# Parameter identifiability of differential delay equations

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**Abstract**

Parameter identifiability is concerned with the question whether the parameters of a specific model can be identified from knowledge about certain solutions of the model, assuming perfect data. In this paper we shall consider conditions for identifiability of parameters and time delays in infinite dimensional dynamical systems, described by linear differential delay equations, assuming knowledge of particular solutions on bounded time intervals. We shall show how methods from operator theory can be exploited to give necessary and sufficient conditions which guarantee that the inverse problem has a unique solution.

## 1 Introduction

A mathematical model often predicts quantities which cannot be measured directly. Therefore, it is not only important to know whether the parameters of the model are uniquely identifiable but, more general, whether there is a unique model in a given class that fits the experimental data, assuming perfect data. For models that can be embedded into finite dimensional dynamical systems, there is a rather complete theory, see, for example, [12, 4, 6]. Also for parabolic partial differential equations, there is a rather complete theory [2, 9, 10, 16], but for differential delay equations there are only a few results known [11].

Before attempting parameter estimation in infinite dimensional dynamical systems the identifiability question has to be addressed. For linear systems controlled by linear feedback laws, time delays arise naturally and represent effects due to communication, transmission, transportation or iternia effects. In this paper we shall address the question whether a specific model, from a given class of differential delay equations, can be identified from information about certain solutions of the model, assuming perfect data.

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We start this paper with a short introduction to differential delay equations. Consider the simplest linear differential delay equation

$$\frac{dx}{dt}(t) = Ax(t) + Bx(t-1), \quad t \geq 0, \tag{1.1}$$

where $A$ and $B$ are $n \times n$-matrices and $x : [0, \infty) \to \mathbb{R}^n$ denotes the solution. A moment of reflection indicates that the minimal amount of information of initial data in order for (1.1) to a unique solution $x$ is to specify a continuous function on the interval $[-1, 0]$. In fact, if $x(t) = \varphi(t)$, $-1 \leq t \leq 0$, then

$$x(t) = e^{At}\varphi(0) + \int_0^t e^{A(t-s)} B\varphi(s-r)\, ds, \qquad 0 \leq t \leq 1, \tag{1.2}$$

is the unique solution of (1.1) on $[0, 1]$. Once $x$ is known on $[0, 1]$ the process can be repeated to find the unique solution on $[1, 2]$, etc. This process is called *the method of steps* and yields a unique, globally defined, solution of (1.1).

The method of steps can be interpreted as follows. The differential delay equation (1.1) can be viewed as an iterative process on a space of continuous functions, iterating a segment of the solution. Indeed, set $x_n(\theta) = x(n+\theta)$, $-1 \leq \theta \leq 0$, $n = 0, 1, 2, \ldots$. Then $x_0 = \varphi$ and for $n = 1, 2, \ldots$

$$x_{n+1}(\theta) = e^{A\theta}x_n(0) - \int_\theta^0 e^{A(\theta-s)} Bx_n(s)\, ds, \quad -1 \leq \theta \leq 0. \tag{1.3}$$

From this approach, it also follows that, in general, solutions of (1.1) are only defined on $[0, \infty)$. Backward continuation of solutions of differential delay equations along the negative real axis requires smoothness of the initial function $\varphi$ defined on $[-1, 0]$. From this approach, it is clear that differential delay equations are infinite dimensional dynamical systems defined by iterating a segment of a function. There are, in general, infinitely many independent exponential solutions $x_j(t) = e^{\lambda_j t}$, $j = 1, 2, 3, \ldots$. Indeed, substitution of an exponential solution in (1.1) yields that for each zero $\lambda$ of the equation

$$\det[\lambda I - A - Be^{-\lambda}] = 0,$$

the exponential $x(t) = e^{\lambda t}$ is a solution of (1.1) for all real $t$.

The dynamical system approach to differential delay equations is to associate with (1.1) a semiflow defined by the time evolution of segments of solutions (see Figure 1). If the solution of (1.1) with initial data $x(\theta) = \varphi(\theta)$, $-1 \leq \theta \leq 0$, is denoted by $x(\,\cdot\,;\varphi)$ and

$$x_t(\theta; \varphi) = x(t+\theta; \varphi), \qquad -1 \leq \theta \leq 0,$$

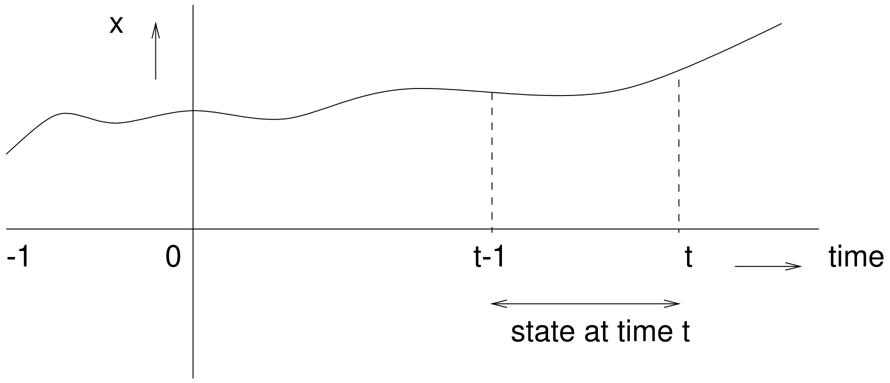


Figure 1: The state space approach

then the semiflow is defined by $T(t)\varphi = x_t(\,\cdot\,;\varphi)$.

An important observation is that the evolution of the state $x_t$ is given by an abstract ordinary differential equation in the infinite dimensional state space of continuous functions defined on the interval $[-1,0]$ (see Section 2 for the details).

Next we shall motivate the problem which we shall address in this paper. Parameter identification using a model system is very important in the control of distributed parameter systems [1]. As an illustration of the main result in this paper, we consider the following inverse problem. Suppose that we have an unknown scalar differential delay equation

$$\frac{dx}{dt}(t) = ax(t) + bx(t-1), \qquad x_0 = \varphi, \tag{1.4}$$

and that we would like to find the system parameters $a$ and $b$ using a model system

$$\frac{dy}{dt}(t) = a^m y(t) + b^m y(t-1), \qquad y_0 = \varphi^m, \tag{1.5}$$

where $a^m$ and $b^m$ are known model parameters and $\varphi^m$ is a known model initial function. Suppose that we can adjust the system parameters in the model system (1.5) and find an initial condition $\varphi_m$ such that the solution of the model system (1.5) and the observed solution of the unknown system (1.4) are identical on a bounded time interval $[t_1, t_2]$, i.e.,

$$y(t;\varphi^m) = x(t;\varphi), \qquad 0 < t_1 \le t \le t_2. \tag{1.6}$$

When does this identity imply that $a = a^m$, $b = b^m$ and $\varphi = \varphi^m$?

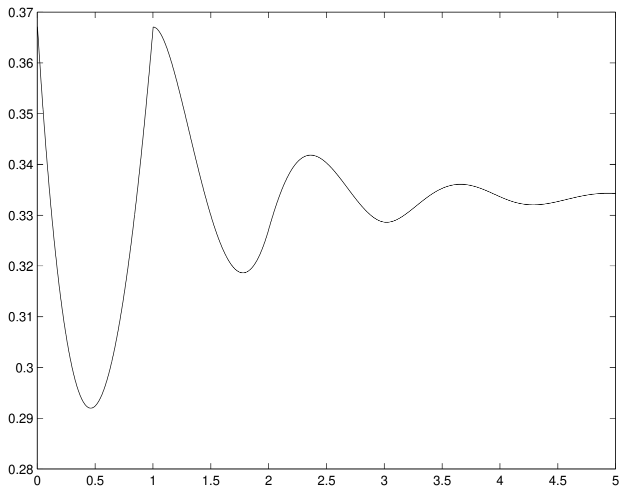


Figure 2: The solution of (1.7) with initial data $\varphi^m(\theta) = e^\theta$, $-1 \le \theta \le 0$

Our main result states that if $\varphi^m(\theta) = e^{\lambda\theta}$ with $\lambda$ such that

$$\lambda - a_m - b_m e^{-\lambda} \neq 0$$

and $t_2 - t_1 > 1$, the length of the maximal time delay in the system, then (1.6) implies that $a = a^m$, $b = b^m$ and $\varphi = \varphi^m$.

