# Strong stabilization of neutral functional differential equations

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**Abstract**

A linear neutral functional differential equation is called strongly exponentially stable if it is exponentially stable when subjected to small variations in the delays. In this paper we present necessary and sufficient conditions for the existence of a feedback control so that the resulting closed-loop neutral functional differential equation is strongly exponentially stable.

## 1 Introduction

In the implementation of any feedback control system, it is very likely that time delays will occur. Therefore, it is of importance to understand the sensitivity of the stability properties of the closed-loop system with respect to variations in the delays. In [6] a first attempt was made for a unifying theory which explains the underlying mechanisms and the role difference equations are playing in these results.

In this paper we investigate the robustness of the stability of the closed-loop neutral functional differential equation. We explain that the difference operator associated with the neutral part of the equation determines whether stabilization by feedback control is a finite or an infinite dimensional problem. If the control problem turns out to be infinite dimensional, then we show that feedback stabilization of the system might be sensitive to variations in the delays in the system. A neutral functional differential equation is called strongly exponentially stable if it is exponentially stable when subjected to small variations in the delays. We present necessary and sufficient conditions for the existence of a linear feedback control so that the resulting closed-loop neutral functional differential equation is strongly exponentially stable.

Before we can introduce the class of neutral functional differential equations that will be considered in this paper, we need some preparations. Let $C = C\big([-h,0],\mathbb{R}^n\big)$ denote the Banach space of continuous functions provided with the supremum norm, i.e.,

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$\|\varphi\| := \max_{-h\le\theta\le 0} |\varphi(\theta)|$ for $\varphi \in C$. From the Riesz representation theorem, it follows that a bounded linear mapping $L : C \to \mathbb{R}^n$ can be represented by a matrix function of bounded variation, i.e.,

$$L\varphi = \int_{-h}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta(\theta)$, $-h \le \theta \le 0$, is an $n \times n$-matrix whose elements are of bounded variation, normalized so that $\eta$ is continuous from the left on $(-h, 0)$ and $\eta(0) = 0$.

For a function $x : [-h, \infty) \to \mathbb{R}^n$, we define $x_t \in C$ by $x_t(\theta) = x(t+\theta)$, $-h \le \theta \le 0$, $t \ge 0$.

An initial value problem for a linear neutral functional differential equation (NFDE) is given by the following relation

$$\frac{d}{dt} Dx_t = Lx_t, \qquad t \ge 0, \tag{1.1}$$

where $D$ and $L$ are bounded linear maps from $C$ into $\mathbb{R}^n$. The initial data for (1.1) are given by $x_0 = \varphi$, where $\varphi$ belongs to $C$.

A continuous function $x : [-h, \infty) \to \mathbb{R}^n$ is called a solution of the initial value problem (1.1) if $x_0 = \varphi$, the function $t \mapsto Dx_t$ is continuously differentiable on $(0, \infty)$ with a continuous right hand derivative at $t = 0$ and (1.1) is satisfied for $t \ge 0$.

If the map $D : C \to \mathbb{R}^n$ is atomic at zero, that is,

$$D\varphi = \varphi(0) - \int_{-h}^{0} d\mu(\theta)\varphi(\theta), \tag{1.2}$$

where $\mu$ is continuous at zero, then the initial value problem (1.1) has a unique solution on $[-h, \infty)$ (see, for example, [5]).

The difference equation associated with (1.1)–(1.2) is given by $Dx_t = 0$, i.e.,

$$x(t) = \int_{-h}^{0} d\mu(\theta)x(t+\theta). \tag{1.3}$$

For any $\varphi \in C$ for which $D\varphi = 0$, there is a unique solution $x(t)$ of (1.3) which is continuous on $[-h, \infty)$ and satisfies $x_0 = \varphi$.

The solution of (1.1) with initial data $x_0 = \varphi$ is denoted by $x(\,\cdot\,;\varphi)$. The state of the solution at time $t$ is the minimal amount of data needed in order to continue the solution of (1.1) forward in time starting at time $t$. In the present setting the state at time $t$ is precisely $x_t$ and we define the solution map $T(t) : C \to C$ to be the linear operator that maps the initial data onto the state at time $t$, i.e.,

$$\big(T(t)\varphi\big)(\theta) = x_t(\theta;\varphi) = x(t+\theta;\varphi) \qquad \text{for } -h \le \theta \le 0. \tag{1.4}$$

Here we have set up the problem with respect to the state space $C$, but, depending on the applications, we can consider other state spaces as well. For example, the product space $M_p = \mathbb{R}^n \times L^p[-h, 0]$, see [1]. In this case, we define a solution of (1.1) to be a locally integrable function such that $Dx_t$ is absolutely continuous on $[0, \infty)$ and (1.1) is satisfied for almost every $t \geq 0$. The solution of (1.1) with initial data $Dx_0 = c$ and $x_0 = \varphi$ for $(c, \varphi) \in M_p$, is denoted by $x(\,\cdot\,; c, \varphi)$. The solution map $S(t) : M_p \to M_p$ is given by

$$S(t)(c, \varphi) = (Dx_t, x_t), \qquad \text{where } x_t = x_t(\,\cdot\,; c, \varphi). \tag{1.5}$$

The solution maps $T(t)$ and $S(t)$ are closely related. In fact, if we define the embedding $j : C \to M_p$ by $j\varphi = (D\varphi, \varphi)$, then the solution map $T(t)$ corresponds to the restriction of $S(t)$ to the subspace $jC$.

Notice that in the definition of a solution, we do only require differentiability properties on $Dx_t$ and not on the solution $x(\,\cdot\,; c, \varphi)$ itself. However, in applications we sometimes need solutions with more regularity. For example, if the feedback control depends on the derivative of the solution. This naturally leads to the state space $W^{1,p}[-h, 0]$. It has been proved by Henry [8] that if $x_0 = \varphi$ with $\varphi \in W^{1,p}[-h, 0]$, then (1.1) has a unique solution such that $x_t \in W^{1,p}[-h, 0]$ for every $t \geq 0$. The corresponding solution map can again be obtained by the restriction of $S(t)$ to the image of $W^{1,p}[-h, 0]$ under the mapping $\varphi \mapsto (D\varphi, \varphi)$.

The zero solution of (1.1) or (1.8) is *stable* if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|\varphi\| < \delta$ implies $\|x_t(\,\cdot\,; \varphi)\| \leq \epsilon$ for $t \geq 0$. The zero solution of (1.1) is *asymptotically stable* if $\|x_t(\,\cdot\,; \varphi)\| \to 0$ as $t \to \infty$ and *exponentially stable* if there exists an $\alpha > 0$ and a positive constant $M$ such that $\|x_t(\,\cdot\,; \varphi)\| \leq M e^{-\alpha t}\|\varphi\|$ for $t \geq 0$.

