POINTS ON SUBVARIETIES OF TORI

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To Professor Alan Baker on his 60th birthday

1. Introduction.

Denote by $\mathbf{G}_{\mathbf{m}}^N$ the N-dimensional torus. Let K be any algebraically closed field of characteristic 0. Further, let Γ be a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^N(K) = (K^*)^N$ and $\overline{\Gamma}$ its division group. We give a survey on results about the structure of sets

$$X \cap \overline{\Gamma},$$

where X is an algebraic subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over K.

We recall that $\mathbf{G}_{\mathbf{m}}^N$ consists of points $(x_1, \dots, x_N)$ with $x_1 \cdots x_N \neq 0$. For $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{y} = (y_1, \dots, y_N) \in \mathbf{G}_{\mathbf{m}}^N$ and $m \in \mathbf{Z}$ we define coordinatewise multiplication $\mathbf{x} * \mathbf{y} = (x_1 y_1, \dots, x_N y_N)$ and exponentiation $\mathbf{x}^m = (x_1^m, \dots, x_N^m)$. By a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over a field K we mean an irreducible Zariski-closed subset of $\mathbf{G}_{\mathbf{m}}^N$, that is a set $\{\mathbf{x} \in \mathbf{G}_{\mathbf{m}}^N : f_1(\mathbf{x}) = 0, \dots, f_M(\mathbf{x}) = 0\}$ where $f_1, \dots, f_M$ are polynomials in $K[x_1, \dots, x_N]$ generating a prime ideal. By a *subtorus* of $\mathbf{G}_{\mathbf{m}}^N$ we mean a subvariety which is a subgroup of $\mathbf{G}_{\mathbf{m}}^N$, i.e., which is closed under coordinatewise multiplication. Thus, a subtorus is the set of solutions of a system of equations $X_1^{a_1} \cdots X_N^{a_N} = X_1^{b_1} \cdots X_N^{b_N}$ where the a_i, b_i are non-negative integers, and a subtorus is isomorphic to $\mathbf{G}_{\mathbf{m}}^{N'}$ for some $N' \leq N$. By a *torus coset* over K we mean a translate of a subtorus, i.e. $\mathbf{u} * H = \{\mathbf{u} * \mathbf{x} : \mathbf{x} \in H\}$ where $\mathbf{u} \in \mathbf{G}_{\mathbf{m}}^N(K)$ and where H is a subtorus. For more basic facts about subtori and torus cosets we refer to [36], Section 2.

As before, let K be an algebraically closed field of characteristic 0, X a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over K, Γ a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^N(K) = (K^*)^N$ and $\overline{\Gamma}$ its

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division group, i.e., the group of $\mathbf{x} \in \mathbf{G}_{\mathbf{m}}^N(K)$ for which there is a positive integer m with $\mathbf{x}^m \in \Gamma$. We define the rank of Γ to be the rank of $\Gamma/\Gamma_{\rm tors}$. In 1938, Chabauty [3] proved the following result about the set $X \cap \Gamma$ (i.e. not with the division group):

Theorem A. *Suppose that $K = \overline{\mathbf{Q}}$ and that $\operatorname{rank}\Gamma < N - \dim X$. Then if $X \cap \Gamma$ is infinite, there is a torus coset $\mathbf{u}{*}H \subset X$ such that $(\mathbf{u}{*}H) \cap \Gamma$ is infinite.*

In his proof, Chabauty used a method based on p-adic power series which was introduced by Skolem.

Chabauty's work inspired Lang to formulate a general conjecture (cf. [23], p. 221) the following special case of which was proved by Laurent in 1984 [24]:

Theorem B. *$X \cap \overline{\Gamma}$ is contained in a finite union of torus cosets $\mathbf{u}_1{*}H_1 \cup \cdots \cup \mathbf{u}_t{*}H_t$ with $\mathbf{u}_i{*}H_i \subset X$ for $i = 1, \ldots, t$.*

Laurent deduced his theorem from a result on linear equations. Let $a_1, \ldots, a_N \in K^*$ and consider the equation

$$a_1x_1 + \cdots + a_Nx_N = 1 \qquad \text{in } \mathbf{x} = (x_1, \ldots, x_N) \in \overline{\Gamma}. \tag{1.1}$$

To avoid easy constructions of infinite sets of solutions, we consider only *non-degenerate* solutions of (1.1), these are solutions with

$$\sum_{i \in I} a_ix_i \neq 0 \quad \text{for each non-empty subset } I \text{ of } \{1, \ldots, N\}. \tag{1.2}$$

It follows from work of the author [7], van der Poorten and Schlickewei [28], and Laurent [24] that equation (1.1) has at most finitely many non-degenerate solutions.

The ingredients going into the proof of this result were W.M. Schmidt's Subspace Theorem, cf. Section 2.3 (with which one can handle equations (1.1) with solutions $\mathbf{x} \in \Gamma$ where $\Gamma \subset \mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}})$), a specialization argument (with which one can extend the result to equations with solutions $\mathbf{x} \in \Gamma$ where $\Gamma \subset \mathbf{G}_{\mathbf{m}}^N(K)$ for some arbitrary field K of characteristic 0) and some Kummer theory (to pass from Γ to $\overline{\Gamma}$). Laurent proved his Theorem

B by taking polynomials $a_1M_1 + \cdots + a_sM_s$ vanishing identically on X, where the a_i are constants and the M_i are monomials, and applying the result on linear equations to $a_1M_1 + \cdots + a_sM_s = 0$ where the M_i are considered to be the unknowns.

We now discuss quantitative versions of Theorem B, i.e., explicit upper bounds for the number of torus cosets t. This is joint work of Schlickewei and the author. We keep our conventions that K is an algebraically closed field of characteristic 0, Γ a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^N(K)$, $\overline{\Gamma}$ its division group, and X a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over K. A linear subvariety of $\mathbf{G}_{\mathbf{m}}^N$ is defined by a set of polynomials of degree 1, which may have constant terms. The degree $\deg X$ of X is the number of points in the intersection of X with a general linear subvariety of $\mathbf{G}_{\mathbf{m}}^N$ of dimension $N - \dim X$. (In other words, if we embed $\mathbf{G}_{\mathbf{m}}^N$ into projective space $\mathbf{P}^N$ by means of the map $\iota : (x_1, \ldots, x_N) \mapsto (1 : x_1 : \cdots : x_N)$ and Y is the Zariski closure of $\iota(X)$ in $\mathbf{P}^N$, then we define $\deg X := \deg Y$, with the usual definition for the latter, cf. [20], p. 52.)

