Travelling waves for fourth-order semilinear parabolic equations*

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Abstract

We study travelling wave solutions of the fourth-order equation

$$u_t = -\gamma u_{xxxx} + u_{xx} + (u+a)(1-u^2),$$

where $0 < a < 1$ and $\gamma > 0$. We prove the existence of a travelling wave connecting the stable equilibrium points $u = -1$ and $u = 1$ for a large range of γ-values. The proof is based on the observation that the stable manifold of $u = 1$ (making the Ansatz $u(x,t) = U(x+ct)$) 'behaves' topologically different for small c and for large c. This allows us to conclude that a transition occurs as we vary c, which corresponds to the existence of a connection between $u = -1$ and $u = 1$. We also investigate travelling waves between $u = -a$ and $u = \pm 1$.

1 Introduction

We consider the fourth-order semilinear parabolic equation

$$u_t = -\gamma u_{xxxx} + u_{xx} + f(u), \qquad \gamma > 0, \tag{1.1a}$$

with the nonlinearity given by the cubic

$$f(u) = (u+a)(1-u^2), \qquad 0 < a < 1. \tag{1.1b}$$

This equation is one of the prototype equations describing pattern formation in physical models [5]. With $\gamma = 0$ it is a standard equation arising in population genetics [2] and admits a travelling wave solution of the form

$$u(x,t) = \tanh\Big(\frac{x + a\sqrt{2}t}{\sqrt{2}}\Big).$$

This travelling wave connects the two stable equilibria $u = -1$ and $u = +1$ of the ordinary differential equation $u' = f(u)$.

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For $a = 0$ Equation (1.1) is known in the literature as the Extended Fisher-Kolmogorov (EFK) equation and has been extensively studied during the last decade. In particular it admits a standing travelling wave (i.e. a stationary solution) connecting $u = -1$ and $u = +1$ for all $\gamma > 0$. This 'kink' solution has been obtained by ODE methods [12] and by variational methods [14]. It is unique (modulo the obvious symmetries) for $\gamma \leq \frac{1}{8}$ [17, 18, 10], while for for $\gamma > \frac{1}{8}$ there is a multitude of (multi-bump) solutions connecting $u = -1$ and $u = +1$ [13, 9]. This is due to the fact that as γ crosses the critical value $\gamma = \frac{1}{8}$, the eigenvalues of the linearised (around $u = \pm 1$) stationary equation become complex.

Returning to the case $0 < a < 1$, it was shown in [1] that the explicit travelling wave for $\gamma = 0$ perturbs to a travelling wave for small γ. Besides, it was shown in [4] that for all $\gamma > 0$ the kink solution for $a = 0$ perturbs to a travelling wave for small a. Both these results are of a local character. In this paper we establish a more global result.

We introduce the *potential* (see Figure 1)

$$F(u) = \int_1^u f(s)ds, \tag{1.2}$$

and state our main theorem.

Theorem 1.1 *For all* $0 < a < 1$ *and all* $0 < \gamma < \sigma(a)$, *where*

$$\sigma(a) \stackrel{\text{def}}{=} \min_{-1<u<-a} \frac{-F(u)}{2f(u)^2} \tag{1.3}$$

there exists a travelling wave solution of (1.1) *connecting* $u = -1$ *and* $u = 1$.

Remark 1.2 Since at the (attained) minimum in (1.3) the equality

$$\frac{-F(u)}{2f(u)^2} = \frac{-1}{4f'(u)}$$

holds, we easily derive that

$$\sigma(a) > \frac{1}{8(1-a)} \qquad \text{for all } 0 < a < 1.$$

We note that although this estimate is sharp for $a \to 0$, it is not sharp at all for larger values of a. In general, for nonlinearities $f(u)$ having the same shape as (1.1b), a lower bound on σ is

$$\sigma \geq \min_{\substack{-1 \leq u \leq -a \\ f'(u)<0}} \frac{-1}{4f'(u)}. \tag{1.4}$$

This estimate is often easier to compute than σ itself, but it is in general a rather blunt estimate. Note that if the minimum in the right-hand side of (1.4) is only attained at $u = -1$, then the inequality in (1.4) is strict (e.g. in the case of $f(u) = (u + a)(1 - u^2)$). Finally, we remark that the critical value σ is also encountered in the study of homoclinic orbits for $c = 0$ (see [11, Theorem B]). This originates from the similarity of that problem with the proof of Lemma 4.2, which is in fact the only instance in our analysis where γ is required to be smaller than σ. •

In Theorem 1.1 we obtain the existence of a travelling wave connecting $u = -1$ and $u = +1$ for some $c > 0$. In this paper we also prove that for all $c > 0$ there exists a travelling wave connecting $u = -a$ to $u = -1$, whereas a travelling wave connecting $u = -a$ to $u = +1$ exists for $c > 0$ sufficiently large (see Theorems 6.1 and 6.2). These results do not depend on the precise shape of the nonlinearity. They hold for all nonlinearities $f(u)$ having the same shape as (1.1b). For the travelling waves connecting $u = -a$ and $u = \pm 1$ the proof goes on unchanged, while for the travelling wave connecting $u = -1$ to $u = +1$ we need (see Theorem 7.1) the uniform bound which is obtained in Lemma 7.3. Finally, in Theorem 6.3 we give an example of a result for nonlinearities with a different shape.

The organisation of this paper is as follows. In Section 2 we give an outline of the proof of Theorem 1.1, and in the Sections 3 to 5 the details of this proof are filled in. In particular, in Section 3 we perform an analysis of the flow "at infinity". Section 6 discusses the existence of travelling waves connecting $u = -a$ to $u = \pm 1$. We conclude with an a priori bound and some remarks on open problems in Section 7.

2 Outline of the proof

Searching for a travelling wave we set

$$u(x,t) = U(x + ct) \quad (c > 0),$$

which, switching to lower case again, reduces (1.1a) to the ordinary differential equation

$$cu' = -\gamma u'''' + u'' + f(u), \tag{2.1}$$

or, written as a four-dimensional system,

$$u' = v; \quad v' = w; \quad w' = z; \quad \gamma z' = w - cv + f(u). \tag{2.2}$$

The equilibria of this system are $(u, v, w, z) = (-1, 0, 0, 0)$, $(u, v, w, z) = (-a, 0, 0, 0)$ and $(u, v, w, z) = (1, 0, 0, 0)$ (for short: $u = -1$, $u = -a$ and $u = 1$). To prove Theorem 1.1 we look for a $c \neq 0$ and a corresponding heteroclinic orbit of (2.2) connecting $u = -1$ to $u = 1$. Linearising around $u = \pm 1$ we find that, irrespective of c, both $u = -1$ and $u = 1$ have two-dimensional stable and unstable manifolds, denoted by $W^s(\pm 1)$ and $W^u(\pm 1)$. Generically $W^s(1)$ and $W^u(-1)$ will not intersect but varying c we expect to pick up a non-empty intersection.

A key observation is the following. For $c = 0$ the eigenvalues of $u = \pm 1$ are real when γ is small, and they become complex when γ crosses the critical values $\gamma = \frac{1}{8(1+a)}$ (for $u = +1$) and $\gamma = \frac{1}{8(1-a)}$ (for $u = -1$). It follows from the analysis in [17] that if $c = 0$ then for $0 < \gamma \leq \frac{1}{8(1+a)}$ all orbits in the stable manifold $W^s(1)$ are unbounded. In this paper we enlarge the range of values of γ for which this holds to $0 < \gamma < \sigma(a)$.

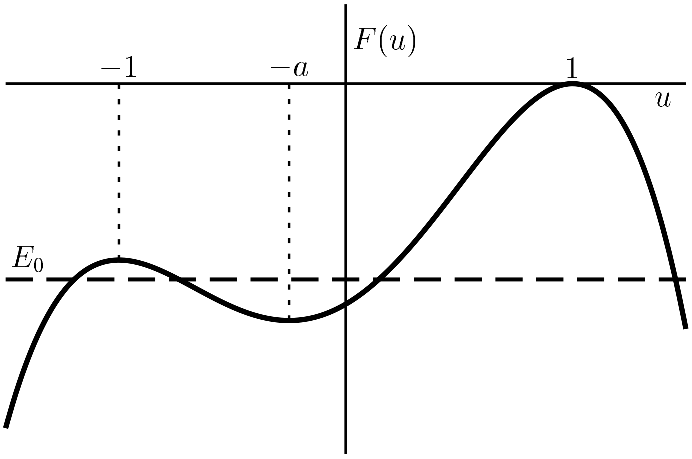

Figure 1: The potential $F(u)$ and the energy level E_0 separating $u = -a$ from $u = \pm 1$.

An important ingredient in our analysis is a conserved quantity for (2.1) when $c = 0$, which is a Lyapunov function when $c \neq 0$. We define

$$\mathcal{E}(u, v, w, z) \stackrel{\text{def}}{=} -\gamma v z + \frac{\gamma}{2} w^2 + \frac{v^2}{2} + F(u), \tag{2.3}$$

where the potential $F(u)$ is given by (1.2), and is depicted in Figure 1. Multiplying (2.1) by u' we have

$$\mathcal{E}'(u, u', u'', u''') = c u'^2, \tag{2.4}$$

so that $\mathcal{E}$, which we shall refer to as the *energy* of the solution, is increasing along orbits if $c > 0$, constant if $c = 0$, and decreasing if $c < 0$. Since we are looking for a solution of (2.1) connecting $u = -1$ to $u = 1$, we see from Figure 1 that we can restrict our attention to $c > 0$.

To separate the equilibrium point $u = a$ from $u = \pm 1$, we choose an energy level E_0 such that (see also Figure 1)

$$F(-a) < E_0 < F(-1) < 0,$$

and we define the set

$$K \stackrel{\text{def}}{=} \{(u, v, w, z) \in \mathbb{R}^4 \mid \mathcal{E}(u, v, w, z) \geq E_0\}. \tag{2.5}$$

This allows us to formulate the following proposition:

Proposition 2.1 *If $c > 0$ is such that $W^s(1) \cap W^u(-1) = \varnothing$, then every orbit in $W^s(1)$ enters K through its boundary δK and $\hat{\Gamma} = W^s(1) \cap \delta K$ is a simple closed curve. The set of positive c for which this property holds is open and $\hat{\Gamma}$ varies continuously with c.*

Proof. In view of (2.4) the intersection of $W^s(1)$ and δK must be transversal. Let us assume that $W^s(1) \cap W^u(-1) = \varnothing$. We need to show that every orbit in $W^s(1)$ can be traced back to

δK, for then there is bijection between $W^s(1) \cap \delta K$ and a smooth simple closed curve in $W^s_{\text{loc}}(1)$ winding around $u = 1$ (in $W^s_{\text{loc}}(1)$). Arguing by contradiction we assume that there is an orbit in $W^s(1)$ which is completely contained in K. Let $u(t)$ be a solution representing this orbit. Then $u(t)$ exists on some maximal time interval $(t_{\min}, \infty)$. Since $u(t)$ has energy larger than E_0, it follows from (2.4) and (2.5) that

$$\int_{t_{\min}}^{\infty} u'^2 \leq \frac{F(1) - E_0}{c} = \frac{-E_0}{c}, \tag{2.6}$$

so that $u(t)$ remains bounded on $(t_{\min}, \infty)$ if $t_{\min}$ is finite. Thus $t_{\min} = -\infty$ and, again by (2.6), $u(t)$ converges to a limit as $t \to -\infty$. Because $u = -1$ is the only equilibrium in K with energy less than the energy of $u = 1$, we infer by standard arguments that $u(t) \in W^u(-1)$. This contradicts the assumption that $W^s(1) \cap W^u(-1) = \varnothing$. The second statement is an immediate consequence of the transversality of $W^s(1) \cap \delta K$. □

Proof of Theorem 1.1. It suffices to show that there is a $c > 0$ for which the assumption of Proposition 2.1 fails. Again arguing by contradiction, we assume that Proposition 2.1 applies to all $c > 0$ and search for a topological obstruction. This requires a description of δK that allows us to form a global picture of this set.

