DIOPHANTINE INEQUALITIES ON PROJECTIVE VARIETIES

JAN-HENDRIK EVERTSE AND ROBERTO G. FERRETTI

Abstract. We will deduce a quantitative version of a Diophantine approximation result of Faltings and Wüstholz [7] dealing with systems of Diophantine inequalities to be solved in algebraic points of a projective variety X. Our method consists of embedding X into a linear variety by means of a suitable Veronese map and then applying a recent quantitative version of the Subspace Theorem due to Evertse and Schlickewei [5]. To construct the Veronese map, we prove a result of independent interest, which gives a lower bound for the m-th normalized Hilbert weight of X in terms of the normalized Chow weight of X.

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1. Introduction

1.1. Let Y be an n-dimensional projective subvariety (i.e., a geometrically irreducible Zariski-closed subset) of $\mathbb{P}^M$ which is defined over a number field K. Let S be a finite set of places of K. For $v \in S$, $i = 0, \dots, n_v$, let f_{iv} be a homogeneous polynomial of degree $k \geqslant 1$ in $M+1$ variables with coefficients in K and let d_{iv} a real $\geqslant 0$. We are interested in systems of inequalities

$$\log\Big(\frac{|f_{iv}(\mathbf{y})|_v}{\|\mathbf{y}\|_v^k}\Big) \leqslant -d_{iv}h(\mathbf{y}) \quad (v \in S, i = 0, \dots, n_v) \quad \text{in } \mathbf{y} \in Y(K), \tag{1.1}$$

where $|\ |_v, \|\ \|_v$ $(v \in S)$ are normalized absolute values and norms and $h(\mathbf{y})$ is the absolute logarithmic height (cf. §2.1 below).

Assume that for $v \in S$, the map $\mathbf{y} \mapsto (f_{0v}(\mathbf{y}) : \cdots : f_{n_v,v}(\mathbf{y}))$ is a finite morphism from Y to $\mathbb{P}^{n_v}$. Then we may reduce (1.1) to a system in which all polynomials involved are coordinates. Indeed, let $\{f_0, \dots, f_N\}$ be the union of the sets $\{f_{0v}, \dots, f_{n_v,v}\}$ $(v \in S)$. Then $\varphi : \mathbf{y} \mapsto (f_0(\mathbf{y}) : \cdots : f_N(\mathbf{y}))$ is a finite morphism from Y to $\mathbb{P}^N$. Let $X = \varphi(Y)$. Then X is a projective subvariety of $\mathbb{P}^N$ defined over

K. Write $x_i = f_i(\mathbf{y})$, $\mathbf{x} = (x_0 : \cdots : x_N) = \varphi(\mathbf{y})$. Then if we ignore the necessary modifications in the norms and the height, we see that $\mathbf{x}$ satisfies the system of inequalities

$$\log\left(\frac{|x_i|_v}{\|\mathbf{x}\|_v}\right) \leqslant -c_{iv} h(\mathbf{x}) \qquad (v \in S, i = 0, \dots, N) \qquad \text{in } \mathbf{x} = (x_0 : \cdots : x_N) \in X(K), \tag{1.2}$$

where $c_{iv} = d_{jv}/k$ if $f_i = f_{jv}$ and $c_{iv} = 0$ if $f_i \notin \{f_{0v}, \dots, f_{n_v,v}\}$. Clearly, φ establishes a finite-to-one map from solutions $\mathbf{y}$ of (1.1) to solutions $\mathbf{x}$ of (1.2). In the sequel we will focus our attention on systems (1.2).

1.2. Let X be a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d which is defined over a number field K. Assume that $1 \leqslant n < N$. Further, let c_{iv} ($v \in S$, $i = 0, \dots, N$) be non-negative reals. Faltings and Wüstholz [7] proved that the set of solutions of (1.2) is contained in the union of finitely many proper subvarieties of X if the expectation of a particular probability distribution is larger than 1. Ferretti [9] showed that this latter condition is equivalent to

$$\frac{1}{(n+1)d} \sum_{v \in S} e_X(\mathbf{c}_v) > 1, \tag{1.3}$$

where $\mathbf{c}_v = (c_{0v}, \dots, c_{Nv})$ and where $e_X(\mathbf{c}_v)$ is the *Chow weight* of X with respect to $\mathbf{c}_v$ (cf. §3.3). If X is a linear variety, then the result of Faltings and Wüstholz is equivalent to Schmidt's Subspace Theorem. Whereas Schmidt's proof of his Subspace Theorem is based on techniques from Diophantine approximation and geometry of numbers, Faltings and Wüstholz developed a totally different method, based on Faltings' Product Theorem (cf. [6], Theorem 3.1, 3.3).

1.3. Starting with Schmidt [18], much work has been done to obtain good quantitative versions of the Subspace Theorem. The sharpest such version to date is due to Evertse and Schlickewei ([5], Theorem 2.1). From their result we will deduce the following for (1.2) in the case that X is a linear variety. Let $X \subseteq \mathbb{P}^N$ be an $n-$dimensional linear subvariety defined over a number field K and denote by $h(X)$ the logarithmic height of X (cf. §2.2). Assume that

$$\frac{1}{n+1} \sum_{v \in S} e_X(\mathbf{c}_v) > 1 + \delta \quad \text{with } \delta > 0. \tag{1.4}$$

Then there are explicitly computable constants c_1, c_2, depending only on N, n, δ, such that the set of solutions $\mathbf{x} \in X(K)$ of (1.2) with $h(\mathbf{x}) \geqslant c_1(1+h(X))$ is contained in the union of at most c_2 proper linear subspaces of X. It has turned out to be crucial for applications that c_1, c_2 are independent of K and S. More generally, the result of Evertse and Schlickewei allows to deduce a similar result for an "absolute" generalization of (1.2) dealing with points in $X(\overline{\mathbb{Q}})$ rather than in $X(K)$. For the precise statement we refer to Theorem 3.2 in Section 3.

1.4. Using the method of Faltings and Wüstholz, Ferretti [8] obtained a quantitative version of their result, an equivalent version of which reads as follows. Let X be a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d which is defined over K, where $1 \leqslant n < N$. Assume that

$$\frac{1}{(n+1)d} \sum_{v \in S} e_X(\mathbf{c}_v) > 1 + \delta \quad \text{with } \delta > 0. \tag{1.5}$$

Then there are explicitly computable constants c_1, c_2, c_3, depending on N, n, δ, K, S and some geometric invariants of X, such that the set of solutions of (1.2) with $h(\mathbf{x}) \geqslant c_1(1+h(X))$ lies in the union of at most c_2 proper subvarieties of X, each of degree $\leqslant c_3$.

1.5. In the present paper we prove another quantitative version of the result of Faltings and Wüstholz, in which the constants c_1, c_2, c_3 depend only on N, n, δ and the degree of X. Further, just as for linear varieties, we prove a similar quantitative version for an absolute generalization of (1.2), dealing with points in $X(\overline{\mathbb{Q}})$. For the precise statement see Theorem 3.4 in Section 3.

We sketch our method which is very different from that of Faltings and Wüstholz. Let $\varphi_m : \mathbb{P}^N \hookrightarrow \mathbb{P}^R$ with $R = \binom{N+m}{m} - 1$ denote the Veronese embedding, which maps $\mathbf{x} \in \mathbb{P}^N$ to the point whose coordinates are the monomials in $\mathbf{x}$ of degree m. Let X_m denote the smallest linear subvariety of $\mathbb{P}^R$ containing $\varphi_m(X)$. We construct from (1.2) a new system of a similar shape, with solutions taken from X_m, which is such that if $\mathbf{x}$ is a solution of (1.2) then $\varphi_m(\mathbf{x})$ is a solution of the new system. The hard core of our paper is to find an explicit value for m such that the analogue of condition (1.4) for the new system is satisfied. Having achieved this, we obtain our quantitative result for the original system (1.2) by applying our previously obtained quantitative result for linear varieties to the new system.

In order to find a suitable value for m, we prove a result which gives, in some well-defined sense, an explicit lower bound of the m-th normalized Hilbert weight of X with respect to a tuple of reals $\mathbf{c}$ in terms of the normalized Chow weight of X with respect to $\mathbf{c}$ (cf. Section 4 for the definitions and the statement of the result). Our result may be viewed as a one-sided explicit version of a result of Mumford ([16], p. 61, Proposition 2.11) which states that the normalized Chow weight of X with respect to $\mathbf{c}$ is the limit of the sequence of its normalized Hilbert weights.

As a by-product of our investigations we obtain that the theorem of Faltings and Wüstholz, which at a first glance seems to be a generalization of the Subspace Theorem, is in fact equivalent to the Subspace Theorem.

1.6. In Section 2 we introduce some notation. In Section 3 we give the precise statements of the results mentioned above related to (1.2) (Theorem 3.2 and Theorem 3.4). In Section 4 we give the definition of the Hilbert weights and Chow weight of X, and state our result concerning these (Theorem 4.6). In Sections 5,6 we prove Theorem 4.6. In Section 7 we prove Theorem 3.2 (the result for linear varieties). In Section 8 we prove an auxiliary result about heights. Finally, in Section 9 we prove Theorem 3.4 (the result for arbitrary varieties).