In particular, knowledge of just the solution, on a time interval of length bigger than the largest time delay in the system, uniquely determines a non-degenerate scalar delay equation.

For example, knowledge of just the solution with initial data $\varphi^m(\theta) = e^\theta$ (see Figure 2), on any time interval of length bigger than 1, uniquely determines the model equation

$$\frac{dx}{dt}(t) = -2x(t) + 2x(t-1). \tag{1.7}$$

The organization of the paper is as follows. In Section 2 we first discuss some basic properties of general linear delay equations and then generalize the identification problem, as considered before by Nakagiri and Yamamoto [11] to general linear systems of differential delay equations. In Section 3 we shall discuss some elements of the theory of semigroups and develop the spectral theory for linear differential delay equations. In Section 4 we prove the main result using the results from Section 3. And, finally, in Section 5 we shall illustrate how to apply the main result to specific problems.

## 2 Parameter identifiability of linear systems of delay equations

An initial value problem for a linear retarded functional differential equation (RFDE) is given by the following equation

$$\frac{dx}{dt}(t) = \int_0^h d\zeta(\theta)x(t-\theta), \quad t \geq 0, \tag{2.1}$$

where $\zeta$ is an $n \times n$-matrix function of bounded variation normalized such that $\zeta(0) = 0$ and $\zeta$ is continuous from the right on the open interval $(0, \infty)$. The initial data is given by $x(\theta) = \varphi(\theta)$ for $-h \leq \theta \leq 0$, where $\varphi$ is a given continuous function on $[-h, 0]$ with values in $\mathbb{R}^n$. The solution with initial data $\varphi$ we shall denote by $x = x(\,\cdot\,, \varphi)$, $\varphi \in C$, where $C$ denotes the space of $\mathbb{R}^n$-valued continuous functions on the interval $[-h, 0]$ provided with the sup-norm, i.e., for $\varphi \in C$, the norm is given by $\|\varphi\| := \max_{-h\leq\theta\leq 0} |\varphi(\theta)|$.

The dynamical systems approach is to associate with (2.1) a semigroup of solution operators acting on the state space $C$. The semigroup is given by translation along the solution, i.e.,

$$T(t)\varphi = x_t(\,\cdot\,;\varphi), \quad t \geq 0,$$

where $x = x(\,\cdot\,;\varphi)$ denotes the solution to (2.1) and $x_t(\theta;\varphi) = x(t+\theta)$ for $-h \leq \theta \leq 0$. See [3] and [5] for further information about the general theory of delay equations.

The infinitesimal generator of the semigroup $T(t)$ is given by

$$\begin{aligned} \mathcal{D}(A) &= \Big\{\varphi \in C \,\Big|\, \frac{d\varphi}{d\theta} \in C,\ \frac{d\varphi}{d\theta}(0) = \int_0^h d\zeta(\theta)\varphi(-\theta)\Big\}, & (2.2)\\ A\varphi &= \frac{d\varphi}{d\theta}, \quad \varphi \in \mathcal{D}(A). & (2.3) \end{aligned}$$

It is a simple computation to show that the spectrum of $A$ consists of eigenvalues only, i.e., complex numbers $\lambda$ with $A\phi = \lambda\phi$ for some nonzero $\phi \in C$, and that the set of eigenvalues of $A$ is given by the zero set of an entire function

$$\sigma(A) = \big\{\lambda \in \mathbb{C} \,|\, \det \Delta(\lambda) = 0\big\}.$$

Here $\Delta(z)$, the so-called characteristic matrix of (2.1), is given by

$$\Delta(z) = zI - \int_0^h e^{-zt} d\zeta(t) \tag{2.4}$$

(see Section 3 for details, in particular, compare formula (3.3) for the resolvent of $A$).

In [11] Nakagiri and Yamamoto consider the identification problem for the following system of equations

$$\frac{dx}{dt}(t) = \sum_{j=1}^{k} A_j x(t-h_j) + \int_0^h A_I(s)x(t-s)\,ds. \tag{2.5}$$

Here $h$ is a given fixed real number. The matrices $A_j$ and $A_I(\,\cdot\,)$ and the time delays $h_j$ $(0 \le h_1 < h_2 < \cdots < h_k \le h)$ are unknown.

Suppose that $N$ solutions $x_i = x(\,\cdot\,;\varphi_i)$ of eq. (2.5) are known on a given bounded time interval $[t_1, t_2]$. The identification problem consists of the determination of the unknown coefficient matrices, time delays and the unknown initial data $\varphi_i \in C$ $(i = 1, \dots, N)$.

The model system which is used for the identifiability of the unknown parameters in (2.5) is given by

$$\frac{dy}{dt}(t) = \sum_{j=1}^{l} A_j^m y(t-h_j^m) + \int_0^h A_I^m(s)y(t-s)\,ds \tag{2.6}$$

with initial data $y_0 = \varphi_i^m \in C$. An '$m$' superscript in the model system means that the matrices $A_j^m$, the integrable matrix-valued-function $A_I^m$, the time delays $h_j^m$ $(0 = h_1^m < h_2^m < \cdots < h_l^m = h)$ and the initial data $\varphi_i^m$, $i = 1, 2, \dots, N$, are known.

For problem (2.6), the kernel $\zeta^m$ consists of finitely many jumps and an absolutely continuous part, i.e.,

$$\zeta^m(s) = \sum_{j=1}^{l} A_j^m H(s-h_j^m) + \int_0^s A_I^m(\theta)\,d\theta, \tag{2.7}$$

where $H$ denotes the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ with $H(s) = 0$ for $s \le 0$ and $H(s) = 1$ for $s > 0$. The characteristic matrix associated with (2.6) is given by

$$\Delta^m(z) = zI - \sum_{j=1}^{l} A_j^m e^{-h_j^m z} - \int_0^h A_I^m(s)e^{-sz}\,ds. \tag{2.8}$$

**Definition 2.1** The unknown quantities $A_I$, $A_j$ and $h_j$ for $1 \le j \le k$ and $\varphi_i$ for $1 \le i \le N$ in (2.5) are said be to *identifiable* on $[t_1, t_2]$ if the equalities

$$\begin{aligned}
&k = l, \quad A_j = A_j^m, \quad h_j = h_j^m \quad (j = 1, \dots, l),\\
&A_I(s) = A_I^m(s) \quad \text{a.e } s \in [0, h],\\
&\varphi_i(s) = \varphi_i^m(s) \quad -h \le s \le 0 \quad (i = 1, \dots, N),
\end{aligned}$$

follow from

$$y(t;\varphi_i^m) = x(t;\varphi_i), \qquad t_1 \le t \le t_2 \ (i = 1, \dots, N).$$

It is clear that some conditions on the solutions that are known on the interval $[t_1, t_2]$ are needed. For example, consider the following scalar model equation

$$\frac{dy}{dt}(t) = y(t) - y(t-1).$$

If the initial data $\varphi^m(\theta) = 1$, then $y(t;\varphi^m) = 1$ for all $t \ge 0$. Therefore, the observation data $y(t,\varphi^m) = x(t,\varphi)$ for $t \in [t_1, t_2]$ holds for any unknown system that has the solution $x \equiv 1$.

This example describes actually an exceptional case. The next surprising corollary is a consequence of our main result.