The main tool is the following result of Schlickewei, Schmidt and the author [12], which gives an explicit upper bound for the number of non-degenerate solutions of the linear equation (1.1):

Theorem 1.1. *Suppose Γ has rank $r \geq 0$. Then equation (1.1) has at most $2^{2^{5N}(r+1)}$ non-degenerate solutions.*

For a historical survey on equation (1.1) we refer to [9].

By making explicit the arguments in Laurent's proof, Schlickewei and the author [11] proved the following quantitative version of Theorem B:

Theorem 1.2. *Suppose $\operatorname{rank}\Gamma = r \geq 0$, $\dim X = n$, $\deg X = d$. Then $X \cap \overline{\Gamma}$ is contained in some union of torus cosets $\mathbf{u}_1 * H_1 \cup \cdots \cup \mathbf{u}_t * H_t$ where $\mathbf{u}_i * H_i \subset X$ for $i = 1, \ldots, t$ and where*

$$t \leq c(n,d)^{r+1} \qquad \text{with } c(n,d) = 2^{2^{6d\binom{n+d}{d}}}. \tag{1.3}$$

The main features of this upper bound are its good dependence on r and its uniform dependence on n and d. It should be noted that the bound depends on $n = \dim X$ and not on N. However, if L is the smallest linear subvariety of $\mathbf{G}_{\mathbf{m}}^N$ containing X and X has codimension δ in L then $d \geq \delta + 1$ (cf. [18], p. 173); hence the upper bound depends implicitly on δ.

Theorem 1.2 is the first result giving an explicit upper bound for the number of torus cosets in the most general case, but such explicit bounds have been given before in certain special cases. Let S be a finite set of places in some number field F. Denote by U_S the group of S-units and by U_S^N the N-fold direct product. From a result of Győry ([19], Thm. 9) it follows that if X is defined over F then $X \cap (U_S)^N$ is contained in the union of at most $c_1(N, d, \#S, [F:\mathbf{Q}])$ torus cosets contained in X, with some explicit expression for c_1. From a result of Schmidt ([36], Thm. 2) it follows that if $\operatorname{rank} \Gamma = 0$, i.e. if $\overline{\Gamma} = U^N$ where U is the group of roots of unity in some algebraically closed field K of characteristic 0, then $X \cap \overline{\Gamma}$ is contained in the union of at most $c_2(N, d)$ torus cosets contained in X, with some explicit expression for c_2.

We deduce some corollaries of Theorem 1.2. We keep the notation of Theorem 1.2. Let $X^{\rm exc}$ be the union of all torus cosets $\mathbf{u} * H$ of dimension ≥ 1 which are contained in X and let $X^0 = X \backslash X^{\rm exc}$. For instance, if X is the variety given by equation (1.1), then X^0 consists precisely of the non-degenerate points of X, i.e., with (1.2). Since zero-dimensional torus cosets are simply points, we obtain at once from Theorem 1.2:

Corollary 1.3. *Let Γ, X be as in Theorem 1.2. Then $X^0 \cap \overline{\Gamma}$ has cardinality at most $c(n,d)^{r+1}$.*

A special case of this is:

Corollary 1.4. *Let Γ be as in Theorem 1.2 and let X be an irreducible curve of degree d in $\mathbf{G}_{\mathbf{m}}^N$ defined over K. Suppose X is not a torus coset. Then $X \cap \overline{\Gamma}$ has cardinality at most $2^{2^{6d(d+1)}(r+1)}$.*

A qualitative version of this result (giving only the finiteness) follows from work of Lang [22] and Liardet [25].

We now consider points that "lie almost in $\overline{\Gamma}$." To make this precise we need heights. Therefore we have to restrict ourselves to the case that X is defined over $\overline{\mathbf{Q}}$ and that $\Gamma \subset \mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}}) = (\overline{\mathbf{Q}}^*)^N$.

Denote by h the usual logarithmic Weil height on $\mathbf{P}^N(\overline{\mathbf{Q}})$ (cf. section 2.1) and for $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}})$ put $h(\mathbf{x}) := h(1 : x_1 : \cdots : x_N)$. Let Γ be a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}})$ and $\overline{\Gamma}$ its division group. For $\varepsilon > 0$, we define the following sets:

$$
\begin{aligned}
T(\overline{\Gamma}, \varepsilon) = \{\mathbf{x} \in \mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}}) : \exists\, \mathbf{y}, \mathbf{z} \text{ with } \mathbf{x} = \mathbf{y} * \mathbf{z}, \\
\mathbf{y} \in \overline{\Gamma},\ \mathbf{z} \in \mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}}),\ h(\mathbf{z}) \leq \varepsilon\}, \quad (1.4) \\
C(\overline{\Gamma}, \varepsilon) = \{\mathbf{x} \in \mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}}) : \exists\, \mathbf{y}, \mathbf{z} \text{ with } \mathbf{x} = \mathbf{y} * \mathbf{z}, \\
\mathbf{y} \in \overline{\Gamma},\ \mathbf{z} \in \mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}}),\ h(\mathbf{z}) \leq \varepsilon \cdot (1 + h(\mathbf{y}))\}. \quad (1.5)
\end{aligned}
$$

We may view $T(\overline{\Gamma}, \varepsilon)$ as a thickening of $\overline{\Gamma}$ and $C(\overline{\Gamma}, \varepsilon)$ as a truncated cone centered around $\overline{\Gamma}$. It is obvious that $T(\overline{\Gamma}, \varepsilon) \subset C(\overline{\Gamma}, \varepsilon)$. For instance, if $\operatorname{rank} \Gamma = 0$ then $T(\overline{\Gamma}, \varepsilon) = C(\overline{\Gamma}, \varepsilon)$ is the set of points of height $\leq \varepsilon$.

