_2 corresponds to u_1 and u_2 respectively, by the change of variables described above. Note that $z_1(t) \neq z_2(t)$ for $t \in (\tilde{u}, \tilde{u} + \varepsilon_0)$, since otherwise $u_1 \equiv u_2$ by Theorem 1. Without loss of generality we may assume that $z_1(t) > z_2(t)$ on $(\tilde{u}, \tilde{u} + \varepsilon_0)$. Since $z_i(t)$ is differentiable on $(\tilde{u}, \tilde{u} + \varepsilon_0)$, there exist a point $t_0 \in (\tilde{u}, \tilde{u} + \varepsilon_0)$, such that $z_1'(t_0) \geq z_2'(t_0)$. Thus, there are x_1 and x_2 in $\mathbb{R}$ such that $u_1(x_1) = u_2(x_2) = t_0$.

After translation we obtain that

$$u_1(0) = u_2(0), \qquad u_1'(0) > u_2'(0) > 0 \qquad \text{and} \qquad u_1''(0) \geq u_2''(0) > 0.$$

We will now show that $u_1'''(0) - \lambda u_1'(0) \geq u_2'''(0) - \lambda u_2'(0)$. A contradiction then follows from the Comparison Lemma.

To simplify notation we write $\lambda = -\frac{q}{2} - C$, where $C = \sqrt{\left(\frac{q}{2}\right)^2 - \omega} \geq 0$. From the energy identity we obtain

$$u''' - \lambda u' = \frac{-\mathcal{E}[u] + F(u) + \frac{1}{2}(u'')^2}{u'} + Cu'. \tag{18}$$

Since u_1 and u_2 tend to $\tilde{u}$ monotonically as $x \to -\infty$, we infer from Lemma 3 that

$$(u_i, u_i', u_i'', u_i''')(x) \to (\tilde{u}, 0, 0, 0) \quad \text{as } x \to -\infty \qquad \text{for } i = 1, 2.$$

Therefore $\mathcal{E}[u_1] = \mathcal{E}[u_2]$. At $x = 0$ we have $F(u_1) = F(u_2)$ and $(u_1'')^2 \geq (u_2'')^2$. By Lemma 5 we see that at $x = 0$

$$-\mathcal{E}[u_2] + F(u_2) + \tfrac{1}{2}(u_2'')^2 \leq -\mathcal{E}[u_1] + F(u_1) + \tfrac{1}{2}(u_1'')^2 \leq 0.$$

Combining this with (18) and the fact that $u_1'(0) > u_2'(0) > 0$, we obtain that

$$u_1'''(0) - \lambda u_1'(0) \geq u_2'''(0) - \lambda u_2'(0).$$

An application of the Comparison Lemma ends the proof. $\square$

Of course a similar result hold for solutions that tend to an equilibrium point as $x \to +\infty$: there are at most two bounded solution in the stable manifold of each equilibrium point.

The next lemma shows that if $\mathcal{E}[u_1]$ would we smaller than $\mathcal{E}[u_2]$, where u_1 and u_2 are solutions obeying the assumptions in Theorem 2, then we could find a point where $u_1 = u_2$ and $u_1'' = u_2''$.

Lemma 7. *Let $u_1, u_2 \in \mathcal{B}(a, b)$ be bounded solutions of (6) for some $q \leq -2\sqrt{\omega(a, b)}$. Suppose that $u_1(0) = u_2(0)$ and $u_1'(0) > u_2'(0) \geq 0$, and $\mathcal{E}[u_1] \leq \mathcal{E}[u_2]$. Then there exists x_1 and x_2 in $\mathbb{R}$ such that $u_1(x_1) = u_2(x_2)$ and $u_1''(x_1) = u_2''(x_2)$.*

Proof. We change variables again. Let z_1 correspond to u_1 on the largest interval, containing $x = 0$, where u_1' is positive, say $[\tilde{t}_a, \tilde{t}_b]$. Let z_2 correspond to u_2 on the largest interval, containing $x = 0$, where u_1' is positive, say $[t_a, t_b]$. If $t_a \notin \mathcal{A}$, then it follows from Theorem 1 that $\tilde{t}_a < t_a$, whereas if $t_a \in \mathcal{A}$ this follows from Lemma 6. Similarly, $\tilde{t}_b > t_b$. Clearly $z_1(t) > z_2(t)$ for all $t \in [t_a, t_b]$, since bounded solution do not cross in the (u, u')-plane.

We have that $z_2(t_a) = 0$ and $z_2'(t_a) \geq 0$. We will now prove that $z_1'(t_a) < z_2'(t_a)$ by showing that $(z_2')^2(t_a) - (z_1')^2(t_a) > 0$. Let x_1^a and x_2^a be the points in the intervals under consideration such that $u_1(x_1^a) = u_2(x_2^a) = t_a$. By the energy identity we have that

$$\begin{aligned}
\frac{(z_2')^2(t_a) - (z_1')^2(t_a)}{8} &= \tfrac{1}{2}(u_2''(x_2^a))^2 - \tfrac{1}{2}(u_1''(x_1^a))^2 \\
&= \mathcal{E}[u_2] - F(t_a) - \left\{\mathcal{E}[u_1] - F(t_a) + u_1'(x_1^a)\left(u_1'''(x_1^a) + \frac{q}{2}u_1'(x_1^a)\right)\right\} \\
&= \mathcal{E}[u_2] - \mathcal{E}[u_1] - u_1'(x_1^a)\left(u_1'''(x_1^a) + \frac{q}{2}u_1'(x_1^a)\right).
\end{aligned}$$

From Lemma 4 and the observation that $u_1'(x_1^a) = \sqrt{z_1(t_a)} > \sqrt{z_2(t_a)} = 0$, we conclude that

$$u_1'(x_1^a)\left(u_1'''(x_1^a)+\frac{q}{2}u_1'(x_1^a)\right)<0.$$

Having assumed that $\mathcal{E}[u_1]\le\mathcal{E}[u_2]$, we now see that $z_1'(t_a)<z_2'(t_a)$.

In the same way we can show that $z_1'(t_b)>z_2'(t_b)$. By continuity there exists a $t_c\in(t_a,t_b)$ such that $z_1'(t_c)=z_2'(t_c)$, which proves the lemma. $\square$

We now complete the proof of Theorem 2. Let u_1 and u_2 satisfy the assumptions in the theorem. The previous lemma shows that if by contradiction $\mathcal{E}[u_1]\le\mathcal{E}[u_2]$, then there exists points x_1 and x_2 such that

$$u_1(x_1)=u_2(x_2),\qquad u_1'(x_1)>u_2'(x_2)\ge 0\qquad\text{and}\qquad u_1''(x_1)=u_2''(x_2).$$

By translation invariance we may take $x_1=x_2=0$. The following lemma now shows that $u_1\equiv u_2$, which contradicts the assumption. Therefore $\mathcal{E}[u_1]>\mathcal{E}[u_2]$, which proves the theorem.

Lemma 8. *Let $u_1,u_2\in\mathcal{B}(a,b)$ be bounded solutions of (6) for some $q\le-2\sqrt{\omega(a,b)}$. Suppose that $\mathcal{E}[u_1]\le\mathcal{E}[u_2]$ and*

$$u_1(0)=u_2(0),\qquad u_1'(0)>u_2'(0)\ge 0\qquad\textit{and}\qquad u_1''(0)=u_2''(0).$$

Then $u_1\equiv u_2$.

Proof. We will show that

$$u_1'''(0)-\lambda u_1'(0)\ge u_2'''(0)-\lambda u_2'(0)\tag{19}$$

and then an application of the Comparison Lemma completes the proof. From the energy identity we obtain at $x=0$

$$u_i'''-\lambda u_i'=\frac{-\mathcal{E}[u_i]+F(u_i)-\frac{1}{2}(u_i'')^2}{u_i'}+Cu_i'\qquad\text{for } i=1,2,$$

where C is a positive constant. By the assumptions and from Lemma 5, it follows that

$$-\mathcal{E}[u_1]+F(u_1(0))-\tfrac{1}{2}(u_1''(0))^2\le-\mathcal{E}[u_2]+F(u_2(0))-\tfrac{1}{2}(u_2''(0))^2\le 0.$$

Inequality (19) is now easily verified. $\square$

4 A priori bounds

In this section we derive a priori estimates for bounded solutions of the EFK equation (9) or (10). Where possible, we will indicate how the methods can be generalised to arbitrary $f(u)$. We will prove Theorem 4 which states that every bounded solution for $q\le-\sqrt{8}$ ($\gamma\in(0,\frac{1}{8}]$) satisfies $|u(x)|\le 1$ for all $x\in\mathbb{R}$. We first derive a weaker bound for all $q\le 0$, which follows from the shape of the potential and the energy identity (7). Subsequently, we sharpen this bound for all $q\le-\sqrt{8}$ with the help of the maximum principle.