**Corollary 2.1** *Suppose that an unknown scalar system is given by*

$$\frac{dx}{dt}(t) = \sum_{j=1}^{k} a_j x(t-h_j) + \int_0^h a_I(s) x(t-s)\, ds, \tag{2.9}$$

*and that the model system is given by*

$$\frac{dy}{dt}(t) = \sum_{j=1}^{l} a_j^m y(t-h_j^m) + \int_0^h a_I^m(s) y(t-s)\, ds. \tag{2.10}$$

*If* $a_l^m \ne 0$, $\varphi^m(\theta) = e^{\lambda_0 \theta}$, $-h \le \theta \le 0$, *where* $\lambda_0$ *is such that* $\Delta^m(\lambda_0) \ne 0$, *then*

$$y(t;\varphi^m) = x(t,\varphi), \quad t_1 \le t \le t_2, \ t_2 - t_1 > h,$$

*implies that the systems* (2.9) *and* (2.10) *are identical.*

The following example shows that the condition $t_2 - t_1 > h$ in Corollary 2.1 cannot be improved. Suppose that the model system is given by the scalar differential delay equation

$$\frac{dy}{dt}(t) = y(t) + y(t-1). \tag{2.11}$$

To satisfy the first condition of Corollary 2.1, we first compute the characteristic matrix of the model system (2.11)

$$\Delta^m(\lambda) = \lambda - 1 - e^{-\lambda}.$$

It follows that we can take $\lambda_0 = 0$ and $\varphi^m(\theta) = 1$, $-1 \le \theta \le 0$. The solution $y(t;\varphi^m)$ of (2.11) on the interval $[0,1]$ is given by

$$y(t;\varphi^m) = 2e^t - 1, \qquad 0 \le t \le 1.$$

However, the solution $x(t;\varphi)$ of the following system

$$\frac{dx}{dt}(t) = 2ex(t-1), \quad x_0 = \varphi, \tag{2.12}$$

with $\varphi(\theta) = e^\theta$ is, on the interval $[0,1]$, also given by

$$x(t;\varphi) = 2e^t - 1, \qquad 0 \le t \le 1.$$

Thus

$$y(t;\varphi^m) = x(t,\varphi), \quad 0 \le t \le 1,$$

but the systems (2.11) and (2.12) are not identical.

Before we state our main result, we present another special case for systems of two equations.

**Corollary 2.2** *Suppose that the model system is given by*

$$\begin{aligned} \frac{dy_1}{dt}(t) &= a_1 y_2(t-1), \\ \frac{dy_2}{dt}(t) &= a_2 y_1(t-1). \end{aligned} \tag{2.13}$$

*If $a_1a_2 \ne 0$ and $\varphi^m = (\varphi_1^m, \varphi_2^m)^T$ is constant with $\sqrt{a_1}\varphi_2^m \pm \sqrt{a_2}\varphi_1^m \ne 0$, then*

$$y(t;\varphi^m) = x(t,\varphi), \quad t_1 \le t \le t_2, \ t_2 - t_1 > 2h,$$

*implies that the unknown quantities $A_I$, $A_j$ and $h_j$ for $1 \le j \le k$ and $\varphi_1$ ($N = 1$) in (2.5) are identifiable on $[t_1, t_2]$.*

The next example shows that the condition $t_2 - t_1 > 2h$ in Corollary 2.2 cannot be improved. Suppose that the model system is given by the differential delay equation

$$\begin{aligned} \frac{dy_1}{dt}(t) &= y_2(t-1), \\ \frac{dy_2}{dt}(t) &= y_1(t-1) \end{aligned} \tag{2.14}$$

and that the initial data $\varphi^m = (\varphi_1^m, \varphi_2^m)^T$ is constant with $\varphi_1^m = 1$ and $\varphi_2^m = 0$.

A direct computation shows that the solution $y(t;\varphi^m) = (y_1(t), y_2(t))$ of (2.14) for $t \in [0,1]$ is given by

$$y_1(t) = 1 \quad \text{and} \quad y_2(t) = t \quad \text{for } 0 \le t \le 1.$$

Using the method of steps we next use the solution on $[0,1]$ to compute the solution $y(t;\varphi^m)$ of (2.14) for $t \in [1,2]$. This yields

$$y_1(t) = \tfrac{1}{2}t^2 - t + \tfrac{3}{2} \quad \text{and} \quad y_2(t) = t \quad \text{for } 1 \le t \le 2.$$

However, the following system

$$\begin{aligned} \frac{dx_1}{dt}(t) &= x_1(t-1) + 2x_2(t-1) - x_2(t), \\ \frac{dx_2}{dt}(t) &= x_2(t) - x_2(t-1), \end{aligned} \tag{2.15}$$

with initial data $\varphi = (\varphi_1, \varphi_2)^T$ and

$$\varphi_1(\theta) = 1 - \theta \quad \text{and} \quad \varphi_2(\theta) = \theta,$$

has the property that

$$y(t;\varphi^m) = x(t,\varphi), \quad 0 \le t \le 2,$$

but the model system (2.14) and the unknown system (2.15) are not identical.

We can now state our main result.

**Theorem 2.1** *Assume that the model system (2.6) is nondegenerate, i.e.,* $\det A_l^m \neq 0$ *or* $\det A_I^m$ *is not identically zero a.e. in a neighbourhood of* $nh$. *Suppose that* $N$ *initial conditions* $\varphi_i^m$, $i = 1, 2, \dots, N$, *are given. Define*

$$p_{\lambda,i}^m := \varphi_i^m(0) + \int_0^h \Big(e^{\lambda\tau} \int_\tau^h e^{-\lambda s}\, d\zeta^m(s)\Big) \varphi_i^m(-\tau)\, d\tau, \quad i = 1, \dots, N. \tag{2.16}$$

*For every* $\lambda$ *with* $\det \Delta^m(\lambda) = 0$, *define* $\{q_{\lambda,j}^m\}_{j=1}^{d_\lambda}$ *to be a basis of row vectors for the left null space* $\{v \in \mathbb{R}^n \mid v\Delta^m(\lambda) = 0\}$. *If*

$$\operatorname{rank}\left(q_{\lambda,j}^m p_{\lambda,i}^m\right)_{1 \le i \le N, 1 \le j \le d_\lambda} = d_\lambda, \tag{2.17}$$

*and*

$$t_2 - t_1 > nh, \tag{2.18}$$

*then the quantities* $A_I$, $A_j$ *and* $h_j$ *for* $1 \le j \le k$ *in (2.5) and the initial data* $\varphi_i$ *for* $1 \le i \le N$ *are identifiable on* $[t_1, t_2]$. *Furthermore, the conditions (2.17) and (2.18) are sharp for* $n = 1$ *and* $n = 2$.

In the result of Nakagiri and Yamamoto [11] the condition $t_2 - t_1 > 2nh$ is needed and the authors raise the question to find the optimal condition. The rank condition (2.17) in Theorem 2.1 is related to the condition obtained by Nakagiri and Yamamoto but is simpler to verify in examples. The examples in this section illustrate that the conditions (2.17) and (2.18) are sharp for $n = 1$ and $n = 2$.

The main idea in the proof of Theorem 2.1 is to use the semigroup approach presented in the beginning of this section. In the sequel we let $T(t)$ with infinitesimal generator $A$ denote the semigroup associated with (2.5) and $T^m(t)$ with infinitesimal generator $A^m$ the semigroup associated with the model system (2.6). To prove the theorem, it suffices to show that $T(t) = T^m(t)$. This implies that the infinitesimal generators $A$ and $A^m$ are equal and hence, from (2.2) that $\zeta = \zeta^m$. An important ingredient in the proof is a result about completeness of the set of eigenvectors and generalized eigenvectors of the generator $A$ [13, 14] which extends earlier work by Manitius [8].

# 3 Spectral theory for differential delay equations

Let $T(t) : C \to C$ be the semigroup given by translation along the solutions of the differential delay equation (2.1). The semigroup $T(t)$ satisfies the following basic properties

(i) $T(0) = I$;

(ii) $T(t + s) = T(t)T(s)$ for $t, s \geq 0$;

(iii) for every $\varphi \in C$, the maps $t \to T(t)\varphi$ are continuous.