We mention the following result of Schlickewei, Schmidt and the author [12]:

Theorem 1.5. *Let $0 < \varepsilon < 2^{-2^{4N}}$. Suppose Γ has rank $r \geq 0$. Then the set of vectors $\mathbf{x} = (x_1, \ldots, x_N)$ satisfying*

$$x_1 + \cdots + x_N = 1\,, \quad \mathbf{x} \in C(\overline{\Gamma}, \varepsilon) \tag{1.6}$$

is contained in the union of at most $2^{2^{4N}(r+1)}$ proper linear subspaces of $\overline{\mathbf{Q}}^N$.

One may wonder whether it is possible to deduce a quantitative result similar to Theorem 1.2 for sets

$$X \cap C(\overline{\Gamma}, \varepsilon)$$

if in the proof of Theorem 1.2 one uses Theorem 1.5 instead of Theorem 1.1. This approach does not work. A problem is that Theorem 1.5 deals only with equations all of whose coefficients are equal to 1, whereas by going through the proof of Theorem 1.2

one arrives at equations of the shape

$$a_1x_1 + \cdots + a_Nx_N = 1 \quad \text{in } \mathbf{x} \in C(\overline{\Gamma}, \varepsilon) \tag{1.7}$$

with coefficients $a_1, \ldots, a_N$ over which one has no control.

One may try to reduce (1.7) to (1.6) by working with the tuple of variables $\mathbf{w} = (w_1, \ldots, w_N)$ where $w_1 = a_1x_1, \ldots, w_N = a_Nx_N$, and with the group Γ' generated by Γ and $\mathbf{a} = (a_1, \ldots, a_N)$. Then Γ' has rank $\leq r+1$. We clearly have $w_1 + \cdots + w_N = 1$. But then the problem remains that in general $\mathbf{x} \in C(\overline{\Gamma}, \varepsilon)$ does not imply that $\mathbf{w} \in C(\overline{\Gamma'}, \varepsilon)$. At this point our argument breaks down.

The situation is quite different if we restrict ourselves to points $\mathbf{x}$ belonging to the smaller set $T(\overline{\Gamma}, \varepsilon)$. Notice that for such $\mathbf{x}$ we do have $\mathbf{w} \in T(\overline{\Gamma'}, \varepsilon)$. Thus, by applying Theorem 1.5 but restricted to solutions in $T(\overline{\Gamma}, \varepsilon)$, it is possible to obtain an analogue of Theorem 1.2 for sets $X \cap T(\overline{\Gamma}, \varepsilon)$. Schlickewei and the author [11] proved the following result:

Theorem 1.6. *Let Γ be a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}})$ of rank $r \geq 0$. Further, let X be a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over $\overline{\mathbf{Q}}$ of dimension n and degree d. Let $c(n,d)$ be the quantity from Theorem 1.2. Suppose that $0 < \varepsilon < c(n,d)^{-1}$.*
Then $X \cap T(\overline{\Gamma}, \varepsilon)$ is contained in a union of torus cosets $\mathbf{u}_1 * H_1 \cup \cdots \cup \mathbf{u}_t * H_t$ where $\mathbf{u}_i * H_i \subset X$ for $i = 1, \ldots, t$ and where $t \leq c(n,d)^{r+1}$.

Theorem 1.6 implies that in particular, $X^0 \cap T(\overline{\Gamma}, \varepsilon)$ has cardinality at most $c(n,d)^{r+1}$. Previously, Bombieri and Zannier [1] (Thm. 1) and in a more precise form Schmidt [36] (Thm. 4) and David and Philippon [5] (Thm. 1.3) obtained a similar result in the special case that $r = 0$, i.e., that $T(\overline{\Gamma}, \varepsilon)$ is just the set of points with small height in $\mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}})$. The result of David and Philippon was one of the ingredients in the proofs of the results mentioned above.

The best one can obtain at present for the set $X \cap C(\overline{\Gamma}, \varepsilon)$ is the following result of Schlickewei and the author [11]. By $h(X)$ we denote the logarithmic height of X (see section 2.1). Given a subtorus H of $\mathbf{G}_{\mathbf{m}}^N$, let X^H denote the union of all torus cosets $\mathbf{u} * H$ contained in X. The set X^H is Zariski-closed in X.

Theorem 1.7. *(i). Let* Γ, X *be as in Theorem 1.6. There are an ineffective constant* $\alpha = \alpha(N, d, \Gamma) > 0$, *depending only on* N, d *and* Γ, *and an effective constant* $\beta = \beta(N, d) > 0$ *depending only on* N *and* d, *such that for every* ε *with* $0 < \varepsilon < \frac{1}{\alpha+\beta h(X)}$, *the set* $X^0 \cap C(\overline{\Gamma}, \varepsilon)$ *is finite.*
(ii). Let H *be a positive-dimensional subtorus such that* $X^H \neq \emptyset$ *and such that* $H \cap \Gamma$ *is not a torsion group. Then for every* $\varepsilon > 0$, $X^H \cap C(\Gamma, \varepsilon)$ *is Zariski-dense in* X^H.

The proof of part (ii) is straightforward. Part (i) is a consequence of a "semi-effective" version of Theorem B proved by Laurent [24]. The dependence on $h(X)$ of the upper bound for ε is necessary. It is an interesting open problem to prove a version of part (i) such that both constants α, β are effective and depend only on N and d.

A semi-abelian variety is a commutative group variety A which has a subgroup variety T such that $T \cong \mathbf{G}_{\mathbf{m}}^N$ for some $N \geq 0$ and such that the factor group variety A/T is an abelian variety. Thus, a semi-abelian variety is a common generalization of a torus and an abelian variety. Lang (cf. [23], p. 221) posed the following conjecture, which includes Theorem B as a special case: *If* A *is a semi-abelian variety defined over an algebraically closed field* K *of characteristic* 0, X *is a subvariety of* A *defined over* K, Γ *is a finitely generated subgroup of* $A(K)$ *and* $\overline{\Gamma}$ *its division group, then* $X \cap \overline{\Gamma}$ *is contained in the union of finitely many translates of semi-abelian subvarieties of* A *which are all contained in* X.