To this end we write δK as

$$\delta K = \left\{ (u, v, w, z) \in \mathbb{R}^4 \;\middle|\; \frac{1}{2}(v - \gamma z)^2 + \frac{\gamma}{2} w^2 = E_0 - F(u) + \frac{\gamma^2 z^2}{2} \right\}. \tag{2.7}$$

In Figure 2 we have plotted the projection of δK onto the (u, z)-plane. For (u, z) lying inside one of the two closed curves (see Figure 2) defined by

$$E_0 - F(u) + \frac{\gamma^2 z^2}{2} = 0, \tag{2.8}$$

every (u, v, w, z) belongs to K, while for (u, z) lying outside the two closed curves we only have that (u, v, w, z) is in K if (v, w) is outside the ellipse defined by

$$\frac{1}{2}(v - \gamma z)^2 + \frac{\gamma}{2} w^2 = 0.$$

The projection of δK onto the (u, z)-plane maps $\hat{\Gamma} = W^s(1) \cap \delta K$, which by assumption exists for all $c > 0$, to a closed but not necessarily simple curve Γ in the (u, z)-plane for which the winding numbers $n(\Gamma, -1)$ and $n(\Gamma, 1)$ around $(u, z) = (-1, 0)$ and $(u, z) = (1, 0)$ respectively, are well-defined and independent of c (by continuity). A contradiction is derived by showing (in the following proposition) that for small $c > 0$ we must have $n(\Gamma, -1) = n(\Gamma, 1) = 1$, while $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$ for large $c > 0$ (this is depicted in Figure 2). □

We may choose the orientation of the simple closed curve in $W^s_{\text{loc}}(1)$ winding around $u = 1$ in such a way that its projection onto the (u, z) plane has winding number equal to $+1$.

Proposition 2.2

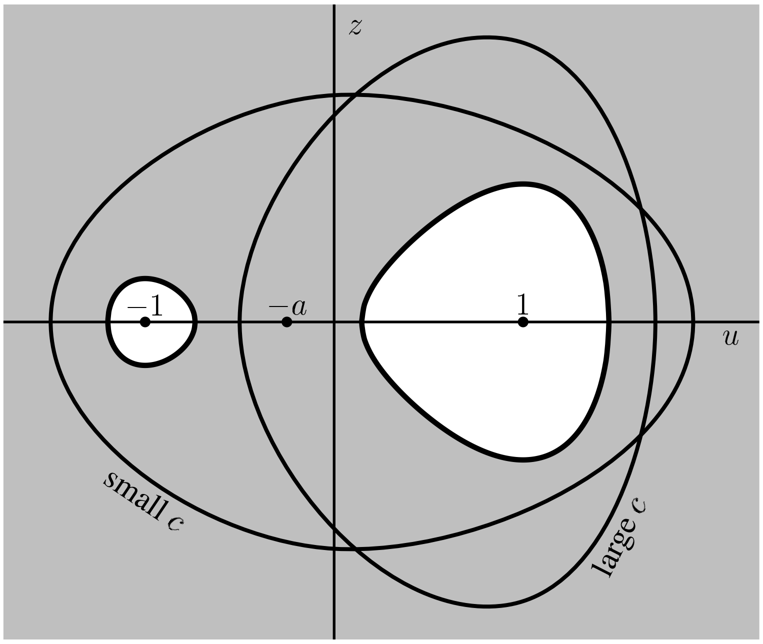

Figure 2: The projection (in grey) of δK onto the (u, z)-plane. The closed curves which form the boundary of the grey area are given by Equation (2.8). The other two curves depict Γ (i.e. the projection of $W^s(1) \cap \delta K$ onto the (u, z)-plane) for small c and large c.

(a) *Let* $0 < \gamma < \sigma(a)$. *Then there exists a* $c_* > 0$ *such that* $n(\Gamma, -1) = 1$ *and* $n(\Gamma, 1) = 1$ *for all* $0 < c < c_*$.

(b) *Let* $\gamma > 0$. *Then there exists a* $c^* > 0$ *such that* $n(\Gamma, -1) = 0$ *and* $n(\Gamma, 1) = 1$ *for all* $c > c^*$.

Note that the restriction on γ in Theorem 1.1 is solely due to the fact that our analysis for $c = 0$ only holds for $\gamma < \sigma(a)$.

The analysis for $c > 0$ large is based on a continuation argument deforming the nonlinearity $f(u)$ into a function which is linear on some interval containing $u = 1$. For an appropriate choice of E_0 this allows for an explicit calculation giving $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$. Using an estimate prohibiting a connection between $u = -1$ and $u = 1$ for $c > 0$ large (uniform in the deformation) we find the same winding numbers for our original f. This will be done in Section 5.

For $c > 0$ small the analysis is much more involved. We already mentioned that the starting point of the analysis in this paper is the fact that for $c = 0$ all orbits in $W^s(1)$ are unbounded as long as $0 < \gamma < \sigma(a)$. For the description of unbounded orbits we use a modified Poincaré transformation which we believe is of independent interest. In Section 3 we investigate orbits which are unbounded if we trace them backwards in time. We shall show that, in an appropriate compactification of the phase space, these orbits must originate from a unique periodic orbit lying at infinity in the phase space. A similar statement holds for orbits unbounded in forward time. The analysis at infinity relies largely on a global analysis of bounded and unbounded solutions of the family of equations

$$u'''' + u^s = 0 \quad \text{with the convention} \quad u^s = |u|^{s-1}u, \ s \geq 1. \tag{2.9}$$

This equation is invariant under the scaling $u(t) \to \lambda u(\lambda^{\frac{s-1}{4}} t)$ for all $\lambda > 0$.

In Section 4 we use the analysis in Section 3 to conclude that $n(\Gamma, -1) = n(\Gamma, 1) = 1$ for small $c > 0$. This part of the analysis also uses a series of deformations of f, resulting in the nonlinearity $-u^3$. Here it is important to keep the deformations within the class of functions for which $W^s(1)$ consists of unbounded orbits only.

We want to remark that our methods are inspired by Conley index theory, but we have chosen an approach in which only the essential topological information required for our analysis is used, and which avoids the use of algebraic invariants.

3 Classification of unbounded solutions

In this section we show that the flow of (2.2) far away from the origin is governed by Equation (2.9) with $s = 3$. In this analysis it will be convenient not to have parameters in the coefficients of the dominating terms in (2.1) and (2.2). Therefore we scale the independent variable by a factor $\gamma^{1/4}$ and arrive at the system

$$x_1' = x_2; \quad x_2' = x_3; \quad x_3' = x_4; \quad x_4' = Ax_3 - cx_2 + (x_1 + a)(1 - x_1^2), \tag{3.1}$$

where x_1, x_2, x_3 and x_4 correspond to u, u', u'' and u''' in the new variable. Note that also c has been scaled and that

$$A = \frac{1}{\sqrt{\gamma}},$$

so that

$$0 < \gamma < \sigma(a) \quad \Leftrightarrow \quad A > \frac{1}{\sqrt{\sigma(a)}}.$$

3.1 The reduced system

We begin by examining the reduced system

$$x_1' = x_2; \quad x_2' = x_3; \quad x_3' = x_4; \quad x_4' = -x_1^s, \tag{3.2}$$

corresponding to (2.9) with the same notational convention for the power function. Note that for this system the energy (or Hamiltonian)

$$H(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} -x_2 x_4 + \frac{x_3^2}{2} - \frac{|x_1|^{s+1}}{s+1} \tag{3.3}$$

is a conserved quantity.

We introduce five new dependent variables X_1, X_2, X_3, X_4 and $X_5 > 0$ by setting

$$x_i = \frac{X_i}{X_5^{a_i}} \qquad (i = 1, 2, 3, 4), \tag{3.4}$$

where the exponents a_i are to be chosen shortly. Unbounded orbits of (3.2) will correspond to orbits in the new variables with X_5 approaching zero. By substituting (3.4) in (3.2) we obtain the four equations

$$\begin{aligned} X_5 X_1' - a_1 X_1 X_5' &= X_2 X_5^{1+a_1-a_2}; && \text{(3.5a)} \\ X_5 X_2' - a_2 X_2 X_5' &= X_3 X_5^{1+a_2-a_3}; && \text{(3.5b)} \\ X_5 X_3' - a_3 X_3 X_5' &= X_4 X_5^{1+a_3-a_4}; && \text{(3.5c)} \\ X_5 X_4' - a_4 X_4 X_5' &= -X_1^s X_5^{1+a_4-sa_1}, && \text{(3.5d)} \end{aligned}$$

with a fifth equation pending. We choose the exponents in such a way that all the exponents in the right hand sides of (3.5) are the same, i.e,

$$b \stackrel{\text{def}}{=} 1 + a_1 - a_2 = 1 + a_2 - a_3 = 1 + a_3 - a_4 = 1 + a_4 - sa_1.$$

Solving for a_1, a_2, a_3, a_4 and b we find

$$a_1 = 4\lambda; \quad a_2 = (s+3)\lambda; \quad a_3 = (2s+2)\lambda; \quad a_4 = (3s+1)\lambda; \quad b = 1 - (s-1)\lambda, \tag{3.6}$$

where λ is still free and, for the moment, positive. We close system (3.5) by imposing as a fifth equation

$$X_1^s X_1' + X_2 X_2' + X_3 X_3' + X_4 X_4' = 0. \tag{3.7}$$

When we multiply (3.5a-3.5d) by X_1^s, X_2, X_3 and X_4 respectively, and add up the resulting equations, we obtain

$$P X_5' = -\frac{1}{\lambda} Q X_5^b. \tag{3.8}$$

Here we have set

$$P \stackrel{\text{def}}{=} 4|X_1|^{s+1} + (3+s)X_2^2 + (2+2s)X_3^2 + (1+3s)X_4^2, \tag{3.9}$$

which is non-negative, and

$$Q \stackrel{\text{def}}{=} X_1^s(X_2 - X_4) + X_3(X_2 + X_4).$$

Introducing a new independent variable, we write

$$\dot{X}_5 = P X_5^{(s-1)\lambda} X_5' = -\frac{1}{\lambda} Q X_5, \tag{3.10}$$

where the dot denotes derivation with respect to this new independent variable from which the old one may be recovered by integration. Thus, combining (3.10) and (3.5), we arrive at the system

$$\begin{aligned} \dot{X}_1 &= X_2P - 4X_1Q\,; && (3.11a)\\ \dot{X}_2 &= X_3P - (3+s)X_2Q\,; && (3.11b)\\ \dot{X}_3 &= X_4P - (2+2s)X_3Q\,; && (3.11c)\\ \dot{X}_4 &= -X_1^sP - (1+3s)X_4Q\,. && (3.11d) \end{aligned}$$

Note that X_5 has been decoupled from the equations and by construction the system (3.11) leaves the surfaces

$$\Sigma \stackrel{\text{def}}{=} \left\{(X_1, X_2, X_3, X_4) \;\middle|\; \frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0\right\} \cong S^3 \qquad (3.12)$$

invariant for all $C_0 > 0$. The free parameter λ only appears in (3.10) and may be discarded.

The Poincaré transformation (3.4) is used here to blow up the flow near "infinity". As will be explained in Section 3.3 this is equivalent to blowing up the flow near the equilibrium point $u = 0$. This blowing-up technique is frequently used in the study of flows in the neighbourhood of non-hyperbolic equilibrium points (see e.g. [6, 7]). The transformation defined by (3.4) and (3.12) is a variant of the standard Poincaré transformation, which has $a_1 = a_2 = a_3 = a_4 = 1$ and imposes as fifth equation that $X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2$ be constant, so that the transformed problem is situated on the Poincaré sphere. The modification presented above, in particular the choice of exponents, is needed to obtain a non-trivial vector field at infinity from which we may derive the qualitative properties of the flow of the full system (3.1) near infinity.

In Equation (3.7) we have chosen not to include a term X_5X_5' and to modify the exponent of X_1. This simplifies the new vector field and allows the decoupling of the $\dot{X}_5$-equation. Note that instead of a Poincaré sphere we now have a Poincaré cylinder Π, namely the topological product of the deformed sphere Σ and the positive X_5-axis:

$$\Pi \stackrel{\text{def}}{=} \{(X_1, X_2, X_3, X_4, X_5) \,|\, (X_1, X_2, X_3, X_4) \in \Sigma,\ X_5 \geq 0\} \cong S^2 \times \mathbb{R}^+.$$

The flow of (3.2) is completely determined by the flow of (3.11) on Σ. Thus we have a reduction from dimension 4 for (3.2) to dimension 3 for (3.11). The role of $X_5 = 0$ and $X_5 = \infty$ can be reversed by changing from positive to negative λ at the expense of a minus sign in (3.10).