2. Notation

2.1. We introduce the notation needed in the statements of our results. We first define absolute values and heights. Let K be a number field. Denote by M_K its set of places. For $v \in M_K$ we define a normalized absolute value $|.|_v$ on K by requiring that for $x \in \mathbb{Q}$:

$$\begin{aligned} |x|_v &= |x|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]} && \text{if } v \text{ is archimedean,} \\ |x|_v &= |x|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]} && \text{if } v \text{ lies above a prime number } p, \end{aligned}$$

where $\mathbb{Q}_p, K_v$ denote the respective completions. These absolute values satisfy the product formula $\prod_{v\in M_K} |x|_v = 1$ for $x \in K^*$.

Given a finite extension L of K we write $w|v$ to indicate that a place w of M_L lies above $v \in M_K$. Further, we denote the completion of L at w by L_w. Then if we define normalized absolute values in the same manner for L, we get the extension formulas

$$|x|_w = |x|_v^{[L_w:K_v]/[L:K]} \quad \text{for } x \in K,\ w \in M_L,\ v \in M_K \text{ with } w|v. \tag{2.1}$$

For $\mathbf{x} = (x_0, \ldots, x_N) \in K^{N+1}$, $v \in M_K$ we put

$$\|\mathbf{x}\|_v := \max\{|x_0|_v, \ldots, |x_N|_v\}.$$

We then define the *absolute logarithmic height* of $\mathbf{x} \in \overline{\mathbb{Q}}^{N+1}$ by taking a number field K with $\mathbf{x} \in K^{N+1}$ and putting

$$h(\mathbf{x}) := \sum_{v \in M_K} \log \|\mathbf{x}\|_v.$$

By the product formula we have $h(\lambda\mathbf{x}) = h(\mathbf{x})$ for $\lambda \in K^*$ and by the extension formulas, this height is independent of the choice of K. Therefore, h defines a height on $\mathbb{P}^N(\overline{\mathbb{Q}})$. For a polynomial P with coefficients in $\overline{\mathbb{Q}}$, we denote by $h(P)$ the absolute logarithmic height of the vector of coefficients of P.

2.2. We define the height of a projective variety. Given any field K, we define the usual scalar product of $\mathbf{x} = (x_0, \ldots, x_r)$, $\mathbf{y} = (y_0, \ldots, y_r) \in K^{r+1}$ by $\mathbf{x} \cdot \mathbf{y} = x_0y_0 + \cdots + x_ry_r$. Further, if $0 \leqslant s \leqslant r$ we define the exterior product $\mathbf{x}_0 \wedge \cdots \wedge \mathbf{x}_s$ of $\mathbf{x}_0 = (x_{00}, \ldots, x_{0r})$,..., $\mathbf{x}_s = (x_{s0}, \ldots, x_{sr}) \in K^{r+1}$ as follows: let $I_0, \ldots, I_R$ with $R = \binom{r+1}{s+1} - 1$ be the subsets of $\{0, \ldots, r\}$ of cardinality $s+1$ in lexicographical order and for $k = 0, \ldots, R$, let $A_k = \det\left(x_{ij}\right)_{0 \leqslant i \leqslant s,\, j \in I_k}$; then $\mathbf{x}_0 \wedge \cdots \wedge \mathbf{x}_s = (A_0, \ldots, A_R)$.

Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension n and degree d, defined over $\overline{\mathbb{Q}}$, where $1 \leqslant n < N$. To X we can associate an up to a constant factor unique polynomial

$$F_X = F_X(\mathbf{h}_0, \ldots, \mathbf{h}_n) = F_X(h_{00}, \ldots, h_{0N}; \ldots; h_{n0}, \ldots, h_{nN})$$

in $n+1$ blocks of variables $\mathbf{h}_i = (h_{i0}, \ldots, h_{iN})$ $(i = 0, \ldots, n)$ which is irreducible in $\overline{\mathbb{Q}}[h_{00}, \ldots, h_{nN}]$ and which is homogeneous of degree d in each block $\mathbf{h}_i$, with the property that $F_X(\mathbf{h}_0, \ldots, \mathbf{h}_n) = 0$ if and only if $X(\overline{\mathbb{Q}})$ and the hyperplanes given by $\mathbf{h}_i \cdot \mathbf{x} = h_{i0}x_0 + \cdots + h_{iN}x_N = 0$ $(i = 0, \ldots, n)$ have a point in common. F_X is called the *(Cayley-Bertini-van der Waerden-)Chow form* of X. We then define the height of X by

$$h(X) := h(F_X). \tag{2.2}$$

For instance, suppose that X is an n-dimensional linear subvariety of $\mathbb{P}^N$ over $\overline{\mathbb{Q}}$. Let $\{\mathbf{a}_0,\dots,\mathbf{a}_n\}$ be any basis of $X(\overline{\mathbb{Q}})$ considered as a vector space. Then

$$F_X(\mathbf{h}_0,\dots,\mathbf{h}_n)=(\mathbf{a}_0\wedge\cdots\wedge\mathbf{a}_n)\cdot(\mathbf{h}_0\wedge\cdots\wedge\mathbf{h}_n), \tag{2.3}$$

and so

$$h(X)=h(\mathbf{a}_0\wedge\cdots\wedge\mathbf{a}_n). \tag{2.4}$$

Faltings ([6], pp. 552, 553) defined another height for projective varieties by means of arithmetic intersection theory. Let $h_{\mathrm{Falt}}(X)$ denote $\frac{1}{[K:\mathbb{Q}]}$ times the height introduced by Faltings where K is any number field over which X is defined. The quantity $h_{\mathrm{Falt}}(X)$ is independent of K and by [1], Theorem 4.3.8, there is an explicitly computable constant $c(N)$ such that $|h(X)-h_{\mathrm{Falt}}(X)|\leqslant c(N)\cdot\deg X$.

3. Statements of the results

3.1. We first state our quantitative result for (1.2) if X is a linear subvariety of $\mathbb{P}^N$.

Let $X\subset\mathbb{P}^N$ be a linear subvariety of dimension n defined over a number field K, where $1\leqslant n<N$. A set of indices $\{i_0,\dots,i_n\}\subset\{0,\dots,N\}$ is called *independent with respect to* X if there is no tuple $(a_{i_0},\dots,a_{i_n})\in\overline{\mathbb{Q}}^{n+1}\backslash\{\mathbf{0}\}$ such that $a_{i_0}x_{i_0}+\cdots+a_{i_n}x_{i_n}$ vanishes identically on X. Denote by $\mathcal{I}_X$ the collection of all subsets of $\{0,\dots,N\}$ of cardinality $n+1$ which are independent with respect to X.

We consider the system of inequalities

$$\log\left(\frac{|x_i|_v}{\|\mathbf{x}\|_v}\right)\leqslant -c_{iv}h(\mathbf{x})\quad(v\in S,\ i=0,\dots,N)\quad\text{in }\mathbf{x}\in X(K) \tag{3.1}$$

with reals $c_{iv}\geqslant 0$, where as before, $(x_0:\cdots:x_N)$ are the homogeneous coordinates of $\mathbf{x}$. More generally, for every finite extension L of K we consider

$$\log\left(\frac{|x_i|_w}{\|\mathbf{x}\|_w}\right)\ \leqslant\ -c_{iw}h(\mathbf{x})\quad(w\in S_L,\ i=0,\dots,N)\quad\text{in }\mathbf{x}\in X(L) \tag{3.2}$$

where S_L is the set of places of L lying above the places in S and where

$$c_{iw}=c_{iv}\cdot\tfrac{[L_w:K_v]}{[L:K]}\quad\text{for } i=0,\dots,N,\ w\in S_L,\ v\in S\text{ with } w|v. \tag{3.3}$$

For a given finite extension L of K denote by $\mathcal{S}_X(L)$ the set of solutions of (3.2). Extension formula (2.1) implies that if $K\subset L_1\subset L_2$ are number fields, then

$\mathcal{S}_X(L_2) \cap X(L_1) = \mathcal{S}_X(L_1)$. We put

$$\mathcal{S}_X(\overline{\mathbb{Q}}) = \bigcup_{L \supseteq K} \mathcal{S}_X(L),$$

where the union is taken over all finite extensions L of K.

Theorem 3.2. *Let $X \subset \mathbb{P}^N$ be a linear subvariety of dimension n defined over K, where $1 \leqslant n < N$. Let S be a finite set of places of K. Further, let $\delta > 0$ and let c_{iv} ($v \in S$, $i = 0, \dots, N$) be reals $\geqslant 0$ such that*

$$\frac{1}{n+1} \sum_{v \in S} \max_{\{i_0,\dots,i_n\} \in \mathcal{I}_X} (c_{i_0,v} + \cdots + c_{i_n,v}) \geqslant 1 + \delta. \tag{3.4}$$

Then there are proper linear subspaces $Y_1, \dots, Y_t$ of X, all defined over K, with

$$t \leqslant 4^{(n+10)^2}(1+\delta^{-1})^{n+5} \log(3N) \log\log(3N), \tag{3.5}$$

such that the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x}) \geqslant (1+\delta^{-1})(N+1)^{n+1} \cdot \big(1 + h(X)\big) \tag{3.6}$$

is contained in $Y_1 \cup \cdots \cup Y_t$.