As is well-known, in 1983 Faltings [13] was the first to give a proof of Mordell's conjecture, which may be viewed as Lang's conjecture in the case that X is a curve of genus ≥ 2 and A is the Jacobian of X. Vojta [39] gave a very different proof of this, thereby introducing new and powerful techniques from Diophantine approximation. By extending Vojta's ideas, Faltings [14], [15] proved Lang's conjecture in the case that A is an abelian variety and with '$X \cap \overline{\Gamma}$' replaced by '$X \cap \Gamma$' . For a more detailed treatment of Faltings' proof, cf. [6]. Vojta [40] generalized Faltings' result to arbitrary semi-abelian varieties, but still only for sets $X \cap \Gamma$. Finally, McQuillan [26] extended this to sets $X \cap \overline{\Gamma}$ and thereby completed the proof of Lang's conjecture. McQuillan combined Vojta's result with ideas of Hindry [21].

Recently, Poonen [27] and Zhang [43] proved a qualitative analogue of Theorem 1.6 (giving only the finiteness result) for semi-abelian varieties that are isogenous to a product of a torus and an abelian variety. Poonen used among others McQuillan's result and the so-called Bogomolov conjecture for semi-abelian varieties. This conjecture states that if A is a semi-abelian variety over $\overline{\mathbf{Q}}$ and X is a subvariety of A over $\overline{\mathbf{Q}}$ not equal to a translate of a semi-abelian subvariety by a torsion point, then there is an $\varepsilon > 0$ such that the set of points in $X(\overline{\mathbf{Q}})$ with height $< \varepsilon$ is not Zariski dense in X. Bogomolov's conjecture is proved for tori by Zhang [41], for abelian varieties by Szpiro, Ullmo and Zhang [38], [42] and for semi-abelian varieties of the type mentioned above by Chambert-Loir [4].

A subject for future research is of course to obtain quantitative analogues of Theorems 1.2–1.7 for (semi-)abelian varieties. Recently, Rémond [29] made important progress in this direction by deriving in a completely explicit form a higher dimensional generalization for abelian varieties of a fundamental inequality of Vojta which was the main tool in his proof of Mordell's conjecture.

In order to give an overview of the main ingredients going into the proofs of the above mentioned results, we will sketch in the next section a proof of a weaker version of Corollary 1.3. We will deduce this weaker version directly from the basic results from Diophantine approximation, and not follow the route via the linear equation (1.1).

2. Proof of a weaker version of Corollary 1.3.

We consider the special case that X is a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over $\overline{\mathbf{Q}}$ and that Γ is a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}})$. We will sketch a proof of the following result:

Theorem 2.1. *Suppose* $\deg X = d$, $\operatorname{rank}\Gamma = r$. *Then the set* $X^0 \cap \overline{\Gamma}$ *has cardinality at most* $c_1(N,d)^{r+1}$, *where* $c_1(N,d)$ *is an effectively computable constant depending only on* N *and* d.

We first show that it suffices to prove the result for the set $X^0 \cap \Gamma$ instead of $X^0 \cap \overline{\Gamma}$. Notice that in order to prove Theorem 2.1 it suffices to prove that every finite subset

M of $X^0 \cap \overline{\Gamma}$ has cardinality at most $c_1(N,d)^{r+1}$. Let Γ' be the multiplicative group generated by M. Then Γ' is finitely generated and has rank $\leq r$. Now assuming Theorem 2.1 to be true for the set $X^0 \cap \Gamma'$ we get the required upper bound for the cardinality of M. Notice that to pass from Γ to $\overline{\Gamma}$ no Kummer theory is needed.

By means of a specialization argument we may extend Theorem 2.1 to the case that X is defined over any field K of characteristic 0 and that $\Gamma \subset \mathbf{G}_{\mathbf{m}}^N(K)$. We shall not work this out.

Theorem 2.1 is deduced from the following result:

Theorem 2.2. *Let X be a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over $\overline{\mathbf{Q}}$ and let Γ be a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^N(\overline{\mathbf{Q}})$. Suppose $\deg X = d$, $\operatorname{rank} \Gamma = r$. Then $X^0 \cap \Gamma$ is contained in the union of at most $c_2(N,d)^{r+1}$ proper subvarieties of X, each of degree at most $c_3(N,d)$, where $c_2(N,d), c_3(N,d)$ are explicitly computable constants depending only on N and d.*

Noticing that for each subvariety Y of X we have $X^0 \cap Y \subset Y^0$, we easily obtain by induction on $\dim X$ that $X^0 \cap \Gamma$ has cardinality at most $c_1(N,d)^{r+1}$. Together with the reduction argument explained above this gives Theorem 2.1. □

2.1. Absolute values and heights.

We give some basic facts about absolute values and heights. Let K be an algebraic number field and denote its ring of integers by O_K. Denote by $\mathcal{M}(K)$ the set of places of K. Every archimedean place $v \in \mathcal{M}(K)$ corresponds either to an isomorphic embedding $\sigma : K \hookrightarrow \mathbf{R}$ or to a pair of complex conjugate embeddings $\{\sigma, \overline{\sigma} : K \hookrightarrow \mathbf{C}\}$. The non-archimedean places of K correspond to the prime ideals of O_K. We define normalized absolute values $|\cdot|_v$ ($v \in \mathcal{M}(K)$) on K by

$$|x|_v = |\sigma(x)|^{1/[K:\mathbf{Q}]} \quad \text{if } v \text{ corresponds to } \sigma : K \hookrightarrow \mathbf{R};$$
$$|x|_v = |\sigma(x)|^{2/[K:\mathbf{Q}]} = |\overline{\sigma}(x)|^{2/[K:\mathbf{Q}]} \quad \text{if } v \text{ corresponds to } \{\sigma, \overline{\sigma} : K \hookrightarrow \mathbf{C}\};$$
$$|x|_v = (N\wp)^{-w_\wp(x)/[K:\mathbf{Q}]} \quad \text{if } v \text{ corresponds to the prime ideal } \wp,$$

where $N\wp = \#O_K/\wp$ denotes the norm of $\wp$ and $w_\wp(x)$ the exponent of $\wp$ in the prime ideal decomposition of x. These absolute values satisfy the product formula

$$\prod_{v \in \mathcal{M}(K)} |x|_v = 1 \quad \text{for } x \in K^*.$$

For $\mathbf{x} = (x_0, \ldots, x_N) \in K^{N+1}$, $v \in \mathcal{M}(K)$ we define

$$||\mathbf{x}||_v = ||x_0, \ldots, x_N||_v := \max(|x_0|_v, \ldots, |x_N|_v)\,.$$