The definition of asymptotically stable that we have given is not the usual one since we did not require that the origin was stable. Because the systems we are considering are linear, stability is implied by the hypothesis that solutions approach zero. In fact, the following result is true.

**Lemma 1.1** *If, for each $\varphi \in C$, the solution $x(\,\cdot\,; \varphi)$ of (1.1) satisfies $\|x_t(\,\cdot\,; \varphi)\| \to 0$ as $t \to \infty$, then*

*(i) the origin is stable,*

*(ii) for any compact set $K \subset C$ and any $\epsilon > 0$, there exists a $t_0 > 0$ such that $\sup_{\varphi \in K}\{x_t(\,\cdot\,; \varphi) \mid t \geq t_0\} < \epsilon$,*

*(iii) the set $\{x_t(\,\cdot\,; \varphi) \mid \varphi \in B,\ t \geq 0\}$ is bounded in $C$ if the set $B \subset C$ is bounded.*

Notice that (ii) is an assertion about each compact set whereas (iii) says that the solutions corresponding to a bounded set are uniformly bounded. If it were true that, for any bounded set $B$ and any $\epsilon > 0$, there is a $t_0 > 0$ such that $\sup_{\varphi \in B}\{x_t(\,\cdot\,; \varphi) \mid t \geq t_0\} < \epsilon$ (uniform asymptotic stability), then we would have exponential stability.

This leads naturally to the question of whether or not there is an equation which is asymptotically stable and not exponentially stable. The following is such an example:

$$\frac{d}{dt}[x(t) + x(t-1)] = -cx(t), \qquad x_0 = \varphi,\ \varphi \in C. \tag{1.6}$$

If $c > 0$, then it will follow from Theorem 2.1 and an analysis of the characteristic equation associated with (1.6) that $\beta = 0$. This implies that (1.6) is not exponentially stable. On the other hand, it is not difficult to verify that the Laplace transform of the solution of (1.6) has an analytic extension to the closed right half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$ but this implies that the solution is integrable along the positive real axis and hence, the solution tends to zero as $t$ tends to infinity (see [7] for details).

For some of our results, we need further restrictions on the operator $D$ and we assume that $D$ is a difference operator given by

$$D\varphi = \varphi(0) - \sum_{j=1}^{M} A_j \varphi(-r_j), \tag{1.7}$$

where the coefficients $A_1, A_2, \ldots, A_M$ are $n \times n$-matrices and the delays satisfy $0 < r_1 < r_2 < \cdots < r_M \leq h$. In this case, the difference equation associated with (1.1)–(1.2) becomes

$$x(t) = \sum_{j=1}^{M} A_j x(t - r_j), \qquad t \geq 0. \tag{1.8}$$

Since we are interested in the sensitivity of stability to variations in the delays, we say that the origin is *strongly stable* (resp. *strongly asymptotically stable*) (resp. *strongly exponentially stable*) if it is stable (resp. asymptotically stable) (resp. exponentially stable) when subjected to small variations in the delays. For the difference equation (1.8), we have that (see Corollary 2.2) strong exponential stability is equivalent to exponential stability for *all* variations in the delays. In this case we say that the operator $D$ is *strongly exponentially stable*. We remark that previous work on this subject have referred to strong exponential stability of $D$ as stability of the $D$ operator (see [5]).

We now consider the problem of feedback stabilization of the neutral functional differential equation (1.1)

$$\frac{d}{dt}\Big[x(t) - \sum_{j=1}^{M} A_j x(t - r_j)\Big] = Lx_t + Bu(t), \tag{1.9}$$

where $B$ is an $n \times m$-matrix and $u : [0, \infty) \to \mathbb{R}^m$ is a vector-valued control. So we consider the case that there are no input delays in the control. Since the eigenvectors and generalized eigenvectors of the solution map associated with (1.1) are smooth functions, it follows that exponential stability of (1.1) is independent of the particular state space $C$, $M_p$ or $W^{1,p}[-h, 0]$ under consideration (see also Section 2). So, in order to construct a feedback control to stabilize (1.9), it suffices to define the control on a dense set of initial data containing the eigenvectors and generalized eigenvectors of the solution map associated with (1.1).

If we close the loop with a proportional and a derivative feedback

$$u(t) = \sum_{j=1}^{M} F_j \frac{dx}{dt}(t - r_j) + Gx_t, \tag{1.10}$$

where $G : C \to \mathbb{R}^m$ is a bounded linear map, then the closed-loop system becomes

$$\frac{d}{dt}\Big[x(t) - \sum_{j=1}^{M} (A_j + BF_j) x(t - r_j)\Big] = Lx_t + BGx_t. \tag{1.11}$$

The control problem (1.9) is called *stabilizable* (resp. *strongly stabilizable*) by a proportional and a derivative feedback control (1.10) if the zero solution of the closed-loop system (1.11) is exponentially stable (resp. strongly exponentially stable).

Our main result gives necessary and sufficient conditions for the strong stabilizability of the zero solution of (1.9) by a feedback control of the form (1.10). These conditions will imply that, if the system is strongly stabilizable by a feedback (1.10), then the $D$-operator associated with the closed-loop system

$$D_{cl}\varphi = \sum_{j=1}^{M}(A_j + BF_j)\varphi(-r_j)$$

is exponentially stable for all values of the delays.

The organization of the paper is as follows. In Section 2 we discuss the basic stability properties of linear neutral functional differential equations. In Section 3 we use the results from Section 2 to prove our main result on feedback stabilization of neutral functional differential equations.

## 2 Stability and spectral theory for neutral equations

The dynamical system approach to functional differential equations is to associate with (1.1) a semigroup $T(t) : C \to C$ defined by (1.4). In other words, the time evolution of the state $x_t$ is given by an abstract ordinary differential equation in the infinite dimensional state space $C$. Given equation (1.1), this ordinary differential equation can be computed explicitly.

Let $v : [0, \infty) \to C$ be a solution of

$$\frac{dv}{dt} = Av, \qquad v(0) = \varphi,\ \varphi \in C, \tag{2.1}$$

where $A : \mathcal{D}(A) \to C$ is an unbounded operator defined by

$$\begin{cases} A\varphi & = \frac{d\varphi}{d\theta} \\ \\ \mathcal{D}(A) & = \{\varphi \in C \mid \frac{d\varphi}{d\theta} \in C \text{ and } D\frac{d\varphi}{d\theta} = L\varphi\}. \end{cases} \tag{2.2}$$

The operator $A$ defined by (2.2) is the infinitesimal generator of the strongly continuous semigroup $T(t) : C \to C$ defined by translation along the solution, viz. (1.4).