We explain the relation with Schmidt's Subspace Theorem, which reads as follows: let $\kappa > n+1$ and let $\{l_{0v}, \dots, l_{nv}\}$ ($v \in S$) be linearly independent set of linear forms in $n+1$ variables with coefficients in K; then the set of solutions of

$$\log\Big(\prod_{v \in S} \prod_{j=0}^{n} \frac{|l_{jv}(\mathbf{y})|_v}{\|\mathbf{y}\|_v}\Big) \leqslant -\kappa h(\mathbf{y}) \quad \text{in } \mathbf{y} \in \mathbb{P}^n(K) \tag{3.7}$$

is contained in the union of finitely many proper linear subspaces of $\mathbb{P}^n$.

Let $\mathbf{x} = (x_0 : \cdots : x_N) \in X(K)$ be a solution of (3.1) and assume that (3.4) holds. For $v \in S$, let I_v be an independent subset of $\{0, \dots, N\}$ of cardinality $n+1$ for which $\sum_{j \in I_v} c_{jv}$ is maximal. Thus for each $v \in S$ the set of linear forms $\{x_j : j \in I_v\}$ is linearly independent on X. Then

$$\log\left(\prod_{v \in S} \prod_{j \in I_v} \frac{|x_j|_v}{\|\mathbf{x}\|_v}\right) \leqslant -\Big(\sum_{v \in S} \sum_{j \in I_v} c_{jv}\Big) \cdot h(\mathbf{x}) \leqslant -(n+1)(1+\delta) \cdot h(\mathbf{x}),$$

and this can be transformed into an inequality of the shape (3.7) by means of a linear isomorphism from X to $\mathbb{P}^n$.

Thus, the Subspace Theorem implies that under hypothesis (3.4), the set of solutions of (3.1) is contained in the union of finitely many proper linear subvarieties of X. Using a standard combinatorial argument originating from Mahler (cf. [5, Section 21]) one may show that conversely the latter statement implies the Subspace Theorem.

3.3. We now state our quantitative result for arbitrary projective subvarieties of $\mathbb{P}^N$.

Let $X \subset \mathbb{P}^N$ be an arbitrary projective variety of dimension n and degree d which is defined over a number field K. We assume again $1 \leqslant n < N$. Let $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^N$ and let t be an auxiliary variable. Write

$$F_X(t^{c_0}h_{00}, \dots, t^{c_N}h_{0N}; \dots; t^{c_0}h_{n0}, \dots, t^{c_N}h_{nN}) = t^{e_0}F_0 + t^{e_1}F_1 + \cdots + t^{e_T}F_T \quad \text{with } F_0, \dots, F_T \in K[h_{00}, \dots, h_{nN}],\ e_0 > e_1 > \cdots > e_T, \tag{3.8}$$

where $F_X = F_X(h_{00}, \dots, h_{0N}; \dots; h_{n0}, \dots, h_{nN})$ is the Chow form of X. Then we define the *Chow weight of X with respect to* $\mathbf{c}$ by

$$e_X(\mathbf{c}) := e_0 \tag{3.9}$$

(cf. (6.4) below for an alternative expression).

Let again S be a finite set of places of K, and c_{iv} ($v \in S$, $i = 0, \dots, N$) non-negative reals. For a finite extension L of K, let S_L be the set of places of L lying above those in S, and let c_{iw} ($w \in S_L$, $i = 0, \dots, N$) be defined by (3.3). Denote by $\mathcal{S}_X(L)$ the set of solutions of

$$\log\left(\frac{|x_i|_w}{\|x\|_w}\right) \leqslant -c_{iw}h(\mathbf{x}) \ \ (w \in S_L, i = 0, \dots, N) \quad \text{in } \mathbf{x} \in X(L)$$

and let

$$\mathcal{S}_X(\overline{\mathbb{Q}}) = \bigcup_{L \supseteq K} \mathcal{S}_X(L),$$

where the union is taken over all finite extensions L of K.

By a proper K-*subvariety* of X we mean a proper Zariski closed subset of X defined over K which is not the union of two strictly smaller Zariski closed subsets defined over K. Then we have:

Theorem 3.4. *Let $X \subset \mathbb{P}^N$ be a projective subvariety of dimension n and degree d defined over a number field K, where $1 \leqslant n < N$. Let S be a finite set of places of K. Further, let $\delta > 0$ and let $\mathbf{c}_v = (c_{0v}, \ldots, c_{Nv})$ $(v \in S)$ be tuples of non-negative reals with*

$$\frac{1}{(n+1)d} \sum_{v \in S} e_X(\mathbf{c}_v) \;\geqslant\; 1 + \delta. \tag{3.10}$$

Put

$$\begin{cases} c_1(N,n,d,\delta) := \exp\Big((10n)^{4n} d^{4n+2} (1+\delta^{-1})^{2n}\Big) \cdot \log(3N) \log\log(3N), \\ c_2(N,n,d,\delta) := (8n+5)(1+\delta^{-1}) d^2 \min\big((n+1)d, N+1\big), \\ c_3(N,n,d,\delta) := \exp\Big((10n)^{2n+2} d^{2n+3} (1+\delta^{-1})^{n+1} \cdot \log(3N)\Big). \end{cases} \tag{3.11}$$

Then there are proper K-subvarieties $Y_1, \ldots, Y_t$ of X with

$$t \;\leqslant\; c_1(N,n,d,\delta), \tag{3.12}$$

$$\deg Y_i \;\leqslant\; c_2(N,n,d,\delta) \quad \textit{for } i = 1, \ldots, t, \tag{3.13}$$

such that the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x}) \geqslant c_3(N,n,d,\delta) \cdot \big(1 + h(X)\big) \tag{3.14}$$

is contained in $Y_1 \cup \cdots \cup Y_t$.

3.5. Let again $\mathbf{h}_i = (h_{i0}, \ldots, h_{iN})$ $(i = 0, \ldots, n)$ be blocks of $N+1$ variables. For each subset $I = \{j_0, \ldots, j_n\}$ of $\{0, \ldots, N\}$ with $j_0 < \ldots < j_n$ we define the *bracket* $[I] = [j_0 \cdots j_n] = \det(h_{i,j_k})_{i,k=0,\ldots,n}$. From [13], p. 41, Thm. IV it follows that the Chow form F_X of an n-dimensional subvariety X of $\mathbb{P}^N$ can be expressed as a polynomial in terms of these brackets. It is easy to show that for $\mathbf{c} = (c_0, \ldots, c_N) \in \mathbb{R}^{N+1}$, the substitution

$$(h_{00}, \ldots, h_{0N}; \ldots; h_{n0}, \ldots, h_{nN}) \leftarrow (t^{c_0} h_{00}, \ldots, t^{c_N} h_{0N}; \ldots; t^{c_0} h_{n0}, \ldots, t^{c_N} h_{nN})$$

transforms $[I]$ into $t^{\sum_{j \in I} c_j} [I]$.

In particular, let $X \subset \mathbb{P}^N$ be a linear subvariety of dimension n. Then from (2.3) it follows that $F_X = \sum_{I \in \mathcal{I}_X} \gamma_I [I]$ with $\gamma_I \neq 0$ for $I \in \mathcal{I}_X$ where as before $\mathcal{I}_X$ is the collection of subsets of $\{0,\dots,N\}$ of cardinality $n+1$ which are independent with respect to X. Hence

$$e_X(\mathbf{c}) = \max_{\{i_0,\dots,i_n\}\in\mathcal{I}_X} c_{i_0}+\cdots+c_{i_n}.$$

So for linear varieties X, (3.10) is equivalent to (3.4).

3.6. Now let $X\subset\mathbb{P}^N$ be the hypersurface given by $f=0$, where

$$f=\sum_{\mathbf{a}\in A}\beta(\mathbf{a})x_0^{a_0}\dots x_N^{a_N}\in K[x_0,\dots,x_N] \tag{3.15}$$

is a homogeneous polynomial of degree d which is irreducible over $\overline{\mathbb{Q}}$. Here A is a finite set of tuples of non-negative integers $\mathbf{a}=(a_0,\dots,a_N)$ with $a_0+\cdots+a_N=d$, and $\beta(\mathbf{a})\neq 0$ for $\mathbf{a}\in A$.