The infinitesimal generator $A$ is defined by

$$A\varphi := \lim_{t\downarrow 0} \frac{1}{t}\big[T(t)\varphi - \varphi\big] \quad \text{for every } \varphi \in \mathcal{D}(A), \tag{3.1}$$

where $\mathcal{D}(A)$ consists of all $\varphi \in C$ for which the limit in (3.1) exists. The growth bound of $T(t)$ is defined by

$$\omega_0 := \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|.$$

It is known that $\omega_0 < \infty$ and that for $\mathrm{Re} z > \omega_0$, the resolvent of $A$ equals the Laplace transform of the semigroup $T(t)$

$$(zI - A)^{-1} = \int_0^\infty e^{-zt}T(t)\,dt \quad \text{for } \mathrm{Re} z > \omega_0.$$

Let $\lambda \in \sigma(A)$ be an eigenvalue of $A$, the kernel $\mathcal{N}\big(\lambda I - A\big)$ is called the eigenspace at $\lambda$ and its dimension $d_\lambda$ the *geometric multiplicity*. The generalized eigenspace $\mathcal{M}_\lambda$ is the smallest closed subspace that contains all $\mathcal{N}\big((\lambda I - A)^j\big)$, $j = 1, 2, \dots$ and its dimension is called the *algebraic multiplicity*. It is known that there is a close connection between the spectral properties of the infinitesimal generator $A$ and the characteristic matrix $\Delta(z)$ associated with (2.1), see [3] and [7]. In particular, the geometric multiplicity $d_\lambda$ equals the dimension of the null space of $\Delta(\lambda)$ at $\lambda$ and the algebraic multiplicity is equal to the multiplicity of $z = \lambda$ as a zero of $\det \Delta(z)$. Furthermore, the generalized eigenspace at $\lambda$ is given by

$$\mathcal{M}_\lambda = \mathcal{N}\big((\lambda I - A)^{k_\lambda}\big),$$

where $k_\lambda$ is the order of $z = \lambda$ as a pole of $\Delta(z)^{-1}$. The eigenvalue $\lambda$ is called *simple* if $m_\lambda = 1$.

Let $\{\varphi_1, \dots, \varphi_{m_\lambda}\}$ be a basis of eigenvectors and generalized eigenvectors of $A$ at $\lambda$. Define the row $m_\lambda$-vector $\Phi = \{\varphi_1, \dots, \varphi_{m_\lambda}\}$. Since $\mathcal{M}_\lambda$ is invariant under $A$, there exists a $m_\lambda \times m_\lambda$ matrix $M$ such that $A\Phi = \Phi M$. The action of the semigroup $T(t)$ on $\Phi$ is given by $T(t)\Phi = \Phi e^{tM}$. In particular, it follows that $\Phi(\theta) = \Phi(0)e^{\theta M}$, $-h \le \theta \le 0$. So, in order to obtain a complete description of the subspace $\mathcal{M}_\lambda$, we have to compute $\Phi(0)$.

From standard spectral theory it follows that the spectral projection onto $\mathcal{M}_\lambda$ along $\mathcal{R}\big((\lambda I - A)^{k_\lambda}\big)$ can be represented by a Dunford integral

$$P_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (zI - A)^{-1}\, dz, \tag{3.2}$$

where $\Gamma_\lambda$ is a small circle such that $\lambda$ is the only singularity of $(zI - A)^{-1}$ inside $\Gamma_\lambda$. A direct computation yields the following representation for the resolvent of $A$:

$$(zI - A)^{-1}\varphi = \frac{1}{\det \Delta(z)} P(z)\varphi, \tag{3.3}$$

where

$$\begin{aligned}\big(P(z)\varphi\big)(\theta) \;&=\; e^{z\theta}\big\{\operatorname{adj}\Delta(z)[\varphi(0) + \int_0^h d\zeta(\tau)\int_0^\tau e^{-z\sigma}\varphi(\sigma - \tau)\, d\sigma] \\ &\qquad + \det \Delta(z)\int_\theta^0 e^{-z\sigma}\varphi(\sigma)\, d\sigma\big\}. \end{aligned}\tag{3.4}$$

For example, if $\lambda$ is a simple eigenvalue, this representation together with formula (3.2) can be used to obtain an explicit representation for the spectral projection onto the eigenspace corresponding to a simple eigenvalue

$$P_\lambda \varphi = e^{\lambda \cdot} H(\lambda)\big(\varphi(0) + \int_0^h d\zeta(\theta) \int_0^\theta e^{-\lambda\sigma} \varphi(\sigma - \theta)\, d\sigma\big),$$

where $H(\lambda) = \lim_{z \to \lambda}(z - \lambda)\Delta(z)^{-1}$. Equivalently, we can write

$$P_\lambda \varphi = \langle \phi_\lambda^*, \varphi \rangle \phi_\lambda,$$

where $\phi_\lambda$ is an eigenvector at $\lambda$ for $A$, $\phi_\lambda^*$ is an eigenvector at $\lambda$ for $A^*$ and $\langle \phi_\lambda^*, \phi_\lambda \rangle = 1$ (here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $C$ and the dual space $C^*$).

Thus $\Phi(0)$ can be found by providing a basis for the range of $P_\lambda$. In [7] and Section IV.4 of [3] a systematic procedure has been developed to construct a canonical basis using Jordan chains. We shall here describe the underlying idea.

**Definition 3.1** A sequence of vectors $\{\phi_0, \phi_1, \dots, \phi_{r-1}\}$ in $C$ with $\phi_0 \neq 0$ is called a *Jordan chain* of $A$ at $\lambda$ if

$$A\phi_0 = \lambda \phi_0, \qquad A\phi_j = \lambda \phi_j + \phi_{j-1},$$

for $j = 1, 2, \dots, r-1$. The integer $r$ is called the *rank* of the Jordan chain. Note that $\phi_0$ is an eigenvector of $A$, the vectors $\phi_1, \dots, \phi_{r-1}$ are called *generalized eigenvectors* at $\lambda$.

One can organize the Jordan chains as follows. Choose an eigenvector, say $\phi_\lambda^{1,0}$, with maximal rank, say $r_1$. Next, choose a Jordan chain

$$(\phi_\lambda^{1,0}, \dots, \phi_\lambda^{1,r_1-1})$$

of length $r_1$ and let $N_1$ be the complement in $\mathcal{N}\big(\lambda I - A\big)$ of the subspace spanned by $\phi_\lambda^{1,0}$. In $N_1$ we choose an eigenvector $\phi_\lambda^{2,0}$ of maximal rank, say $r_2$, and let

$$(\phi_\lambda^{2,0}, \dots, \phi_\lambda^{2,r_2-1})$$

be a corresponding Jordan chain of length $r_2$. Next let $N_2$ be the complement in $N_1$ of the subspace spanned by $\phi_\lambda^{2,0}$ and replace $N_1$ by $N_2$ in the above-described procedure.

This construction yields a basis $\{\phi_\lambda^{1,0}, \dots, \phi_\lambda^{d_\lambda,0}\}$ for $\mathcal{N}\big(\lambda I - A\big)$ and a corresponding canonical system of Jordan chains

$$\phi_\lambda^{1,0}, \dots, \phi_\lambda^{1,r_1-1}, \dots, \phi_\lambda^{p,0}, \dots, \phi_\lambda^{d_\lambda, r_{d_\lambda}-1}.$$

Note that $\max\{r_j \mid j = 1, \ldots, d_\lambda\} = k_\lambda$ and that

$$\sum_{j=1}^{d_\lambda} r_j = m_\lambda.$$

Because of the construction, the canonical system of Jordan chains is a basis for the generalized eigenspace $\mathcal{M}_\lambda$. With respect to this basis, the matrix $M$ has Jordan normal form. We shall call such a basis a canonical basis of eigenvectors and generalized eigenvectors for $A$ at $\lambda$ (see [3] for much more information).