Remark 3.1 The choice of $C_0 > 0$ in (3.12) is arbitrary, because the flows on all spheres Σ are C^1-conjugated (modulo the introduction of the new independent variable in Equation (3.10)). This is in fact the very idea of Poincaré transformations, namely that we divide out the invariance of (3.1) and focus on the resulting flow. To be explicit, let X_i and Y_i be two sets of Poincaré coordinates, i.e.,

$$x_i = \frac{X_i}{X_5^{a_i}} = \frac{Y_i}{Y_5^{a_i}} \qquad \text{for } i = 1, 2, 3, 4,$$

with constraints

$$\frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0, \tag{3.13a}$$

$$\frac{|Y_1|^{s+1}}{s+1} + \frac{Y_2^2}{2} + \frac{Y_3^2}{2} + \frac{Y_4^2}{2} = C_1. \tag{3.13b}$$

When we define $\lambda = \frac{X_5}{Y_5}$, then the two sets of coordinates are related by

$$X_5 = \lambda Y_5 \qquad \text{and} \qquad X_i = \lambda^{a_i} Y_i \quad \text{for } i = 1, 2, 3, 4. \tag{3.14}$$

Substituting this into (3.13a) we obtain

$$G(Y_1, Y_2, Y_3, Y_4, \lambda) \equiv \lambda^{(s+1)a_1} \frac{|Y_1|^{s+1}}{s+1} + \lambda^{2a_2} \frac{Y_2^2}{2} + \lambda^{2a_3} \frac{Y_3^2}{2} + \lambda^{2a_4} \frac{Y_4^2}{2} = C_0.$$

Since $\frac{\partial G}{\partial \lambda} > 0$ for all Y_i that obey (3.13b), it follows from the implicit function theorem that $\lambda(Y_1, Y_2, Y_3, Y_4)$ is a differentiable function. It is now easily seen from (3.14) that X_i and Y_i are related by a C^1-conjugacy. Therefore, we may choose the constant C_0 according to our liking to obtain a description of the flow that is most suitable to our needs. •

For the analysis of (3.11) we first observe

Lemma 3.2 *System* (3.11) *has no stationary points on* Σ *for any* $C_0 > 0$.

Proof. Since $X_1 = X_2 = X_3 = X_4 = 0$ is excluded we have that P defined by (3.9) is positive. Equating the right hand sides of (3.11) to zero and considering the resulting equations as linear equations in P and Q, it follows that we can only have solutions if every determinant of every pair of two equations vanishes. This would give for instance that

$$\begin{aligned} 0 &\leq (2+2s)X_3^2 = (3+s)X_2X_4; \\ 0 &\leq 4|X_1|^{s+1} = -(1+3s)X_2X_4. \end{aligned}$$

We conclude that $X_2X_4 = 0$ and with any of the $X_i = 0$ the others follow immediately. □

We next use the conserved quantity to obtain a further reduction from dimension 3 to dimension 2 for the limit sets of orbits of (3.5) which approach infinity ($X_5 \to 0$) or the origin ($X_5 \to \infty$). In the new variables the Hamiltonian is

$$H = \Big(-X_2X_4 + \frac{X_3^2}{2} - \frac{|X_1|^{s+1}}{s+1}\Big) X_5^{-4\lambda(s+1)}, \tag{3.15}$$

from which we conclude that for $\lambda > 0$

$$X_5 \to 0 \iff -X_2X_4 + \frac{X_3^2}{2} - \frac{|X_1|^{s+1}}{s+1} \to 0. \tag{3.16}$$

For the classification of unbounded orbits we have to analyse the flow restricted to the invariant set given by

$$T \stackrel{\text{def}}{=} \Big\{(X_1, X_2, X_3, X_4) \;\Big|\; \frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0,\; \frac{X_3^2}{2} = X_2 X_4 + \frac{|X_1|^{s+1}}{s+1}\Big\}.$$

This set is a topological torus as can be seen by setting

$$X_1 = \xi_1; \quad X_2 = \frac{\xi_2 + \xi_4}{\sqrt{2}}; \quad X_3 = \xi_3; \quad X_4 = \frac{\xi_2 - \xi_4}{\sqrt{2}}, \tag{3.17}$$

so that, in terms of the ξ-variables,

$$T = \Big\{(\xi_1, \xi_2, \xi_3, \xi_4) \;\Big|\; \frac{2}{s+1}|\xi_1|^{s+1} + \xi_2^2 = \xi_3^2 + \xi_4^2 = C_0\Big\} \cong S^1 \times S^1. \tag{3.18}$$

Clearly we have that T is the product of two topological circles, one in the (ξ_1, ξ_2)-plane, the other in the (ξ_3, ξ_4)-plane.

Theorem 3.3 *Let $s \geq 1$ and fix the constant $C_0 > 0$.*

(a) *Then there exist precisely two periodic orbits Λ_- and Λ_+ of (3.11) on the torus T.*

(b) *The three non-trivial Floquet multipliers of Λ_+ are contained in the interval $(0, 1)$. The three non-trivial Floquet multipliers of Λ_- are contained in the interval $(1, \infty)$.*

(c) *Every other orbit on the sphere Σ has Λ_- as α-limit set and Λ_+ as ω-limit set.*

Proof. The proof is based on the observation that the coefficient Q in (3.10), which after transforming by (3.17) reads

$$Q = \sqrt{2}(\xi_1^s \xi_4 + \xi_2 \xi_3), \tag{3.19}$$

plays a double role. Obviously it determines which parts of infinity attract solutions towards $X_5 = 0$, in forward and in backward time. To prove part (a) we begin by showing that Q can also be seen as minus the divergence of the vector field restricted to the invariant torus T. From (3.11) and (3.17) we derive

$$\dot{\xi}_1 = \frac{\xi_2 + \xi_4}{\sqrt{2}} P - 4\xi_1 Q\,; \tag{3.20a}$$

$$\dot{\xi}_2 = \frac{\xi_3 - \xi_1^s}{\sqrt{2}} P - ((2 + 2s)\xi_2 + (1 - s)\xi_4) Q\,; \tag{3.20b}$$

$$\dot{\xi}_3 = \frac{\xi_2 - \xi_4}{\sqrt{2}} P - (2 + 2s)\xi_3 Q\,; \tag{3.20c}$$

$$\dot{\xi}_4 = \frac{\xi_3 + \xi_1^s}{\sqrt{2}} P - ((1 - s)\xi_2 + (2 + 2s)\xi_4) Q\,. \tag{3.20d}$$

We parametrise T by 'polar coordinates'

$$\xi_1 = f_1(\phi); \quad \xi_2 = g_1(\phi); \quad \xi_3 = f_2(\theta); \quad \xi_4 = g_2(\theta), \tag{3.21}$$

satisfying

$$f_1' = -g_1; \quad g_1' = f_1^s; \quad f_2' = -g_2; \quad g_2' = f_2. \tag{3.22}$$

Note that when $C_0 = 1$ and $s = 1$ we just have

$$\xi_1 = \cos\phi; \quad \xi_2 = \sin\phi; \quad \xi_3 = \cos\theta; \quad \xi_4 = \sin\theta.$$

From (3.20a), (3.20c), (3.21) and (3.22) we derive that on T the flow is given by .

$$\begin{aligned}
\dot\phi &= \frac{P}{\sqrt{2}}(-1 - \frac{g_2}{g_1}) + 4Q\frac{f_1}{g_1} \equiv w_1(\phi,\theta); \\
\dot\theta &= \frac{P}{\sqrt{2}}(1 - \frac{g_1}{g_2}) + 2(s+1)Q\frac{f_2}{g_2} \equiv w_2(\phi,\theta),
\end{aligned}$$

where in terms of f_1, g_1, f_2, g_2,

$$P = 4(s+1)C_0 + 2(1-s)g_1g_2, \qquad \text{and} \qquad Q = \sqrt{2}(f_1^s g_2 + f_2 g_1).$$

The functions w_1 and w_2, defined in (3.23), appear to have singularities, but using (3.18) they can be written as

$$\begin{aligned}
w_1(\phi,\theta) &= \sqrt{2}\big[-2(s+1)C_0 - (s+3)g_1g_2 + (s-1)g_2^2 + 4f_1f_2\big], \\
w_2(\phi,\theta) &= \sqrt{2}\big[2(s+1)C_0 - (3s+1)g_1g_2 + (s-1)g_1^2 + 2(s+1)f_1^s f_2\big].
\end{aligned}$$

Taking the divergence of the vector field w we obtain (using (3.22),

$$\nabla\cdot w = \frac{\partial w_1}{\partial\phi} + \frac{\partial w_2}{\partial\theta} = \sqrt{2}(-5-3s)(f_1^s g_2 + f_2 g_1) = -(3s+5)Q.$$

Next, we split T into

$$T_+ = \{(X_1,X_2,X_3,X_4)\,|\,Q>0\} \quad \text{and} \quad T_- = \{(X_1,X_2,X_3,X_4)\,|\,Q<0\}.$$

These two sets share the boundary

$$T_0 = \{(X_1,X_2,X_3,X_4)\,|\,Q=0\},$$

which, in view of (3.18) and (3.19), consists of two topological circles, which both wind one time around the two homotopically distinct simple loops on the torus (see Figure 3). If we can show that an orbit can only pass through T_0 from T_- to T_+, then it follows from the negativity of $\nabla\cdot w$ in T_+ and the winding properties of T_0 on T that T_+ contains precisely one periodic orbit. The same statement follows for T_- with respect to the backward flow on T.

To be precise, it follows from (3.21), (3.22) and (3.18) that we may choose $\xi_3 = f_2(\theta) = \sqrt{C_0}\cos\theta$. We define the set $S \stackrel{\text{def}}{=} \{(\theta,\phi)\in T\,|\,\theta=\frac{\pi}{2}\}$, and it follows that

$$\dot\theta\Big|_S = \sqrt{2}[2(s+1)C_0 - (3s+1)\sqrt{C_0}g_1 + (s-1)g_1^2].$$

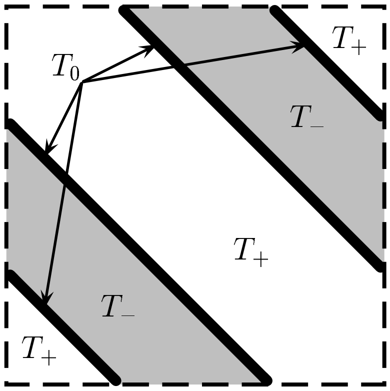

Figure 3: A fundamental domain of the torus, in which T_-, T_+ and T_0 are indicated (schematically).

Since $|g_1| \le \sqrt{C_0}$, it is easy to check that $\dot{\theta}\big|_S \ge 0$, and equality only holds when $g_1 = \sqrt{C_0}$. By continuity arguments the orbit through this point also crosses S in the direction of increasing θ. Thus S is a global section for the flow on T. Moreover, the return map is well-defined, since there is no point in T for which the forward orbit is contained in $T \setminus S$. Otherwise such a forward orbit would either be contained in T_- or eventually be in T_+, because T_+ is positively invariant and orbits can only pass through T_0 from T_- to T_+. In the absence of equilibrium points (Lemma 3.2) its ω-limit set would be a periodic orbit. However, there would have to be an equilibrium point inside this periodic orbit, contradicting Lemma 3.2. Hence the return map is well-defined. The intersection $S \cap (T_+ \cup T_0)$ consists of the line segment $\{(\theta, \phi) \in T \mid \theta = \frac{\pi}{2}, f_1(\phi) \ge 0\}$. The return map maps this line segment into itself, which implies the existence of a periodic orbit in T_+. Similarly there exists a periodic orbit in T_-. The return map is contracting in T_+ and expanding in T_-, since the divergence of the vector field is negative in T_+ and positive in T_-. This proves the uniqueness of the two period orbits and shows that all other orbits on the torus T have Λ_- as α-limit set and Λ_+ as ω-limit set.

We remark that the same conclusion can be reached by combining the Poincaré-Bendixson theorem for flows on the torus and the Morse theory for Morse-Smale flows.