Thus, the neutral functional differential equations (1.1) with time delays can be viewed as a transport equation (2.1) with nonlocal boundary conditions. Furthermore, the solutions of (1.1) are in one-to-one correspondence with the solutions of the infinite dimensional ordinary differential equation (2.1) and this correspondence is given by

$$v(t)(\theta) = x(t + \theta). \tag{2.3}$$

This observation originated with Krasovskii and has been crucial in the development of the qualitative theory of functional differential equations [3, 5].

If $M : \mathcal{D}(M) \subset C \to C$ is a linear operator, then the spectrum $\sigma(M)$ is the set of complex numbers $\mu$ for which the operator $\mu I - M$ is not invertible. A complex number $\mu$ is an eigenvalue of $M$ if there is a nonzero $\varphi \in \mathcal{D}(M)$ such that $M\varphi = \mu\varphi$. The kernel $\mathcal{N}\big(\,\mu I - M\,\big)$ is called the eigenspace at $\mu$ and its dimension $d_\mu$ is called the geometric multiplicity of $\mu$. The generalized eigenspace $\mathcal{M}_\mu$ is the smallest closed subspace that contains all $\mathcal{N}\big(\,(\mu I - A)^j\,\big)$, $j = 1, 2, \ldots$. The dimension of $\mathcal{M}_\mu$ is called the algebraic multiplicity of $\mu$.

It is not difficult to show that the operator $A$ in (2.2) has compact resolvent and, therefore, the spectrum $\sigma(A)$ of $A$ consists only of eigenvalues of finite multiplicity. To be precise, the complex number is an eigenvalue of $A$ if and only if $\lambda$ is the root of the determinant of the characteristic matrix $\Delta(z)$, where

$$\Delta(z) = z\big(I - \sum_{j=1}^{M} A_j e^{-zr_j}\big) - \int_{-h}^{0} e^{z\theta} d\eta(\theta). \tag{2.4}$$

In this representation, we have used (1.2) and the Riesz representation to write the bounded linear map $L : C \to \mathbb{C}^n$ as a Riemann-Stieltjes integral with respect to a real matrix-valued function $\eta$ of bounded variation.

It is known that there is a close connection between the spectral properties of the operator $A$ and the characteristic matrix $\Delta(z)$ associated with (1.1). In particular, the spectral properties of the generator $A$ are independent of the state space $C$, $M_p$ or $W^{1,p}[-h, 0]$, see [10]. In fact, if $\lambda \in \sigma(A)$, then the geometric multiplicity $d_\lambda$ equals the dimension of the null space of $\Delta(\lambda)$ at $\lambda$ and the algebraic multiplicity is equal to the multiplicity of $z = \lambda$ as a zero of $\det \Delta(z)$. Furthermore, the generalized eigenspace at $\lambda$ is given by

$$\mathcal{M}_\lambda = \mathcal{N}\big(\,(\lambda I - A)^{k_\lambda}\,\big),$$

where $k_\lambda$ is the order of $z = \lambda$ as a pole of $\Delta(z)^{-1}$.

From a result by Henry [9], see also [11] for related results, we also know that

$$\sigma(T(t)) = \text{ closure}\big\{e^{\lambda t} \mid \lambda \in \sigma(A)\big\} \text{ plus possibly } \{0\}. \tag{2.5}$$

From (2.5), it follows that the spectral radius of $T(1)$ is given by

$$\begin{aligned} r_\sigma(T(1)) \;&:=\; \max\big\{|\mu| \in \mathbb{C} : \mu \in \sigma(T(1))\big\} \\ &=\; \sup\big\{e^{\operatorname{Re}\lambda} \mid \lambda \in \sigma(A)\big\}. \end{aligned} \tag{2.6}$$

The number $r_\sigma(T(1))$ determines the asymptotic behaviour of the semigroup $T(t)$. If $r_\sigma(T(1)) < 1$, then zero is exponentially stable. If $r_\sigma(T(1)) > 1$, then there are exponentially unbounded orbits and $T(t)$ is unstable.

Next we define the essential spectrum $\sigma_e(T(1))$ of $T(1)$. We will show that the behaviour of the essential spectrum plays a fundamental role whether feedback stabilization is sensitive to small variations in the delays. The essential spectrum $\sigma_e(T(1))$ of $T(1)$ is that part of the spectrum of $T(1)$ that cannot be removed by compact perturbations

$$\sigma_e(T(1)) = \bigcap_{K \text{ compact}} \sigma(T(1) + K). \tag{2.7}$$

The essential spectral radius is defined to be

$$r_e(T(1)) := \max\{|\mu| \in \mathbb{C} : \mu \in \sigma_e(T(1))\}. \tag{2.8}$$

The following result is due to Henry [8].

**Theorem 2.1** *The spectral radius and the essential spectral radius of the semigroup $T(t) = T_{D,L}(t)$ associated with the neutral functional differential equation* (1.1) *are given by*

*(i)* $r_\sigma(T(1)) = e^{\beta t}$, *where $\beta$ is defined by*

$$\beta = \sup\{\operatorname{Re}\lambda \mid \det \Delta(\lambda) = 0\}. \tag{2.9}$$

*(ii)* $r_e(T(1)) = e^{\alpha t}$, *where $\alpha$ is defined by*

$$\alpha = \sup\{\operatorname{Re}\lambda \mid \det \Delta_0(\lambda) = 0\}, \tag{2.10}$$

*where $\Delta_0(\lambda)$ denotes the characteristic matrix*

$$\Delta_0(\lambda) = I - \sum_{j=1}^{M} A_j e^{-\lambda r_j} \tag{2.11}$$

*associated with the difference equation* (1.8).

From Theorem 2.1, it becomes clear that the eigenvalues of the difference equation (1.8) play a fundamental role in the asymptotic behaviour of the solutions of the neutral equation (1.1). If $r_e(T(1)) < 1$, then there are only finitely many independent unstable solutions. On the other hand, $r_e(T(1)) \geq 1$ implies that there is an infinite dimensional linear subspace $U$ of $C$ such that $U$ is invariant under $T(1)$ and any solution with initial data in $U$ has norm approaching $\infty$ as $t \to \infty$; that is, the zero solution is not only unstable, but it is unstable for every element of an infinite dimensional subspace of $C$. If the unstable subspace of zero were finite dimensional, then $r_e(T(1)) \leq 1$ since the unstable part could be removed by a compact perturbation. If $r_e(T(1)) = 1$, then we may or may not have any unbounded solutions. The example (1.6) from Section 1 has the property that $r_e(T(1)) = 1$, but all solutions approach zero. If we reverse the sign of $c$ in that example, we would have at least a dense set of solutions which become unbounded as $t \to \infty$.