The generalized eigenspace $\mathcal{M}$ is the smallest subspace that contains all $\mathcal{M}_\lambda$, $\lambda \in \sigma(A)$. If $\mathcal{M}$ is dense in $C$, we call the system of eigenvectors and generalized eigenvectors of $A$ *complete*. A Jordan chain of $A$ gives rise to a special solution of (2.1). If $\{\varphi_0, \varphi_1, \ldots, \varphi_{r-1}\}$ is a Jordan chain of $A$ at $\lambda$, then

$$x(t) = e^{\lambda t} \sum_{\nu=0}^{r-1} \frac{1}{\nu!} t^\nu \varphi_{r-1-\nu}(0) \tag{3.5}$$

is a solution of (2.1) for all $t$. So if the linear space spanned by all eigenvectors and generalized eigenvectors is dense in $C$ then each solution can be approximated by a linear combination of solutions of the form (3.5) (in a sense that can be made precise). In this case we say that the operator $A$ has a complete span of eigenvectors and generalized eigenvectors.

To present a simple necessary and sufficient condition for completeness, we need the following definition.

**Definition 3.2** An entire function $f : \mathbb{C} \to \mathbb{C}$ is of *exponential type* if

$$\limsup_{r\to\infty} \frac{1}{r} \log \max_{0 \le \theta \le 2\pi} |f(re^{i\theta})| = \mathrm{E(f)},$$

where $0 \le \mathrm{E(f)} < \infty$. In that case, $\mathrm{E(f)}$ is called the exponential type of $f$.

In is known (see [3]) that the system of eigenvectors and generalized eigenvectors of $A$ is dense in the space $C$ if and only if the semigroup $T(t)$ is one-to-one if and only if $\mathrm{E}(\det \Delta(\mathrm{z})) = \mathrm{nh}$. For example, the system

$$
\begin{aligned}
\frac{dx_1}{dt}(t) &= x_1(t) + x_2(t-1) \\
\frac{dx_2}{dt}(t) &= x_1(t-1)
\end{aligned}
\tag{3.6}
$$

has the following characteristic equation

$$\Delta(z) = \begin{pmatrix} z-1 & -e^{-z} \\ -e^{-z} & z \end{pmatrix}.$$

So $\det \Delta(z) = z^2 - z - e^{-2z}$ and $\mathrm{E}(\det \Delta(\mathrm{z})) = 2$. Therefore, the system of eigenvectors and generalized eigenvectors corresponding to (3.6) is complete.

The condition in Theorem 2.1 that $\det A_l^m \neq 0$ or $\det A_I^m$ is not identically zero in a neighbourhood of $h$ implies that $\mathrm{E}(\det \Delta^{\mathrm{m}}(\mathrm{z})) = \mathrm{nh}$. This condition cannot be omitted in the theorem. Suppose that $\mathrm{E}(\det \Delta^{\mathrm{m}}(\mathrm{z})) < \mathrm{nh}$ and, hence, $T^m(t)$ is not one-to-one. If $\varphi^m$ is such that $T^m(t_0)\varphi^m$ is identically zero for certain $t_0$ then the condition $y(t;\varphi^m) = x(t;\varphi)$ only implies that the unknown system has a solution of this type as well. No information is obtained about the unknown parameters in (2.5).

# 4 Proof of the main result

We start this section with the formulation of a more general version of Theorem 2.1.

**Theorem 4.1** *Consider an unknown equation (2.1) and assume that the model system*

$$\frac{dy}{dt}(t) = \int_0^h d\zeta^m(\theta) y(t-\theta), \quad t \geq 0, \tag{4.1}$$

*has the property that* $\mathrm{E}(\det \Delta^{\mathrm{m}}(\mathrm{z})) = \mathrm{nh}$. *For every* $\lambda \in \sigma(A^m)$, *define* $\phi_\lambda^{m,*,j,0}$, $j = 1, \ldots, d_\lambda$ *to be a basis for* $\mathcal{N}\big((\lambda I - (A^m)^*\big)$. *Suppose that* $N$ *initial conditions* $\varphi_i^m$, $i = 1, 2, \ldots, N$, *are given such that*

$$\operatorname{rank}\big(\langle \phi_\lambda^{m,*,j,0}, \varphi_i^m \rangle\big)_{1\leq i\leq N, 1\leq j\leq d_\lambda} = d_\lambda \tag{4.2}$$

*(here* $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $C$ *and the dual space* $C^*$*) and such that there exists* $t_1, t_2 \geq 0$ *with*

$$T^m(t)\varphi_i^m = T(t)\varphi_i \quad t_1 + h \leq t \leq t_2, \ i = 1, 2, \ldots, N. \tag{4.3}$$

*If $t_2 - t_1 > nh$, then $\zeta = \zeta^m$ and the unknown and the model system are identical.*

In the proof of Theorem 4.1 we shall exploit the fact that the system is time-invariant. Define

$$\psi_i^m := T^m(t_1 + h)\varphi_i^m \quad \text{for } i = 1, 2, \dots, N.$$

Using the semigroup property of $T^m(t)$, it follows that condition (4.3) can be rewritten as

$$T^m(t)\psi_i^m = T(t)\psi_i^m \quad 0 \leq t \leq \tau,\ i = 1, 2, \dots, N, \tag{4.4}$$

where $\tau > (n-1)h$. Furthermore, since

$$\begin{aligned} \langle \phi_\lambda^{m,*,j,0}, T^m(t_1 + h)\varphi_i^m \rangle &= \langle T^m(t_1 + h)^* \phi_\lambda^{m,*,j,0}, \varphi_i^m \rangle \\ &= e^{\lambda(t_1 + h)} \langle \phi_\lambda^{m,*,j,0}, \varphi_i^m \rangle, \end{aligned}$$

we have

$$\operatorname{rank} \left( \langle \phi_\lambda^{m,*,j,0}, \psi_i^m \rangle \right)_{1 \leq i \leq N, 1 \leq j \leq d_\lambda} = d_\lambda. \tag{4.5}$$

Thus in the proof of Theorem 4.1 we can, without loss of generality, assume that $t_1 = -h$, $\tau = t_2 - t_1 > (n-1)h$ and $\varphi_i = \varphi_i^m$ for $i = 1, 2, \dots, N$.

The following lemma is essential and allows us to use techniques from the theory of the Laplace transform.

**Lemma 4.1** *Suppose that* $\mathrm{E}(\det \Delta^{\mathrm{m}}(\mathrm{z})) = \mathrm{nh}$. *If*

$$T^m(t)\psi^m = T(t)\psi^m \quad \textit{for } 0 \leq t \leq \tau$$

*and $\tau > (n-1)h$, then*

$$T^m(t)\psi^m = T(t)\psi^m \quad \textit{for all } t \geq 0.$$

**Proof.** From basic properties of the Laplace transform, it follows that

$$\int_0^\infty e^{-zt} T^m(t)\psi^m \, dt = (zI - A^m)^{-1}\psi^m, \quad \mathrm{Re} z > \omega_0,$$

where $A^m$ denotes the infinitesimal generator and $\omega_0$ denotes the growth bound of $T^m(t)$. The resolvent of $A^m$ admits an explicit representation

$$(zI - A^m)^{-1}\psi^m = \frac{1}{\det \Delta^m(z)} P^m(z)\psi^m, \tag{4.6}$$

where $P^m(z)$ is given by (3.4).