The variety X has dimension $n=N-1$ and degree d, and its Chow form is equal to

$$\begin{aligned} F_X &= f([1\ 2\cdots N],-[0\ 2\cdots N],\dots,(-1)^{N-1}[0\ 1\cdots N-1]) \\ &= \sum_{\mathbf{a}\in A}\pm\beta(\mathbf{a})[1\ 2\cdots N]^{a_0}\cdots[0\ 1\cdots N-1]^{a_N}. \end{aligned} \tag{3.16}$$

This implies that for $\mathbf{c}=(c_0,\dots,c_N)\in\mathbb{R}^{N+1}$ we have

$$\begin{aligned} e_X(\mathbf{c}) &= \max_{\mathbf{a}\in A}\sum_{j=0}^N a_j\Big(\sum_{k=0,\,k\neq j}^N c_k\Big) \\ &= d(c_0+\cdots+c_N)-\min_{\mathbf{a}\in A}(a_0c_0+\cdots+a_Nc_N). \end{aligned} \tag{3.17}$$

Now we have:

Corollary 3.7. *Let $X\subset\mathbb{P}^N$ be the irreducible hypersurface defined by $f=0$, where f is given by (3.15). Let S, δ be as in Theorem 3.4, and let $\mathbf{c}_v=(c_{0v},\dots,c_{Nv})$ $(v\in S)$ be tuples of non-negative reals with*

$$\frac{1}{N}\sum_{v\in S}\sum_{i=0}^N c_{iv}-\frac{1}{Nd}\sum_{v\in S}\min_{\mathbf{a}\in A}(a_0c_{0v}+\cdots+a_Nc_{Nv})\geqslant 1+\delta. \tag{3.18}$$

Further, let

$$\begin{aligned} c_1^*(N,d,\delta) &:= \exp\left((10N)^{4N}d^{4N-2}(1+\delta^{-1})^{2N-2}\right), \\ c_2^*(N,d,\delta) &:= (8N-3)(N+1)d^2(1+\delta^{-1}), \\ c_3^*(N,d,\delta) &:= \exp\left((10N)^{2N+1}d^{2N+1}(1+\delta^{-1})^{N}\right). \end{aligned}$$

Then there are proper K-subvarieties $Y_1,\dots,Y_t$ of X with

$$t \leqslant c_1^*(N,d,\delta), \quad \deg Y_i \leqslant c_2^*(N,d,\delta) \text{ for } i=1,\dots,t$$

such that the set of $\mathbf{x}\in\mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x}) \geqslant c_3^*(N,d,\delta)\cdot(1+h(X))$$

is contained in $Y_1\cup\cdots\cup Y_t$.

Proof. We apply Theorem 3.4 with $n=N-1$ to X. In view of (3.17), condition (3.18) is equivalent to (3.10). Further we have $c_i(N,N-1,d,\delta)\leqslant c_i^*(N,d,\delta)$ for $i=1,2,3$. □

Lastly, we give a consequence of Theorem 3.4 for curves. For $\mathbf{x}\in\mathbb{P}^N(\overline{\mathbb{Q}})$, denote by $K(\mathbf{x})$ the smallest extension of K containing a set of homogeneous coordinates for $\mathbf{x}$.

Corollary 3.8. *Let $X\subset\mathbb{P}^N$ be an irreducible projective curve of degree d defined over K. Further, S,δ be as in Theorem 3.4 and let $\mathbf{c}_v=(c_{0v},\dots,c_{Nv})$ $(v\in S)$ be tuples of non-negative reals satisfying*

$$\frac{1}{2d}\sum_{v\in S} e_X(\mathbf{c}_v) \geqslant 1+\delta\,. \tag{3.19}$$

Put

$$\begin{aligned} c_1^{**}(N,d,\delta) &:= \exp\left(10^5 d^7(1+\delta^{-1})^3\right)\cdot\log(3N)\log\log(3N), \\ c_2^{**}(N,d,\delta) &:= 13(1+\delta^{-1})d^2\min(2d,N+1), \\ c_3^{**}(N,d,\delta) &:= \exp\left(10^4 d^5(1+\delta^{-1})^2\log(3N)\right). \end{aligned}$$

Then there are at most $c_1^{**}(N,d,\delta)$ points $\mathbf{x}\in\mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x}) \geqslant c_3^{**}(N,d,\delta)\cdot(1+h(X))\,.$$

Moreover, for each of these points we have

$$[K(\mathbf{x}) : K] \leqslant c_2^{**}(N, d, \delta).$$

Proof. Notice that if Y is a proper K-subvariety of X of degree D then Y consists of D points which are conjugate to one another over K and have degree D over K. By applying Theorem 3.4 with $n = 1$ to X, we obtain that $\mathcal{S}_X(\overline{\mathbb{Q}})$ contains at most $c_1(N, 1, d, \delta) \cdot c_2(N, 1, d, \delta)$ points $\mathbf{x}$ with $h(\mathbf{x}) \geqslant c_3(N, 1, d, \delta) \cdot (1 + h(X))$ and moreover, that for each of these points we have $[K(\mathbf{x}) : K] \leqslant c_2(N, 1, d, \delta)$. Now Corollary 3.8 follows on observing that $c_1(N, 1, d, \delta) \cdot c_2(N, 1, d, \delta) \leqslant c_1^{**}(N, d, \delta)$ and $c_i(N, 1, d, \delta) \leqslant c_i^{**}(N, d, \delta)$ for $i = 2, 3$. □

We mention that computing the Chow weights $e_X(\mathbf{c})$ for arbitrary projective varieties X is in general quite difficult. In [9], [10] Ferretti discussed various methods to compute Chow weights, and computed them for certain varieties other than linear varieties or hypersurfaces.

4. Hilbert weights and Chow weights

4.1. Denote by $\mathbb{Z}_{\geqslant 0}^{N+1}$, $\mathbb{R}_{\geqslant 0}^{N+1}$ the sets of $(N+1)$-tuples consisting of non-negative integers, non-negative reals, respectively. For $\mathbf{a} = (a_0, \ldots, a_N) \in \mathbb{Z}_{\geqslant 0}^{N+1}$ we write $\mathbf{x}^{\mathbf{a}}$ for the monomial $x_0^{a_0} \cdots x_N^{a_N}$. In this section, K is an algebraically closed field of characteristic 0. A homogeneous ideal I of $K[x_0, \ldots, x_N]$ is said to be *relevant* if $I \neq (0)$ and if there is no integer $k \geqslant 0$ such that $x_0^k, \ldots, x_N^k \in I$.

4.2. For a positive integer m, let $K[x_0, \ldots, x_N]_m$ denote the vector space of homogeneous polynomials in $K[x_0, \ldots, x_N]$ of degree m (including 0). Let I be a relevant homogeneous ideal of $K[x_0, \ldots, x_N]$. Put $I_m := K[x_0, \ldots, x_N]_m \cap I$ and define the Hilbert function H_I of I by

$$H_I(m) := \dim_K \Big(K[x_0, \ldots, x_N]_m / I_m \Big) \quad \text{for } m = 1, 2, \ldots . \tag{4.1}$$

Then there are integers $n \geqslant 0$, $d > 0$ such that

$$H_I(m) = d \cdot \frac{m^n}{n!} + O(m^{n-1}) \quad \text{as } m \to \infty. \tag{4.2}$$

We call n the *dimension* of I, notation $\dim I$, and d the *degree* of I, notation $\deg I$.

Let $P_1, \dots, P_g$ be the prime ideals of maximal dimension associated to I. For $i = 1, \dots, g$, let $O_{P_i,I}$ be the localization of $K[x_0, \dots, x_N]/I$ at P_i and let $\mu_{P_i,I} := l_{O_{P_i,I}}(O_{P_i,I})$ be the length of $O_{P_i,I}$ as a $O_{P_i,I}$-module. This quantity is known to be finite. We call $\mu_{P_i,I}$ the multiplicity of I with respect to P_i. Then

$$\dim I = \dim P_1 = \cdots = \dim P_g\,, \quad \deg I = \sum_{i=1}^{g} \mu_{P_i,I} \deg P_i. \tag{4.3}$$

4.3. We define the *m-th Hilbert weight* $s_I(m, \mathbf{c})$ of I with respect to a tuple $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$ by

$$s_I(m, \mathbf{c}) = \max(\mathbf{a}_1 + \cdots + \mathbf{a}_{H_I(m)}) \cdot \mathbf{c}, \tag{4.4}$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{H_I(m)}}$ whose residue classes modulo I form a basis of the K-vector space $K[x_0, \dots, x_N]_m/I_m$.

4.4. We define the Chow form of a homogeneous prime ideal P of $K[x_0, \dots, x_N]$ by $F_P := F_X$, where X is the variety defined by P and F_X is the Chow form of X as defined in §2.2 (with K in place of $\overline{\mathbb{Q}}$). Further, we define the Chow form of an arbitrary relevant homogeneous ideal I of $K[x_0, \dots, x_N]$ by

$$F_I := \prod_{i=1}^{g} F_{P_i}^{\mu_{P_i,I}}, \tag{4.5}$$

where $P_1, \dots, P_g$ are the prime ideals of maximal dimension associated to I and where $\mu_{P_i,I}$ is the multiplicity of I with respect to P_i.