In order to use Theorem 2.1 efficiently in applications, we have to study how the values $\alpha$ and $\beta$ depend on the matrices $A_j$ and the delays $r_1, r_2, \ldots, r_M$. Similarly as for neutral functional differential equations, we can associate with the difference equation (1.8) a semigroup of bounded linear operators by translation along the solution. If we define the closed subspace $C_D$ of $C$ by

$$C_D = \{\varphi \in C \mid \varphi(0) = \sum_{j=1}^{M} A_j \varphi(-r_j)\},$$

then translation along the solutions of (1.8), given by,

$$T_D(t)\varphi = x_t(\,\cdot\,;\varphi)$$

defines a strongly continuous semigroup on $C_D$. An application of Theorem 2.1 with $L = 0$ yields that $r_\sigma(T_D(1)) = r_e(T_D(1)) = e^{\alpha t}$, where $\alpha$ is defined by (2.10).

Thus, if a difference equation is not exponentially stable, then $r_e(T_D(1)) \geq 1$ and there is the possibility that there are infinitely many independent unstable solutions. This is always the case if $r_e(T_D(1)) > 1$. We next show that this property leads to rather strong conditions in order to have robustness of stability with respect to variations in the delays.

As an example, we consider the difference equation

$$x(t) = a_1 x(t-h), \tag{2.12}$$

where $a_1 > 0$ and $h > 0$. The characteristic equation $\Delta_0(z)$ associated with (2.12) is given by

$$\Delta_0(z) = 1 - a_1 e^{-zh}$$

and the roots $\lambda_k$ of $\Delta_0(z)$ are given by $\lambda_k = h^{-1}\log a_1 + 2k\pi i$. Therefore, the difference equation (2.12) is not exponentially stable if and only if $a_1 \geq 1$. Note that this condition is independent of the delays. This is actually a rather general phenomenon for difference equations, see Corollary 2.2.

The following theorem is a slight generalization of Theorem 9.6.1 of [5]. The presentation of the proof is new.

**Theorem 2.2** *Let $T_D(t)$ denote the semigroup associated with (1.8). Define*

$$\gamma_0 = \max\{r_\sigma(\sum_{j=1}^{M} e^{i\theta_j} A_j) \mid \theta_j \in [0, 2\pi],\ 1 \leq j \leq M\} =: r_\sigma\big(\sum_{j=1}^{M} e^{i\theta_j^*} A_j\big). \tag{2.13}$$

*If the components of $r = (r_1, r_2, \ldots, r_M)$ in the difference operator are rationally independent, then $T_D(t)$ is exponentially stable if and only if $\gamma_0 < 1$. Furthermore, if $\gamma_0 > 1$, then $T_D(t)$ is exponentially unstable.*

We divide the proof of Theorem 2.2 into two lemmas which are of independent interest.

**Lemma 2.1** *If the components of $r = (r_1, r_2, \ldots, r_M)$ are rationally independent and*

$$\rho_0 = \max\big\{\rho \in \mathbb{R} \mid r_\sigma\big(\sum_{j=1}^{M} e^{i\theta_j^*} A_j e^{-\rho r_j}\big) = 1\big\}, \tag{2.14}$$

*where $\theta_1^*, \theta_2^*, \ldots, \theta_M^*$ are defined as in (2.13), then*

$$\rho_0 \leq \sup\big\{\operatorname{Re}\lambda \mid \det\big(I - \sum_{j=1}^{M} A_j e^{-\lambda r_j}\big) = 0\big\}. \tag{2.15}$$

**Proof.** Note that, by definition, $\rho_0$ is the largest real number with the property that there exists a $\nu \in [0, 2\pi]$ such that

$$\det\big(I - e^{-i\nu} \sum_{j=1}^{M} e^{i\theta_j^*} A_j e^{-\rho_0 r_j}\big) = 0.$$

To prove (2.15) we have to show that $\rho_0 \leq \alpha$, where $\alpha$ is defined by (2.10). The proof follows from Kronecker's theorem and basic properties of almost periodic functions.

In particular, we use the following result for almost periodic functions that are analytic in a strip (see p. 268 of [14]). For given real numbers $a \leq b$, there exists a number $N$ such that, for all real $t$, the function $\det \Delta_0(z)$, where $\Delta_0(z)$ is defined in (2.11), has no more than $N$ zeros in the box

$$\{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b,\ t \leq \operatorname{Im} z \leq t+1\}.$$

Furthermore, for every $\delta > 0$, there exists an $m = m(\delta)$ such that if $z$ is at distance at least $\delta$ from every zero of $\det \Delta_0(z)$, then

$$|\det\, \Delta_0(z)| \geq m(\delta). \tag{2.16}$$

Since the $r_j$ are rationally independent, it follows from Kronecker theorem (see, for example, [2]) that, for every $\delta > 0$, we can find a $\tau$ such that

$$|\tau r_j - \nu + \theta_j^*| < \delta \mod 2\pi, \quad j = 1, 2, \ldots, M.$$

So, for a given sequence $\delta_n \downarrow 0$, there exists a sequence $\tau_n$ such that

$$\lim_{n\to\infty} e^{i\tau_n r_j} = e^{i(\nu - \theta_j^*)}, \quad j = 1, 2, \ldots, M.$$

As a consequence

$$\lim_{n\to\infty} \det \Delta_0(\rho_0 + i\tau_n) = 0,$$

where $\Delta_0$ is given by (2.11).

On the other hand, since $|e^{i\tau_n}| = 1$, there exists a subsequence $\tau_{n_k}$ of $\tau_n$ such that $e^{i\tau_{n_k}} \to e^{i\tau^*}$ as $k \to \infty$. By continuity, it follows that, for $j = 1, 2, \ldots, M$, $e^{i\tau_{n_k} r_j} \to e^{i\tau^* r_j}$ as $k \to \infty$. Thus

$$\lim_{n\to\infty} \det \Delta_0(\rho_0 + i\tau_n) = \det \Delta_0(\rho_0 + i\tau^*).$$

Together with (2.16), this implies that $\det \Delta_0(\rho_0 + i\tau^*) = 0$. This proves that $\rho_0 \leq \alpha$ and completes the proof of (2.15). □

**Lemma 2.2** *Let $T_D(t)$ denote the semigroup associated with* (1.8) *and let $\gamma_0$ be defined by* (2.13). *If $\gamma_0 < 1$, then $T_D(t)$ is exponentially stable.*