The first question we address, is whether solutions can go to infinity monotonically.

Remark 7. It was proved in [PT4, PT3] that if $u(x)$ is a solution of (10) on its maximal interval of existence (x_a, x_b), then for any $x_0 \in (x_a, x_b)$, there either exists an infinite number of extrema of $u(x)$ for $x > x_0$, or $u(x)$ eventually tends to a finite limit monotonically as $x \to \infty$. For bounded solutions this is obvious. Notice that the energy of the solution must be equal to the energy of the equilibrium point towards which it converges. Besides, if an equilibrium point is a saddle-focus (complex eigenvalues) then no solution can tend to it monotonically. •

We now prove a slight variation of an important lemma from [PT3], which shows that when a solution of (10) becomes larger than $\sqrt{2}$, then it will oscillate towards infinity, and thus is unbounded. The proof can be easily extended to more general potentials F, as is done in [PRT]. The value $\sqrt{2}$ is directly related to the fact that

$$\max\bigl\{x_0 > 0 \bigm| F(x) \le F(x_0) \text{ for all } x \in [-x_0, x_0]\bigr\} = \sqrt{2}.$$

Lemma 9. *For any $q \le 0$, let $u(x)$ be a solution of (10). Suppose that there exists a point $x_0 \in \mathbb{R}$, such that*

$$u(x_0) \ge \sqrt{2}, \qquad u'(x_0) = 0, \qquad u''(x_0) \le 0, \qquad \text{and} \qquad u'''(x_0) \le 0. \tag{20}$$

Then, there exists a first critical point of u on (x_0, ∞), say y_0, and we have

$$u(y_0) < -u(x_0) \le -\sqrt{2}, \qquad u'(y_0) = 0, \qquad u''(y_0) > 0, \qquad \text{and} \qquad u'''(y_0) > 0.$$

Besides, $F(u(y_0)) < F(u(x_0))$, and the following estimate holds:

$$F(u(y_0)) - F(u(x_0)) < -\frac{5\sqrt{2}}{3} \frac{[\mathcal{E}[u] - F(u(x_0))]}{u(y_0) - u^3(y_0)}. \tag{21}$$

Proof. We write $f(u) \equiv u - u^3$. Since $f(u(x_0)) < 0$ and $u''(x_0) \le 0$, we see that $u^{(iv)}(x_0) = -qu''(x_0) + f(u(x_0)) < 0$ and thus $u''' < 0$ in a right neighbourhood of x_0. We now conclude that

$$x_1 \stackrel{\text{def}}{=} \sup\bigl\{x > x_0 \bigm| u''' < 0 \text{ on } (x_0, x)\bigr\}$$

is well-defined. By Remark 7 we know that either $u(x)$ attains a critical point on (x_0, ∞), or $u(x)$ tends to a limit monotonically. In both cases we conclude that x_1 is finite. Since $u''' < 0$ on (x_0, x_1), we have that $u''(x_1) < u''(x_0) \le 0$. Using the energy identity and the fact that $u'''(x_1) = 0$ and $u'(x_0) = 0$, we obtain

$$\begin{aligned} F(u(x_0)) &= \mathcal{E}[u] - \frac{1}{2}(u''(x_0))^2 \\ &> \mathcal{E}[u] - \frac{1}{2}(u''(x_1))^2 \\ &> \mathcal{E}[u] - \frac{1}{2}(u''(x_1))^2 + \frac{q}{2}(u'(x_1))^2 = F(u(x_1)). \end{aligned}$$

It follows from the definition of x_1 and the initial data at x_0, that $u''' < 0$, $u'' < 0$ and $u' < 0$ on

(x_0, x_1), and thus $u(x_1) < u(x_0)$. It is seen from the shape of the potential that

$$F(u(x_0)) \le F(s) \qquad \text{for all } s \in [-u(x_0), u(x_0)].$$

Therefore, $F(u(x_0)) > F(u(x_1))$ and $u(x_1) < u(x_0)$ imply that $u(x_1) < -u(x_0) \le -\sqrt{2}$. From Lemma 3 we now conclude that $u(x_0)$ does not decrease monotonically to some finite limit and therefore there exists a first critical point of u on (x_1, ∞), say y_0. We now define

$$x_2 \stackrel{\text{def}}{=} \sup\big\{x > x_1 \;\big|\; u'' < 0 \text{ on } (x_1, x)\big\},$$

which is well-defined since $u''(x_1) < 0$, and x_2 is finite because $x_2 \le y_0 < \infty$. Clearly

$$u(x_2) < u(x_1) < -u(x_0), \quad u''(x_2) = 0, \quad u'''(x_2) \ge 0, \quad \text{and} \quad u^{(iv)}(x_2) = f(u(x_2)) > 0.$$

It is not too difficult to see that, since $f(u(x)) < f(u(x_2))$ on $(x_2, y_0]$,

$$u'' > 0, \quad u''' > 0 \quad \text{and} \quad u^{(iv)} = -qu'' + f(u) > 0 \qquad \text{on } (x_2, y_0].$$

To summarise, we have that

$$u(y_0) < -u(x_0), \quad u'(y_0) = 0, \quad u''(y_0) > 0, \quad u'''(y_0) > 0 \quad \text{and} \quad F(u(y_0)) < F(u(x_0)).$$

We still have to prove the estimate (21). By the energy identity (7) we have that $F(u(x_0)) \le \mathcal{E}[u]$. For $F(u(x_0)) = \mathcal{E}[u]$ the estimate has already been proved. Therefore we may assume that $F(u(x_0)) < \mathcal{E}[u]$, so that $u''(x_0) = -\sqrt{2[\mathcal{E}[u] - F(u(x_0))]} \stackrel{\text{def}}{=} -\beta < 0$.

From the definition of x_1 and x_2 we see that $u''(x_1) < -\beta$, $u'''(x_1) = 0$, and

$$u^{(iv)} = -qu'' + f(u) < f(u(y_0)) \text{ on } (x_1, x_2).$$

Therefore

$$u''(x) < -\beta + \tfrac{1}{2} f(u(y_0)(x - x_1)^2 \qquad \text{for } x \in (x_1, x_2]. \tag{22}$$

By definition, x_2 is the first zero of $u''(x)$, thus $x_2 - x_1 > \sqrt{\frac{2\beta}{f(u(y_0))}} \stackrel{\text{def}}{=} \xi_2$. Integrating (22) twice and using the fact that $u'(x_1) < 0$, we obtain

$$\begin{aligned} u(x_1 + \xi_2) - u(x_1) \quad &< \quad -\beta\frac{\xi_2^2}{2} + f(u(y_0))\frac{\xi_2^4}{24} \\ &< \quad -\frac{5}{6}\frac{\beta^2}{f(u(y_0))}. \end{aligned}$$

Because $u' < 0$ on $[x_1 + \xi_2, x_2]$, we see that

$$u(x_2) - u(x_1) < u(x_1 + \xi_2) - u(x_1) < -\frac{5}{6} \frac{\beta^2}{f(u(y_0))} \stackrel{\text{def}}{=} -\alpha.$$

Since $F'(u) = f(u) > 0$ for $u < -1$ and $u_2 < u_1 - \alpha < u_1 < -\sqrt{2}$ ($u_i \equiv u(x_i)$), we have that

$$F(u(y_0)) < F(u_2) < F(u_1 - \alpha).$$

Moreover, $F''(u) = f'(u) > 0$ for $u < -\sqrt{2}$, and we finally obtain that

$$F(u(y_0)) < F(u_1 - \alpha) < F(u_1) - \frac{dF(u_1)}{du}\alpha < F(u_1) - f(-\sqrt{2})\alpha.$$

Since $f(-\sqrt{2}) = \sqrt{2}$, it is seen from the definitions of α and β, and the fact that $F(u(x_1)) > F(u(x_0))$, that (21) holds. □

Remark 8. Notice that the estimate (21) is by no means sharp. We will use the estimate to show that once a solution gets larger than $\sqrt{2}$ it will start oscillating, and the amplitude of the oscillations tends to infinity. For the EFK potential we have given the explicit estimate (21), but in general it suffices that $F(u)$ strictly decreases to $-\infty$ as $|u| \to \infty$. In this paper we do not need any information on the speed at which the solution tends to infinity, and therefore we are satisfied with this rather weak estimate. It can in fact be shown that if a solution of (10) obeys (20) at some $x_0 \in \mathbb{R}$, then the solution blows up in finite time (i.e., the maximal interval of existence for $x > x_0$ is finite) [HV]. •

Remark 9. The following symmetric counterpart of Lemma 9 holds. If there exists a point $x_0 \in \mathbb{R}$, such that

$$u(x_0) \le -\sqrt{2}, \qquad u'(x_0) = 0, \qquad u''(x_0) \ge 0, \qquad \text{and} \qquad u'''(x_0) \ge 0,$$

then, there exists a first critical point of u on (x_0, ∞), say y_0, and we have

$$u(y_0) > -u(x_0) \ge \sqrt{2}, \qquad u'(y_0) = 0, \qquad u''(y_0) < 0, \qquad \text{and} \qquad u'''(y_0) < 0.$$

Besides, $F(u(y_0)) < F(u(x_0))$, and an estimate similar of (21) holds. •

The next lemma implies that if a solution $u(x)$ obeys (20) then it becomes wildly oscillatory for $x > x_0$. The function $u(x)$ then has an infinite number of oscillations on the right-hand side of x_0 and the amplitude of these oscillations grows unlimited. The function sweeps from one side of the potential to the other.