Let $\dim I = n$, $\deg I = d$. Then it follows from §2.2 and (4.2) that $F_I = F_I(h_{00}, \dots, h_{0N}; \dots; h_{n0}, \dots, h_{nN})$ is a polynomial in $n + 1$ blocks of $N + 1$ variables $\mathbf{h}_i = (h_{i0}, \dots, h_{iN})$ $(i = 0, \dots, n)$ such that F_I is homogeneous of degree d in each block $\mathbf{h}_i$. Given $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$, we write similarly as in (3.8), (3.9)

$$F_I(t^{c_0}h_{00}, \dots, t^{c_N}h_{0N}; \dots; t^{c_0}h_{n0}, \dots, t^{c_N}h_{nN}) = \sum_{k=0}^{T} t^{e_j} F_j$$

with $F_0, \dots, F_T \in K[h_{00}, \dots, h_{nN}]$, $e_0 > e_1 > \cdots > e_T$ and define the Chow weight of I with respect to $\mathbf{c}$ by

$$e_I(\mathbf{c}) = e_0. \tag{4.6}$$

4.5. According to Mumford [16], p.61, Proposition 2.11 we have

$$s_I(m,\mathbf{c}) = e_I(\mathbf{c})\cdot\frac{m^{n+1}}{(n+1)!}+O(m^n)\quad\text{as } m\to\infty .$$

Together with (4.2) this implies

$$\lim_{m\to\infty}\frac{1}{mH_I(m)}\cdot s_I(m,\mathbf{c}) = \frac{1}{(n+1)d}\cdot e_I(\mathbf{c}).$$

We call $\frac{1}{mH_I(m)}\cdot s_I(m,\mathbf{c})$ the m-th normalized Hilbert weight and $\frac{1}{(n+1)d}\cdot e_I(\mathbf{c})$ the normalized Chow weight of I.

For a projective subvariety X of $\mathbb{P}^N$ defined over K, denote by P_X the prime ideal of $K[x_0,\dots,x_N]$ consisting of all polynomials vanishing identically on X. Then we put $\dim X := \dim P_X$, $\deg X := \deg P_X$, $H_X(m) := H_{P_X}(m)$, $s_X(m,\mathbf{c}) := s_{P_X}(m,\mathbf{c})$, $e_X(\mathbf{c}) := e_{P_X}(\mathbf{c})$. This coincides with earlier given definitions. We deduce an explicit lower bound for the m-th normalized Hilbert weight of X in terms of the normalized Chow weight of X.

Theorem 4.6. *Let X be a subvariety of $\mathbb{P}^N$ of dimension n and degree d, defined over an algebraically closed field K of characteristic 0. Let $m > d$ be an integer. Further, let $\mathbf{c}=(c_0,\dots,c_N)\in\mathbb{R}^{N+1}_{\geqslant 0}$. Then*

$$\frac{1}{mH_X(m)}\,s_X(m,\mathbf{c}) \geqslant \frac{1}{(n+1)d}\,e_X(\mathbf{c}) - \frac{(2n+1)d}{m}\cdot\Big(\max_{i=0,\dots,N} c_i\Big). \tag{4.7}$$

Inequality (4.7) is sufficient for our purposes. It is probably more difficult to prove an inequality in the other direction. In the proof of Theorem 4.6 we use some ideas of Kapranov, Sturmfels and Zelevinsky [14] which were also implicit in Mumford's paper [16]: in Section 5 we deduce an auxiliary result for monomial ideals (i.e., ideals generated by monomials) and in Section 6 we deduce from this Theorem 4.6.

5. Monomial ideals

5.1. We keep the notation introduced in the previous section, so in particular K is an algebraically closed field of characteristic 0. In addition, for $\mathbf{a}=(a_0,\dots,a_N)$,

$\mathbf{b} = (b_0, \ldots, b_N) \in \mathbb{R}^{N+1}$ we write $\mathbf{a} \leqslant \mathbf{b}$ or $\mathbf{b} \geqslant \mathbf{a}$ if $a_i \leqslant b_i$ for all $i = 0, \ldots, N$. For $\mathbf{a} = (a_0, \ldots, a_N) \in \mathbb{R}^{N+1}$ we define the norm $\|\mathbf{a}\| := \sum_{i=0}^{N} |a_i|$ and the support $\operatorname{supp} \mathbf{a} = \{i : 0 \leqslant i \leqslant N,\ a_i \neq 0\}$; further, for $W \subset \{0, \ldots, N\}$ let $\mathbf{a}_W$ be the vector obtained by setting the coordinates of $\mathbf{a}$ with indices outside W to 0, i.e., $\mathbf{a}_W := (b_0, \ldots, b_N)$ with $b_i = a_i$ for $i \in W$, $b_i = 0$ for $i \notin W$. For $f_1, \ldots, f_T \in K[x_0, \ldots, x_N]$ let $(f_1, \ldots, f_T)$ denote the ideal in $K[x_0, \ldots, x_N]$ generated by $f_1, \ldots, f_T$ and for $W \subset \{0, \ldots, N\}$, let $P_W := (x_i : i \in W)$ denote the ideal in $K[x_0, \ldots, x_N]$ generated by x_i $(i \in W)$.

5.2. Throughout this section, let

$$I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_T}) \tag{5.1}$$

be the ideal generated by the monomials $\mathbf{x}^{\mathbf{a}_i}$ $(i = 1, \ldots, T)$, where $\mathbf{a}_i = (a_{i0}, \ldots, a_{iN}) \in \mathbb{Z}_{\geqslant 0}^{N+1}$. We assume that I is relevant. Note that $\mathbf{x}^{\mathbf{a}} \in I$ if and only if $\mathbf{a} \geqslant \mathbf{a}_i$ for some $i \in \{1, \ldots, T\}$. Let $S(I)$ be the collection of sets $W \subseteq \{0, \ldots, N\}$ with the property that for every $i \in \{1, \ldots, T\}$ there is a $j \in W$ with $a_{ij} > 0$. Given $W \in S(I)$, let

$$A_W(I) := \{\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{N+1} :\ \operatorname{supp} \mathbf{a} \subseteq W,\ \mathbf{a} \not\geqslant \mathbf{a}_{i,W} \text{ for all } i = 1, \ldots, T\}. \tag{5.2}$$

We have included a proof of the following simple lemma (see also [19], Proposition 3.4).

Lemma 5.3. *Let $W_1, \ldots, W_g$ be the non-empty sets in $S(I)$ of minimal cardinality. Then $P_{W_1}, \ldots, P_{W_g}$ are the prime ideals of maximal dimension associated to I. Further, for $i = 1, \ldots, g$, the multiplicity $\mu_{P_{W_i}, I}$ of I with respect to P_{W_i} is equal to the cardinality of $A_{W_i}(I)$.*

Proof. For $\mathbf{x} = (x_0 : \cdots : x_N) \in \mathbb{P}^N(K)$ we have that $\mathbf{x}^{\mathbf{a}_i} = 0$ for $i = 1, \ldots, T$ if and only if there is a set $W \in S(I)$ such that $x_j = 0$ for $j \in W$. Hence the radical of I is $\cap_{W \in S(I)} P_W$. Since $\dim P_W = N - \#W$, it follows that the prime ideals of maximal dimension associated to I are precisely $P_{W_1}, \ldots, P_{W_g}$. Let $W \in \{W_1, \ldots, W_g\}$ and suppose that $W = \{0, \ldots, r\}$. Let $K' = K(x_{r+1}, \ldots, x_N)$, $R = K'[x_0, \ldots, x_r]$, $I' = (\mathbf{x}^{\mathbf{a}_{1,W}}, \ldots, \mathbf{x}^{\mathbf{a}_{T,W}})$. Then $O_{P_W, I} = R/I'$. The latter is a K'-vector space with basis $\{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in A_W(I)\}$. Therefore, $\mu_{P_W, I} = l_{O_{P_W, I}}(O_{P_W, I}) = \dim_{K'} R/I'$ is equal to the cardinality of $A_W(I)$. $\square$

We make some further observations. Let I be as in (5.1) and let $W_1, \dots, W_g$ be the sets from Lemma 5.3. Let $n = \dim I$, $d = \deg I$, $\mu_i = \mu_{P_{W_i}, I}$ $(i = 1, \dots, g)$. Then

$$\#W_i = N - n \quad (i = 1, \dots, g). \tag{5.3}$$

Further, by (4.3) we have

$$\sum_{i=1}^{g} \mu_i = d. \tag{5.4}$$

Lastly,

$$\|\mathbf{a}\| \leqslant \mu_i \quad \text{for } \mathbf{a} \in A_{W_i}(I),\ i = 1, \dots, g. \tag{5.5}$$

Indeed, let $\mathbf{a} \in A_{W_i}(I)$. Then every $\mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1}$ with $\mathbf{b} \leqslant \mathbf{a}$ belongs to $A_{W_i}(I)$. The number of $\mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1}$ with $\mathbf{b} \leqslant \mathbf{a}$ is at least $\|\mathbf{a}\|$, and so $\|\mathbf{a}\|$ is at most the cardinality of $A_{W_i}(I)$.