Finally we define the logarithmic Weil height $h(\mathbf{x}) = h(x_0, \ldots, x_N)$ of $\mathbf{x} \in \overline{\mathbf{Q}}^{N+1}$ by picking a number field K with $\mathbf{x} \in K^{N+1}$ and putting

$$h(\mathbf{x}) := \sum_{v \in \mathcal{M}(K)} \log ||\mathbf{x}||_v\,.$$

This is independent of the choice of K. Further by the product formula it defines a height on $\mathbf{P}^N(\overline{\mathbf{Q}})$.

For a polynomial P with coefficients in $\overline{\mathbf{Q}}$ we define $h(P) := h(\mathbf{p})$, where $\mathbf{p}$ is the vector consisting of all coefficients of P.

We define the height of $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbf{G}_\mathbf{m}^N(\overline{\mathbf{Q}})$ by $h(\mathbf{x}) = h(1 : x_1 : \cdots : x_N)$. We introduce also another height $\hat{h}(\mathbf{x}) := \sum_{i=1}^N h(1 : x_i)$. This latter height has the convenient properties

$$\hat{h}(\mathbf{x}) = 0 \Longleftrightarrow \mathbf{x} \text{ is torsion}, \quad \hat{h}(\mathbf{x}^m) = |m|\hat{h}(\mathbf{x}), \quad \hat{h}(\mathbf{x}*\mathbf{y}) \le \hat{h}(\mathbf{x}) + \hat{h}(\mathbf{y}) \tag{2.1.1}$$

for $\mathbf{x}, \mathbf{y} \in \mathbf{G}_\mathbf{m}^N(\overline{\mathbf{Q}})$, $m \in \mathbf{Z}$. Further we have

$$h(\mathbf{x}) \le \hat{h}(\mathbf{x}) \le N \cdot h(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbf{G}_\mathbf{m}^N(\overline{\mathbf{Q}}). \tag{2.1.2}$$

Let Y be a projective subvariety (i.e., irreducible and Zariski closed) of $\mathbf{P}^N$ defined over $\overline{\mathbf{Q}}$. Let $\dim Y = n$, $\deg Y = d$. Denote by F_Y the Chow form of Y (cf. [37], pp. 65–69). We define the height of Y by $h(Y) := h(F_Y)$. In particular, if Y is linear then we have

$$h(Y) = h(\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n)\,, \tag{2.1.3}$$

where $\mathbf{a}_0, \ldots, \mathbf{a}_n$ is a basis of $Y(\overline{\mathbf{Q}})$ considered as a vector space and where $\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n$ denotes the usual exterior product.

There is a more advanced height h_F for varieties, introduced by Faltings in [14], which is defined by means of arithmetic intersection theory. We need only (cf. [2], Thm. 4.3.8) that there is a constant $C_1(N)$ depending only on N such that

$$|h_F(Y) - h(Y)| \leq C_1(N) \deg Y \,. \tag{2.1.4}$$

Let ι be the map of $\mathbf{G}_{\mathbf{m}}^N$ into $\mathbf{P}^N$ given by $(x_1, \ldots, x_N) \mapsto (1 : x_1 : \ldots : x_N)$. Let X be a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ of dimension n and degree d defined over $\overline{\mathbf{Q}}$. Let Y denote the Zariski closure of $\iota(X)$ in $\mathbf{P}^N$. We define $h(X) := h(Y)$, $h_F(X) := h_F(Y)$. David and Philippon introduced in [5] another, more natural height $h_{DP}(X)$, which has the property that $h_{DP}(X) = 0$ if and only if X is the translate of a subtorus by a torsion point of $\mathbf{G}_{\mathbf{m}}^N$. By (2.1.4) and [5], Prop. 2.1.(v) there is a constant $C_2(N)$ depending only on N such that

$$|h_{DP}(X) - h(X)| \leq C_2(N) \deg X \,. \tag{2.1.5}$$

A much more involved result of David and Philippon (cf. [5], Thm. 1.2) states, that if X is not a torus coset, then

$$h_{DP}(X) \geq \frac{1}{2^{41} (\deg X)^2 \{\log(\deg X + 1)\}^2} \,. \tag{2.1.6}$$

2.2. Points of small height.

Let Y be an n-dimensional linear subvariety of $\mathbf{P}^N$ defined over $\overline{\mathbf{Q}}$. Take a basis $\mathbf{a}_0, \ldots, \mathbf{a}_n$ of $Y(\overline{\mathbf{Q}})$ considered as vector space. Then from (2.1.3) and elementary height computations it follows that

$$h(Y) \leq c(n) + h(\mathbf{a}_0) + \cdots + h(\mathbf{a}_n)$$

where $c(n)$ is some constant depending only on n. This implies that if $\lambda < \frac{1}{n+1}$ and $h(Y)$ is sufficiently large, then the set of $\mathbf{y} \in Y(\overline{\mathbf{Q}})$ with $h(\mathbf{y}) < \lambda \cdot h(Y)$ is contained in a proper linear subspace of Y.