Lemma 10. *For any $q \le 0$, let $u(x)$ be a solution of* (10)*. Suppose that there exists a $\xi_0 \in \mathbb{R}$ such that*

$$u(\xi_0) \ge \sqrt{2}, \qquad u'(\xi_0) = 0, \qquad u''(\xi_0) \le 0, \qquad \text{and} \qquad u'''(\xi_0) \le 0. \tag{23}$$

Then $u(x)$ has for $x \ge \xi_0$ an infinite, increasing sequence of local maxima $\{\xi_k\}_{k=0}^{\infty}$ and minima $\{\eta_k\}_{k=1}^{\infty}$, where $\xi_k < \eta_{k+1} < \xi_{k+1}$ for every $k \ge 0$. The extrema are ordered: $u(\xi_{k+1}) > -u(\eta_{k+1}) > u(\xi_k) \ge \sqrt{2}$, and $u(\xi_k) \to \infty$ as $k \to \infty$.

Proof. Combining Lemma 9 and Remark 9 we obtain the infinite sequences of local maxima and minima and the ordering $u(\xi_{k+1}) > -u(\eta_{k+1}) > u(\xi_k) \geq \sqrt{2}$ is immediate. Clearly $\{u(\xi_k)\}_{k=0}^{\infty}$ is an increasing sequence.

It also follows that $\{F(u(\xi_k))\}$ is a decreasing sequence, and we assert that $F(u(\xi_k)) \to -\infty$ and thus $u(\xi_k) \to \infty$ as $k \to \infty$. Suppose by contradiction that $\{F(u(\xi_k))\}$ is bounded, then $\{F(u(\eta_k))\}$ is bounded as well. Hence $u(x)$ is bounded for $x > \xi_0$. However, estimate (21) then ensures that $F(u(\xi_k))$ tends to $-\infty$ as $k \to \infty$, contradicting the assumption that $\{F(u(\xi_k))\}$ is bounded. □

Note that if $u(x)$ attains a maximum at $x = 0$ above the line $u = \sqrt{2}$ then (23) holds with $\xi_0 = 0$ for either $u(x)$ or $u(-x)$. The next lemma states our first a priori bound.

Lemma 11. *For any $q \leq 0$, let $u(x)$ be a bounded solution of (10). Then $|u(x)| < \sqrt{2}$ for all $x \in \mathbb{R}$.*

Proof. We argue by contradiction and thus suppose that $u(x) \geq \sqrt{2}$ for some $x \in \mathbb{R}$. Since $u(x)$ is bounded, we infer from Lemma 3 that $u(x)$ attains a local maximum larger then $\sqrt{2}$, say at $x_0 \in \mathbb{R}$. By translation invariance we may assume that $x_0 = 0$. Clearly $u(0) \geq \sqrt{2}$, $u'(0) = 0$ and $u''(0) \leq 0$. Without loss of generality we may assume that $u'''(0) \leq 0$ (otherwise we switch to $v(x) \equiv u(-x)$, which also is a bounded solution of (10)). We are now in the setting of Lemma 10. Thus $u(x)$ is unbounded if $u(x_0) \geq \sqrt{2}$ for some $x_0 \in \mathbb{R}$. The case where $u(x_0) \leq -\sqrt{2}$ for some $x_0 \in \mathbb{R}$ is excluded in a similar manner. □

Remark 10. This method of obtaining an a priori estimate on all bounded solutions is applicable to a class of non-symmetric potentials which strictly decrease to $-\infty$ as $|u| \to \infty$. In that case we can find $-\infty < a \leq b < \infty$ such that

$$F(a) = F(b),$$
$$F(u) > F(a) = F(b) \qquad \text{for all } u \in (a, b),$$
$$F'(u) > 0 \text{ for all } u < a \quad \text{ and } \quad F'(u) < 0 \text{ for all } u > b.$$

Then every bounded solutions $u(x)$ of (1) for $\gamma > 0$ satisfies $a \leq u(x) \leq b$.

For the potential in (5) a lower bound can be found in an analogous manner. In general, if for some $b \in \mathbb{R}$

$$F(u) > F(b) \text{ for all } u < b \qquad \text{and} \qquad F'(u) < 0 \text{ for all } u > b,$$

then b is an upper bound on the set of bounded solutions. •

We are now going to use the maximum principle to get sharper a priori bounds for the EFK equation. The following lemma shows that if a bounded solution has two local minima below the line $u = 1$, then the solution stays below this line between these minima. To shorten notation, we will write $u(\infty)$ instead of $\lim_{x\to\infty} u(x)$.

Lemma 12. *For any $q \le -\sqrt{8}$, let $u(x)$ and $v(x)$ be solutions of (10), and let $-\infty \le x_a < x_b \le \infty$. Suppose that $u(x_a)$, $u(x_b) \le 1$ and $u''(x_a)$, $u''(x_b) \ge 0$. If $u(x) \ge -2$ for $x \in (x_a, x_b)$, then either $u \equiv 1$ or $u(x) < 1$ on (x_a, x_b).*

Proof. The proof is based on repeated application of the maximum principle. Let $v(x) \equiv u(x) - 1$. The function $v(x)$ obeys, for $x \in (x_a, x_b)$,

$$v^{(iv)} + qv'' + 2v = u - u^3 + 2(u-1) = -(u+2)(u-1)^2 \le 0,$$

where the inequality is ensured by the hypothesis that $u(x) \ge -2$. Now we define $w(x) \equiv v''(x) - \lambda v(x)$. From the definition of λ and μ we see that

$$w'' - \mu w = v^{(iv)} - (\lambda + \mu)v'' + \lambda\mu v = v^{(iv)} + qv'' + 2v.$$

By the hypotheses on u in x_a and x_b we find that $w(x)$ obeys the system

$$\begin{cases} w'' - \mu w = -(u+2)(u-1)^2 \le 0 & \text{on } (x_a, x_b), \\ w(x_a) = v''(x_a) - \lambda v(x_a) \ge 0, \\ w(x_b) = v''(x_b) - \lambda v(x_b) \ge 0. \end{cases}$$

By the maximum principle we have that $w(x) \ge 0$ on (x_a, x_b). Finally, $v(x)$ obeys the system

$$\begin{cases} v'' - \lambda v \equiv w \ge 0 & \text{on } (x_a, x_b), \\ v(x_a) = u(x_a) - 1 \le 0, \\ v(x_b) = u(x_b) - 1 \le 0. \end{cases}$$

By the strong maximum principle we obtain that either $v \equiv 0$, or $v(x) < 0$ on (x_a, x_b). This proves Lemma 12. $\square$

Remark 11. The symmetric counterpart of the previous lemma shows that if a solution $u(x)$ of (10) has two local maxima above -1 and $u(x) \le 2$ between the maxima, then we have $u(x) > -1$ between the maxima. $\bullet$

Note that for bounded solutions the condition that $-2 \le u(x) \le 2$ is automatically satisfied (Lemma 11). For heteroclinic solutions the previous lemma and remark (with $x_a = -\infty$ and $x_b = +\infty$) imply that every heteroclinic solution is uniformly bounded from above by 1 and from below by -1.

For the case of a general bounded solution, let us look at the consecutive extrema for $x > 0$ (and similarly for $x < 0$) of a bounded solution $u(x)$. Suppose that u is a bounded solution which does not tend to a limit. In that case we will prove that arbitrarily large negative x_a and arbitrarily large positive x_b can be found, such that $u(x_a)$ and $u(x_b)$ are local minima below the line $u = 1$, and thus the conditions in Lemma 12 are satisfied. We will need the following lemma, which has two related consequences. Firstly, it shows that if $u(x)$ has a maximum above the line $u = 1$, then the first minimum on at least one of the sides of this maximum lies below the line $u = 1$. Secondly, we infer that a solution does not have two consecutive minima above the line $u = 1$.

Lemma 13. *For any* $q \leq 0$ *let* $u(x)$ *be a solution of* (10). *Suppose that there exists a point* $x_0 \in \mathbb{R}$*, such that*

$$u(x_0) > 1, \qquad u'(x_0) = 0, \qquad u''(x_0) \leq 0, \qquad \text{and} \qquad u'''(x_0) \leq 0.$$

Then there exists a $y_0 \in (x_0, \infty)$ *such that* $u(y_0) = 1$ *and* $u'(x) < 0$ *on* $(x_0, y_0]$.