Define $Z : [0, \infty) \to C$ by

$$Z(t) := T^m(t)\psi^m - T(t)\psi^m,$$

then $Z(t) = 0$ on $[0, \tau]$. From the definition of $Z(t)$, it follows

$$\int_0^\infty e^{-zt} Z(t)\,dt = \frac{1}{\det \Delta^m(z)} P^m(z)\psi^m - \frac{1}{\det \Delta(z)} P(z)\psi^m. \tag{4.7}$$

On the other hand

$$\begin{aligned} \int_0^\infty e^{-zt} Z(t)\,dt &= \int_\tau^\infty e^{-zt} Z(t)\,dt \\ &= e^{-z\tau} \int_0^\infty e^{-zt} Z(t+\tau)\,dt. \end{aligned} \tag{4.8}$$

Since

$$\begin{aligned} Z(t+\tau) &= T^m(t+\tau)\psi^m - T(t+\tau)\psi^m \\ &= T^m(t)T^m(\tau)\psi^m - T(t)T^m(\tau)\psi^m, \end{aligned} \tag{4.9}$$

it follows that

$$\begin{aligned} \int_0^\infty e^{-zt} Z(t+\tau)\,dt &= \frac{1}{\det \Delta^m(z)} P^m(z)T^m(\tau)\psi^m \\ &\quad - \frac{1}{\det \Delta(z)} P(z)T^m(\tau)\psi^m. \end{aligned} \tag{4.10}$$

Now define $H : \mathbb{C} \to C$

$$H(z) := \det \Delta(z) \det \Delta^m(z) \int_0^\infty e^{-zt} Z(t)\,dt. \tag{4.11}$$

Observe that

$$H(z) = \det \Delta(z) P^m(z)\psi^m - \det \Delta^m(z) P(z)\psi^m \tag{4.12}$$

and the function $H$ is an entire vector-valued function of finite exponential type.

Suppose that $H$ is not identically zero. Using the exponential type calculus (see Section V.2 of [3]) we can compute the exponential type of $H$ in the left half plane

$$\mathrm{E(H)} = \max\big\{\mathrm{E}(\det \Delta(\mathrm{z})) + \mathrm{E}(\mathrm{P^m(z)}\psi^{\mathrm{m}}), \mathrm{E}(\det \Delta^{\mathrm{m}}(\mathrm{z})) + \mathrm{E}(\mathrm{P(z)}\psi^{\mathrm{m}})\big\}.$$

From Lemma V.3.1 of [3], it follows that

$$\mathrm{E}(\mathrm{P}^{\mathrm{m}}(\mathrm{z})\psi^{\mathrm{m}}) \le \mathrm{nh} \quad \text{and} \quad \mathrm{E}(\mathrm{P}(\mathrm{z})\psi^{\mathrm{m}}) \le \mathrm{nh}.$$

Furthermore, $\mathrm{E}(\det\Delta(\mathrm{z})) \le \mathrm{nh}$ and hence

$$\mathrm{E}(\mathrm{H}) \le 2\mathrm{nh}.$$

On the other hand, using (4.8) and (4.10), we have

$$\begin{aligned} \mathrm{E}(\mathrm{H}) \;=\; \max\{&\tau + \mathrm{E}(\det\Delta(\mathrm{z})) + \mathrm{E}(\mathrm{P}^{\mathrm{m}}(\mathrm{z})\mathrm{T}^{\mathrm{m}}(\tau)\psi^{\mathrm{m}}), \\ &\tau + \mathrm{E}(\det\Delta^{\mathrm{m}}(\mathrm{z})) + \mathrm{E}(\mathrm{P}(\mathrm{z})\mathrm{T}^{\mathrm{m}}(\tau)\psi^{\mathrm{m}})\}. \end{aligned} \tag{4.13}$$

Since $\mathrm{E}(\det\Delta^{\mathrm{m}}(\mathrm{z})) = \mathrm{nh}$, it follows that $T^m(t)$ is one-to-one and therefore $T^m(\tau)\psi^m \neq 0$. Together with the definition of $P(z)$ (see (3.4)) this implies that $\mathrm{E}(\mathrm{P}^{\mathrm{m}}(\mathrm{z})\mathrm{T}^{\mathrm{m}}(\tau)\psi^{\mathrm{m}}) \ge \mathrm{h}$. Thus (4.13) yields the estimate

$$\begin{aligned} 2nh \ge \mathrm{E}(\mathrm{H}) &\ge \tau + \mathrm{E}(\det\Delta^{\mathrm{m}}(\mathrm{z})) + \mathrm{E}(\mathrm{P}(\mathrm{z})\mathrm{T}^{\mathrm{m}}(\tau)\psi^{\mathrm{m}}) \\ &\ge \tau + nh + h > 2nh. \end{aligned}$$

A contradiction to the assumption that $H$ is nonzero. Thus $H(z) = 0$ and the Laplace transform of $Z(\cdot+\tau)$ is zero. Since $Z$ is continuous, this implies that $Z$ is identically zero and

$$T^m(t)\psi^m = T(t)\psi^m \quad \text{for } t \ge 0.$$

This completes the proof of the lemma. □

In the following lemma we use the Laplace transform and the Dunford representation of the spectral projection to derive a useful identity.

**Lemma 4.2** *If $T^m(t)\psi^m = T(t)\psi^m$ for all $t \ge 0$, then for every $\lambda \in \mathbb{C}$ and $s \ge 0$*

$$T^m(s)(A^m - \lambda I)^j P^m_\lambda \psi^m = T(s)(A - \lambda I)^j P_\lambda \psi^m, \quad j = 1, 2, \ldots. \tag{4.14}$$

**Proof.** From the assumption and the semigroup property, it follows that for all $t, s \ge 0$

$$T^m(t)T^m(s)\psi^m = T(t)T(s)\psi^m. \tag{4.15}$$

Applying the Laplace transform to (4.15) with respect to $t$ yields

$$(zI - A^m)^{-1}T^m(s)\psi^m = (zI - A)^{-1}T(s)\psi^m \quad \text{for } \operatorname{Re} z > \omega_0. \tag{4.16}$$

From the representation (3.3) for the resolvent of $A$ it follows, by analytic continuation, that we can extend identity (4.16) to hold for all $z \in \mathbb{C}$ except at the poles of

$$(zI - A^m)^{-1} T^m(s)\psi^m.$$

For every $\lambda \in \sigma(A^m)$, we choose a small circle $\Gamma_\lambda$ such that $\lambda$ is the only singularity of $(zI - A^m)^{-1} T^m(s)\psi^m$ inside $\Gamma_\lambda$. A direct computation, using the extended identity (4.16) and representation (3.2) for the spectral projection, yields for every $\lambda \in \sigma(A^m)$ and $t_0 \geq 0$

$$T^m(s) P^m_\lambda \psi^m = T(s) P_\lambda \psi^m. \tag{4.17}$$

Again, using the semigroup property, we have for all $t, s \geq 0$

$$T^m(t) T^m(s) P^m_\lambda \psi^m = T(t) T(s) P_\lambda \psi^m. \tag{4.18}$$

From this identity and the definition

$$A\varphi = \lim_{t \downarrow 0} \frac{1}{t} \left[ T(t)\varphi - \varphi \right]$$

for $\varphi \in \mathcal{D}(A)$, it follows that

$$A^m T^m(s) P^m_\lambda \psi^m = A T(s) P_\lambda \psi^m \quad s \geq 0. \tag{4.19}$$

Since $A$ commutes with $T(t)$ the identity (4.14) follows by induction. $\square$

**Lemma 4.3** *Suppose that $\mathcal{N}\big( (\lambda I - A^m)^k_\lambda \big) = \{\phi^{m,1,0}_\lambda, \dots, \phi^{m,1,r_1-1}_\lambda\}$ consists of one Jordan chain only ($d_\lambda = 1$). If $\phi^{m,*,1,0}_\lambda$ is an eigenvector of $(A^m)^*$ at $\lambda$ and*

$$\langle \phi^{m,*,1,0}_\lambda, \psi^m \rangle \neq 0, \tag{4.20}$$

*then*

$$\operatorname*{span}_{j=0,1,\dots,k_\lambda - 1} (A^m - \lambda I)^j P^m_\lambda \psi^m = \mathcal{M}^m_\lambda. \tag{4.21}$$

**Proof.** If $\phi^{m,*,1,j}_\lambda$, $j = 0, \dots, r_1 - 1$, denotes a canonical basis for $\mathcal{N}\big( (\lambda I - (A^m)^*)^{k_\lambda} \big)$, we have

$$((A^m)^* - \lambda I)\phi^{m,*,1,j}_\lambda = \phi^{m,*,1,j-1}_\lambda \tag{4.22}$$

and

$$((A^m)^* - \lambda I)\phi^{m,*,1,0}_\lambda = 0.$$

Using induction with respect to $j$, we conclude that (4.20) implies

$$\langle \phi_\lambda^{m,*,1,j}, (A^m - \lambda)^j \psi^m \rangle \neq 0 \tag{4.23}$$

for $j = 0, \ldots, m_\lambda - 1$. Using identity (4.23) we shall show that the vectors

$$(A^m - \lambda I)^j P_\lambda^m \psi^m, \quad j = 0, \ldots, r_1 - 1, \tag{4.24}$$

are linearly independent.