5.4. Let $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}_{\geqslant 0}^{N+1}$. Note that $K[x_0, \dots, x_n]_m / I_m$ has a unique monomial basis consisting of those monomials $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{a} \not\geqslant \mathbf{a}_i$ for $i = 1, \dots, T$ and $\|\mathbf{a}\| = m$. This implies

$$s_I(m, \mathbf{c}) = \sum_{\substack{\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{N+1},\ \|\mathbf{a}\| = m, \\ \mathbf{a} \not\geqslant \mathbf{a}_i \text{ for } i = 1, \dots, T}} \mathbf{a} \cdot \mathbf{c}. \tag{5.6}$$

Let $W_k^c = \{0, \dots, N\} \backslash W_k$ for $k = 1, \dots, g$. Then by (4.5) we have, with the bracket notation from §3.5,

$$F_I = \prod_{k=1}^{g} F_{P_{W_k}}^{\mu_k} = \prod_{k=1}^{g} [W_k^c]^{\mu_k}.$$

Hence

$$e_I(\mathbf{c}) = \sum_{k=1}^{g} \mu_k \Big(\sum_{j \in W_k^c} c_j \Big). \tag{5.7}$$

We prove:

Lemma 5.5. *Let m be an integer $> d$ and $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}_{\geqslant 0}^{N+1}$. Then*

$$s_I(m, \mathbf{c}) \geqslant \tfrac{m-d}{n+1} \binom{m-d+n}{n} \cdot e_I(\mathbf{c}) - d^2 m \binom{m+n-1}{n-1} \cdot \Big(\max_{0 \leqslant i \leqslant N} c_i \Big). \tag{5.8}$$

Proof. For a finite subset S of $\mathbb{Z}_{\geqslant 0}^{N+1}$ put $\Sigma_{\mathbf{c}}(S) := \sum_{\mathbf{a}\in S} \mathbf{a}\cdot\mathbf{c}$. Write A_k for the set $A_{W_k}(I)$ given by (5.2). For $k = 1,\dots,g$, $\mathbf{a}\in A_k$, let $S_k(\mathbf{a})$ be the set of vectors $\mathbf{r}$ such that

$$\begin{cases} \mathbf{r} = \mathbf{a}+\mathbf{b} \quad \text{for some } \mathbf{b}\in\mathbb{Z}_{\geqslant 0}^{N+1} \text{ with } \operatorname{supp}\mathbf{b}\subseteq W_k^c, \\ \|\mathbf{r}\| = m. \end{cases}$$

We estimate from below $s_I(m,\mathbf{c})$ using (5.6). Using that for $\mathbf{r}\in S_k(\mathbf{a})$ we have $\mathbf{r}\not\geqslant \mathbf{a}_i$ for $i = 1,\dots,T$, and applying the principle of inclusion and exclusion we obtain

$$\begin{aligned} s_I(m,\mathbf{c}) &\geqslant \Sigma_{\mathbf{c}}\Big(\bigcup_{k=1}^{g}\bigcup_{\mathbf{a}\in A_k} S_k(\mathbf{a})\Big) \\ &\geqslant \sum_{k=1}^{g}\sum_{\mathbf{a}\in A_k}\Sigma_{\mathbf{c}}(S_k(\mathbf{a})) - \sum_{(k,\mathbf{a}')\neq(l,\mathbf{a}'')}\Sigma_{\mathbf{c}}(S_k(\mathbf{a}')\cap S_l(\mathbf{a}'')), \end{aligned} \tag{5.9}$$

where the last summation is over all quadruples $(k,l,\mathbf{a}',\mathbf{a}'')$ with $k,l = 1,\dots,g$, $\mathbf{a}'\in A_k$, $\mathbf{a}''\in A_l$ and $(k,\mathbf{a}')\neq(l,\mathbf{a}'')$.

Let $k\in\{1,\dots,g\}$, $\mathbf{a}\in A_k$. By (5.3) we have $\#W_k^c = n+1$ and by (5.4), (5.5) we have $\|\mathbf{a}\|\leqslant d$. Hence

$$\begin{aligned} \Sigma_{\mathbf{c}}(S_k(\mathbf{a})) &\geqslant \Sigma_{\mathbf{c}}(\{\mathbf{b}\in\mathbb{Z}_{\geqslant 0}^{N+1} : \operatorname{supp}\mathbf{b}\subseteq W_k^c,\ \|\mathbf{b}\| = m-\|\mathbf{a}\|\}) \\ &= \binom{m-\|\mathbf{a}\|+\#W_k^c-1}{\#W_k^c-1}\cdot\frac{m-\|\mathbf{a}\|}{\#W_k^c}\sum_{j\in W_k^c} c_j \\ &\geqslant \binom{m-d+n}{n}\cdot\frac{m-d}{n+1}\sum_{j\in W_k^c} c_j. \end{aligned}$$

Summing over $k = 1,\dots,g$, $\mathbf{a}\in A_k$ we obtain, using that $\#A_k = \mu_k$ by Lemma 5.3 and using (5.7),

$$\begin{aligned} \sum_{k=1}^{g}\sum_{\mathbf{a}\in A_k}\Sigma_{\mathbf{c}}(S_k(\mathbf{a})) &\geqslant \frac{m-d}{n+1}\binom{m-d+n}{n}\sum_{k=1}^{g}\mu_k\big(\sum_{j\in W_k^c} c_j\big) \\ &= \frac{m-d}{n+1}\binom{m-d+n}{n}\cdot e_I(\mathbf{c}). \end{aligned} \tag{5.10}$$

Let $(k,l,\mathbf{a}',\mathbf{a}'')$ be a quadruple with $k,l\in\{1,\dots,g\}$, $\mathbf{a}'\in A_k$, $\mathbf{a}''\in A_l$ and $(k,\mathbf{a}')\neq (l,\mathbf{a}'')$. If $k = l$ then $S_k(\mathbf{a}')\cap S_l(\mathbf{a}'') = \emptyset$. Assume $k\neq l$. Write $\mathbf{a}' = (a_0',\dots,a_N')$,

$\mathbf{a}'' = (a_0'', \dots, a_N'')$ and put $\max(\mathbf{a}', \mathbf{a}'') := (\max(a_0', a_0''), \dots, \max(a_0', a_0''))$. Then $S_k(\mathbf{a}') \cap S_l(\mathbf{a}'')$ consists of all vectors $\mathbf{r}$ such that

$$\begin{cases} \mathbf{r} = \max(\mathbf{a}', \mathbf{a}'') + \mathbf{b} \quad \text{for some } \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1} \text{ with } \operatorname{supp} \mathbf{b} \subseteq W_k^c \cap W_l^c, \\ \|\mathbf{r}\| = m. \end{cases}$$

By (5.3) we have $\#(W_k^c \cap W_l^c) \leqslant n$, hence

$$\#(S_k(\mathbf{a}') \cap S_l(\mathbf{a}'')) \leqslant \binom{m - \|\max(\mathbf{a}', \mathbf{a}'')\| + n - 1}{n-1} \leqslant \binom{m+n-1}{n-1}.$$

Further, for each $\mathbf{r} \in S_k(\mathbf{a}') \cap S_l(\mathbf{a}'')$ we have $\mathbf{r} \cdot \mathbf{c} \leqslant m \cdot \big(\max_{0 \leqslant i \leqslant N} c_i\big)$. Therefore,

$$\|\Sigma_{\mathbf{c}}(S_k(\mathbf{a}') \cap S_l(\mathbf{a}''))\| \leqslant m \binom{m+n-1}{n-1} \cdot \Big(\max_{0 \leqslant i \leqslant N} c_i\Big).$$

By Lemma 5.3 and (5.4), the number of pairs $(k, \mathbf{a})$ with $k = 1, \dots, g$, $\mathbf{a} \in A_k$ is equal to $\mu_1 + \cdots + \mu_g = d$. Therefore,

$$\|\sum_{(k,\mathbf{a}') \neq (l, \mathbf{a}'')} \Sigma_{\mathbf{c}}(S_k(\mathbf{a}') \cap S_l(\mathbf{a}''))\| \leqslant d^2 m \binom{m+n-1}{n-1} \cdot \Big(\max_{0 \leqslant i \leqslant N} c_i\Big).$$

By inserting this and (5.10) into (5.9) we arrive at (5.8). □

6. Proof of Theorem 4.6

6.1. Much of the material in this section can be found in bits and pieces in the literature, in particular in [14], [3, Chapter 15], [17]. For convenience of the unspecialized reader we have worked out more details. We keep the previously introduced notation; in particular K is an algebraically closed field of characteristic 0. Further, in what follows I is a relevant homogeneous ideal of $K[x_0, \dots, x_N]$ of dimension n and degree d and $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$.

Let t be a parameter. For $f \in K[x_0, \dots, x_N]$, $f \neq 0$, define the number $w_{\mathbf{c}}(f)$ and the polynomial $in_{\mathbf{c}}(f) \in K[x_0, \dots, x_N]$ (the initial part of f with respect to $\mathbf{c}$) by

$$f(t^{c_0} x_0, \dots, t^{c_N} x_N) = t^{w_{\mathbf{c}}(f)} \cdot in_{\mathbf{c}}(f) + (\text{terms with higher powers of } t). \tag{6.1}$$

Alternatively, if we write $f = \sum_{\mathbf{a}\in A} \beta(\mathbf{a})\mathbf{x}^{\mathbf{a}}$ with $\beta(\mathbf{a}) \neq 0$ for $\mathbf{a} \in A$, then $w_{\mathbf{c}}(f) = \min\{\mathbf{a}\cdot\mathbf{c} :\ \mathbf{a} \in A\}$ and

$$in_{\mathbf{c}}(f) = \sum_{\substack{\mathbf{a}\in A \\ \mathbf{a}\cdot\mathbf{c}=w_{\mathbf{c}}(f)}} \beta(\mathbf{a})\mathbf{x}^{\mathbf{a}}. \tag{6.2}$$

We denote by $in_{\mathbf{c}}(I)$ the ideal generated by $in_{\mathbf{c}}(f)$ $(f \in I)$. The following lemma is implicit in [3], Chapter 15.