Proof. The proof is along the same lines as the proof of Lemma 9. Since $f(u(x_0)) < 0$ and $u''(x_0) \leq 0$, we see that $u^{(iv)}(x_0) = -qu''(x_0) + f(u(x_0)) < 0$ and thus $u''' < 0$ in a right neighbourhood of x_0. We now conclude that

$$x_1 \stackrel{\text{def}}{=} \sup\bigl\{x > x_0 \bigm| u''' < 0 \text{ on } (x, x_0)\bigr\}$$

is well-defined. By Remark 7 we conclude that x_1 is finite. Since $u''' < 0$ on (x_0, x_1), we have that $u''(x_1) < u''(x_0) \leq 0$. Using the energy identity and the facts that $u'''(x_1) = 0$ and $u'(x_0) = 0$, we obtain

$$F(u(x_0)) = \mathcal{E}[u] - \frac{1}{2}(u''(x_0))^2 > \mathcal{E}[u] - \frac{1}{2}(u''(x_1))^2 > F(u(x_1)).$$

It follows from the definition of x_1 and the initial data at x_0, that $u''' < 0$, $u'' < 0$ and $u' < 0$ on (x_0, x_1), and so $u(x_1) < u(x_0)$. It is easily seen from the shape of the potential, that $F(s) > F(u(x_0))$ for all $s \in [1, u(x_0))$, so that $u(x_1) < 1$. This proves the lemma. □

We can now apply Lemma 12 to prove Theorem 4. We will only prove that $u(x) < 1$ for all $x \in \mathbb{R}$ (the proof of the assertion that $u(x) > -1$, is analogous). We argue by contradiction. Suppose there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) \geq 1$. We will show that there exists a constant $x_a \in [-\infty, x_0)$ such that

$$u(x_a) \leq 1, \qquad u'(x_a) = 0 \qquad \text{and} \qquad u''(x_a) \geq 0. \tag{24}$$

Similarly we obtain a constant $x_b \in (x_0, \infty]$ such that

$$u(x_b) \leq 1, \qquad u'(x_b) = 0 \qquad \text{and} \qquad u''(x_b) \geq 0.$$

From Lemmas 11 and 12 we then conclude that $u(x) < 1$ on (x_a, x_b), which contradicts the fact that $u(x_0) \geq 1$. We will only prove the existence of x_a. The proof of the existence of x_b is similar. By Remark 7 we see that either $u(x)$ has an infinite number of local minima on the left-hand side of x_0 or $u(x)$ tends to a limit monotonically as $x \to -\infty$. In the second case Lemma 3 guarantees that $x_a = -\infty$ satisfies (24). In the first case we prove that at least one of the minima on the left-hand side of x_0 lies below the line $u = 1$. By contradiction, suppose there exist two consecutive local minima y_0 and y_1 above the line $u = 1$ ($y_0 < y_1 < x_0$). Then there clearly exists a local maximum $x_1 \in (y_0, y_1)$. By translation invariance we may assume that $x_1 = 0$. We now have that $u(0) > 1$, $u'(0) = 0$ and $u''(0) \leq 0$. We now first assume that

$u'''(0) \leq 0$. Then we are in the setting of Lemma 13 and we conclude that $u(y_1) < 1$, thus we have reached a contradiction. On the other hand, if $u'''(0) \geq 0$, we switch to $v(x) \equiv u(-x)$ and, by the same argument, we conclude that $u(y_0) < 1$. This completes the proof of Theorem 4.

Remark 12. The method employed in this section to obtain a better a priori bound from a weaker one has a nice geometrical interpretation, which makes it easy to apply the method to (1) with general $f(u)$ (a similar idea is used in [PRT]). Let us assume that we have an a priori bound, i.e., for some $\gamma > 0$ all bounded solutions of (1) are in $\mathcal{B}(a, b)$. Suppose now that we can find constants $A > a$ and $0 < \Omega \leq \frac{1}{4\gamma}$ (i.e., $\gamma \in (0, \frac{1}{4\Omega}]$), such that

$$-\Omega(u - A) \leq f(u) \qquad \text{for all } u \in [a, b],$$

which means that the line $-\Omega(u - A)$ stays below $f(u)$ on the interval under consideration. Then A is a new (improved) lower bound on the set of bounded solutions.

Similarly, when we can find constants $B < b$ and $0 < \Omega \leq \frac{1}{4\gamma}$, such that

$$-\Omega(u - B) \geq f(u) \qquad \text{for all } u \in [a, b],$$

then B is a new (improved) upper bound on the set of bounded solutions. Remark that a new upper bound might allow us to find an improved lower bound, and vice versa. •

5 Conclusions for the EFK equation

We first make the observation that every bounded solution (except $u \equiv \pm 1$) has a zero.

Lemma 14. *For any $q \leq -\sqrt{8}$, let $u(x) \not\equiv 1$ be a bounded solution of (10). Then $u(x)$ has at least one zero.*

Proof. Suppose $u(x)$ does not have a zero. We may assume that $u(x) > 0$ for all $x \in \mathbb{R}$. Either $u(x)$ has a local minimum in the range $(0, 1)$, or $u(x)$ is homoclinic to 0. The latter would imply that $\mathcal{E}[u] = -\frac{1}{4}$, and besides $u(x)$ must attain a local maximum in the range $(0, 1)$. It is easily seen from the energy identity that these two observations lead to a contradiction. We complete the proof by showing that $u(x)$ cannot have a local minimum in the range $(0, 1)$.

Suppose that after translation we have

$$u(0) \in (0, 1), \qquad u'(0) = 0 \qquad \text{and} \qquad u''(0) \geq 0.$$

We may suppose that in addition $u'''(0) \geq 0$ (otherwise we switch to $v(x) = u(-x)$). In a manner that is completely analogous to the proof of Lemma 13, it can be shown that there exists a $y_0 \in (0, \infty)$ such that $u(y_0) = 1$. This contradicts Theorem 4. □

Lemma 2 shows that the only possible bounded solutions are equilibrium points, monotone heteroclinic solutions, homoclinic solutions with a unique extremum and periodic solutions with a unique maximum and minimum. Lemma 14 shows that any non-constant bounded solution has a zero, which means that except for the equilibrium points and the decreasing kink, every bounded solution has a zero at which it has a positive slope. Excluding the equilibrium points and the decreasing kink from these considerations, we conclude from Theorem 1 that no two solutions can have the same positive slope at their zeros, and from Theorem 2 that the solution with the larger slope has the higher energy. From these considerations we draw the following conclusions, to finish the proof of Theorem 5.

- Starting at low energies, it follows from the energy identity that solutions which lie in the levels $E < -\frac{1}{4}$ have no extrema in the range $[-\sqrt{2}, \sqrt{2}]$, and thus are unbounded.
- Similarly, for $E = -\frac{1}{4}$ the equilibrium solution $u \equiv 0$ is the only bounded solution, since any other would have a zero and this would contradict Theorem 2.
- There are no equilibrium points (and thus no connecting orbits) in the energy levels $E \in (-\frac{1}{4}, 0)$. Hence, it follows immediately from Lemma 14 and Theorem 2 that in each of these energy levels the periodic solution which have been proved to exist in [PT4], is the only bounded solution.
- For the energy level $E = 0$ we derive that beside the equilibrium points $u \equiv \pm 1$, the only bounded solutions are a unique monotonically increasing and a unique monotonically decreasing heteroclinic solution, of which the existence has been proved in [PT1]. In particular there exist no homoclinic connections to ± 1.
- Finally, there are no equilibrium points and thus no connecting orbits in the energy levels $E > 0$. Periodic solutions in these energy levels cannot have maxima smaller than 1 by Theorem 2 (comparing them to the increasing kink). Therefore, Theorem 4 excludes the existence of periodic solutions for energies $E > 0$.

We recall how crucially these arguments depend on the real-saddle character of the equilibrium points. Both Theorem 4 and the Comparison Lemma do not hold when $\gamma > \frac{1}{8}$. The variety of solutions which exist for $\gamma > \frac{1}{8}$, shows that this bound is sharp.

Up to now, we did not use in an essential manner the invariance of (10) under the transformation $u \to -u$. This invariance can be used to obtain further information on the shape of bounded solutions of (9). The next lemma states that every bounded solution is antisymmetric with respect to its zeros.

Lemma 15. *For any* $\gamma \in (0, \frac{1}{8}]$, *let* $u(x)$ *be a bounded solution of* (9). *Suppose that* $u(x_0) = 0$ *for some* $x_0 \in \mathbb{R}$. *Then* $u(x_0 + x) = u(x_0 - x)$ *for all* $x \in \mathbb{R}$.