We still have to show that an orbit can only pass through T_0 from T_- to T_+. Thus we calculate from (3.19) and (3.20) that

$$\dot{Q}\Big|_{Q=0} = P\Big(|\xi_1|^{2s} + \xi_2^2 + \xi_3^2 + \xi_4^2 + (s|\xi_1|^{s-1} - 1)(\xi_2 + \xi_4)\xi_4\Big). \tag{3.24}$$

Notice that P is positive on T (see (3.9)), thus $\dot{Q}\big|_{Q=0} > 0$ on T for $s = 1$. For $s > 1$ we define R as the second part of the right hand side of (3.24) and simplify it using the expression (3.18) for T:

$$\begin{aligned} R &\stackrel{\text{def}}{=} |\xi_1|^{2s} + \xi_2^2 + \xi_3^2 + \xi_4^2 + (s|\xi_1|^{s-1} - 1)(\xi_2 + \xi_4)\xi_4 \\ &= 2C_0 + |\xi_1|^{2s} - \frac{2}{s+1}|\xi_1|^{s+1} - (1 - s|\xi_1|^{s-1})(\xi_2 + \xi_4)\xi_4. \end{aligned} \tag{3.25}$$

We notice that by (3.18)

$$(\xi_2+\xi_4)\xi_4 \leq ((C_0 - \frac{2}{s+1}|\xi_1|^{s+1})^{\frac{1}{2}} + C_0^{\frac{1}{2}})C_0^{\frac{1}{2}} = C_0(1 + (1 - \frac{2}{C_0(s+1)}|\xi_1|^{s+1})^{\frac{1}{2}}),$$

and we fix

$$C_0 = \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}. \tag{3.26}$$

Setting

$$|\xi_1| = x\Big(\frac{1}{s}\Big)^{\frac{1}{s-1}}, \text{ where } 0 \leq x \leq 1,$$

it follows that

$$\begin{aligned} R &\geq \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}\Big(2 + \frac{s+1}{2s}x^{2s} - x^{s+1} - (1-x^{s-1})(1+(1-x^{s+1})^{\frac{1}{2}})\Big) \\ &= \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}\Big(1 + \frac{s+1}{2s}x^{2s} - x^{s+1} + x^{s-1} - (1-x^{s-1})(1-x^{s+1})^{\frac{1}{2}}\Big) \\ &= \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}\Big((1-x^{s+1})^{\frac{1}{2}}\big((1-x^{s+1})^{\frac{1}{2}} - (1-x^{s-1})^{\frac{1}{2}}\big) + x^{s-1} + \frac{s+1}{2}x^{2s}\Big). \end{aligned}$$

Since $0 \leq x \leq 1$ we see that $R > 0$ unless $x = 0$. Looking at (3.25) we infer that R can only be zero if $\xi_1 = \xi_3 = 0$ and $\xi_2 = \xi_4 = \pm\sqrt{C_0}$, or, in terms of the X_i, if $X_1 = X_3 = X_4 = 0$. By continuity arguments it follows that also in these two points the orbits go from T_- to T_+. Thus, with the particular choice of C_0 given by (3.26) we have indeed that T_+ is positively invariant and T_- is negatively invariant. This completes the proof of part (a) of the theorem.

Note that, although the preceding proof needs C_0 to be given by (3.26), the statements in Theorem 3.3 are true for any choice of $C_0 > 0$ (see Remark 3.1). Another observation is that the linear case $s = 1$ may be treated directly by transforming the general solution of the then linear equation (2.9) to the X-variables.

Next we prove part (b) of the theorem. Restricted to T the nontrivial Floquet multiplier of Λ_+ equals (see e.g. [16, p. 198])

$$\exp\Big(\oint_{\Lambda_+} \nabla \cdot w\Big) = \exp\Big(\oint_{\Lambda_+} -(3s+5)Q\Big).$$

Since Q is uniformly positive on Λ_+, this Floquet multiplier is in $(0,1)$. Close to the periodic orbit Λ_+ we choose ϕ, θ, X_5 and H_0 as coordinates on the Poincaré cylinder Π, where H_0 is the first factor of the Hamiltonian H in (3.3):

$$H_0 \stackrel{\text{def}}{=} -X_2X_4 + \frac{X_3^2}{2} - \frac{|X_1|^{s+1}}{s+1}.$$

Since $H = H_0X_5^{-4\lambda(s+1)}$ is a conserved quantity on Π, it follows from (3.10) that

$$\dot{H}_0 = -4(s+1)Q\,H_0.$$

Together with (3.10) this implies that the other Floquet multipliers are

$$\exp\Big(\oint_{\Lambda_+} -4(s+1)Q\Big) \quad \text{and} \quad \exp\Big(\oint_{\Lambda_+} -\frac{1}{\lambda}Q\Big),$$

which are in $(0,1)$ as before. Thus Λ_+ is exponentially stable. The statement for Λ_- is obtained by time reversal.

Finally, we prove part (c) of the theorem. We have already dealt with the flow on the torus T in part (a). Orbits of the flow on the complement $\Sigma \setminus T$ of the torus T in the sphere Σ correspond to solutions of (2.9) with non-zero Hamiltonian. Since $H = H_0 X_5^{-4\lambda(s+1)}$ and H_0 is bounded on Σ (because Σ is compact), it follows that for such orbits X_5 remains bounded. Thus orbits in $\Sigma \setminus T$ are bounded in the X-variables and hence have nonempty compact invariant α- and ω-limit sets, consisting of bounded orbits. We have to show that these limit sets can only be the two periodic orbits Λ_- and Λ_+ provided by part (a) of the theorem. To this end it suffices to show that all solutions with $H \neq 0$ are unbounded, i.e., $X_5 \to 0$ along a subsequence, since this also implies $H_0 \to 0$ by (3.16), so that an unbounded orbit comes arbitrary close to the torus T.

Postponing the proof of the unboundedness of solutions with $H \neq 0$, we then choose an open tubular neighbourhood Λ_-^ε of Λ_- in T, such that $Q < 0$ in Λ_-^ε. Clearly all orbits starting in $T \setminus \Lambda_-^\varepsilon$ tend to Λ_+ in forward time. Note that $T_0 \cup T_+ \subset T \setminus \Lambda_-^\varepsilon$. By compactness of T and by part (b) of the theorem, there exists a open neighbourhood T^ε of $T \setminus \Lambda_-^\varepsilon$ in Π such that all orbits starting in T^ε tend to Λ_+ in forward time. Since an orbit which comes close to $X_5 = 0$ (and thus close to T), can only do so with non-negative Q, it enters T^ε and hence tends to Λ_+.

We still have to prove that any orbit with non-zero Hamiltonian is unbounded. It has been proved in [10] that (2.9) admits no bounded solutions except $u \equiv 0$. Here we present a different proof because we need to extend this result to more general situations (see Remark 3.4).

Arguing by contradiction we assume that there is a bounded orbit which has X_5 bounded away both from zero and infinity. Such an orbit corresponds to a bounded orbit of (3.2) which stays away from the origin, i.e. to a globally defined bounded solution u of (2.9) which in phase space stays away from the zero state. It is a simple exercise to show that such solutions u are oscillatory and must attain positive local maxima. In particular any such solution must have a point t_0 in which

$$u(t_0) > 0, \quad u'(t_0) = 0, \quad u''(t_0) \le 0.$$

Changing from t to $-t$ if necessary, we may assume that $u'''(t_0) \le 0$ and apply an oscillation argument from [17] which we repeat here for the sake of completeness. There exists a $t^* > t_0$ such that $u'''(t) < 0$ for $t_0 < t < t^*$ and $u'''(t^*) = 0$. Using the fact that,

$$H = -u'u''' + \frac{1}{2}u''^2 - \frac{1}{s+1}|u|^{s+1}$$

is constant, it follows that $u(t^*) < -u(t_0)$ and that the next minimum must occur at $t_1 > t^*$ with $u(t_1) < u(t^*) < -u(t_0)$ and both $u''(t_1)$ and $u'''(t_1)$ positive. Repeating this argument we obtain a sequence $t_1 < t_2 < t_3 < \dots$, in which $u(t)$ has non-degenerate extrema with $|u(t_1)| < |u(t_2)| < |u(t_3)| < \dots$. By assumption these extrema remain bounded, say $\lim_{i\to\infty} |u(t_i)| = a \in \mathbb{R}^+$. By the energy identity $u''(t_i)$ is then also bounded, and it is not difficult to see from the differential equation (3.2) that $u'''(t_i)$ must be bounded as well. A compactness argument now shows that there must be a solution $\tilde{u}$ of (3.2) in the ω-limit set of u with

$$\tilde{u}(t_0) = a, \quad \tilde{u}'(t_0) = 0, \quad \tilde{u}''(t_0) < 0, \quad \text{and} \quad \tilde{u}'''(t_0) \le 0 \qquad \text{at some } t_0 \in \mathbb{R},$$

and such that $|\tilde{u}| \le a$ on $\mathbb{R}$. However, this is impossible by the very same argument. This completes the proof of part (c) of the theorem. $\square$

Remark 3.4 The oscillation argument above will be applied several times in this paper to differential equations that differ from the present one. It holds that any solution of (2.1) with $c = 0$ which does not have its range contained in

$$\{u \in \mathbb{R} \,|\, F(u) \ge F(-a)\}$$

oscillates towards infinity exactly as described above (the additional second order term does not cause any difficulties). For more details we refer to [17]. •

3.2 The reduced system in the linear limit

The transformation of the reduced equation (2.9) to the system (3.11) also allows us to take the limit $s \downarrow 1$. Let us rewrite this system as

$$\dot{X} = V(X; s), \quad X = (X_1, X_2, X_3, X_4). \tag{3.27}$$

Then the vector field $V(\cdot, s)$ is continuously differentiable for every $s \ge 1$ and the first order partial derivatives are bounded on compact sets, uniformly in $s \ge 1$. We do not have that $V(\cdot, s) \to V(\cdot, 1)$ in C^1_{loc} because of the term X_1^s appearing in V, but we do have that $V(\cdot, s) \to V(\cdot, 1)$ uniformly on compact sets. Therefore the orbits of (3.27) with $s > 1$, which are bounded uniformly in s in view of (3.12), converge to orbits of (3.27) with $s = 1$ as $s \to 1$. More precisely, the solution map

$$(\tau, \xi, s) \to X(\tau; \xi, s),$$

where $X(\tau; \xi, s)$ is the solution $X(\tau)$ of (3.27) with $X(0) = \xi$, is continuous on $\mathbb{R} \times \mathbb{R}^4 \times [1, \infty)$. In particular, this implies that the two periodic orbits Λ_- and Λ_+ depend continuously on s for $s \in [1, \infty)$.

In the limit case $s = 1$ the two periodic orbits on

$$T = \{(\xi_1, \xi_2, \xi_3, \xi_4) \,|\, \xi_1^2 + \xi_2^2 = \xi_3^2 + \xi_4^2 = C_0\}$$

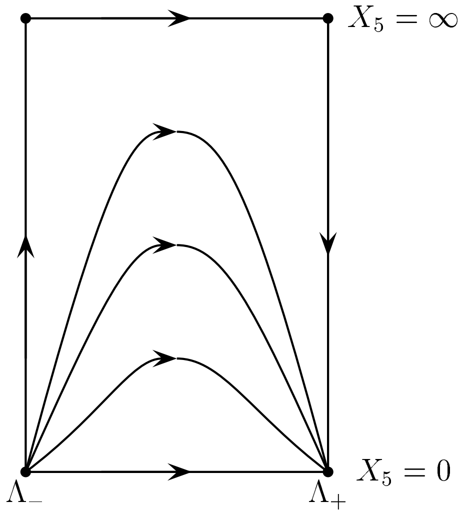

Figure 4: A schematic view of the flow on the Poincaré cylinder Π for the equation $u'''' + u^s = 0$. The role of $X_5 = 0$ and $X_5 = \infty$ is reversed when λ is negative.

are given by

$$\xi_1 \xi_3 - \xi_2 \xi_4 = 0, \tag{3.28}$$

or in terms of (3.21), by $\phi - \theta = \pm\frac{\pi}{2}$. This can be seen from a second conservation law that exists in the linear case: multiplying $u'''' + u = 0$ by u''' we have that $\frac{1}{2}u'''^2 + uu'' - \frac{1}{2}u'^2$ is constant, so that in particular, after transforming to the X-variables,

$$\frac{1}{2}X_4^2 + X_1 X_3 - \frac{1}{2}X_2^2 = 0$$

is invariant, whence (3.28), which defines two circles on the torus T.