**Proof.** The proof is based on an approximation argument. First we approximate the time delays $r_1, r_2, \ldots, r_M$ by rational numbers. For every $\delta > 0$, there exist positive integers $m_j$, $j = 1, 2, \ldots, M$, and $q$ such that $|r_j - m_j/q| < \delta$. Set $s_j = m_j/q$ and consider the difference equation

$$y(t) = \sum_{j=1}^{M} A_j y(t - s_j), \qquad t \geq 0. \tag{2.17}$$

If $T$ is a bounded linear operator, then the spectral radius $r_\sigma(T)$ is an upper semicontinuous function of $T$, i.e., the spectral radius of $T$ cannot increase suddenly when $T$ is varied continuously (see Thm II.4.1 of [4] or Thm IV.3.1 of [12]). Since the semigroup $T(1)$ depends continuously on $r = (r_1, r_2, \ldots, r_M)$, see [13], it follows that the spectral radius $r_\sigma(T)$ is upper semicontinuous with respect to the delays $r_1, r_2, \ldots, r_M$. Thus we obtain that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|r_j - s_j| < \delta$ and

$$\alpha - \epsilon < \alpha^* = \sup \big\{ \operatorname{Re} \lambda \mid \det (I - \sum_{j=1}^{M} A_j e^{-s_j \lambda}) = 0 \big\}. \tag{2.18}$$

Consider the following equation in $z$ and $\lambda$

$$\det (zI - \sum_{j=1}^{M} A_j e^{-s_j \lambda}) = 0 \tag{2.19}$$

(for $z = 1$ this is the characteristic equation of (1.8)). If we define $\mu = e^{-\lambda/q}$, then (2.19) becomes a polynomial in $z$ and $\mu$:

$$\det(zI - \sum_{j=1}^{M} A_j \mu^{m_j}) = 0. \tag{2.20}$$

Suppose that $\alpha^* \geq 0$, where $\alpha^*$ is defined in (2.18). Since $|\mu| = e^{-\operatorname{Re}\lambda/q}$, we obtain that (2.20) has solutions $z = 1$ and $\mu = \mu_n$ with $\mu_n \to e^{-\alpha^*/q}$ and $|\mu_n| \leq 1$. Let $\mu_0$ be such a solution with $|\mu_0| \leq 1$. Since (2.20) is a polynomial in $z$ and $\mu$, we can use the implicit function theorem to select a continuous branch of solutions $[1, \infty) \ni z \mapsto \mu(z) \in \mathbb{C}$ with $\mu(1) = \mu_0$. Since $|\mu(z)| \to \infty$ as $z \to \infty$, there exists a real $z^* \geq 1$ such that $|\mu(z^*)| = 1$. If $\theta_0 \in [0, 2\pi]$ is such that $\mu(z^*) = e^{i\theta_0}$, then

$$r_\sigma(\sum_{j=1}^{M} A_j e^{i m_j \theta_0}) \geq z^* \geq 1,$$

but this is a contradiction to the assumption that $\gamma_0 < 1$.

Thus $\alpha^* < 0$ and hence, for $\epsilon$ sufficiently small, we conclude that $\alpha < 0$ and Theorem 2.1 implies that $T_D(t)$ is exponentially stable. □

**Proof of Theorem 2.2.** From Lemma 2.2 it follows that, if $\gamma_0 < 1$, then the semigroup $T_D(t)$ is exponentially stable. Next we consider the case that $\gamma_0 = 1$. By definition of $\rho_0$ in (2.14), it follows that $\rho_0 \geq 0$ and hence, by Lemma 2.1, we obtain $\alpha \geq 0$. Thus, if $\gamma = 1$, then $T_D(t)$ cannot be exponentially stable. Furthermore, it follows by definition of $\rho_0$ that actually $\rho_0 > 0$ if $\gamma_0 > 1$. Hence another application of Lemma 2.1 yields that if $\gamma_0 > 1$, then $\alpha > 0$ and $T_D(t)$ is exponentially unstable. This completes the proof of the theorem. □

**Corollary 2.1** *Exponential stability of the neutral functional differential equation* (1.1) *is preserved under small perturbations in the delays if and only if* $\gamma_0 < 1$, *where* $\gamma$ *is defined by* (2.13).

**Proof.** If the stability properties of (1.1) are preserved under small perturbations in the delays, then we can assume that the components of $r = (r_1, r_2, \ldots, r_M)$ in the difference operator are rationally independent. If the components of $r$ are rationally independent, then we can apply Theorem 2.2 to obtain that exponential stability of (1.8) is equivalent to $\gamma_0 < 1$. Since exponential stability of (1.8) is necessary for exponential stability of (1.1), we derive that the condition $\gamma_0 < 1$ is necessary. Next we assume that $\gamma_0 < 1$ and hence, by Theorem 2.2, we obtain that the difference equation is exponentially stable. For neutral equations with a stable $D$-operator, it is not difficult to show that there exists an $\epsilon > 0$ such that the number of roots in the right half-plane $\{z \mid \operatorname{Re} z \geq -\epsilon\}$ are located in a compact subset of the plane and vary continuously with respect to the parameters. Thus exponential stability of the neutral functional differential equation (1.1) is preserved under small perturbations in the delays. □

Note that the condition $\gamma_0 < 1$, where $\gamma_0$ is defined by (2.13) is independent of the delays $r_j$. So, in fact, if (1.8) is locally stable in the delays, then (1.8) is globally stable in the delays. This proves the following corollary.

**Corollary 2.2** *The difference equation* (1.8) *is strongly exponentially stable if and only if* $\gamma_0 < 1$, *where* $\gamma_0$ *is defined by* (2.13). *In particular, for difference equations, strong exponential stability is equivalent to exponential stability for all variations in the delays.*

In the scalar case, Lemma 2.1 actually gives more information.

**Lemma 2.3** *Let* $\gamma_0$ *be defined by* (2.13). *If, in the scalar case, we write* $A_j = a_j$ *for* $j = 1, \ldots, M$, *then*

$$\gamma_0 = \sum_{j=1}^{M} |a_j|. \tag{2.21}$$

*Furthermore, if the components of* $r = (r_1, r_2, \ldots, r_M)$ *are rationally independent, then*

$$\rho_0 = \alpha, \tag{2.22}$$

*where* $\rho_0$ *is defined by* (2.15) *and* $\alpha$ *is defined by* (2.10).