Proof. The proof is analogous to the proof of Lemma 2. Without loss of generality we may assume that $x_0 = 0$. Define $v(x) = -u(-x)$. By the symmetry of (9), $v(x)$ is also a bounded solution of (9). Clearly $u(0) = v(0)$ and $u'(0) = v'(0)$. From Theorem 1 we conclude that $u(x) \equiv v(x)$. $\square$

We already saw that the periodic solutions of (9) can be parametrised by the energy. The next lemma shows that they can also be parametrised by their period.

Lemma 16. *Let* $\gamma \in (0, \frac{1}{8}]$*. Then the periodic solutions of* (9) *can be parametrised by the period* $L \in (L_0, \infty)$*, where*

$$L_0 \stackrel{\text{def}}{=} 2\pi \sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}\,.$$

Proof. By Lemma 15 any periodic solution, of period L, is antisymmetric with respect to its zeros, and thus has exactly two zeros on the interval $[0, L)$. Using a variational method, it has been proved in [PTV] that for every period $L \in (L_0, \infty)$ there exists at least one periodic solution $u(x)$ of (9) with exactly two zeros on the interval $[0, L)$. Therefore, we only need to show the uniqueness of these solutions.

We argue by contradiction. Suppose there are two periodic solutions $u_1 \not\equiv u_2$ of (9) with period L. By Lemmas 2 and 15 we have that (after translation), for $i = 1, 2$

$$u_i'(0) = 0, \qquad u_i(\pm\tfrac{L}{4}) = 0, \qquad \text{and} \qquad u_i(x) > 0 \qquad \text{for } x \in (-\tfrac{L}{4}, \tfrac{L}{4}).$$

Clearly, both solutions are increasing on $(-\frac{L}{4}, 0)$.

We see from Theorem 1 that $u_1'(-\frac{L}{4}) \neq u_2'(-\frac{L}{4})$, and without loss of generality we may assume that $u_1'(-\frac{L}{4}) > u_2'(-\frac{L}{4})$. Let

$$x_0 \stackrel{\text{def}}{=} \sup\bigl\{x > -\tfrac{L}{4} \bigm| u_2 < u_1 \text{ on } (-\tfrac{L}{4}, x)\bigr\}.$$

We assert that $x_0 = \frac{L}{4}$. Suppose that $x_0 < \frac{L}{4}$. Then $x_0 < 0$ since the solutions are symmetric with respect to $x = 0$. However, u_1 and u_2 are increasing on $(-\frac{L}{4}, x_0)$, and $u_1(-\frac{L}{4}) = u_2(-\frac{L}{4})$ and $u_1(x_0) = u_2(x_0)$. This implies that there exist x_1 and x_2 in $(0, x_0)$ such that $u_1(x_1) = u_2(x_2)$ and $u_1'(x_1) = u_2'(x_2)$, contradicting Theorem 1.

Hence, we have established that

$$u_1(x) > u_2(x) > 0 \qquad \text{for } x \in (-\tfrac{L}{4}, \tfrac{L}{4}). \tag{25}$$

When we multiply the differential equation of u_1 by u_2 and integrate over $(-\frac{L}{4}, \frac{L}{4})$, then we obtain

$$\begin{aligned} 0 &= \int_{-\frac{L}{4}}^{\frac{L}{4}} \Bigl\{u_2(-\gamma u_1^{(iv)} + u_1'' + u_1 - u_1^3)\Bigr\}\,dx \\ &= \int_{-\frac{L}{4}}^{\frac{L}{4}} \Bigl\{u_1(-\gamma u_2^{(iv)} + u_2'' + u_2) - u_2 u_1^3\Bigr\}\,dx. \end{aligned}$$

Here we have used partial integration and the fact that $u_i''(\pm\frac{L}{4}) = 0$ (by Lemma 15). Since u_2 is a solution of (9), this implies that

$$0 = \int_{-\frac{L}{4}}^{\frac{L}{4}} \{u_1 u_2 (u_2^2 - u_1^2)\}\,dx,$$

which contradicts (25). $\square$

6 Transversality

The unique monotonically increasing heteroclinic solution $v(x)$ of (10) for $q \le -\sqrt{8}$ is antisymmetric by Lemma 15. Removing the translational invariance by taking the unique zero of $v(x)$ at the origin, we have

$$v(0) = 0, \qquad v'(0) > 0 \qquad \text{and} \qquad v''(0) = 0.$$

In this section we will apply a technique similar to the one in [BCT] to prove that $v(x)$ is a transverse intersection of $W^u(-1)$ and $W^s(+1)$ in the zero energy set (here we write $W^{u,s}(\pm 1)$ instead of $W^{u,s}(\pm 1, 0, 0, 0)$). If the intersection would not be transversal, then it follows from the symmetry of the potential that there are only two possibilities. We will exclude these possibilities with the help of the Comparison Lemma and some delicate and rather technical estimates. When the potential is not symmetric we still expect the intersection to be transversal, but a proof along the same lines seems more involved.

The following lemma provides a bound the orbits $u(x)$ in the stable manifold of $+1$ that lie close to the kink $v(x)$. This bound will be useful later on, since it enables the application of the Comparison Lemma to these solutions.

Lemma 17. *For any $q \le -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solutions of (10) with its zero at the origin. Suppose that $u(x)$ is a solution of (10) such that $u \in W^s(+1)$, and*

$$|u^{(i)}(x) - v^{(i)}(x)| < \delta \qquad \text{for } i = 0, 1, 2, 3 \text{ and } x \in [0, \infty). \tag{26}$$

Then for $\delta > 0$ sufficiently small we have $|u(x)| < 1$ for all $x > 0$.

Proof. Recall that $v(x)$ increases monotonically from -1 to $+1$. The fact that $u(x) > -1$ on $[0, \infty)$ is immediate from (26). It is easily seen that the monotone kink $v(x)$ obeys the system

$$\begin{cases} v'''' + qv'' = v - v^3 < 0 \le 0 & \text{on } (-\infty, 0), \\ v''(0) = 0, \\ v''(-\infty) = 0. \end{cases}$$

Since $q \le 0$, it follows from the strong maximum principle that $v''(x) > 0$ on $(-\infty, 0)$, and in particular $v''(-1) > 0$. Let $u(x)$ obey (26), then this implies that

$$u''(-1) > 0, \qquad u(-1) < 1 \qquad \text{and} \qquad u(x) > -2 \text{ on } [-1, \infty),$$

for δ sufficiently small. Besides, $u(\infty) = 1$ and $u''(\infty) = 0$. It now follows from Lemma 12 that $u(x) < 1$ on $[-1, \infty)$. $\square$

We now start the proof of Theorem 6. We emphasise that we assume that the potential F is symmetric, which greatly reduces the number of possibilities that we have to check in order to conclude that the intersection of $W^u(-1)$ and $W^s(+1)$ is transversal.

For any $q \leq -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solution of (10). Since $v(x)$ is antisymmetric by Lemma 15, we have that

$$v(0) = 0, \qquad v'(0) > 0 \qquad \text{and} \qquad v''(0) = 0,$$

and by Lemma 4 we have $v'''(0) - \lambda v'(0) < 0$. Besides, v lies in the zero-energy manifold

$$v'v''' - \frac{1}{2}(v'')^2 + \frac{q}{2}(v')^2 - F(v) = 0,$$

where $F(v) = -\frac{1}{4}(v^2-1)^2$. Therefore

$$v'''(0) + qv'(0) = \frac{q}{2}v'(0) - \frac{1}{4v'(0)} = (-\lambda - C)v'(0) - \frac{1}{4v'(0)}, \tag{27}$$

where $C = \sqrt{\left(\frac{q}{2}\right)^2 - 2} \geq 0$. The tangent space to the zero energy manifold at the point $P = (0, v'(0), 0, v'''(0))$ is

$$\big(0, u'''(0) + qu'(0), 0, u'(0)\big)^{\perp} \subset \mathbb{R}^4.$$

Now the tangent spaces to the two-dimensional manifolds $W^u(-1)$ and $W^s(+1)$ at this point both contain the vector

$$X = \big(v'(0), 0, v'''(0), 0\big), \tag{28}$$

because of the differential equation.