3.3 Small solutions

We observed in Section 3.1 that the role of $X_5 = 0$ and $X_5 = \infty$ may be reversed. This is a direct consequence of the scaling invariance of (2.9). Thus we may also use (3.4) for the analysis of small solutions to (2.9). The situation is depicted schematically in Figure 4. We simply apply (3.4) with a negative λ so that $X_5 \to 0$ corresponds to $u \to 0$. This only changes the sign in the equation (3.10) for X_5 and means that the orbit Λ_+ now lies in the part of $X_5 = 0$ which repels solutions with $X_5 > 0$. Hence the stable manifold of Λ_+ is contained in $\Pi \cap \{X_5 = 0\}$. The unstable manifold of Λ_+ is given by the direct product $\Lambda_+ \times \{X_5 \,|\, X_5 > 0\}$ and has dimension 2. In the original variables it is the unstable manifold of $u = 0$ if $s = 1$ and the center-unstable manifold if $s > 1$. Likewise, the stable manifold of Λ_- is $\Lambda_- \times \{X_5 \,|\, X_5 > 0\}$, i.e., the direct product of Λ_- and the positive X_5-axis. As we have seen in Section 3.2, the limit $s \to 1$ is well behaved in the X-variables.

3.4 The full system

Applying the Poincaré transformation (3.4) with exponents (3.6) to the differential equation (3.1), or, more generally, to

$$x_1' = x_2; \quad x_2' = x_3; \quad x_3' = x_4; \quad x_4' = \Phi(x_1, x_2, x_3, x_4),$$

we arrive at

$$\dot{X}_1 = X_2 P - 4X_1 Q\,; \tag{3.29a}$$
$$\dot{X}_2 = X_3 P - (3+s)X_2 Q\,; \tag{3.29b}$$
$$\dot{X}_3 = X_4 P - (2+2s)X_3 Q\,; \tag{3.29c}$$
$$\dot{X}_4 = \Psi P - (1+3s)X_4 Q\,; \tag{3.29d}$$
$$\dot{X}_5 = -\frac{1}{\lambda} X_5 Q\,, \tag{3.29e}$$

where

$$\Psi = X_5^{4\lambda s}\Phi\Big(\frac{X_1}{X_5^{4\lambda}}, \frac{X_2}{X_5^{(3+s)\lambda}}, \frac{X_3}{X_5^{(2+2s)\lambda}}, \frac{X_4}{X_5^{(1+3s)\lambda}}\Big), \tag{3.30}$$

and

$$Q = X_1^s X_2 + X_4\Psi + X_3(X_2 + X_4). \tag{3.31}$$

In the case of (3.1) we have

$$\Phi(x_1, x_2, x_3) = Ax_3 - cx_2 + (x_1 + a)(1 - x_1^2),$$

so that, with $s = 3$ and $\lambda = \frac{1}{2}$,

$$\Psi = -X_1^3 + (AX_3 - aX_1^2)X_5^2 - cX_2X_5^3 + X_1X_5^4 + aX_5^6. \tag{3.32}$$

Clearly the extra terms are at least quadratic in X_5. Thus the local analysis near $X_5 = 0$ and in particular the Floquet multipliers of $\Lambda_\pm$ in the previous section are completely unaffected. The flow on the sphere Σ (at infinity) is identical to the flow for the reduced equation (3.1). Only the flow on $\Pi \setminus \Sigma$ is different. Note that in this analysis it is essential that the exponent s is larger than 1. We have the following theorem (compare Theorem 3.3).

Theorem 3.5 *Let $s = 3$, $\lambda = \frac{1}{2}$ and let Q and Ψ be given by* (3.31) *and* (3.32).

(a) *The stable periodic orbit Λ_+ of* (3.11) *is an asymptotically stable periodic orbit of* (3.29) *with non-trivial Floquet multipliers in $(0, 1)$. Every solution of* (3.1) *which is unbounded in forward time corresponds to a solution of* (3.29) *having Λ_+ as ω-limit set. A similar statement holds for solutions unbounded in backward time and Λ_-.*

(b) *Unbounded solutions of* (3.1) *blow-up oscillatory in finite time.*

(c) *If $c \neq 0$ the energy $\mathcal{E}$ also blows up.*

Proof. By Theorem 3.3(c) all solutions of (3.29) which lie in the invariant set $\Pi \cap \{X_5 = 0\} \setminus \Lambda_- \subset \Pi$ tend to Λ_+ in forward time. Reminiscent of the proof of Theorem 3.3(c) we choose a small negatively invariant open tubular neighbourhood Λ_-^ε of Λ_- in Π. By compactness of $\Pi \cap \{X_5 = 0\}$ there exists an open neighbourhood Σ^ε of $\Pi \cap \{X_5 = 0\} \setminus \Lambda_-^\varepsilon$ in Π such that all orbits with starting point in Σ^ε tend to Λ_+ in forward time. Clearly every unbounded solution of (2.1) enters Σ^ε and thus tends to Λ_+.

For part (b) we observe that the exponent b in (3.8) is smaller than 1 so that in the old time variable X_5 can only go to zero in finite time. Finally we have that the energy $\mathcal{E}$ can only remain bounded if its derivative is integrable. For $c \neq 0$ this implies that u' is square integrable (see (2.4)) and thus u itself is (locally) bounded, which prohibits finite time blow-up, a contradiction. $\square$

Remark 3.6 Theorem 3.5 establishes that large solutions of (2.1) are really described by oscillating solutions of $u'''' + u^3 = 0$. Thus large solutions do not "see" the other terms in (2.1) as they oscillate away to infinity. This is not only true for lower order polynomial perturbations of $f(u)$, but also for localised perturbations, which will be used in Section 4. E.g., when

$$f(u) = -u^3 - pu - g_\mu(u)$$

with g_μ a smooth family of functions vanishing outside a fixed interval and p a parameter, then (3.30) reads, setting $s = 3$ and $\lambda = \frac{1}{4}$,

$$\Psi = -X_1^3 - pX_1X_5^2 + X_5^3 g_\mu\Big(\frac{X_1}{X_5}\Big). \tag{3.33}$$

The last term in (3.33) is C^2 and has its derivatives up to second order vanishing in $X_5 = 0$. Theorem 3.5 applies here equally well. •

4 The winding number for small speeds

In this section we examine the winding properties of the intersection $W \cap \delta K$ of the stable manifold $W \equiv W^s(1)$ and the boundary of the set K defined by (2.5) when c is small. We shall switch notation between (2.1), (2.2), (3.1) and (3.29) without further comment.

Theorem 4.1 *Let $0 < \gamma < \sigma(a)$. For $F(-a) < E_0 < F(-1)$ let K be defined by (2.5) and let W be the stable manifold of the equilibrium $u = 1$. Then, provided $c > 0$ is sufficiently small, $W \cap \delta K$ is a topological circle. Its projection Γ on the (u, u''')-plane winds exactly once around a disk containing both closed curves defined by $C - F(u) + \frac{\gamma^2 u'''^2}{2} = 0$ (see also Figure 2), i.e., $n(\Gamma, -1) = n(\Gamma, 1) = 1$.*

This theorem establishes Proposition 2.2(a). Before we can prove this theorem we first need a description of the global behaviour of W for $c = 0$. In the following lemma we show that for $0 < \gamma < \sigma(a)$ all orbits in the stable manifold W are unbounded, and, after transforming to the X-variables in Section 3, they all have Λ_- as α-limit set. It is proved in [17] that for $0 < \gamma \le \frac{1}{8(1+a)}$ the bounded solutions of (2.1) with $c = 0$ correspond exactly to the bounded solutions of the second order equation ($\gamma = 0$). Here we want to exclude bounded solutions in the stable manifold of $u = 1$, i.e., we can restrict the analysis to the energy level $\mathcal{E} = 0$. This allows us to cover a larger range of γ-values.

Lemma 4.2 *Let* $0 < \gamma < \sigma(a)$ *and* $c = 0$*. Then* W *consists of unbounded orbits only, all of which connect* Λ_- *to* $u = 1$.

Proof. The proof is a combination of arguments also used in [15]. Any bounded solution must have its range in the set

$$V = \{u \in \mathbb{R} \,|\, F(u) \ge F(-a)\},$$

because a solution reaching outside this interval oscillates away towards infinity, as mentioned in Remark 3.4. Besides, any bounded solution must have at least one minimum below the line $u = -a$, again basically by the same oscillation argument as in the proof of part (b) of Theorem 3.3. We now assume, arguing by contradiction, that u is a bounded orbit in W. We will show that the range of u is not contained in V, so that u is in fact unbounded. It then follows from Theorem 3.5 that u tends to Λ_- as $t \to -\infty$.

Thus, suppose that u is a bounded solution in W. Changing from t to $-t$ if necessary we have that in such a minimum (using the fact that $\mathcal{E}(u, u', u'', u''') = 0$)

$$u(t_0) \le -a, \quad u'(t_0) = 0, \quad u''(t_0) = \sqrt{-\tfrac{2}{\gamma}F(u(t_0))} > 0, \quad u'''(t_0) \ge 0. \tag{4.1}$$

We will show that $u(t)$ increases to a value outside V for $t > t_0$, which immediately leads to a contradiction.

We define an auxiliary function

$$G(t) \stackrel{\text{def}}{=} u''(t) - \sqrt{-\tfrac{2}{\gamma}F(u(t))}.$$

The following line of reasoning is depicted in Figure 5. Firstly, $G(t_0) = 0$ and we show that $G(t) > 0$ in a right neighbourhood of t_0. It is seen from the condition on γ and the observation that $f(u) > 0$ on $(-\infty, -1) \cup (-a, 1)$, that

$$f(u) > -\sqrt{\frac{-F(u)}{2\gamma}} \qquad \text{for } u < 1. \tag{4.2}$$

If $u'''(t_0) > 0$ then clearly $G'(t_0) > 0$, whereas when $u'''(t_0) = 0$ then $G'(t_0) = 0$, and (since $u'(t_0) = 0$)

$$G''(t_0) = u''''(t_0) + \frac{f(u(t_0))}{\sqrt{-2\gamma F(u(t_0))}}u''(t_0) = \frac{2}{\gamma}\left(\sqrt{\frac{-F(u(t_0))}{2\gamma}} + f(u(t_0))\right) > 0$$

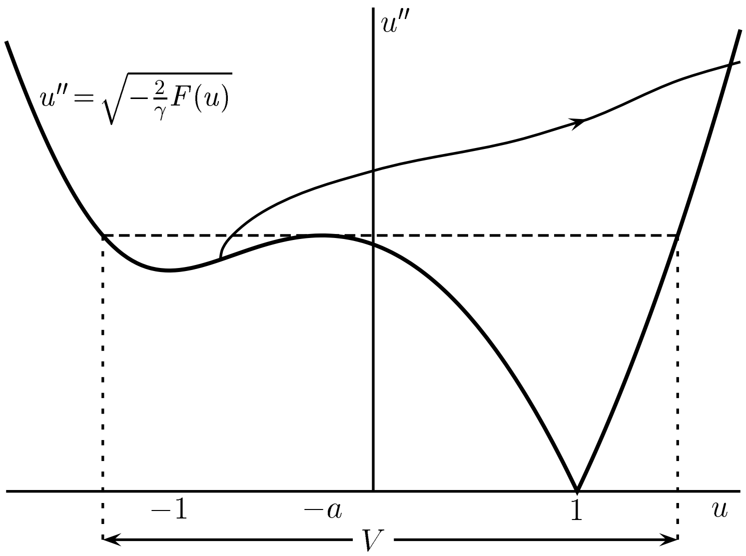

Figure 5: The (u, u'')-plane with the curve $u'' = \sqrt{-\frac{2}{\gamma}F(u)}$. We have sketched the orbit of u for $t \geq t_0$, which is discussed in the proof of Lemma 4.2. We have also indicated the set V, in which every bounded solution has its range.

by the differential equation, and (4.1) and (4.2). Thus $G(t) > 0$ in a right neighbourhood of t_0.