**Proof.** First observe that, in the scalar case, the $\theta_j^*$ defined in (2.13) can be chosen such that

$$a_j e^{i\theta_j^*} = |a_j| \qquad \text{for } 1 \le j \le M. \tag{2.23}$$

By definition of $\gamma_0$, the identity (2.21) follows from (2.23). To prove (2.22) observe that, if $\Delta_0(\omega + i\tau) = 0$, then $\sum_{j=1}^M a_j e^{-\omega r_j} e^{-i\tau r_j} = 1$ and, hence,

$$1 \le \sum_{j=1}^M |a_j| e^{-\omega r_j}. \tag{2.24}$$

Therefore, the larger $\omega$, the smaller the right hand side of (2.24) becomes. This shows that $\alpha \le \rho_0$, where $\rho_0$ is defined by (2.14) and $\alpha$ is defined by (2.10). But, in general, Lemma 2.1 implies that $\rho_0 \le \alpha$. Thus (2.22) follows. □

The next three corollaries follows from Lemma 2.3 and Corollary 2.1.

**Corollary 2.3** *Consider the scalar difference equation*

$$x(t) - \sum_{j=1}^M a_j x(t - r_j) = 0. \tag{2.25}$$

*If the components of* $r = (r_1, r_2, \ldots, r_M)$ *are rationally independent, then we have*

$$r_\sigma(T_D(1)) = e^{\rho_0},$$

*where* $\rho_0$ *is the solution to the equation*

$$\sum_{j=1}^M |a_j| e^{-\rho_0 r_j} = 1. \tag{2.26}$$

**Corollary 2.4** *In the scalar case, if the components of* $r = (r_1, r_2, \ldots, r_M)$ *are rationally independent, then the zero solution of* (2.25) *is exponentially stable if and only if*

$$\sum_{j=1}^M |a_j| < 1.$$

**Corollary 2.5** *In the scalar case, exponential stability of the neutral functional differential equation* (1.1) *is preserved under small perturbations in the delays if and only if*

$$\sum_{j=1}^M |a_j| < 1.$$

**Example 2.1** Consider the system

$$x(t) - a_1 x(t - r_1) - a_2 x(t - r_2) = 0, \tag{2.27}$$

where $r_1$ and $r_2$ are rationally independent. According to Corollary 2.4, this equation is not exponentially stable if and only if

$$|a_1| + |a_2| \geq 1.$$

Given $|a_1| + |a_2| \geq 1$, we can apply a feedback control to stabilize the equation

$$x(t) - (a_1 + f_1)x(t - r_1) - (a_2 + f_2)x(t - r_2) = 0.$$

Because of Corollary 2.4, this system is exponentially stable if and only if

$$|a_1 + f_1| + |a_2 + f_2| < 1.$$

Since $|a_1| + |a_2| \geq 1$, we must have $|f_1| + |f_2| > 0$. Suppose now that the feedback control cannot be applied instantaneously and that there is a small time delay in the feedback:

$$x(t) - a_1x(t - r_1) - a_2x(t - r_2) - f_1x(t - r_1 - \epsilon_1) - f_2x(t - r_2 - \epsilon_2) = 0. \tag{2.28}$$

We claim that, although this equation is exponentially stable for $\epsilon_1 = \epsilon_2 = 0$, there is a sequence $(\epsilon_1^j, \epsilon_2^j)$ tending to zero so that (2.28) is exponentially unstable. To prove the claim, we choose $\epsilon_1$ and $\epsilon_2$ such that $r_1$, $r_2$, $r_1 + \epsilon_1$ and $r_2 + \epsilon_2$ are rationally independent. From Corollary 2.4 it follows that (2.28) is always exponentially unstable, since

$$|a_1| + |f_1| + |a_2| + |f_2| > |a_1| + |a_2| \geq 1.$$

This example illustrates a general fact, see [6].

# 3 Strong stabilization of neutral equations

Although problems of feedback stabilization for retarded equations have been widely studied, little work has been done on feedback stabilization of neutral functional differential equations. The reason for this being the fact that feedback stabilization of retarded equations is a finite dimensional problem, see [17], while feedback stabilization for neutral equations with a $D$-operator that is not exponentially stable is an infinite dimensional problem [19].

To consider the neutral case more closely, let $\alpha$ be defined by (2.10). If $\alpha \geq 0$, then the neutral equation (1.1) has to be stabilized in two steps.

First one has to apply a control law to stabilize the difference equation (1.8) associated with (1.1) and then, if the difference equation (1.8) is exponentially stable, it follows from Theorem 2.1 that (1.1) has at most finitely many unstable modes. In this case, the stabilization of (1.1) reduces to a finite pole shifting problem [17]. Note that the stabilization of the difference equation (1.8) is equivalent to changing the radius of the essential spectrum of the semigroup $T(t)$ associated with (1.1) (see Theorem 2.1).

To analyse the first step further, we consider the control problem for the difference equation (1.8),

$$x(t) = \sum_{j=1}^{M} A_j x(t - r_j) + Bu(t), \tag{3.1}$$

where $A_1, \ldots, A_M$ are $n \times n$-matrices, $B$ is an $n \times m$-matrix and $u : [0, \infty) \to \mathbb{R}^m$ is a vector-valued control. If we close the loop by

$$u(t) = \sum_{j=1}^{M} F_j x(t - r_j), \tag{3.2}$$

then the closed-loop system becomes

$$x(t) = \sum_{j=1}^{M} A_j x(t - r_j) + B \sum_{j=1}^{M} F_j x(t - r_j). \tag{3.3}$$

The characteristic matrix of (3.3) is given by

$$\Delta_{0,cl}(\lambda) = \Delta_0(\lambda) - B \sum_{j=1}^{M} F_j e^{-\lambda r_j}, \tag{3.4}$$

where $\Delta_0(\lambda)$ is given by (2.11).

From Theorem 2.1, it follows that (3.3) is exponentially stable if and only if the roots of the characteristic equation $\det \Delta_{0,cl}(\lambda) = 0$ are in a left half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq -\epsilon < 0\}$ for some $\epsilon > 0$. In particular, a necessary condition for the stabilizability of (3.1) is given by

$$\operatorname{rank}\, [\Delta_0(\lambda)\ B] = n \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq -\epsilon \tag{3.5}$$

for some $\epsilon > 0$. The necessity of this condition has been observed in [19] for general difference equations and in [15, 16] for systems with a single point delay. To explain this condition further, we make a digression to ordinary differential equations.

Consider the closed-loop control system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad u(t) = Fx(t).$$

The pair $(A, B)$ is controllable if the rank $[A\ AB\ \cdots A^{n-1}B] = n$ and this is equivalent to

$$\operatorname{rank}\, [\lambda I - A\ B] = n \qquad \text{for all } \lambda \in \mathbb{C}. \tag{3.6}$$

If the pair $(A, B)$ is controllable, then the pole shifting theorem implies that the eigenvalues of $A + BF$ can be put into arbitrary position by an appropriate choice of $F$, see Chapter 5 of [20] for the details.