Let us suppose, seeking a contradiction, that these stable and unstable manifolds do not intersect transversally in the zero energy set. Then their tangent spaces, which are two-dimensional, coincide. We denote this two-dimensional tangent space by T_P. Because of the symmetry of F and reversibility, $(\alpha, \beta, \gamma, \delta)$ lies in $W^u(-1)$ if and only if $(-\alpha, \beta, -\gamma, \delta)$ lies in $W^s(+1)$. It then follows that

$$(\alpha, \beta, \gamma, \delta) \in T_P \;\Leftrightarrow\; (-\alpha, \beta, -\gamma, \delta) \in T_P. \tag{29}$$

This symmetry relation implies that there are only two possibilities for T_P. Namely, let T_P be spanned by $Y = (\alpha, \beta, \gamma, \delta)$ and (28). We may assume that $\alpha = 0$ (replacing Y by $Y - \frac{\alpha}{v'(0)}X$). If $\beta \neq 0$, then we see from (29) that $\gamma = 0$ (otherwise $(v'(0), 0, v'''(0), 0)$, $(0, \beta, \gamma, \delta)$ and $(0, \beta, -\gamma, \delta)$ would be three linearly independent vectors in T_P). Besides, δ is directly related to β since $T_P \in \big(0, u'''(0) + qu'(0), 0, u'(0)\big)^{\perp}$. On the other hand, if $\beta = 0$, then also $\delta = 0$. Thus, we are left with two possibilities:

$$\begin{aligned} \text{case A:} \quad T_P &= \big\{(\xi, 0, \eta, 0) \;\big|\; (\xi, \eta) \in \mathbb{R}^2\big\}, \qquad \text{or} \\ \text{case B:} \quad T_P &= \big\{\big(\xi v'(0), -\eta v'(0), \xi v'''(0), \eta(v'''(0) + qv'(0))\big) \;\big|\; (\xi, \eta) \in \mathbb{R}^2\big\}. \end{aligned}$$

Note that the symmetry of the potential has reduced the number of possibilities enormously.

In case A let $\xi = 1$, $\eta = 1 + \lambda$ and consider the point on $W^s(+1)$ given by

$$(u, u', u'', u''')(0) = \big(\varepsilon + O(\varepsilon^2), v'(0) + O(\varepsilon^2), (1+\lambda)\varepsilon + O(\varepsilon^2), v'''(0) + O(\varepsilon^2)\big).$$

Moreover, it should be clear that for ε small enough the conditions of Lemma 17 are satisfied, so that $|u(x)| < 1$ on $[0, \infty)$. We will deal with this case in Lemma 19, where we show that under the present conditions, $u(x) \notin W^s(+1)$, which contradicts the assumption.

Now suppose that case B holds and let $\xi = 0$, $\eta = -1$. Then there is a point $(u, u', u'', u''')(0)$ on $W^s(+1)$ of the form

$$\big(O(\varepsilon^2), v'(0) + \varepsilon v'(0) + O(\varepsilon^2), O(\varepsilon^2), v'''(0) - \varepsilon(v'''(0) + qv'(0)) + O(\varepsilon^2)\big).$$

Now

$$(u' - v')(0) = \varepsilon v'(0) + O(\varepsilon^2),$$

and, using (27),

$$\begin{aligned}(u''' - v''')(0) &= -\varepsilon(v'''(0) + qv'(0)) + O(\varepsilon^2) \\ &= \varepsilon(\lambda + C)v'(0) + \frac{\varepsilon}{4v'(0)} + O(\varepsilon^2) \\ &= \lambda(u' - v')(0) + \varepsilon C v'(0) + \frac{\varepsilon}{4v'(0)} + O(\varepsilon^2).\end{aligned}$$

Because $C \geq 0$ we infer that

$$(u''' - \lambda u')(0) - (v''' - \lambda v')(0) \geq \varepsilon\left(Cv'(0) + \frac{1}{4v'(0)}\right) + O(\varepsilon^2).$$

Besides, it should be clear that for ε small enough the conditions of Lemma 17 are satisfied, so that $|u(x)| < 1$ on $[0, \infty)$. We will deal with this case in Lemma 18, where we show that under the present conditions, $u(x) \notin W^s(+1)$, which contradicts the assumption.

We now prove two technical lemmas (adopted from [BCT] to the case of an antisymmetric heteroclinic orbit) to exclude the two possibilities which could occur if the intersection of $W^u(-1)$ and $W^s(+1)$ is not transversal. We show that in both case A and case B the initial data of u and v are such that for some small positive x, we arrive in the situation of the Comparison Lemma. We then conclude that u cannot be in the stable manifold of 1.

The next lemma deals with case B.

Lemma 18. *For any $q \leq -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solutions of (10) with its zero at the origin. Suppose that $u(x)$ is a solution of (10) with $|u(x)| < 1$ on $[0, \infty)$, satisfying (for some $\varepsilon > 0$)*

$$k\varepsilon \geq (u''' - \lambda u')(0) - (v''' - \lambda v')(0) \geq \alpha\varepsilon \qquad \text{and} \qquad k\varepsilon \geq (u'(0) - v'(0)) \geq \varepsilon,$$

and

$$|u(0)| + |u''(0)| \leq \beta\varepsilon^2,$$

where $k, \alpha, \beta > 0$ are constant. Then, for ε sufficiently small, $u(0) \notin W^s(+1)$.

Proof. The solution of v exists on $[0,\infty)$ and the initial data of u are ε-close to those of v. Therefore there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0,\varepsilon_0)$ the function u and its derivatives of all orders exist and are uniformly bounded on $[0,1]$, independent of $\varepsilon \in (0,\varepsilon_0)$ and for all u satisfying the assumptions. Consequently, by Taylor's theorem we infer that for some $M > 0$ and for all $x \in [0,1]$

$$\begin{aligned}
u(x) - v(x) &\geq u(0) - v(0) + \{u'(0) - v'(0)\}x \\
&\quad + \tfrac{1}{2}\{u''(0) - v''(0)\}x^2 - Mx^3 \\
&\geq -\beta\varepsilon^2 + \varepsilon x - \tfrac{1}{2}\beta\varepsilon^2 x^2 - Mx^3. \\
u'(x) - v'(x) &\geq u'(0) - v'(0) + \{u''(0) - v''(0)\}x - Mx^2 \\
&\geq \varepsilon - \beta\varepsilon^2 x - Mx^2. \\
(u'' - \lambda u)(x) - (v'' - \lambda v)(x) &\geq (u'' - \lambda u)(0) - (v'' - \lambda v)(0) \\
&\quad + \{(u''' - \lambda u')(0) - (v''' - \lambda v')(0)\}x \\
&\quad + \tfrac{1}{2}\{(u'''' - \lambda u'')(0) - (v'''' - \lambda v'')(0)\}x^2 - Mx^3 \\
&\geq -(1+\lambda)\beta\varepsilon^2 + \alpha\varepsilon x - \tfrac{1}{2}K\beta\varepsilon^2 x^2 - Mx^3. \\
(u''' - \lambda u')(x) - (v''' - \lambda v')(x) &\geq (u''' - \lambda u')(0) - (v''' - \lambda v')(0) \\
&\quad + \{(u'''' - \lambda u'')(0) - (v'''' - \lambda v'')(0)\}x - Mx^2 \\
&\geq \alpha\varepsilon - K\beta\varepsilon^2 x - Mx^2.
\end{aligned}$$

Here we have used the fact that for some constant $K > 0$,

$$|u''''(0) - \lambda u''(0)| = |(-q-\lambda)u''(0) + u(0) - u^3(0) \leq K\beta\varepsilon^2.$$

Let $\tilde{\alpha} = \min\{1,\alpha\}$ and $\tilde{K} = \max\{1,K\}$, and we define

$$\Gamma(\varepsilon) = \sqrt{\varepsilon}\left\{\sqrt{\frac{\tilde{\alpha}}{M}} - \frac{\tilde{K}\beta\varepsilon^{3/2}}{2M}\right\}.$$

Then, on $[0,\Gamma(\varepsilon)]$, we have

$$u'(x) - v'(x) \geq 0 \qquad \text{and} \qquad (u''' - \lambda u')(x) - (v''' - \lambda v')(x) \geq 0.$$

We now introduce $\tau(\varepsilon) = \varepsilon^{2/3}$. Then $\tau(\varepsilon) \in [0,\Gamma(\varepsilon)] \cap [0,1]$ for $\varepsilon > 0$ sufficiently small. We obtain that

$$(u-v)(\tau(\varepsilon)) \geq -\beta\varepsilon^2 + \varepsilon^{5/3} - \tfrac{1}{2}\beta\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for $\varepsilon > 0$ sufficiently small, and

$$(u'' - \lambda u)(\tau(\varepsilon)) - (v'' - \lambda v)(\tau(\varepsilon)) \geq -(1+\lambda)\beta\varepsilon^2 + \alpha\varepsilon^{5/3} - \tfrac{1}{2}K\beta\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for $\varepsilon > 0$ sufficiently small.

We can now apply the Comparison Lemma to conclude that $u(x)$ does not tend to 1 as $x \to \infty$, which proves the lemma. $\square$

The following lemma excludes case A, and thus completes the proof of Theorem 6.

Lemma 19. *For any $q \leq -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solutions of (10) with its zero at the origin. Suppose that $u(x)$ is a solution of (10) with $|u(x)| < 1$ on $[0, \infty)$, satisfying (for some $\varepsilon > 0$)*

$$k\varepsilon \geq u''(0) - \lambda u(0) \geq \alpha\varepsilon \qquad \text{and} \qquad k\varepsilon \geq u(0) \geq \varepsilon,$$

and

$$|u'(0) - v'(0)| + |u'''(0) - v'''(0)| \leq \beta\varepsilon^2,$$

where $k, \alpha, \beta > 0$ are constant. Then, for ε sufficiently small, $u(0) \notin W^s(+1)$.