The following generalization is due to Zhang ([41], Theorem 5.8):

Theorem 2.2.1. *Let Y be a projective subvariety of $\mathbf{P}^N$ defined over $\overline{\mathbf{Q}}$ with $\dim Y = n$, $\deg Y = d$. Then for every $\lambda < \frac{1}{(n+1)d}$ the set of $\mathbf{y} \in Y(\overline{\mathbf{Q}})$ with $h(\mathbf{y}) < \lambda \cdot h_F(Y)$ is not Zariski-dense in Y.*

David and Philippon ([5], Prop. 5.4) proved the following result for subvarieties of $\mathbf{G}_{\mathbf{m}}^N$, which is basically a quantitative version of Theorem 2.2.1 for small λ:

Theorem 2.2.2. *Let X be a subvariety of $\mathbf{G}_{\mathbf{m}}^N$ defined over $\overline{\mathbf{Q}}$ which is not a torus coset. Suppose $\dim X = n$, $\deg X = d$. Put*

$$\alpha(n,d) = 2(4e)^{n+1}d\,, \quad \beta(N,n,d) = 2^{N+46}(4e)^{n+1}(n+1)^2 \cdot d^4 \log(d+1)^2\,.$$

Then the set of $\mathbf{x} \in X(\overline{\mathbf{Q}})$ with $h(\mathbf{x}) \leq \alpha(n,d)^{-1} h_{DP}(X)$ is contained in a proper Zariski-closed subset of X, the sum of the degrees of the irreducible components of which is at most $\beta(N,n,d)$.

We apply Theorem 2.2.2 to the set $X^0 \cap \Gamma$, where X, Γ are as in Theorem 2.2, i.e., with $\dim X = n$, $\deg X = d$, $\operatorname{rank}\Gamma = r$. We observe that for any translate $\mathbf{u}{*}X = \{\mathbf{u}{*}\mathbf{x} : \mathbf{x} \in X\}$ we have $\deg(\mathbf{u}{*}X) = \deg X$. This implies that the statement of Theorem 2.2 does not change if we replace X by a translate $\mathbf{u}{*}X$ with $\mathbf{u} \in \Gamma$. We replace X by such a translate of minimal height. Thus, we may assume without loss of generality that

$$h_{DP}(\mathbf{u}{*}X) \geq h_{DP}(X) \quad \text{for every } \mathbf{u} \in \Gamma. \tag{2.2.1}$$

The following lemma is more or less routine:

Lemma 2.2.3. *Assume (2.2.1). Then for every $C \geq 1$, the set of points $\mathbf{x} \in X^0 \cap \Gamma$ with*

$$h(\mathbf{x}) \leq C \cdot h_{DP}(X)$$

is contained in the union of at most $c_4(N,d)\big(c_4(N,d) \cdot C\big)^r$ proper subvarieties of X, each of degree at most $c_5(N,d)$, where $c_4(N,d)$ and $c_5(N,d)$ are constants depending only on N and d.

Proof. We may assume that X is not a torus coset since otherwise X^0 is empty. It is slightly more convenient to work with the height $\hat{h}(\mathbf{x})$ introduced in Section 2.1. Define the distance function $\delta(\mathbf{u}_1, \mathbf{u}_2) := \hat{h}(\mathbf{u}_1 * \mathbf{u}_2^{-1})$. Let $\alpha(n,d)$, $\beta(N,n,d)$ have the meaning of Theorem 2.2.2.

In view of (2.1.2), we have to consider the set of points $\mathbf{x} \in X^0 \cap \Gamma$ with $\hat{h}(\mathbf{x}) \le B$ with $B = NC \cdot h_{DP}(X)$. Let $\mathcal{S}$ be a maximal subset of this set, with the property that any two distinct points $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{S}$ satisfy $\delta(\mathbf{u}_1, \mathbf{u}_2) \ge \varepsilon$ where $\varepsilon = \alpha(n,d)^{-1} h_{DP}(X)$. According to, e.g., Lemma 4 of [35] (which is valid for any function with properties (2.1.1) defined on an abelian group of rank r) the set $\mathcal{S}$ has cardinality at most $(1 + (2B/\varepsilon))^r \le (3N\alpha(n,d) \cdot C)^r$.

Our choice of $\mathcal{S}$ implies that for every $\mathbf{x} \in X^0 \cap \Gamma$ with $\hat{h}(\mathbf{x}) \le B$, there is a $\mathbf{u} \in \mathcal{S}$ with $\delta(\mathbf{x}, \mathbf{u}) < \varepsilon$. Consider the points $\mathbf{x}$ corresponding to a fixed $\mathbf{u} \in \mathcal{S}$. By (2.1.2) and (2.2.1) we have $h(\mathbf{u}^{-1} * \mathbf{x}) \le \alpha(n,d)^{-1} h_{DP}(\mathbf{u}^{-1} * X)$. By applying Theorem 2.2.2 with $\mathbf{u}^{-1} * X$ and the points $\mathbf{u}^{-1} * \mathbf{x}$ and then passing from $\mathbf{u}^{-1} * \mathbf{x}$ to $\mathbf{x}$ we infer that the set of vectors $\mathbf{x}$ under consideration lies in a finite union of proper subvarieties of X, the sum of the degrees of which is at most $\beta(N,n,d)$. Together with our estimate for the cardinality of $\mathcal{S}$ this implies Lemma 2.2.3. $\square$

2.3. The Subspace Theorem.

Let K be an algebraic number field. Let S be a finite set of places of K. For $v \in S$, let $L_0^{(v)}, \ldots, L_n^{(v)}$ be linearly independent linear forms in $K[x_0, \ldots, x_n]$. The Subspace Theorem, first proved by Schmidt for archimedean absolute values [32] and later extended by Schlickewei [31] to arbitrary sets of absolute values, reads as follows:

For every $\kappa > n+1$ the set of points $\mathbf{x} = (x_0 : \cdots : x_n) \in \mathbf{P}^n(K)$ satisfying

$$\log\left(\prod_{i=0}^{n} \prod_{v \in S} \frac{|L_i^{(v)}(\mathbf{x})|_v}{||\mathbf{x}||_v}\right) \le -\kappa h(\mathbf{x}) \tag{2.3.1}$$

is contained in the union of finitely many proper linear subspaces of $\mathbf{P}^N$.