Secondly, we show that $G(t) > 0$ as long as $u(t) \leq 1$. We define $t_1 > t_0$ as the first maximum of $u(t)$ and $t_2 > t_0$ as the first point where $G(t_2) = 0$ (a priori, both t_1 and t_2 may be ∞). Then $t_2 < t_1$ since $u''(t) > 0$ as long as $G(t) > 0$. It now follows from the expression (2.3) for the energy and by (4.2) that

$$\begin{aligned} G'(t) &= u'''(t) + \frac{f(u(t))}{\sqrt{-2\gamma F(u(t))}} u'(t) \\ &= \frac{\frac{1}{2}u''^2(t) + \frac{1}{\gamma}F(u(t))}{u'(t)} + \left(\frac{1}{2\gamma} + \frac{f(u(t))}{\sqrt{-2\gamma F(u(t))}} \right) u'(t) \\ &> 0, \end{aligned}$$

as long as $G(t) > 0$ and $u(t) \leq 1$. Since $G(t) > 0$ in a right neighbourhood of t_0 this implies that $G(t) > 0$ and $G'(t) > 0$ as long as $u(t) \leq 1$, and thus $u(t_2) > 1$.

Finally, we define $t_3 > t_0$ as the first point where $u(t) = -a$. It is easily seen that $t_3 < t_2$. By the energy expression we have that $u'''(t) > 0$ as long as $G(t) > 0$, thus $u''(t_2) > u''(t_3) > \sqrt{-\frac{2}{\gamma}F(-a)}$. Combining the inequalities $u(t_2) > 1$ and $F(u(t_2)) = -\frac{\gamma}{2}u''^2(t_2) < F(-a)$, we infer that $u(t_2)$ lies outside V, so that u is unbounded. $\square$

Remark 4.3 Because all the non-trivial Floquet multipliers of Λ_- lie in $(1, \infty)$ (see Theorem 3.5(a)), Lemma 4.2 remains true for $c > 0$ sufficiently small. $\bullet$

Proof of Theorem 4.1. Our strategy is to deform $f(u)$ in several steps to the pure cubic $-u^3$ and let A go to zero. We have to do this in such a way that for each intermediate f the conclusion of Lemma 4.2 remains valid. We can then analyse the flow for the reduced equation (3.2), and obtain the results of Theorem 4.1 for Equation (3.1) via continuation arguments.

Step 1. We first deform $f(u)$ to a cubic which changes sign at $u = 1$ only. Let

$$f(u;\mu) \equiv f(u) + \mu(u-1)^2 = -(u-1)^3 + (\mu - a - 3)(u-1)^2 - 2(a+1)(u-1), \quad (4.3)$$

then $f(u,0) = (u+a)(1-u^2)$, and as we increase μ the two other zeros disappear for $\mu = 3 + a - \sqrt{8(1+a)}$. When $\mu = 3 + a$ we simply have

$$f_1(u) \equiv f(u; 3+a) = -(u-1)^3 - 2(a+1)(u-1).$$

For $f(u;\mu)$ we still have the result of Lemma 4.2.

Lemma 4.4 *Let $0 < \gamma < \sigma(a)$ and replace $f(u)$ by (4.3) with $0 \le \mu \le 3 + a$. Then W still consists of unbounded orbits only, all of which connect Λ_- to $u = 1$.*

Proof. For any $0 < \mu < 3 + a - \sqrt{8(1+a)}$ the argument is exactly the same as in the proof of Lemma 4.2. Note that $\sigma(a) < \frac{-F(u;\mu)}{2f(u;\mu)^2}$ in the interval where $f(u;\mu) < 0$ for all $\mu \in [0, 3+a]$. For $3 + a - \sqrt{8(1+a)} \le \mu \le 3 + a$ the result also holds, but by a different and less restrictive oscillation argument, which applies to any $f(u)$ with a single zero at which it goes from positive to negative and all $\gamma > 0$ (or, which is equivalent, for all $A > 0$). We already used this argument in the proof of part (c) of Theorem 3.3. The arguments showing that every solution $u \not\equiv 1$ oscillates towards infinity are almost identical (for $A > 0$ the second order term does not cause any difficulties). This completes the proof of the lemma. □

Continuing with the proof of Theorem 4.1, we change f to f_1 by letting μ go from $\mu = 0$ to $\mu = 3 + a$. This leaves the local structure near $X_5 = 0$, and in particular near Λ_-, unaffected, because the lower order terms of $f(u)$ correspond to terms in (3.29) which are at least quadratic in X_5.

Step 2. Next we let the coefficient A go to zero. This makes the eigenvalues of $u = 1$ go from real to complex without any of them going through the imaginary axis. Thus W varies smoothly, its dimension remaining two. The local structure near Λ_- again remains unaffected because A only appears in terms quadratic in X_5. For $A = 0$ we have arrived, shifting the zero of f to the origin, at the equation

$$u'''' - f_2(u) = 0, \quad \text{with } f_2(u) = -2(a+1)u - u^3 = 0.$$

Since we have shifted the zero of f to the origin, we now have that $W = W^s(0)$. All orbits in W are still unbounded and tend to Λ_- as $t \to -\infty$, since $f(u)$ has a single zero at which it goes from positive to negative (see the proof of Lemma 4.4).

Step 3. We deform $f_2(u)$ to $f_3(u)$ satisfying

$$f_3(u) = \begin{cases} -u & \text{for } u \text{ near } 0, \\ -u^3 & \text{for } u \text{ near } \pm\infty, \end{cases}$$

using deformations which change sign only in $u = 0$. Again $W = W^s(0)$ changes smoothly and by Remark 3.6 the local analysis near Λ_- is unaffected. Also, by the same arguments as above (see the proof of Lemma 4.4), every solution in W still comes out of Λ_- for every intermediate $f(u)$.

Step 4. We change f_3 using a family of functions satisfying

$$f(u; s) = \begin{cases} -u^s & \text{for } u \text{ near } 0, \\ -u^3 & \text{for } u \text{ near } \pm\infty. \end{cases} \tag{4.4}$$

Letting s increase from $s = 1$ to $s = 3$ we obtain a function $f_4(u)$ satisfying (4.4) with $s = 3$ (again using deformations which change sign only in $u = 0$). We note that for $s > 1$ the manifold W is the center-stable manifold of 0. Here we use Section 3.2 to conclude that in this process W changes continuously. The manifold $W = W^{cs}(0)$ still consists of orbits coming out of Λ_-.

Step 5. We change $f_4(u)$ to the pure power $-u^3$. Clearly our requirements concerning W and Λ_- are still met.

By Sections 3.1 and 3.3 we have that, after going through Steps 1–5, W is the product of Λ_- and the X_5-axis. In view of the non-trivial Floquet multipliers of Λ_- being in $(1, \infty)$, it holds that for any small $\varepsilon > 0$ there exists a negatively invariant tubular neighbourhood Λ_-^ε of Λ_- in Π with

$$\Lambda_-^\varepsilon \subset \{X = (X_1, X_2, X_3, X_4, X_5) \in \Pi \mid d(X, \Lambda_-) < \varepsilon\}.$$

We can choose this neighbourhood such that

$$\overline{\Lambda_-^\varepsilon} \cap \{X_5 = \varepsilon\} \;=\; \{(X_1, X_2, X_3, X_4) \in \Lambda_-,\ X_5 = \varepsilon\}. \tag{4.5}$$

Besides, we can choose Λ_ε such that the flow for our final equation $u'''' + u^3$ is transversal to $\delta\Lambda_-^\varepsilon$. Moreover, for $\varepsilon > 0$ sufficiently small, we can choose Λ_ε such the flow is transversal to $\delta\Lambda_-^\varepsilon$ for every intermediate $f(u)$ and A in the deformation process of Steps 1–5 above, hence also for the original system (3.1).

For any given $r > 0$ we can choose $\varepsilon > 0$ so small that the projection Γ_ε of $W \cap \delta\Lambda_-^\varepsilon$ on the (x_1, x_4)-plane (or, equivalently, on the (u, u''')-plane) is a curve with minimal distance to the origin at least r. To see this, we observe that the solution of (2.9) represented by Λ_- cannot have a point where $u = u''' = 0$, for in such a point also $u'' = 0$ in view of the energy $\mathcal{E}$ being zero. This would contradict the fact that $Q < 0$ on Λ_-. Thus in the X-variables Λ_- is uniformly bounded away from $(X_1, X_4) = (0, 0)$, so that for any $r > 0$ we can find an $\varepsilon > 0$ such that the projection of Λ_-^ε on the (u, u''')-plane has a distance larger than r from the origin. Therefore, the winding numbers around $u = \pm 1$ of the projection Γ_ε of $W \cap \delta\Lambda_-^\varepsilon$ on the (u, u''')-plane are well-defined for ε sufficiently small.

It follows from (4.5) that for our final equation $u'''' + u^3 = 0$ we have

$$W \cap \delta\Lambda_-^\varepsilon = \{(X_1, X_2, X_3, X_4, X_5) \,|\, (X_1, X_2, X_3, X_4) \in \Lambda_-,\ X_5 = \varepsilon\},$$

so that, choosing r large, $n(\Gamma_\varepsilon, -1) = n(\Gamma_\varepsilon, 1) = 1$. By continuity the winding numbers of Γ_ε do not change if we reverse Steps 1–5, and again by continuity arguments and Remark 4.3 this remains true for $c > 0$ sufficiently small.

Finally, we know that, tracing back orbits in W until they hit $\delta\Lambda_-^\varepsilon$, their energy $\mathcal{E}$ remains close to 0, provided we keep $c > 0$ sufficiently small. Thus $W \cap \delta K$ is contained in Λ_-^ε for small $c > 0$. Following $W \cap \delta\Lambda_-^\varepsilon$ backwards along the flow to $W \cap \delta K$ (which is a transversal intersection for $c > 0$), we see that the winding numbers $n(\Gamma, \pm 1)$ of the projection of $W \cap \delta K$ are also 1. This completes the proof of Theorem 4.1. □

5 The winding number for large speeds

In this section we examine $W \cap \delta K$ when c is large. We show that it is a topological circle and that its projection Γ on the (u, u''')-plane in Figure 2 has winding numbers $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$.

Lemma 5.1 *Let $f(u)$ be any smooth nonlinearity with zeros for $u \in \{-1, -a, 1\}$, having the same sign change behaviour as* (1.1b) *and suppose $f'(u)$ is bounded from above. Let $c > 0$ be a speed for which there exists a travelling wave connecting $u = -1$ to $u = 1$. Then c is bounded by a constant which depends only on γ, $F(-1)$, the difference $F(-1) - F(-a)$, and the upper bound M for $f'(u)$.*

Proof. Suppose u is a solution of (2.1) connecting $u = -1$ to $u = 1$. Integrating (2.4), we have

$$-F(-1) = F(1) - F(-1) = c \int_{-\infty}^{\infty} u'^2. \tag{5.1}$$

Multiplying (2.1) by u'' and integrating (by parts) we obtain

$$\gamma \int_{-\infty}^{\infty} u'''^2 + \int_{-\infty}^{\infty} u''^2 = \int_{-\infty}^{\infty} (f(u))' u' = \int_{-\infty}^{\infty} f'(u) u'^2 \le M \int_{-\infty}^{\infty} u'^2 = M \frac{-F(-1)}{c}. \tag{5.2}$$

Let $u_1 \in (-a, 1)$ be defined by

$$F(u_1) = \frac{F(-a) + F(-1)}{2}.$$

There must be points t_0, $t_1 \in \mathbb{R}$, $t_0 < t_1$, such that $u(t_0) = -a$, $u(t_1) = u_1$ and $u(t) \in [-a, u_1]$ for $t \in [t_0, t_1]$. The length of this interval is estimated from below by

$$(u_1 + a)^2 = \Big(\int_{t_0}^{t_1} u'(t)dt\Big)^2 \le (t_1 - t_0)^2 \int_{t_0}^{t_1} u'(t)^2 dt \le (t_1 - t_0)^2 \frac{-F(-1)}{c}.$$

On the one hand

$$\begin{aligned}
\int_{t_0}^{t_1} \big(-\gamma u'''(t)u'(t) + \frac{\gamma}{2}u''(t)^2 + \frac{1}{2}u'(t)^2\big)dt \\
&\ge \int_{t_0}^{t_1} \big(F(-1) - F(u(t))\big)dt \\
&\ge (F(-1) - F(u_1))(t_1 - t_0) = \frac{F(-1) - F(-a)}{2}(t_1 - t_0) \\
&\ge \frac{F(-1) - F(-a)}{2}(u_1 + a)\sqrt{\frac{c}{-F(-1)}}, \qquad (5.3)
\end{aligned}$$

while on the other hand, estimating using (5.1) and (5.2),

$$\begin{aligned}
\int_{t_0}^{t_1} \big(-\gamma u'''(t)u'(t) + \frac{\gamma}{2}u''(t)^2 + \frac{1}{2}u'(t)^2\big)dt \\
&\le \int_{t_0}^{t_1} \Big(\frac{\gamma}{2}\big(u'''(t)^2 + u''(t)^2\big) + \frac{1+\gamma}{2}u'(t)^2\Big)dt \\
&\le \big(M \max\{\gamma, 1\} + 1 + \gamma\big)\frac{-F(-1)}{2c}. \qquad (5.4)
\end{aligned}$$

By combining (5.3) and (5.4) we obtain

$$\frac{F(-1) - F(-a)}{2}(u_1 + a)\sqrt{\frac{c}{-F(-1)}} \le \big(M \max\{\gamma, 1\} + 1 + \gamma\big)\frac{-F(-1)}{2c}.$$

Since also

$$\frac{F(-a) + F(-1)}{2} = F(u_1) - F(-a) \le \frac{M}{2}(u_1 + a)^2,$$

it follows that

$$c \le M^{\frac{1}{3}}\big(M \max\{\gamma, 1\} + 1 + \gamma\big)^{\frac{2}{3}} \frac{-F(-1)}{F(-1) - F(-a)}.$$

This completes the proof of the lemma. □

Remark 5.2 Lemma 5.1 also holds for Equation (3.1) for all $A \in \mathbb{R}$. For $A > 0$ this is obvious, but for $A \le 0$ the proof is slightly more subtle (in estimate (5.4) one has to use the fact that $\int_{-\infty}^{\infty} u''^2 \le \lambda \int_{-\infty}^{\infty} u'''^2 + \frac{1}{4\lambda}\int_{-\infty}^{\infty} u'^2$ for all $\lambda > 0$). •

We now turn to the proof of Proposition 2.2(b).