Proof. We proceed as in the proof of Lemma 18. We find, by Taylor's theorem, that for some $M > 0$, $K > 0$ and $x \in [0, 1]$,

$$\begin{aligned} u(x) - v(x) &\geq \varepsilon - \beta\varepsilon^2 x - Mx^2 \\ u'(x) - v'(x) &\geq -\beta\varepsilon^2 + (\alpha + \lambda)\varepsilon x - \tfrac{1}{2}\beta\varepsilon^2 x^2 - Mx^3 \\ (u'' - \lambda u)(x) - (v'' - \lambda v)(x) &\geq \alpha\varepsilon - (1 + \lambda)\beta\varepsilon^2 x - Mx^2 \end{aligned}$$

and

$$\begin{aligned} (u''' - \lambda u')(x) - (v''' - \lambda v')(x) &\geq (u''' - \lambda u')(0) - (v''' - \lambda v')(0) \\ &\quad + \left\{(u^{(iv)} - \lambda u'')(0) - (v^{(iv)} - \lambda v'')(0)\right\} x \\ &\quad + \tfrac{1}{2}\left\{(u^{(v)} - \lambda u''')(0) - (v^{(v)} - \lambda v''')(0)\right\} x^2 - Mx^3 \\ &\geq -2\beta\varepsilon^2 + (2 + \mu\alpha)\varepsilon x - \tfrac{1}{2}K\varepsilon^2 x^2 - Mx^3. \end{aligned}$$

Here we have used the following facts. Firstly, $v^{(iv)}(0) = 0$ by (10) and

$$\begin{aligned} (u^{(iv)} - \lambda u'')(0) &= \mu(u'' - \lambda u)(0) + 3u(0) - u^3(0) \\ &\geq \mu\alpha\varepsilon + 3\varepsilon - k^3\varepsilon^3 \\ &\geq (2 + \mu\alpha)\varepsilon \end{aligned}$$

for ε sufficiently small. Secondly, differentiating (10), we obtain

$$u^{(v)} + qu''' + u'(3u^2 - 1) = 0,$$

from which we deduce that

$$\begin{aligned} (u^{(v)} - \lambda u''')(0) - (v^{(v)} - \lambda v''')(0) &= \mu(u''' - \lambda u')(0) - \mu(v''' - \lambda v')(0) \\ &\quad + 3(u'(0) - v'(0)) - 3u'(0)u^2(0) \\ &\geq -\mu(1 + \lambda)\beta\varepsilon^2 - 3\beta\varepsilon^2 - 6v'(0)k^2\varepsilon^2 \equiv -K\varepsilon^2, \end{aligned}$$

since $u'(0) \leq v'(0) + \beta\varepsilon^2 \leq 2v'(0)$, for ε sufficiently small.

Let $\tilde{\alpha} = \min\{1, \alpha\}$ and we define

$$\Gamma(\varepsilon) = \sqrt{\varepsilon}\left\{\sqrt{\frac{\tilde{\alpha}}{M}} - \frac{(1+\lambda)\beta\varepsilon^{3/2}}{2M}\right\}.$$

Then, on $[0, \Gamma(\varepsilon)]$, we have

$$u(x) - v(x) \geq 0 \qquad \text{and} \qquad (u'' - \lambda u)(x) - (v'' - \lambda v)(x) \geq 0.$$

If $\tau(\varepsilon) = \varepsilon^{2/3}$, then $\tau(\varepsilon) \in [0, \Gamma(\varepsilon)] \cap [0, 1]$ for $\varepsilon > 0$ sufficiently small and

$$(u' - v')(\tau(\varepsilon)) \geq -\beta\varepsilon^2 + (\alpha + \lambda)\varepsilon^{5/3} - \tfrac{1}{2}\beta\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for ε sufficiently small, and

$$(u'' - \lambda u)(\tau(\varepsilon)) - (v'' - \lambda v)(\tau(\varepsilon)) \geq -2\beta\varepsilon^2 + (2 + \mu\alpha)\varepsilon^{5/3} - \tfrac{1}{2}K\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for ε sufficiently small.

We can now apply the Comparison Lemma to conclude that $u(x)$ does not tend to 1 as $x \to \infty$, which proves the lemma. □

Remark 13. The special symmetry of $u - u^3$ has enabled us to prove that the heteroclinic solution is transversal. For general $f(u)$ transversality of heteroclinic solutions is much harder to check. However, for homoclinic solutions this difficulty does not arise, since every homoclinic solution (for $\gamma \in (0, \frac{1}{4\omega}]$) is symmetric with respect to its extremum. We will give an outline of the proof that every homoclinic solution is a transversal intersection.

Without loss of generality we may assume that $v(x)$ is a positive homoclinic solution of (1) to 0 with a unique maximum at $x = 0$. As usual, we suppose that $\gamma \in (0, \frac{1}{4\omega(0,v(0))}]$. The method in [BCT] for homoclinic solutions can be extended to general $f(u)$, as was done above for heteroclinic solutions. To be able to apply the Comparison Lemma to $v(x)$ a solution in $W^s(0)$ close to $v(x)$, we need a very mild assumption on $f(u)$, but only in a special case (when $\gamma = \frac{1}{4\omega(0,v(0))}$, then we need that $f'(u) \neq -\omega(0, v(0))$ in some left neighbourhood of $u = 0$). The only fairly specific condition in the rest of the proof is that $\frac{f(v(0))}{-v''(0)} > \lambda$, which follows directly from Remark 6. •

7 Stability of the kink

In this section we look at the stability of the kink for the EFK equation (12) and prove Theorem 7. To fix ideas, for $\gamma \leq \frac{1}{8}$ let $v(x)$ be the unique monotonically increasing heteroclinic solution of (9), such that $v(0) = 0$ (removing the translational invariance). The existence of this solution can be proved by a shooting method [PT1], but it can also be found as the minimiser of the functional

$$J[u] = \int_{\mathbb{R}} \left\{\frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 + \frac{1}{4}(u^2 - 1)^2\right\} dx.$$

The minimum is taken over all functions $u(x)$ with $u - \chi \in H^2(\mathbb{R})$, where $\chi \in C^\infty(\mathbb{R})$ is an antisymmetric function such that such that $\chi(x) = -1$ for $x \leq -1$. (see [KKV, PTV]).

The minimising property of the kink $v(x)$ and its transversality in the zero energy set allow us to conclude that for $\gamma \leq \frac{1}{8}$ the kink is asymptotically stable in $H^1(\mathbb{R})$. Another possible choice is to work in the space of bounded uniformly continuous functions. The analysis below applies to both function spaces.

To study the stability of the kink, we write $u(x,t) = v(x) + \phi(x,t)$. The differential equation for the perturbation $\phi(x,t)$ is then

$$\frac{\partial \phi}{\partial t} = -\gamma \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^2 \phi}{\partial x^2} + (1 - 3v^2)\phi - 3v\phi^2 - \phi^3.$$

Note that the nonlinear term $-3v(x)\phi^2 - \phi^3$ is C^1 from H^1 to H^1.

We have to investigate the spectrum of the linearised operator

$$\mathcal{L}\phi \stackrel{\text{def}}{=} -\gamma\phi'''' + \phi'' - 2\phi + g(x)\phi,$$

where

$$g(x) \equiv 3 - 3v^2(x) \to 0 \qquad \text{as } x \to \pm\infty.$$

We consider $\mathcal{L}$ as an unbounded operator from $D(\mathcal{L}) = H^5(\mathbb{R}) \subset H^1(\mathbb{R})$ to $H^1(\mathbb{R})$. It is well-known that the essential spectrum of $\mathcal{L}$ is

$$\sigma_e(\mathcal{L}) = (-\infty, -2],$$

and that the remaining part of the spectrum $\sigma(\mathcal{L}) \setminus \sigma_e(\mathcal{L})$ consists entirely of isolated real eigenvalues of finite multiplicity [He].

The minimising property of the kink,

$$J[v] = \inf\{J[u] \mid u - \chi \in H^2\},$$

implies that

$$(\mathcal{L}\phi, \phi)_{L^2} \leq 0 \qquad \text{for all } \phi \in H^4. \tag{30}$$

Any eigenfunction of $\mathcal{L}$ in H^1 is in H^4 since it obeys a regularising differential equation. By substituting eigenfunctions in (30) we see that all eigenvalues of $\mathcal{L}$ are in $(-\infty, 0]$, and we conclude that the linear operator $\mathcal{L}$ generates a C_0-semigroup on H^1.