Comparing condition (3.6) and (3.5), we see that (3.5) is the appropriate generalization of the notion of controllable pair in the infinite dimensional context of difference equations. However, for difference equations it is an open problem whether condition (3.5) is sufficient for the stabilizability of (3.1). In the special case that the delays in (3.1) are commensurable, i.e., $r_j = j\sigma$, $j = 1, 2, \ldots, M$, one can say more.

In the case that the delays in (3.1) are commensurable, the stabilizability of (3.1) can be reduced to a finite dimensional problem and condition (3.5) turns out to be necessary and sufficient for the stabilizability of the difference equation (3.1), see [19]. Since we need this result in the sequel, we present a proof of this fact.

If $r_j = j\sigma$, then the characteristic matrix $\Delta_0(\lambda)$ of (3.1) with $B = 0$ becomes

$$\Delta_0(\lambda) = I - \sum_{j=1}^{M} A_j \left[e^{-\lambda\sigma}\right]^j. \tag{3.7}$$

So, with respect to the new variable

$$s = e^{\lambda\sigma}, \tag{3.8}$$

the characteristic matrix (3.7) becomes a rational function in $s$. In this case, condition (3.5) can be rewritten as

$$\operatorname{rank}\,[s^n I - \sum_{j=1}^{M} A_j s^{n-j} \;\; B] = n \qquad \text{for all } s \in \mathbb{C},\ |s| \geq 1. \tag{3.9}$$

Condition (3.9) can be rewritten as

$$\operatorname{rank} \begin{bmatrix} sI - A_1 & -A_2 & -A_3 & \cdots & -A_M & B \\ -I & sI & 0 & \cdots & 0 & 0 \\ 0 & -I & sI & 0 & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -I & sI & 0 \end{bmatrix} = nM. \tag{3.10}$$

Define

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 & A_3 & \cdots & A_M \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \end{bmatrix} \qquad \text{and} \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Condition (3.10) is equivalent to the condition that $(\mathcal{A}, \mathcal{B})$ is a controllable pair and the pole-shifting theorem implies that for any $\rho > 0$, there exists a matrix $\mathcal{F}$ such that the eigenvalues of

$$\mathcal{A} + \mathcal{B}\mathcal{F} = \begin{bmatrix} A_1 + BF_1 & A_2 + BF_2 & A_3 + BF_3 & \cdots & A_M + BF_M \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \end{bmatrix}$$

are inside the open disk $\{z \in \mathbb{C} \mid 0 < |z| < \rho\}$. But this implies that the roots of the determinant of the characteristic matrix (3.4) of the closed-loop system (3.3) are in the open left half-plane

$$\{z \in \mathbb{C} \mid \operatorname{Re} z \leq \log \rho < 0\}. \tag{3.11}$$

This shows that, in the case that the delays are commensurable, condition (3.5) implies the stabilizability of the difference equation (3.1) and that by an appropriate choice of the control matrices $F_1, F_2, \ldots, F_M$ the decay rate of the closed-loop system (3.3) can be made arbitrary fast.

To answer the question whether (3.5) is a sufficient condition in case of a general difference equation (3.1) seems to be a difficult problem. This is due to the sensitivity of the stability properties of difference equations with variations in the delays [7].

In the extreme case that all the delays in (3.1) are rationally independent, it should be the goal of the feedback to make the difference equation strongly stable, i.e., to make $\gamma_0 < 1$ where $\gamma_0$ is defined by (2.13). From Theorem 2.1, it follows that (3.3) is exponentially stable independent of the delays if and only if or any choice of the delays $r_1, r_2, \ldots, r_M$, the roots of the determinant of the characteristic matrix (3.4)

$$\Delta_{0,cl}(\lambda) = I - \sum_{j=1}^{M} (A_j + BF_j) e^{-\lambda r_j},$$

are in the open left half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq -\epsilon < 0\}$ for some positive $\epsilon$ depending on $r = (r_1, r_2, \ldots, r_M)$.

Thus a necessary condition for strong stabilizability of (3.1) is that, for any choice of the delays $r_1, r_2, \ldots, r_M$,

$$\operatorname{rank}\,[I - \sum_{j=1}^{M} A_j e^{-\lambda r_j} \;\; B] = n \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq -\epsilon \tag{3.12}$$

for some positive $\epsilon = \epsilon(r_1, r_2, \ldots, r_M)$. Our main result states that condition (3.12) is not only sufficient but also necessary for strong stabilizability of (3.1).

**Theorem 3.1** *The control problem*

$$x(t) = \sum_{j=1}^{M} A_j x(t - r_j) + Bu(t),$$

*where $A_1, \ldots, A_M$ are $n \times n$-matrices, $B$ is an $n \times m$-matrix and $u : [0, \infty) \to \mathbb{R}^m$ is a vector-valued control is strongly stabilizable by a feedback control*

$$u(t) = \sum_{j=1}^{M} F_j x(t - r_j)$$

*if and only if for any choice of the delays $r_1, r_2, \ldots, r_M$,*

$$\operatorname{rank}\,[I - \sum_{j=1}^{M} A_j e^{-\lambda r_j} \;\; B] = n \qquad \textit{for all } \lambda \in \mathbb{C} \textit{ with } \operatorname{Re} \lambda \geq -\epsilon \tag{3.13}$$

*for some positive $\epsilon = \epsilon(r_1, r_2, \ldots, r_M)$.*

**Proof.** We have already seen that the condition (3.12) is necessary. So it suffices to prove that (3.12) implies that the closed-loop difference equation (3.3) is strongly stabilizable.

We argue by contradiction and assume that, for any choice of the control matrices $F_1, F_2, \ldots, F_M$, the closed-loop difference equation (3.3) is not strongly stable. According to Theorem 2.2, this implies that, for any choice of the matrices $F_1, F_2, \ldots, F_M$,

$$\gamma_0 = \sup\{r_\sigma\big(\sum_{j=1}^{M}(A_j + BF_j)e^{i\theta_j}\big) \mid 0 \leq \theta \leq 2\pi\} \geq 1. \tag{3.14}$$

Fix the matrices $F_1, F_2, \ldots, F_M$ and choose the delays $r_1, r_2, \ldots, r_M$ to be rationally independent. The closed-loop difference equation is given by

$$x(t) = \sum_{j=1}^{M}(A_j + BF_j)x(t - r_j). \tag{3.15}$$

Because of (3.14), it follows that (3.15) is not exponentially stable. Thus, an application of Theorem 2.1 yields

$$\alpha = \sup\{\operatorname{Re}\lambda \mid \det\big(I - \sum_{j=1}^{M}(A_j + BF_j)e^{-\lambda r_j}\big) = 0\} \geq 0. \tag{3.16}$$