Theorem 5.3 *For $c > 0$ sufficiently large the intersection of the stable manifold W of $u = 1$ and the boundary δK of K is a smooth simple closed curve which projects on a closed curve Γ in the (u, z)-plane with $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$.*

Proof. We first prove the theorem for a deformation of $f(u)$. We choose the nonlinearity $\tilde{f}(u)$ to satisfy

$$\tilde{f}(u) = f'(1)(u-1) \quad \text{in a neighbourhood } B_\varepsilon(1) \text{ of } u = 1.$$

For this deformed nonlinearity $\tilde{f}$ we compute the energy $\tilde{\mathcal{E}}$ on a closed curve in $\tilde{W}$ winding once around $u = 1$ with u-values contained in $B_\varepsilon(1)$. The equation is now linear near $u = 1$, and the characteristic equation

$$-\gamma\mu^4 + \mu^2 + f'(1) = c\mu$$

has two eigenvalues $-\mu_1$ and $-\mu_2$ with negative real part. For $c > 0$ large enough we have that μ_1 and μ_2 are real, and asymptotically

$$\mu_1 \sim \left(\frac{c}{\gamma}\right)^{\frac{1}{3}} \quad \text{and} \quad \mu_2 \sim \frac{-f'(1)}{c} \qquad \text{as } c \to \infty. \tag{5.5}$$

Since the equation is linear, $\tilde{W}$ is given by (for c large enough)

$$\tilde{W} = \{(u,v,w,z) \mid u = u(t) = 1 + A_1 e^{-\mu_1 t} + A_2 e^{-\mu_2 t},\ v = u'(t),\ w = u''(t),\ z = u'''(t)\} \tag{5.6}$$

We may choose a curve $S_1 \subset W$ around $u = 1$ parametrised by $\phi \in [0, 2\pi)$, by taking $t = 0$ and $A_1 = r\cos\phi$, $A_2 = r\sin\phi$ in (5.6) for some fixed $r > 0$. The projection of S_1 on the (u, u''')-plane is given by

$$\{(u,z) \mid u = 1 + r(\cos\phi + \sin\phi),\ z = -r(\mu_1^3\cos\phi + \mu_2^3\sin\phi),\ 0 \le \phi < 2\pi\}.$$

The energy on S_1 is given by

$$\begin{aligned}
-\mathcal{E} &= \int_0^\infty c u'(t)^2 dt = c\int_0^\infty (A_1\mu_1 e^{-\mu_1 t} + A_2\mu_2 e^{-\mu_2 t})^2 dt \\
&= c\left(\frac{A_1^2\mu_1}{2} + \frac{2A_1A_2\mu_1\mu_2}{\mu_1+\mu_2} + \frac{A_2^2\mu_2}{2}\right) = c\mu_2\left(\frac{A_1^2\mu_1}{2\mu_2} + \frac{2A_1A_2\mu_1}{\mu_1+\mu_2} + \frac{A_2^2}{2}\right).
\end{aligned} \tag{5.7}$$

Using (5.5) and estimating (5.7) from below we have, for c sufficiently large,

$$\mathcal{E} \le \frac{f'(1)}{4} r^2 < 0 \ \text{ on } S_1.$$

Thus, choosing an energy level $0 > \tilde{E}_0 > \frac{f'(1)}{4}r^2$, we have that S_1 lies in the complement of K. Let $\tilde{S} = \tilde{W} \cap \delta\tilde{K}$. Then $\tilde{S}$ lies inside S_1 and is obtained by tracing solutions in (5.6) of the linear equation forwards in time until they enter $\tilde{K}$. It follows that $\tilde{S}$ winds around $u = 1$ in $\tilde{W}$ exactly once and therefore its projection $\tilde{\Gamma}$ on the (u,z)-plane winds once around $(u,z) = (1,0)$.

The calculations above only involve u-values between $1 - r\sqrt{2}$ and $1 + r\sqrt{2}$ so we may change the definition of $\tilde{f}(u)$ outside this range. In particular, taking r small, we may choose $\tilde{f}(u)$ such that $\tilde{F}(u)$ has a minimum $\tilde{F}(-a) < \tilde{E}_0$ and a maximum $\tilde{F}(-1) \in (\tilde{E}_0, \tilde{F}(1))$, with $-1 < a < 1 - r\sqrt{2}$. Clearly $\tilde{\Gamma}$ does not wind around the point $(u,z) = (-1,0)$.

We continue $\tilde{f}$ to f and $\tilde{E}_0$ to E_0, taking c large enough as to stay within a class of nonlinearities for which there does not exist a connection between $u = -1$ and $u = 1$ (see Lemma 5.1). By continuity we still have that $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$. □

6 Other travelling waves

We consider again the travelling wave equation

$$cu' = -u'''' + Au'' + (u+a)(1-u^2), \tag{6.1}$$

the corresponding system being (3.1). In this section we focus on travelling waves that connect the unstable state $u = -a$ to one of the two stable state $u = \pm 1$.

Theorem 6.1 *Let $-1 < -a < 0$ and $A \in \mathbb{R}$. For every $c > 0$ there exists a travelling wave solution of (6.1) connecting $u = -a$ to $u = -1$.*

Proof. For all $c > 0$ we have that the three equilibria are hyperbolic and

$$\dim W^s(\pm 1) = \dim W^u(\pm 1) = 2, \quad \dim W^u(-a) = 3, \quad \dim W^s(-a) = 1.$$

Travelling wave solutions connecting $u = -a$ and $u = -1$ correspond to a nonempty intersection of $W^u(-a)$ and $\dim W^s(-1)$. Recall that

$$E(u, u', u'', u''') = -u'u''' + \frac{1}{2}u''^2 + \frac{A}{2}u'^2 + F(u), \text{ where } F(u) = \int_1^u (s+a)(1-s^2)ds,$$

satisfies (2.4). We take $F(-1) < E_1 < F(1)$ and consider the set

$$\tilde{K} = \{(u, v, w, z) \mid E(u, v, w, z) = -vz + \frac{1}{2}w^2 + \frac{A}{2}v^2 + F(u) \leq E_1\}.$$

Now suppose that for some $c > 0$ the theorem is false. Then all orbits in $W^u(-a)$ have to leave $\tilde{K}$ through $\delta\tilde{K}$, because an orbit with bounded energy has no other choice than to converge to an equilibrium, see the proof Proposition 2.1, and $u = -1$, the only equilibrium in $\tilde{K}$ with energy larger than $E(-a)$, is excluded by assumption. Thus we have that the intersection of $W^u(-a)$ and $\delta\tilde{K}$ is diffeomorphic to a 2-sphere S^2.

For the moment we consider the case that $A > 0$. Since $\delta\tilde{K}$ is given by

$$A(v - \frac{z}{A})^2 + w^2 = 2E_1 - 2F(u) + \frac{z^2}{A}, \tag{6.2}$$

we may deform it smoothly into

$$\{(u, v, w, z) \mid u^2 + z^2 = 1 + v^2 + w^2\},$$

which defines a 3-manifold diffeomorphic to $\mathbb{R}^2 \times S^1$. As deformations we use

$$(\lambda A + 1 - \lambda)(v - \lambda\frac{z}{A})^2 + w^2 = G(u, \lambda) + (1 - \lambda + \frac{\lambda}{A})z^2,$$

with λ running from 1 to 0, $G(u, 1) = 2E_1 - 2F(u)$ and $G(u, 0) = -1 + u^2$. Singularities can only appear in points on these manifolds where $G_u = v = w = z = 0$ and can thus be avoided by the choice of E_1.

It follows that $\delta\tilde{K}$ is diffeomorphic to $\mathbb{R}^2 \times S^1$, or, equivalently, to the open solid torus. The intersection $W^u(-a) \cap \delta\tilde{K}$, being diffeomorphic to S^2, divides $\delta\tilde{K}$ into two components, one bounded and diffeomorphic to an open ball in $\mathbb{R}^3$, the other unbounded (this is in fact not completely straightforward.* One needs to lift a neighbourhood of $W^u(-a)\cap\delta\tilde{K}$ to the universal covering space $\mathbb{R}^3$ of $\tilde{K}$ and invoke the generalized Schoenflies Theorem (see [3, page 236]). It follows from this theorem that the lift of $W^u(-a) \cap \delta\tilde{K}$ indeed divides $\mathbb{R}^3$ into an unbounded component and a bounded component, which is homeomorphic to an open ball in $\mathbb{R}^3$. Besides, the unbounded component of the complement of the countable union of lifts of $W^u(-a)\cap\delta\tilde{K}$ is path-connected, since it is homotopic to $\mathbb{R}^3 \setminus \mathbb{Z}$).

Now consider the piecewise smooth 3-manifold formed by the disjoint union of $W^u(-a)\cap\tilde{K}$ and this bounded component of $\delta\tilde{K}\setminus(W^u(-a)\cap\delta\tilde{K})$. This 3-manifold is homeomorphic to two closed balls in $\mathbb{R}^3$ sharing an S^2, namely $W^u(-a)\cap\delta\tilde{K}$, as boundary and is therefore homeomorphic to an S^3. By the Jordan-Brouwer theorem this 3-manifold divides $\mathbb{R}^4$ to two components, one bounded, the other unbounded. We notice that the bounded component is negatively invariant. Clearly both components contain exactly one of the two orbits which together form $W^s(-a)$. Now consider the orbit in $W^s(-a)$ contained in the bounded component (which is negatively invariant). Since its energy is bounded we may, again by the argument in the proof of Proposition 2.1, conclude that, tracing it backwards, it must go to an equilibrium with energy less than the energy of $u = -a$. Since such an equilibrium does not exist, we have arrived at a contradiction.

The cases $A < 0$ and $A = 0$ are similar, the only changes being that we deform $\delta\tilde{K}$, given by (6.2), to $u^2 + v^2 = 1 + z^2 + w^2$ if $A < 0$, and that for $A = 0$ we rewrite $\delta\tilde{K}$ as $-2vz + w^2 = 2E_1 - 2F(u)$ which deforms into $-2vz + w^2 = -1 + u^2$ or $\frac{1}{2}(v+z)^2 + u^2 = \frac{1}{2}(v-z)^2 + w^2 + 1$. This completes the proof of the theorem. □

We now show that for large $c > 0$ there exists a travelling wave connecting $u = -a$ to $u = +1$.