The EFK equation is autonomous, thus $v'(x)$ is an eigenfunction with eigenvalue 0. In fact, the zero eigenvalue is simple, which follows from the transversality of $W^s(+1)$ and $W^u(-1)$. To see this, we note that the flow of the tangent plane $TW^s(x)$ of the stable manifold of $+1$ at points $(v, v', v'', v''')(x)$ on the heteroclinic orbit, is given by the linearised equation around the kink. Since $W^s(+1)$ is two-dimensional this implies that there are exactly two linearly independent solutions of $\mathcal{L}\phi = 0$ which tend to 0 as $t \to \infty$, corresponding to two independent directions in the tangent planes $TW^s(x)$. A similar statement holds for the tangent plane $TW^u(x)$ of the unstable manifold of -1. Because an eigenfunction with eigenvalue 0 obeys this linearised equation $\mathcal{L}\phi = 0$ and tends to 0 as $x \to \pm\infty$, it corresponds to a common direction in the tangent planes $TW^s(x)$ and $TW^u(x)$. Therefore, a second independent eigenfunction with eigenvalue 0 would imply that the stable and unstable manifolds do not intersect transversely in the zero energy set, which contradicts Theorem 6.

We note that this reasoning also applies to the space of bounded uniformly continuous functions since there is an exponential dichotomy when $x \to \pm\infty$.

Following [He, BJ] we exploit the translation invariance to conclude that the local stable manifolds (having co-dimension 1) of the translates of the kink $v(x)$ fill a tubular neighbourhood of $\{v(x + x_0) \mid x_0 \in \mathbb{R}\}$ in function space. The family of kinks $\{v(x + x_0) \mid x_0 \in \mathbb{R}\}$ forms the one-dimensional center manifold of v and its translates. This implies asymptotic stability and thus proves Theorem 7.

8 Continuation and existence of solutions

This section is devoted to the continuation of bounded solutions of (6) for values of q that are sufficiently negative. The results in this section show that not only do bounded solutions not intersect each other in the (u, u')-plane, but they also completely fill up part of the (u, u')-plane. The fact that solutions can be continued also implies the existence of solutions, and we will make some general remarks about that.

The main result of this section is Theorem 3. In the proof of this theorem we use the notation of Equation (6). Let $u_0(x)$ be a periodic solution of (6) for $q = q_0$. We define $a \equiv \min u_0(x)$ and $b \equiv \max u_0(x)$. Suppose that $q_0 \le -2\sqrt{\omega(a, b)}$. Then this periodic solution is part of a continuous one-parameter family of periodic solutions. We will use the implicit function theorem to prove this assertion. In Theorem 3 the energy is taken as parameter. Here we first take the maximum value of solutions as parameter and then we show that the energy can be used as parameter equally well.

Without loss of generality we may assume that u_0 attains a maximum at $x = 0$. Then $u_0'(0) = 0$ and $u_0'''(0)$, and from Remark 6 we see that

$$u_0''(0) < 0 \quad \text{and} \quad u_0''''(0) - \lambda u_0''(0) > 0.$$

Let $\xi_0 > 0$ be the first point where u_0 attains a minimum.

We now look at a family of solutions $u(x; \alpha, \beta)$ of (6) with initial data

$$(u, u', u'', u''')(0; \alpha, \beta) = (\alpha, 0, \beta, 0),$$

where (α, β) is in a small neighbourhood of $(\alpha_0, \beta_0) \stackrel{\text{def}}{=} (u_0(0), u_0''(0))$. Note that $u(x; \alpha_0, \beta_0)$ is the periodic solution $u_0(x)$.

To show that u_0 is part of a continuous family it suffices to prove that there exists a one-parameter family of points (ξ, α, β) in a neighbourhood of $(\xi_0; \alpha_0, \beta_0)$ such that

$$u'(\xi, \alpha, \beta) = 0 \quad \text{and} \quad u'''(\xi, \alpha, \beta) = 0.$$

Let α be the parameter, then to be able to apply the implicit function theorem we have to show that the determinant

$$D \stackrel{\text{def}}{=} \det \begin{pmatrix} \frac{\partial u'}{\partial x} & \frac{\partial u'''}{\partial x} \\ \frac{\partial u'}{\partial \beta} & \frac{\partial u'''}{\partial \beta} \end{pmatrix} (\xi_0; \alpha_0, \beta_0)$$

is non-zero.

It follows from Remark 6 that

$$u''(\xi_0; \alpha_0, \beta_0) > 0 \qquad \text{and} \qquad u''''(\xi_0; \alpha_0, \beta_0) - \lambda u''(\xi_0; \alpha_0, \beta_0) < 0. \tag{31}$$

Taking $v(x) = \frac{\partial u}{\partial \beta}(x; \alpha_0, \beta_0)$, we see that

$$v(0) = 0, \quad v'(0) = 0, \quad v''(0) - \lambda v(0) = 1 \quad \text{and} \quad v'''(0) - \lambda v'(0) = 0,$$

and

$$v'''' + qv'' = f'(u)v.$$

Following the proof of the Comparison Lemma we conclude that for all $x > 0$

$$v > 0, \quad v' > 0, \quad v'' - \lambda v > 0, \quad \text{and} \quad v''' - \lambda v' > 0. \tag{32}$$

We now see from (31) and (32) that

$$\det \begin{pmatrix} \frac{\partial u'}{\partial x} & \frac{\partial (u''' - \lambda u')}{\partial x} \\ \frac{\partial u'}{\partial \beta} & \frac{\partial (u''' - \lambda u')}{\partial \beta} \end{pmatrix} (\xi_0; \alpha_0, \beta_0) = \det \begin{pmatrix} > 0 & < 0 \\ > 0 & > 0 \end{pmatrix} > 0,$$

which immediately implies that $D \neq 0$.

Above we have used the amplitude of the periodic solution as a parameter. We can also use the energy E as a parameter, taking x and α as variables. In that case we look at a family of solutions $u(x; \alpha, E)$ of (6) with initial data

$$(u, u', u'', u''')(0; \alpha, E) = (\alpha, 0, -\sqrt{2E - 2F(\alpha)}, 0),$$

where (α, E) is in a small neighbourhood of $(\alpha_0, \mathcal{E}[u_0])$. We define $v(x) = \frac{\partial u}{\partial \alpha}(x; \alpha_0, \mathcal{E}[u_0])$, and we notice that $v(0) = 1$, $v'(0) = 0$, $v'''(0) - \lambda v'(0) = 0$ and

$$\begin{aligned} v''(0) - \lambda v(0) &= \left. \frac{\partial\big(-\sqrt{2\mathcal{E}[u_0] - 2F(\alpha)} - \lambda \alpha\big)}{\partial \alpha} \right|_{\alpha = \alpha_0} \\ &= \frac{-F'(\alpha_0)}{-\sqrt{2\mathcal{E}[u_0] - 2F(\alpha_0)}} - \lambda \\ &\geq \frac{-f(u_0(0))}{u_0''(0)} - \mu = \frac{-u_0''''(0) + \lambda u_0''(0)}{u_0''} > 0, \end{aligned}$$

by (6) and Remark 6. The previous analysis now applies once more and we conclude that Theorem 3 holds.

Another possibility for continuation of solutions is to fix the energy level E, take q as a parameter and use x and α as variables. Finally, instead of changing q as a parameter we can also deform the potential $F(u)$. This offers the possibility to obtain periodic solutions via continuation starting from a linear equation and then deforming the potential.

A different possible starting point for the continuation of bounded solutions is the second order equation ($\gamma = 0$), because for small positive γ the bounded solutions of (1) can be obtained from the second order equation by means of singular perturbation theory [F, J, AH].

The continuation of periodic solutions can come to an end in a limited number of ways:

- the value of q becomes too large compared the critical value $\omega(\min u, \max u)$, i.e., $q > -2\sqrt{\omega(\min u, \max u)}$. This may either happen when we increase q, or when we deform the potential, or when the range of $u(x)$ expands.
- the amplitude of the periodic solutions goes to infinity.
- the amplitude of the periodic solutions goes to zero, i.e., the periodic orbits converge to an equilibrium point.
- the periodic solutions converge to a chain of connecting orbits (homoclinic and/or heteroclinic).

Considering homoclinic solutions we note that it follows from Remark 13 that under very weak assumptions on the potential homoclinic solutions are transversal intersections and thus can be continued (for example starting at $\gamma = 0$ using singular perturbation theory). Another possibility is to obtain the homoclinic solutions as a limit of periodic solutions.

Finally, with regard to heteroclinic solutions there is an important result from [KKV], which states that if there are two equilibrium points u_0 and u_1 ($u_0 < u_1$) such that

$$\begin{gathered} F(u_0) = F(u_1), \\ F''(u_0) < 0 \quad \text{and} \quad F''(u_1) < 0, \\ F(u) < F(u_0) \text{ for all } u \in (u_0, u_1), \end{gathered}$$

then for all $\gamma > 0$ there exists a heteroclinic solution of (1) connecting these equilibrium points. On the other hand, the heteroclinic connections can also be obtained as a limit of periodic solutions, and when the potential is symmetric then the heteroclinic solution is a transversal intersection and thus can be continued.

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