Lemma 6.2. *Let $m \geqslant 1$ be an integer. Further, let $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_I(m)}}\}$ be a basis of $K[x_0, \ldots, x_N]_m/I_m$ for which $(\mathbf{a}_1+\cdots+\mathbf{a}_{H_I(m)})\cdot\mathbf{c}$ is maximal. Then $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_I(m)}}\}$ is a basis of $K[x_0, \ldots, x_N]_m/in_{\mathbf{c}}(I)_m$.*
Consequently, $in_{\mathbf{c}}(I)$ has the same Hilbert function as I.

Proof. Write $H := H_I(m)$, $R = \binom{N+m}{N}$. Let $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_R}$ be all monomials of degree m in $x_0, \ldots, x_N$, ordered such that $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_H}$ are the monomials from the statement of the lemma. Then I_m is generated by

$$f_i = \mathbf{x}^{\mathbf{a}_i} - \sum_{j\in B_i} \beta_{ij}\mathbf{x}^{\mathbf{a}_j} \quad (i = H+1, \ldots, R),$$

where $B_i \subseteq \{1, \ldots, H\}$ and $\beta_{ij} \neq 0$ for $j \in B_i$. For $i \in \{H+1, \ldots, R\}$, $j \in B_i$ we can make a new basis of $K[x_0, \ldots, x_N]_m/I_m$ by replacing $\mathbf{x}^{\mathbf{a}_j}$ by $\mathbf{x}^{\mathbf{a}_i}$ in $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_H}\}$, therefore $\mathbf{a}_i \cdot \mathbf{c} \leqslant \mathbf{a}_j \cdot \mathbf{c}$. By (6.2) we have $in_{\mathbf{c}}(f_i) = \mathbf{x}^{\mathbf{a}_i} - \sum_{j\in B_i'} \beta_{ij}\mathbf{x}^{\mathbf{a}_j}$ where B_i' is the set of indices $j \in B_i$ for which $\mathbf{a}_j \cdot \mathbf{c} = \mathbf{a}_i \cdot \mathbf{c}$ for $i = H+1, \ldots, R$.

We claim that $in_{\mathbf{c}}(I)_m$ is generated by the polynomials $in_{\mathbf{c}}(f_i)$ $(i = H+1, \ldots, R)$. Let $f \in in_{\mathbf{c}}(I)_m$. We may write f as a linear combination of terms $\mathbf{x}^{\mathbf{a}} in_{\mathbf{c}}(g)$ with $g \in I$. We have $\mathbf{x}^{\mathbf{a}} in_{\mathbf{c}}(g) = in_{\mathbf{c}}(h)$ with $h = \mathbf{x}^{\mathbf{a}} g \in I_m$. Now h is a linear combination of the polynomials f_i, therefore $in_{\mathbf{c}}(h)$ is a linear combination of the polynomials $in_{\mathbf{c}}(f_i)$, and so f is a linear combination of these polynomials. This proves our claim.

Now Lemma 6.2 follows by observing that $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_H}$, $in_{\mathbf{c}}(f_{H+1}), \ldots, in_{\mathbf{c}}(f_R)$ form a basis of $K[x_0, \ldots, x_N]$. $\square$

Let as before F_I be the Chow form of I. From the definition of the Chow weight it follows that there is a polynomial $fin_{\mathbf{c}}(F_I) \in K[h_{00}, \ldots, h_{nN}]$ (the final part of F_I

with respect to $\mathbf{c}$) such that

$$F_I(t^{c_0}h_{00},\dots,t^{c_N}h_{0N};\dots;t^{c_0}h_{n0},\dots,t^{c_N}h_{nN}) \tag{6.3}$$
$$= t^{e_I(\mathbf{c})}\mathit{fin}_{\mathbf{c}}(F_I) + (\text{terms with smaller powers of } t).$$

Alternatively, for $\mathbf{a}_i = (a_{i0},\dots,a_{iN}) \in \mathbb{Z}_{\geqslant 0}^{N+1}$ $(i = 0,\dots,n)$ put $\mathbf{h}_0^{\mathbf{a}_0}\cdots\mathbf{h}_n^{\mathbf{a}_n} := \prod_{i=0}^n\prod_{j=0}^N h_{ij}^{a_{ij}}$. Then if we write $F_I = \sum_{(\mathbf{a}_0,\dots,\mathbf{a}_n)\in B}\gamma(\mathbf{a}_0,\dots,\mathbf{a}_n)\mathbf{h}_0^{\mathbf{a}_0}\cdots\mathbf{h}_n^{\mathbf{a}_n}$ with $\gamma(\mathbf{a}_0,\dots,\mathbf{a}_n) \neq 0$ for $(\mathbf{a}_0,\dots,\mathbf{a}_n) \in B$, we have

$$e_I(\mathbf{c}) = \max\{(\mathbf{a}_0+\cdots+\mathbf{a}_n)\cdot\mathbf{c} :\ (\mathbf{a}_0,\dots,\mathbf{a}_n)\in B\}, \tag{6.4}$$

$$\mathit{fin}_{\mathbf{c}}(F_I) = \sum_{\substack{(\mathbf{a}_0,\dots,\mathbf{a}_n)\in B\\ (\mathbf{a}_0+\cdots+\mathbf{a}_n)\cdot\mathbf{c}=e_I(\mathbf{c})}} \gamma(\mathbf{a}_0,\dots,\mathbf{a}_n)\mathbf{h}_0^{\mathbf{a}_0}\cdots\mathbf{h}_n^{\mathbf{a}_n}\,. \tag{6.5}$$

Lemma 6.3. *Apart from a constant factor, $F_{\mathit{in}_{\mathbf{c}}(I)} = \mathit{fin}_{\mathbf{c}}(F_I)$.*

Proof. We first reduce the lemma to the case that $\mathbf{c} \in \mathbb{Z}^{N+1}$. Let $\mathbf{c} \in \mathbb{R}^{N+1}$ be arbitrary. Let M be a sufficiently large integer. Let $\mathbf{b}_1,\dots,\mathbf{b}_R$ be the vectors in $\mathbb{Z}_{\geqslant 0}^{N+1}$ with sum of coordinates at most M, ordered such that $\mathbf{b}_1\cdot\mathbf{c} \leqslant \mathbf{b}_2\cdot\mathbf{c} \leqslant \cdots \leqslant \mathbf{b}_R\cdot\mathbf{c}$. Then there is a vector $\mathbf{c}' \in \mathbb{Z}^{N+1}$ such that for $i = 1,\dots,R-1$ we have the following: if $\mathbf{b}_i\cdot\mathbf{c} < \mathbf{b}_{i+1}\cdot\mathbf{c}$ then $\mathbf{b}_i\cdot\mathbf{c}' < \mathbf{b}_{i+1}\cdot\mathbf{c}'$, while if $\mathbf{b}_i\cdot\mathbf{c} = \mathbf{b}_{i+1}\cdot\mathbf{c}$ then $\mathbf{b}_i\cdot\mathbf{c}' = \mathbf{b}_{i+1}\cdot\mathbf{c}'$. (To obtain such $\mathbf{c}'$, let $V \subseteq \mathbb{R}^{N+1}$ be the smallest linear subspace defined over $\mathbb{Q}$ which contains $\mathbf{c}$, choose $\mathbf{c}'' \in V\cap\mathbb{Q}^{N+1}$ very close to $\mathbf{c}$, and clear the denominators of $\mathbf{c}''$). Now choose polynomials $f_1,\dots,f_s \in K[x_0,\dots,x_N]$ such that $I = (f_1,\dots,f_s)$, $\mathit{in}_{\mathbf{c}}(I) = (\mathit{in}_{\mathbf{c}}(f_1),\dots,\mathit{in}_{\mathbf{c}}(f_s))$. Taking M sufficiently large, it follows from (6.5) that $\mathit{fin}_{\mathbf{c}}(F_I) = \mathit{fin}_{\mathbf{c}'}(F_I)$ and from (6.2) that $\mathit{in}_{\mathbf{c}}(f_i) = \mathit{in}_{\mathbf{c}'}(f_i)$ for $i = 1,\dots,s$. The latter implies that $\mathit{in}_{\mathbf{c}}(I) \subseteq \mathit{in}_{\mathbf{c}'}(I)$. But by Lemma 6.2 these two ideals have the same Hilbert function, and so they must be equal. Therefore, it suffices to prove Lemma 6.3 for $\mathbf{c}'$ instead of $\mathbf{c}$.