Theorem 6.2 *Let $-1 < -a < 0$ and $A \in \mathbb{R}$.*

(a) *There exists a $c_* \geq 0$ such that for all $c > c_*$ there is a travelling wave solution of (2.1) connecting $u = -a$ to $u = 1$.*

(b) *Suppose that $A > \frac{1}{\sqrt{\sigma(a)}}$. Then there exists a $c^* > 0$ such that c^* is the largest speed for which there exists a travelling wave solution of (2.1) connecting $u = -1$ to $u = 1$. For all $c > c^*$ there exists a connection between $u = -a$ and $u = 1$.*

Proof. We consider the stable manifold $W = W^s(1)$ of $u = 1$. We have shown in Theorem 5.3 that for $c > 0$ large enough the intersection of the stable manifold W of $u = -1$ and the boundary δK of K (defined in (2.5)) is a smooth simple closed curve which projects on a closed curve Γ in the (u, z)-plane with $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$. We note that Lemma 5.1 (see Remark 5.2)

*We gratefully acknowledge the help of H. Geiges in dealing with this topological problem.

and Theorem 5.3 still hold for $A \leq 0$. For $A < 0$ we need to consider the $(u, v - z/A)$-plane in stead of the (u, z)-plane, and for $A = 0$ we have to consider the $(u, v + z)$-plane, as already explained at the end of the proof of Theorem 6.1.

Let us now assume, by contradiction, that for such a c there is no connection between $u = -a$ and $u = 1$. The intersection between W and δK depends continuously on the energy level E as long as we do not encounter an equilibrium point. Assuming there is no connection between between $u = -a$ and $u = 1$, we let E decrease from $F(-1) > E_0 > F(-a)$ to $E_2 < F(-a)$. The projection Γ in the (u, z)-plane then depends continuously on E, as do the winding numbers, so that $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$ for all $E_0 \leq E \leq E_2$. However, for the energy level E_2 we have that $(-1, 0)$ and $(1, 0)$ lie in the same component of the complement of the projection of δK onto the (u, z) plane. Therefore $n(\Gamma, -1) = n(\Gamma, 1)$, a contradiction. This completes the proof of part (a).

To prove part (b) we suppose that $A > \frac{1}{\sqrt{\sigma(a)}}$. We again take an energy level $F(-1) > E_0 > F(-a)$. We define the following sets

$$
\begin{aligned}
\mathcal{C}_1 &\stackrel{\text{def}}{=} \{c > 0 \,|\, \text{there is no connection between } u = -1 \text{ and } u = 1\}, \\
\mathcal{C}_2 &\stackrel{\text{def}}{=} \{c > 0 \,|\, c \in \mathcal{C}_1 \text{ and } n(\Gamma, -1) = 0 \text{ and } n(\Gamma, 1) = 1\}, \\
\mathcal{C}_3 &\stackrel{\text{def}}{=} \{c > 0 \,|\, c \in \mathcal{C}_1 \text{ and either } n(\Gamma, -1) \neq 0 \text{ or } n(\Gamma, 1) \neq 1\}.
\end{aligned}
$$

We have that $\mathcal{C}_1 = \mathcal{C}_2 \cup \mathcal{C}_3$ and we notice that by compactness of Γ and transversality of $W \cap \delta K$ the sets $\mathcal{C}_2$ and $\mathcal{C}_3$ are open.

On the one hand we know from the proof of part (a) that whenever there is no connection between $u = -1$ and $u = 1$ and when in addition $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$, then a connection between $u = -a$ and $u = 1$ always exists. This situation occurs for $c > 0$ large enough, so that $\mathcal{C}_2 \neq \varnothing$, and

$$c^* \stackrel{\text{def}}{=} \inf\{\tilde{c} > 0 \,|\, c \in \mathcal{C}_2 \text{ for all } c > \tilde{c}\}$$

is well-defined. On the other hand for small c and $A > \frac{1}{\sqrt{\sigma(a)}}$ there is no connection of $u = -a$ or $u = -1$ with $u = 1$ since W consists of unbounded orbits only (see Lemma 4.2 and Remark 4.3), and $n(\Gamma, -1) = 1$ and $n(\Gamma, 1) = 1$. Thus c^* is strictly positive. Clearly $c^* \notin \mathcal{C}_1$ since c^* is a boundary point of $\mathcal{C}_2$ and both $\mathcal{C}_2$ and $\mathcal{C}_3$ are disjoint open sets which together constitute the whole of $\mathcal{C}_1$. This proves that for $c = c^*$ a connection exists between $u = -1$ and $u = 1$, and it follows from the definition of c^* that this is the largest speed for which such a connection exists. $\square$

We note that the method in the proof of Theorem 6.2(a) can also be used to prove Theorem 6.1. Besides, if $a = 0$ then the same method shows that there exist travelling waves connecting $u = 0$ to $u = \pm 1$ for all $c > 0$ and all $A \in \mathbb{R}$. We have a similar result for

$$cu' = -u'''' + Au'' + f(u) \qquad \text{with } f(u) = u(1 - u). \tag{6.3}$$

Theorem 6.3 *For every $c > 0$ and all $A \in \mathbb{R}$ there exists a travelling wave solution of* (6.3) *connecting the unstable state $u = 0$ to the stable state $u = 1$.*

The proof is a minor modification of the previous ones, and for once we leave the details to the reader.

7 Concluding remarks

7.1 Non-polynomial nonlinearities

The analysis in this paper applies to more general nonlinearities $f(u)$. For travelling waves connecting $u = -a$ to $u = \pm 1$ the analysis in Section 6 goes on unchanged, and for travelling waves connecting $u = -1$ to $u = +1$ we have the following theorem.

Theorem 7.1 *For some $a \in (-1, 1)$ and $M > 0$, let $f(u) \in C^1(\mathbb{R})$ satisfy the assumptions*

$$\begin{aligned} f(u) = 0 \;\Leftrightarrow\; u \in \{-1, -a, 1\}, &\qquad F(-1) < F(1), \\ f'(u) \leq M \;\; \text{for all } u \in \mathbb{R}, &\qquad F(u) \to -\infty \;\; \text{as } u \to \pm\infty. \end{aligned}$$

Then (1.1a) *has a travelling wave solution connecting $u = -1$ and $u = 1$ for all $\gamma < \sigma$, where σ is given by* (1.3).

Proof. The analysis of Sections 2 to 5 is applicable. The only point of care is that we have used in the analysis at infinity (Section 3) the fact that f is polynomial for large u. We accommodate for non-polynomial nonlinearities by changing the nonlinearity for large u, which is without consequences for the existence of travelling waves as is seen from Lemma 7.3, since an a priori bound on $F(u)$ implies an a priori bound on $|u|$. □

Remark 7.2 When $F(-1) = F(1)$ there is a standing wave between $u = -1$ and $u = 1$ for all $\gamma > 0$, as follows from the minimisation procedure in [9]. •

Lemma 7.3 *Let $f(u)$ satisfy the assumptions in Theorem 7.1. Then there exists a $C > 0$ such that for every connection $u(t)$ between $u = -1$ and $u = +1$, we have $F(u(t)) \geq -C$ for all $t \in \mathbb{R}$. The constant C depends on γ, M, $F(-a)$, and $F(-1)$ only.*

Proof. We may suppose that there is a connection u with range not contained in the bounded interval $\{u \in \mathbb{R} \,|\, F(u) \geq F(-a)\}$, otherwise we already have our desired uniform bound. Therefore, without loss of generality we may assume that

$$F(u(0)) = \min_{t \in \mathbb{R}} F(u(t)) < F(-a). \tag{7.1}$$

It follows from Lemma 5.1 that the wave speed c is bounded above by a constant c_0, which depends on γ, M, $F(-a)$ and $F(-1)$ only.

We consider the case where $u(0) < -1$ (the case $u(0) > 1$ is completely analogous). Since

$$\mathcal{E}(u, u', u'', u''')(t) \in (F(-1), F(1)) = (F(-1), 0) \quad \text{for all } t \in \mathbb{R}, \tag{7.2}$$

we clearly have that

$$u(0) < -1, \quad u'(0) = 0, \quad 0 < \sqrt{-\frac{2}{\gamma}\big(F(-1) - F(u(0))\big)} < u''(0) < \sqrt{-\frac{2}{\gamma}F(u(0))},$$

and, changing t to $-t$ (and c to $-c$) if necessary, we may assume that $u'''(0) \geq 0$. Since $u''''(0) > 0$ by the differential equation, and since $u(t)$ tends to an equilibrium point as $t \to \infty$, there exists a $t_1 > 0$ such that $u'''(t) > 0$ for $0 < t < t_1$ and $u'''(t_1) = 0$. Equation (2.4) implies that

$$-\gamma u'(t)u'''(t) + F(u(t)) - F(u(0)) = -\frac{\gamma}{2}\big(u''(t)^2 - u''(0)^2\big) - \frac{1}{2}u'(t)^2 + c\int_0^t u'(s)^2 ds.$$

By (7.1) we know that $F(u(t_1)) \geq F(u(0))$, so that

$$\frac{\gamma}{2}\big(u''(t_1)^2 - u''(0)^2\big) + \frac{1}{2}u'(t_1)^2 \leq c\int_0^{t_1} u'(s)^2 ds. \tag{7.3}$$

Since $u''(t)$ increases on $(0, t_1)$ it is immediate that $c > 0$. We also obtain that

$$\frac{\gamma}{2}\big(u''(t)^2 - u''(0)^2\big) \leq c\int_0^t u'(s)^2 ds < \alpha, \tag{7.4}$$

where $\alpha \overset{\text{def}}{=} -F(-1) > 0$. Estimating $u''(0)t \leq u'(t) \leq u''(t_1)t$ on $(0, t_1)$ gives

$$u'(t_1)^2 \geq t_1^2 u''(0)^2 \quad \text{and} \quad \frac{t_1^3}{3}u''(0)^2 \leq \int_0^{t_1} u'(s)^2 ds \leq \frac{t_1^3}{3}u''(t_1)^2.$$

On the one hand it follows from (7.3) and (7.4) that

$$t_1 \geq \frac{3u''(0)^2}{2cu''(t_1)^2} \geq \frac{3u''(0)^2}{2c(\frac{2\alpha}{\gamma} + u''(0)^2)}. \tag{7.5}$$

On the other hand it follows that

$$\frac{ct_1^3}{3}u''(0)^2 \leq \alpha. \tag{7.6}$$

Combining (7.5) and (7.6) we obtain

$$u''(0)^8 \leq \frac{8\alpha c^2}{9}\left(\frac{2\alpha}{\gamma} + u''(0)^2\right)^3.$$

Since c was already uniformly bounded by c_0, this gives a bound on $u''(0)^2$. By (7.1) and (7.2) we have

$$F(u(t)) \geq F(u(0)) \geq F(-1) - \frac{\gamma}{2}u''(0)^2 \qquad \text{for all } t \in \mathbb{R}.$$

This completes the proof of Lemma 7.3. $\square$

7.2 Open questions

The most apparent (and attractive) open problem is the existence of a travelling wave connecting $u = -1$ to $u = 1$ for large values of γ. Equation (1.1) with $a = 0$ admits an abundance of standing wave solutions for $\gamma > \frac{1}{8}$. It has been proved in [4] that there are many travelling waves for large γ and very small $a > 0$, and these include travelling waves with arbitrary small c, which rules out the possibility of a statement analogous to Proposition 2.2(a). A different method, exploiting the more complicated dynamics, seems to be required to extend the existence results to all $0 < a < 1$ for large γ.

A second question is whether for small γ the travelling wave is unique, as one might expect from the limiting second order case ($\gamma = 0$). Besides, the method used in this paper does not give any information about the shape of the solution. For example, we would like to know for which values of γ and a the solution is monotone. Since we do not know the value of c for which a traveling wave occurs, we do not even know whether the connected equilibrium points are approached monotonically or in an oscillatory manner.

Finally, the question arises to what extent the travelling wave solution is of importance to the dynamics of the PDE. It might be a limit profile for a class of initial conditions as is the case for the second order equation. Since travelling waves connecting $u = -a$ to $u = \pm 1$ exist for large ranges of c, it would be interesting to know which of these waves is generally encountered. In [5, 8] the wave selection mechanism has been investigated for a propagating front which is formed from localised initial data (i.e., $u + a$ is localised). Using the physically motivated assumption that the linearised equation (around $u = -a$) drives the system, it is argued that for $\gamma < \frac{1}{12(1-a^2)}$ one of the travelling waves is selected (and the wave speed is calculated), while for $\gamma > \frac{1}{12(1-a^2)}$ the propagating front is argued not to have a fixed profile. However, it is in fact unknown whether the selected travelling wave is stable.

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