We argue by contradiction and suppose that the vectors in (4.24) are linearly dependent. Then there exist constants $c_j$ not all zero such that

$$\sum_{j=0}^{m_\lambda - 1} c_j (A^m - \lambda I)^j P_\lambda^m \psi^m = 0$$

and hence, using (4.22)

$$\begin{aligned} 0 &= \langle \phi_\lambda^{m,*,1,k}, \sum_{j=0}^{r_1 - 1} c_j (A^m - \lambda I)^j P_\lambda^m \psi^m \rangle \\ &= \sum_{j=0}^{k} c_j \langle \phi_\lambda^{m,*,1,k}, (A^m - \lambda I)^j \psi^m \rangle \end{aligned}$$

for $k = 0, 1, \ldots, m_\lambda - 1$. For $k = 0$, condition (4.20) implies that $c_0 = 0$. For $k = 1$, it follows from (4.23) and $c_0 = 0$ that $c_1 = 0$. We proceed by induction and conclude that $c_j = 0$ for $j = 0, \ldots, r_1 - 1$. A contradiction to the assumption that the constants $c_j$ are not all zero. This proves that the vectors in (4.24) are linearly independent.

Therefore there are $r_1$ linearly independent vectors of the form (4.24) which belong to $\mathcal{M}_\lambda$. Since $\dim \mathcal{M}_\lambda^m = m_\lambda = r_1$, the conclusion of the lemma follows. $\square$

**Proof of Theorem 4.1.** First we consider the case $d_\lambda = 1$. From the earlier observation and Lemma 4.1, it follows that without loss of generality we can assume that

$$T^m(t)\psi^m = T(t)\psi^m \quad \text{for all} \quad t \geq 0.$$

From Lemma 4.2 and Lemma 4.3 we derive that (see (4.14) and (4.21))

$$T^m(t)|_{\mathcal{M}_\lambda^m} = T(t)|_{\mathcal{M}_\lambda^m}.$$

This procedure can be repeated for every $\lambda \in \sigma(A^m)$ to obtain

$$T^m(t)|_{\mathcal{M}^m} = T(t)|_{\mathcal{M}^m}.$$

From the assumption that $\mathrm{E}(\det \Delta^{\mathrm{m}}(\mathrm{z})) = \mathrm{nh}$, it follows that $\mathcal{M}^m$ is a dense subspace of $C$ and thus

$$T^m(t) = T(t).$$

This proves the theorem in case $d_\lambda = 1$.

Next we consider the case $d_\lambda > 1$. Let $\{\phi_\lambda^{m,*,j,0}\}_{j=1}^{d_\lambda}$ denote a basis for $\mathcal{N}\big(\lambda I - (A^m)^*\big)$ and

$$\{\phi_\lambda^{m,*,j,i} \mid j = 1, \dots, d_\lambda,\ i = 0, 1, \dots, r_j - 1\}$$

the canonical basis for $\mathcal{N}\big((\lambda I - (A^m)^*)^{k_\lambda}\big)$. Using Lemma 4.1, it follows that without loss of generality we can assume that for $i = 1, 2, \dots, d_\lambda$

$$T(t)\psi_i^m = T^m(t)\psi_i^m \quad t \geq 0 \tag{4.25}$$

and the matrix

$$\big(\langle \phi_\lambda^{m,*,j,0}, \psi_i^m \rangle\big)_{1 \leq i \leq d_\lambda, 1 \leq j \leq d_\lambda} \quad \text{is invertible.} \tag{4.26}$$

For $k = 1, 2, \dots, d_\lambda$, define

$$\psi^{m,k} = c_1^k \psi_1^m + \cdots c_{d_\lambda}^k \psi_{d_\lambda}^m, \quad c_j^k \in \mathbb{R}.$$

Then it follows from (4.25) that

$$T^m(t)\psi^{m,k} = T(t)\psi^{m,k}, \quad t \geq 0.$$

Using (4.26), we can choose the $c_j^k$ such that

$$\langle \phi_\lambda^{m,*,j,0}, \psi^{m,k} \rangle \neq 0 \quad \text{for } j = k$$

and

$$\langle \phi_\lambda^{m,*,j,0}, \psi^{m,k} \rangle = 0 \quad \text{for } j \neq k.$$

So, for every fixed $k$, we are in a situation similar to the case $d_\lambda = 1$ and $\psi^m = \psi^{m,k}$. Using Lemma 4.2 and Lemma 4.3, we obtain

$$\dim \operatorname*{span}_{j=0,1,\dots,r_k-1} (A - \lambda I)^j P_\lambda^m \psi^{m,k} = r_k. \tag{4.27}$$

This identity implies that

$$\dim \bigoplus_{k=1}^{d_\lambda} \operatorname*{span}_{j=0,1,\dots,r_k-1} (A^m - \lambda I)^j P_\lambda^m \psi^{m,k} = \sum_{k=1}^{d_\lambda} r_k = m_\lambda.$$

Thus

$$\bigoplus_{k=1}^{d_\lambda} \operatorname*{span}_{j=0,1,\ldots,r_k-1} (A^m - \lambda I)^j P_\lambda^m \psi^{m,k} = \mathcal{M}_\lambda^m. \tag{4.28}$$

From the identities (4.14) and (4.28), we derive that

$$T^m(t)|_{\mathcal{M}_\lambda^m} = T(t)|_{\mathcal{M}_\lambda^m}.$$

This procedure can be repeated for every $\lambda \in \sigma(A^m)$ to obtain

$$T^m(t)|_{\mathcal{M}^m} = T(t)|_{\mathcal{M}^m}.$$

From the assumption $\mathrm{E}(\det \Delta^{\mathrm{m}}(\mathrm{z})) = \mathrm{nh}$, it follows that $\mathcal{M}^m$ is a dense subspace of $C$ and thus

$$T^m(t) = T(t).$$

This completes the proof of Theorem 4.1. □

To derive Theorem 2.1 from Theorem 4.1, it suffices to compute the eigenvectors of $(A^m)^*$. In [3], Corollary 5.12, one can find the following explicit representation for a basis of eigenvectors of $(A^m)^*$ at $\lambda \in \sigma(A^m)$

$$\phi_\lambda^{m,*,j,0}(\tau) = q_\lambda^{m,j}(I + \int_0^\tau \int_\sigma^h e^{\lambda(\sigma - s)}\, d\zeta^m(s)\, d\sigma), \quad j = 1, 2, \ldots, d_\lambda,$$

where $q_\lambda^{m,j}$, $j = 1, 2, \ldots, d_\lambda$, is a basis for the left null space of $\Delta^m(\lambda)$. Using this representation, it follows directly that condition (2.17) and (4.2) are equivalent.

For further generalizations to neutral functional differential equations and to models describing the dynamics of age-dependent populations we refer to [15].

# 5 Proofs of the corollaries

In this section we shall prove Corollary 2.1 and Corollary 2.2. The proofs of the corollaries illustrate how to apply Theorem 4.1 and how to adapt the main result to other specific examples.