Since $\alpha$ is upper semicontinuous with respect to $(r_1, r_2, \ldots, r_M)$ (see the proof of Theorem 2.2), we conclude that for any $\epsilon > 0$ there exists a $\delta > 0$ and rational numbers $s_1, s_2, \ldots, s_M$ such that

$$|r_j - s_j| < \delta \quad \text{and} \quad s_j = \frac{m_j}{q},\ m_j \in \mathbb{N}$$

and

$$\alpha - \epsilon < \alpha^* := \sup\{\operatorname{Re}\lambda \mid \det\big(I - \sum_{j=1}^{M}(A_j + BF_j)e^{-\lambda s_j}\big) = 0\}. \tag{3.17}$$

On the other hand, we have seen that, if the delays are commensurable, then (3.9) implies that for any $\rho > 0$, there exist matrices $\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_M$ such that

$$\sup\{\operatorname{Re}\lambda \mid \det\big(I - \sum_{j=1}^{M}(A_j + B\widetilde{F}_j)e^{-\lambda s_j}\big) = 0\} < \log\rho < 0.$$

Next we apply (3.17) with respect to the matrices $\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_M$ to conclude that

$$\alpha - \epsilon < \alpha^* := \sup\{\operatorname{Re}\lambda \mid \det(I - \sum_{j=1}^{M}(A_j + B\widetilde{F}_j)e^{-\lambda s_j} = 0\} < \log\rho < 0.$$

Since $\alpha \geq 0$ is fixed and $\epsilon > 0$ and $\rho > 0$ are arbitrary, we arrive at a contradiction to our assumption.

Thus there exists matrices $F_1, F_2, \ldots, F_M$ such that, the difference equation (3.3) is strongly stabilizable. □

To discuss the second step in the stabilization of neutral equations, we first formulate the control problem for the neutral functional differential equation (1.1)

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{M} A_j x(t - r_j)\right] = Lx_t + Bu(t), \tag{3.18}$$

where $L : C \to \mathbb{C}^n$ is a bounded linear map, $B$ is an $n \times m$-matrix and $u : [0, \infty) \to \mathbb{R}^m$ is a vector-valued control. If we close the loop with a proportional and a derivative feedback

$$u(t) = \sum_{j=1}^{M} F_j \frac{dx}{dt}(t - r_j) + Gx_t, \tag{3.19}$$

where $G : C \to \mathbb{C}^m$ is a bounded linear map, then the closed-loop system becomes

$$\frac{dx}{dt}\left[x(t) - \sum_{j=1}^{M} (A_j + BF_j)x(t - r_j)\right] = Lx_t + BGx_t. \tag{3.20}$$

The characteristic matrix of (3.20) is given by

$$\Delta_{cl}(\lambda) = \lambda\Delta_{0,cl}(\lambda) - \int_0^h e^{-z\theta} d[\zeta(\theta) + B\eta(\theta)], \tag{3.21}$$

where $\Delta_{0,cl}(\lambda)$ is given by (3.3) and where we have used the Riesz representation to write the bounded linear map $G : C \to \mathbb{C}^m$ as a Riemann-Stieltjes integral with respect to a real $n \times m$-matrix-valued function $\eta$ of bounded variation.

From Theorem 2.1, it follows that (3.20) is exponentially stable if and only if the roots of the characteristic equation $\det \Delta_{cl}(\lambda) = 0$ are in a left half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq -\epsilon < 0\}$ for some $\epsilon > 0$. In particular, a necessary condition for stabilizability is given by

$$\begin{cases} \operatorname{rank}\,[\Delta(\lambda)\ B] &= n \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geq -\epsilon \\ \operatorname{rank}\,[\Delta_0(\lambda)\ B] &= n \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geq -\epsilon \end{cases} \tag{3.22}$$

for some $\epsilon > 0$. The necessity of these conditions and the fact that these two conditions are independent has been observed in [19] for general neutral equations. It is an open problem whether (3.22) is sufficient for stabilizability of neutral functional differential equations.

In the case that the second condition of (3.22) holds for any choice of $r_1, r_2, \ldots, r_M$, we have seen in Theorem 3.1 that this condition is necessary and sufficient for strong feedback stabilization of the difference equation associated with the closed-loop neutral equation (3.20). So let us suppose that, the matrices $F_1, F_2, \ldots, F_M$ are chosen such that the difference equation

$$x(t) = \sum_{j=1}^{M} (A_j + BF_j)x(t - r_j).$$

is exponentially stable independent of the delays. This implies that the radius of the essential spectrum of the semigroup associated with closed-loop neutral equation (3.20) is less than one and hence (3.20) has only finitely many unstable modes. But now we are in a position to apply the finite pole shifting method of [17, 18] to stabilize the closed-loop neutral equation (3.20), see also Section V.3 of [19]. Therefore, it follows that the first condition of (3.22) is necessary and sufficient to strongly stabilize the difference equation of the closed-loop neutral equation (3.20) and the second condition of (3.22) is necessary and sufficient to stabilize the remaining finitely many unstable modes. Since the retarded part is robust with variations in the delays and the difference equation corresponding to the neutral part is strongly stable, it follows that the stability properties of the closed-loop system (3.20) are robust with small variations in the delays. We summarize our main result in the following theorem.

**Theorem 3.2** *The control problem*

$$\frac{d}{dt}\big[x(t) - \sum_{j=1}^{M} A_j x(t-r_j)\big] = \int_{-h}^{0} d[\eta(\theta)]x(t+\theta) + Bu(t),$$

*where $B$ is an $n \times m$-matrix and $u : [0,\infty) \to \mathbb{R}^m$ is a vector-valued control, is strongly stabilizable by a proportional and a derivative feedback if and only if for any choice of the delays $r_1, r_2, \ldots, r_M$,*

$$\begin{cases} \operatorname{rank}\,[\lambda(I - \sum_{j=1}^{M} A_j^{-\lambda r_j}) - \int_{-h}^{0} e^{\lambda\theta} d\eta(\theta)\ \ B] & = n \qquad \textit{for all } \lambda \in \mathbb{C} \textit{ with } \operatorname{Re}\lambda \geq -\epsilon \\ \operatorname{rank}\,[I - \sum_{j=1}^{M} A_j^{-\lambda r_j}\ \ B] & = n \qquad \textit{for all } \lambda \in \mathbb{C} \textit{ with } \operatorname{Re}\lambda \geq -\epsilon \end{cases}$$

*for some positive $\epsilon = \epsilon(r_1, r_2, \ldots, r_M)$.*

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