Instead of (2.3.1) we deal with systems of inequalities

$$\log\Big(\frac{|L_i^{(v)}(\mathbf{x})|_v}{||\mathbf{x}||_v}\Big) \le -c_{iv}h(\mathbf{x}) \quad (v \in S, i = 0, \ldots, n) \qquad \text{in } \mathbf{x} \in \mathbf{P}^n(K). \tag{2.3.2}$$

Clearly, the solutions of (2.3.2) lie in only finitely many proper linear subspaces if

$$\sum_{i=0}^{n}\sum_{v\in S} c_{iv} > n+1\,. \tag{2.3.3}$$

Let $\{L_0, \ldots, L_N\}$ be the union of the sets of linear forms $\{L_1^{(v)}, \ldots, L_n^{(v)}\}$ $(v \in S)$. Define the map $\mathbf{x} \mapsto \mathbf{y} = (y_0 : \cdots : y_N)$ by $y_i = L_i(\mathbf{x})$ for $i = 0, \ldots, N$ and let Y be the image of $\mathbf{P}^n$ under this map. Then Y is an n-dimensional linear projective subvariety of $\mathbf{P}^N$ defined over K. For $\mathbf{x} \in \mathbf{P}^n$, we have that $\mathbf{y} \in Y(K)$ and that $L_i^{(v)}(\mathbf{x})$ is a coordinate of $\mathbf{y}$. This leads us to consider systems of inequalities

$$\log\Big(\frac{|y_i|_v}{||\mathbf{y}||_v}\Big) \le -c_{iv}h(\mathbf{y}) \quad (i = 0, \ldots, N, v \in S) \qquad \text{in } \mathbf{y} \in Y(K). \tag{2.3.4}$$

Let $\mathcal{I}(Y)$ be the set of $(n+1)$-tuples $\mathbf{i} = \{i_0, \ldots, i_n\}$ such that the variables $y_{i_0}, \ldots, y_{i_n}$ are linearly independent on Y, i.e., there is no non-trivial linear combination $\sum_{k=0}^{n} c_k y_{i_k}$ vanishing identically on Y. Notice that condition (2.3.3) translates into

$$\frac{1}{n+1}\left(\sum_{v\in S}\max_{\mathbf{i}\in\mathcal{I}(Y)}\sum_{i\in\mathbf{i}} c_{iv}\right) = 1+\delta \quad \text{with } \delta > 0. \tag{2.3.5}$$

Schmidt [34] was the first to prove a quantitative version of his Subspace Theorem, giving an explicit upper bound of the number of subspaces. For an overview of further history we refer to the survey paper [9]. Below we state a consequence of a result of Schlickewei and the author ([10], Theorem 2.1).

Theorem 2.3.1. *Let Y be a linear projective subvariety of $\mathbf{P}^N$ of dimension n defined over the number field K. Assume (2.3.5). Then the set of solutions $\mathbf{y} \in Y(K)$ of (2.3.4) with*

$$h(\mathbf{y}) > (1+\delta^{-1})(N+1)^n \cdot (1+h(Y)) \tag{2.3.6}$$

lies in some finite union $T_1 \cup \cdots \cup T_t$ of proper linear subspaces of Y where

$$t \leq 4^{(n+6)^2}(1+\delta^{-1})^{n+4} \log 4N \log\log 4N \,. \tag{2.3.7}$$

We would like to emphasize that for applications it is very crucial that the quantities in (2.3.6) and (2.3.7) are independent of K and S and that the quantity in (2.3.7) is independent of Y.

The method of proof of Theorem 2.3.1 is basically Schmidt's (cf. [33]), but with some technical innovations. Instead of Roth's lemma (a non-vanishing result for polynomials) used by Schmidt, the proof of Theorem 2.3.1 uses a very special case of an explicit version of Faltings' Product Theorem ([14], Thms. 3.1,3.3). This led to a considerable improvement upon the upper bound for the number of subspaces given by Schmidt [34]. Further, the basic geometry of numbers used by Schmidt was replaced by the "geometry of numbers over $\overline{\mathbf{Q}}$" developed independently by Roy and Thunder [30] (Thm. 6.3) and Zhang [41] (Thm. 5.8). This was of crucial importance to remove the dependence on the number field K which was still present in earlier versions of Theorem 2.3.1. For further comments we refer to [9].

In their fundamental paper [16], Faltings and Wüstholz gave a proof of the Subspace Theorem very different from Schmidt's. Their argument does not use geometry of numbers, but instead the full power of Faltings' Product Theorem. Moreover, Faltings and Wüstholz treated systems of inequalities (2.3.4) where the solutions $\mathbf{y}$ may be taken from an arbitrary projective subvariety Y of $\mathbf{P}^N$, not just a linear subvariety.

Ferretti [17] obtained a quantitative version of the result of Faltings and Wüstholz. Among others, Ferretti considered systems (2.3.4) for arbitrary projective varieties Y. Under suitable conditions imposed on the exponents c_{iv} he gave explicit constants C_1, C_2, C_3 such that the set of solutions $\mathbf{y}$ of (2.3.4) with $h(\mathbf{y}) \geq C_1$ lies in the union of at most C_2 proper subvarieties of Y, each of degree $\leq C_3$. Unfortunately, Ferretti's constants C_1, C_2, C_3 depend on K and S which is an obstacle for applications. It seems to be within reach to prove a version of Ferretti's result with constants C_1, C_2, C_3 independent of K and S but this still requires some work.