**Proof of Corollary 2.1.** In order to apply Theorem 2.1, it suffices to verify (2.17). Since in the scalar case $d_\lambda = 1$ and we can choose $q_\lambda^m = 1$, it remains to show that $p^m_\lambda$ in (2.16) with $\varphi^m = e^{\lambda_0 \theta}$, $\Delta^m(\lambda_0) \neq 0$, is not equal to zero. From the definition of $p^m_\lambda$ we derive

$$\begin{aligned}
p^m_\lambda &= 1 + \int_0^h (e^{\lambda\tau} \int_\tau^h e^{-\lambda s} d\zeta^m(s)) e^{-\lambda_0 \tau}\, d\tau \\
&= 1 + \int_0^h (\int_0^s e^{\lambda(\tau - s)} e^{-\lambda_0 \tau}\, d\tau)\, d\zeta^m(s) \\
&= 1 + \int_0^h (\int_0^s e^{(\lambda-\lambda_0)\tau}\, d\tau) e^{-\lambda s}\, d\zeta^m(s) \\
&= 1 + \int_0^h (\tfrac{1}{\lambda-\lambda_0}[e^{(\lambda-\lambda_0)s} - 1] e^{-\lambda s}\, d\zeta^m(s) \\
&= 1 + \tfrac{1}{\lambda-\lambda_0} \int_0^h e^{-\lambda_0 s} d\zeta^m(s) - \tfrac{1}{\lambda-\lambda_0} \int_0^h e^{-\lambda s} d\zeta^m(s) \\
&= \tfrac{1}{\lambda-\lambda_0} \big[\Delta^m(\lambda) - \Delta^m(\lambda_0)\big] = -\tfrac{1}{\lambda-\lambda_0} \Delta^m(\lambda_0) \neq 0.
\end{aligned}$$

In the computation we have reversed the order of integration and used the fact that $\lambda$ is a zero of $\Delta^m(z)$. □

**Proof of Corollary 2.2.** In order to apply Theorem 2.1, it suffices to verify (2.17). We first compute $d_\lambda$ and the vectors $q^m_\lambda$ for every $\lambda$ with $\det \Delta^m(\lambda) = 0$.

The model system is given by

$$\begin{aligned}
\frac{dy_1}{dt}(t) &= a_1 y_2(t-1) \\
\frac{dy_2}{dt}(t) &= a_2 y_1(t-1)
\end{aligned}$$

and hence the characteristic matrix is given by

$$\Delta^m(z) = \begin{pmatrix} z & -a_1 e^{-z} \\ -a_2 e^{-z} & z \end{pmatrix}.$$

The determinant satisfies

$$\begin{aligned}
\det \Delta^m(z) &= z^2 - a_1 a_2 e^{-2z} \\
&= (z - \sqrt{a_1 a_2} e^{-z})(z + \sqrt{a_1 a_2} e^{-z}).
\end{aligned}$$

If $\lambda = \sqrt{a_1 a_2} e^{-\lambda}$, then

$$\Delta^m(\lambda) = e^{-\lambda} \begin{pmatrix} \sqrt{a_1 a_2} & -a_1 \\ -a_2 & \sqrt{a_1 a_2} \end{pmatrix}.$$

So for every $\lambda$ with

$$\lambda - \sqrt{a_1 a_2} e^{-\lambda} = 0,$$

we have $\dim \mathcal{N}\big(\Delta^m(\lambda)\big) = d_\lambda = 1$ and

$$q_\lambda^{m,+} = \begin{pmatrix} \sqrt{a_1 a_2} & a_1 \end{pmatrix}. \tag{5.1}$$

If $\lambda = -\sqrt{a_1 a_2} e^{-\lambda}$, then

$$\Delta^m(\lambda) = -e^{-\lambda} \begin{pmatrix} \sqrt{a_1 a_2} & a_1 \\ a_2 & \sqrt{a_1 a_2} \end{pmatrix}.$$

So for every $\lambda$ with

$$\lambda + \sqrt{a_1 a_2} e^{-\lambda} = 0,$$

we have $\dim \mathcal{N}\big(\Delta^m(\lambda)\big) = d_\lambda = 1$ and

$$q_\lambda^{m,-} = \begin{pmatrix} \sqrt{a_1 a_2} & -a_1 \end{pmatrix}. \tag{5.2}$$

The matrix $\zeta^m$ in (2.7) is given by

$$\zeta^m = \begin{pmatrix} 0 & a_1 H_1 \\ a_2 H_1 & 0 \end{pmatrix},$$

where

$$H_1(\theta) = \begin{cases} 0 & \theta < 1 \\ 1 & \theta \geq 1. \end{cases}$$

In order to compute $p_\lambda^m$ in (2.16) we first compute

$$\begin{aligned} \int_0^1 e^{\lambda\tau} \int_\tau^1 e^{-\lambda s} d\zeta^m(s) &= \int_0^1 e^{\lambda\tau} \begin{pmatrix} 0 & a_1 e^{-\lambda} \\ a_2 e^{-\lambda} & 0 \end{pmatrix} \\ &= \frac{1}{\lambda} \begin{pmatrix} 0 & a_1(1-e^{-\lambda}) \\ a_2(1-e^{-\lambda}) & 0 \end{pmatrix}. \end{aligned}$$

Thus if $\varphi^m = (\varphi_1^m, \varphi_2^m)^T$ is constant

$$\begin{aligned} p_\lambda^m &= \begin{pmatrix} \varphi_1^m \\ \varphi_2^m \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & a_1(1-e^{-\lambda}) \\ a_2(1-e^{-\lambda}) & 0 \end{pmatrix} \begin{pmatrix} \varphi_1^m \\ \varphi_2^m \end{pmatrix} \\ &= \frac{1}{\lambda} \left[ \lambda I - \begin{pmatrix} 0 & a_1 e^{-\lambda} \\ a_2 e^{-\lambda} & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \right] \begin{pmatrix} \varphi_1^m \\ \varphi_2^m \end{pmatrix} \\ &= \frac{1}{\lambda} \left[ \Delta^m(\lambda) - \Delta^m(0) \right] \begin{pmatrix} \varphi_1^m \\ \varphi_2^m \end{pmatrix}. \end{aligned}$$

Since $d_\lambda = 1$ for every $\lambda$ with $\det \Delta^m(\lambda) = 0$, it suffices to show that

$$q_\lambda^{m,\pm} p_\lambda^m \neq 0.$$

Note that $q_\lambda^{m,\pm} \Delta^m(\lambda) = 0$. Thus

$$\begin{aligned} q_\lambda^{m,\pm} p_\lambda^m &= -\frac{1}{\lambda} q_\lambda^{m,\pm} \Delta^m(0) \begin{pmatrix} \varphi_1^m \\ \varphi_2^m \end{pmatrix} \\ &= -\frac{1}{\lambda}(a_1\sqrt{a_1 a_2}\varphi_2^m \pm a_1 a_2 \varphi_1^m) \\ &= -\frac{1}{\lambda} a_1 \sqrt{a_2}(\sqrt{a_1}\varphi_2^m \pm \sqrt{a_2}\varphi_1^m) \neq 0, \end{aligned}$$

since $\sqrt{a_1}\varphi_2^m \pm \sqrt{a_2}\varphi_1^m \neq 0$. □

# 6 Conclusions

We have presented a result on identification of unknown system parameters and time delays in models described by linear differential delay equations from knowledge of particular solutions on bounded time intervals. We made clear how methods from operator theory can be exploited to give necessary and sufficient conditions that guarantee that this inverse problem has a unique solution. From the proof of Lemma 4.1, it follows that the condition $t_2 - t_1 > nh$ can only be improved if there is more information available about the unknown system. So, in general, the condition that $t_2 - t_1 > nh$ is sharp. However, we do not know an easy procedure to generalize the examples in Section 1 for nondegenerate model equations of arbitrary dimension.

It is surprising that for scalar differential delay equations, in general, knowing one solution on a time interval of length bigger than the largest time delay in the system uniquely determines the infinite dimensional system. Important open problems concern the identifiability of nonlinear equations, and even for linear systems, parameter identifiability in case there is only indirect knowledge about certain particular solutions. For example, the case that only certain components of the solutions can be observed. Another important open problem concerns robustness. Suppose that the observed solutions are very close to the corresponding solutions of the model system. When does this imply that the system parameters and the delays in the unknown system and the model system are close? The present proof completely breaks down in this case.

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