We have to apply Theorem 2.3.1 to the set $X^0 \cap \Gamma$. Recall that for a number field K and a finite set of places $S \subset \mathcal{M}(K)$ containing the archimedean places, the group of S-units is given by $U_S = \{x \in K^* : |x|_v = 1 \text{ for } v \notin S\}$. Let X, Γ be as in Theorem 2.2 but assume that X is linear. Choose the number field K and the set of places $S \subset \mathcal{M}(K)$ such that X is defined over K and $\Gamma \subset U_S^N$. Let Y be the Zariski closure of $\iota(X)$ in $\mathbf{P}^N$ (where as before $\iota : (x_1, \dots, x_N) \mapsto (1 : x_1 : \cdots : x_N)$) so that Y is also linear. Given $\mathbf{x} = (x_1, \dots, x_N) \in X^0 \cap \Gamma$ with $h(\mathbf{x}) > 0$ put $y_0 := 1$, $y_i := x_i$ for $i = 1, \dots, N$ and $\mathbf{y} = (y_0, \dots, y_N)$. Then by definition, $h(\mathbf{y}) = h(\mathbf{x})$. Define reals c_{iv} by

$$\log\left(\frac{|y_i|_v}{||\mathbf{y}||_v}\right) = -c_{iv} h(\mathbf{y}) \quad \text{for } v \in S,\ i = 0, \dots, N. \tag{2.3.8}$$

The following result is a consequence of [8], Lemma 15. Its proof involves only elementary combinatorics.

Lemma 2.3.2. *Assume that X is linear and that $\mathrm{Stab}(X) := \{\mathbf{u} \in \mathbf{G}_{\mathbf{m}}^N : \mathbf{u}{*}X = X\}$ is trivial. Then there are constants $c_6(N), c_7(N) \geq 1$ depending only on N such that for every $\mathbf{x} \in X^0 \cap \Gamma$ with $h(\mathbf{x}) \geq c_6(N) \cdot (1 + h(X))$, the reals c_{iv} defined by (2.3.8) satisfy (2.3.5) with $\delta \geq c_7(N)^{-1}$.*

2.4. Proof of Theorem 2.2.

Let K be the number field and S the finite set of places introduced in Section 2.3. For the moment we assume that X is linear, i.e., $d = 1$. Further we may assume that $\mathrm{Stab}(X)$ is trivial (and in particular that X is not a torus coset) since otherwise $X^0 = \emptyset$. Lastly, we assume (2.2.1) which is no loss of generality. Let $c_8(N), c_9(N), \dots$ denote explicitly computable constants, depending only on N.

In view of (2.1.5) and (2.1.6) there is a constant $c_8(N)$ such that $c_8(N) h_{DP}(X)$ exceeds the lower bounds for $h(\mathbf{x})$ required in Theorem 2.3.1 and Lemma 2.3.2. Take $\mathbf{x} = (x_1, \dots, x_N) \in X^0 \cap \Gamma$ with $h(\mathbf{x}) \geq c_8(N) h_{DP}(X)$. Let $y_0 = 1$, $y_i = x_i$ for $i = 1, \dots, N$, $\mathbf{y} = (y_0, \dots, y_N)$. By Lemma 2.3.2 the reals c_{iv} ($v \in S, i = 0, \dots, N$) defined by (2.3.8) satisfy (2.3.5) with $\delta \geq c_7(N)^{-1}$. A problem is that the c_{iv} depend on $\mathbf{x}$. But we may approximate the c_{iv} by reals c'_{iv} from a finite set independent of $\mathbf{x}$ which still satisfy (2.3.5) with a slightly smaller lower bound for δ. By means of an elementary combinatorial argument which we do not work out one can show that there is a set $\mathcal{C} \subset \mathbf{R}^{(N+1)\#S}$ of cardinality at most $c_9(N)^r$ independent of $\mathbf{x}$ with the following property: there is a tuple $(c'_{iv} : v \in S, i = 0, \ldots, N) \in \mathcal{C}$ such that $c'_{iv} \leq c_{iv}$ for $v \in S$, $i = 0, \ldots, N$, and which satisfies (2.3.5) with $\delta \geq c_{10}(N)^{-1}$. (One has to use that the tuple $(c_{iv} : v \in S, i = 0, \ldots, N)$ belongs to a translate of an r-dimensional linear subspace of $\mathbf{R}^{(N+1)\#S}$.) This means that for every $\mathbf{x} \in X^0 \cap \Gamma$ with $h(\mathbf{x}) \geq c_8(N) h_{DP}(X)$ the corresponding vector $\mathbf{y}$ satisfies one of at most $c_9(N)^r$ systems of inequalities (2.3.4), with $\delta \geq c_{10}(N)^{-1}$.

By applying Theorem 2.3.1 to the systems just mentioned, we obtain that the set of $\mathbf{x} \in X^0 \cap \Gamma$ with $h(\mathbf{x}) \geq c_8(N) \cdot h_{DP}(X)$ lies in the union of at most $c_{11}(N) \cdot c_9(N)^r$ proper linear subvarieties of X. Further, Lemma 2.2.3 implies that the set of $\mathbf{x} \in X^0 \cap \Gamma$ with $h(\mathbf{x}) < c_8(N) \cdot h_{DP}(X)$ lies in the union of at most $c_{12}(N)^{r+1}$ proper subvarieties of X of degree at most $c_{13}(N)$. By combining these two estimates we get Theorem 2.2 in the case that X is linear.

Now assume that X has degree $d > 1$. E.g., by [14], Prop. 2.1, X is the set of zeros of a set of polynomials in $K[x_1, \ldots, x_N]$ of degree at most d. Let φ_d be the Veronese embedding from $\mathbf{G}_{\mathbf{m}}^N$ into $\mathbf{G}_{\mathbf{m}}^{N'}$ with $N' = \binom{N+d}{d}$, mapping $\mathbf{x}$ to the vector consisting of all monomials of degree $\leq d$. Then $\varphi_d(X^0 \cap \Gamma) \subset \tilde{X}^0 \cap \tilde{\Gamma}$, where $\tilde{X}$ is a linear subvariety defined over K of $\mathbf{G}_{\mathbf{m}}^{N'}$ and where $\tilde{\Gamma}$ is a finitely generated subgroup of $\mathbf{G}_{\mathbf{m}}^{N'}(K)$ of rank r. We know already that Theorem 2.2 holds for the set $\tilde{X}^0 \cap \tilde{\Gamma}$. By applying φ_d^{-1} we get Theorem 2.2 for $X^0 \cap \Gamma$. □

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