So assume $\mathbf{c} \in \mathbb{Z}^{N+1}$. For $f \in K[x_0,\dots,x_N]$, $t \in K$ define
$f_t = t^{-w_{\mathbf{c}}(f)}f(t^{c_0}x_0,\dots,t^{c_N}x_N)$. Let I_t be the ideal in $K[x_0,\dots,x_N]$ generated by the polynomials f_t $(f \in I)$. Further, let $Z_t = \mathrm{Proj}(K[x_0,\dots,x_N]/I_t)$ be the corresponding closed subscheme of $\mathbb{P}^N$. Then $I_0 = \mathit{in}_{\mathbf{c}}(I)$ by (6.1). From e.g., [3], p. 343, Theorem 15.17 it follows that the schemes Z_t form a family which is flat over

$\mathbb{A}^1_K = \mathrm{Spec}(K[t])$. Further, for $t \in K$ define

$$F_{I,t} = t^{e_I(\mathbf{c})} F_I(t^{-c_0}h_{00}, \ldots, t^{-c_N}h_{0N}; \ldots ; t^{-c_0}h_{n0}, \ldots, t^{-c_N}h_{nN}).$$

Then $F_{I,0} = \mathit{fin}_{\mathbf{c}}(F_I)$ by (6.3). Let C_t be the subscheme of $\mathbb{P}^N \times \cdots \times \mathbb{P}^N$ ($n+1$ times) defined by $F_{I,t}$. Then, again [3], p. 343, Theorem 15.17 implies that the schemes C_t form a flat family over $\mathbb{A}^1_K$. For $t \in K$, let D_t be the subscheme of $\mathbb{P}^N \times \cdots \times \mathbb{P}^N$ ($n+1$ times) defined by the Chow form F_{I_t} of I_t. For instance by [17], sections 5.2, 5.4, the Chow forms of the closed subschemes of $\mathbb{P}^N$ from a family which is flat over some Noetherian scheme S form themselves a flat family over S. So in particular, the schemes D_t form a flat family over $\mathbb{A}^1_K$. From the definition of Chow form, i.e., §2.2 and (4.5), it follows that if $A \in \mathrm{GL}_{N+1}(K)$ and if I_A is the ideal generated by the polynomials $f(A\mathbf{x})$, ($f \in A$), then I_A has Chow form $F_{I_A} = F_I((A^{-1})^T\mathbf{h}_0, \ldots, (A^{-1})^T\mathbf{h}_n)$, where $(A^{-1})^T$ is the transpose of the inverse of A. In particular, for $t \neq 0$ we have (up to a constant), $F_{I,t} = F_{I_t}$, i.e., $C_t = D_t$. Using [12], p. 258, Prop. 9.8 and the flatness of the families C_t, D_t, it follows that then also $C_0 = D_0$, which means that $F_{I,0} = F_{I_0}$ apart from a constant factor. This proves Lemma 6.3. □

Lemma 6.4. *We have (i)* $\dim \mathit{in}_{\mathbf{c}}(I) = \dim I$, *(ii)* $\deg \mathit{in}_{\mathbf{c}}(I) = \deg I$, *(iii)* $s_{\mathit{in}_{\mathbf{c}}(I)}(m, \mathbf{c}) = s_I(m, \mathbf{c})$, *(iv)* $e_{\mathit{in}_{\mathbf{c}}(I)}(\mathbf{c}) = e_I(\mathbf{c})$.

Proof. (i) and (ii) follow at once from Lemma 6.2. To prove (iii), choose a basis $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_H}\}$ of $K[x_0, \ldots, x_N]_m/I_m$ such that $(\mathbf{a}_1 + \cdots + \mathbf{a}_m) \cdot \mathbf{c}$ is maximal. By Lemma 6.2, $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_H}\}$ is then also a basis of $K[x_0, \ldots, x_N]_m/\mathit{in}_{\mathbf{c}}(I)_m$. So by definition (4.4), $s_{\mathit{in}_{\mathbf{c}}(I)}(m, \mathbf{c}') \geqslant s_I(m, \mathbf{c})$. On the other hand, if $\{\mathbf{x}^{\mathbf{b}_1}, \ldots, \mathbf{x}^{\mathbf{b}_H}\}$ is a monomial basis of $K[x_0, \ldots, x_N]_m/\mathit{in}_{\mathbf{c}}(I)_m$, then it is also a monomial basis of $K[x_0, \ldots, x_N]_m/I_m$. For otherwise, there are γ_i ($i = 1, \ldots, H$), not all zero, such that $f := \sum_{i=1}^H \gamma_i \mathbf{x}^{\mathbf{b}_i} \in I$. But then, $\mathit{in}_{\mathbf{c}}(f) = \sum_{i\in B} \gamma_i \mathbf{x}^{\mathbf{b}_i} \in \mathit{in}_{\mathbf{c}}(I)$ for some non-empty set B with $\gamma_i \neq 0$ for $i \in B$, which is impossible. Therefore, again by (4.4), $s_{\mathit{in}_{\mathbf{c}}(I)}(m, \mathbf{c}') \leqslant s_I(m, \mathbf{c})$. This proves (iii). By (6.3), $e_I(\mathbf{c})$ is equal to the single exponent on t occurring in the expression obtained by substituting $t^{c_j}h_{ij}$ for h_{ij} in $\mathit{fin}_{\mathbf{c}}(F_I)$ for $i = 0, \ldots, n$, $j = 0, \ldots, N$. Together with Lemma 6.3 this implies (iv). □

We are now ready to prove the following result:

Lemma 6.5. *Let $m > d$ be an integer, and let $\mathbf{c} \in \mathbb{R}_{\geqslant 0}^{N+1}$. Then*

$$s_I(m,\mathbf{c}) \geqslant \tfrac{m-d}{n+1}\tbinom{m+n-d}{n}\cdot e_I(\mathbf{c}) - d^2 m\tbinom{m+n-1}{n-1}\cdot\Big(\max_{0\leqslant i\leqslant N} c_i\Big). \tag{6.6}$$

Proof. We first assume that $\mathbf{c} = (c_0,\dots,c_N)$ with $c_0,\dots,c_N$ linearly independent over $\mathbb{Q}$. Thus $\mathbf{b}_1\cdot\mathbf{c} \neq \mathbf{b}_2\cdot\mathbf{c}$ for any pair $\mathbf{b}_1 \neq \mathbf{b}_2 \in \mathbb{Z}^{N+1}$. So by (6.2), for each non-zero $f \in K[x_0,\dots,x_N]$, $in_{\mathbf{c}}(f)$ is a monomial, therefore, $in_{\mathbf{c}}(I)$ is a monomial ideal. In this case, Lemma 6.5 is an immediate consequence of Lemma 5.5 and Lemma 6.4. The lemma for arbitrary $\mathbf{c} \in \mathbb{R}^{N+1}$ now follows by approximating $\mathbf{c}$ by a tuple with $\mathbb{Q}$-linearly independent coordinates and using continuity arguments. □

Our last auxiliary result is an upper bound for the Hilbert function of a projective variety, due to Chardin [2], Théorème 1. In what follows, X is a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d defined over K.

Lemma 6.6. *$H_X(m) \leqslant d\binom{m+n}{n}$ for $m \geqslant 1$.*

6.7. *Proof of Theorem 4.6.*
Let $m > d$. Put $C := \max_{0\leqslant i\leqslant N} c_i$. By Lemma 6.5, Lemma 6.6 we have

$$\begin{aligned}\tfrac{1}{mH_X(m)}\cdot s_X(m,\mathbf{c}) &\geqslant \max\Big\{0,\ \tfrac{1}{mH_X(m)}\cdot\Big(\tfrac{m-d}{n+1}\tbinom{m+n-d}{n}\cdot e_X(\mathbf{c}) - d^2 m\tbinom{m+n-1}{n-1}\cdot C\Big)\Big\}\\ &\geqslant \tfrac{(m-d)\binom{m+n-d}{n}}{m\binom{m+n}{n}}\cdot\tfrac{1}{(n+1)d}\,e_X(\mathbf{c}) \;-\; d\cdot\tfrac{n}{m+n}\cdot C.\end{aligned}$$

Together with

$$\frac{(m-d)\binom{m+n-d}{n}}{m\binom{m+n}{n}} = \prod_{i=0}^{n}\frac{m+i-d}{m+i} \geqslant \Big(1-\frac{d}{m}\Big)^{n+1} \geqslant 1-\frac{(n+1)d}{m}$$

and $\frac{1}{(n+1)d}e_X(\mathbf{c}) \leqslant C$ (which follows from (6.4)) this implies

$$\begin{aligned}\tfrac{1}{mH_X(m)}\cdot s_X(m,\mathbf{c}) &\geqslant \tfrac{1}{(n+1)d}\cdot e_X(\mathbf{c}) - \Big(\tfrac{(n+1)d}{m}+\tfrac{nd}{m+n}\Big)\cdot C\\ &\geqslant \tfrac{1}{(n+1)d}\cdot e_X(\mathbf{c}) - \tfrac{(2n+1)d}{m}\cdot C.\end{aligned}$$

This completes the proof of Theorem 4.6. □

7. Proof of Theorem 3.2 (linear case)

7.1. We recall Theorem 2.1 of Evertse and Schlickewei [5] which is the main tool in the proof of our Theorem 3.2.

Let K be an algebraic number field. Let $N > n \geqslant 1$ be integers. Let $\mathcal{L} = \{l_0, \dots, l_N\}$ be a family (i.e., an unordered tuple with possibly repetitions) of linear forms in $K[x_0, \dots, x_n]$. Suppose that $\mathcal{L}$ has rank $n+1$. For every place $v \in M_K$, let I_v be a subset of $\{0, \dots, N\}$ of cardinality $n+1$ such that $\{l_i : i \in I_v\}$ is linearly independent. Let d_{iv} $(v \in M_K, i \in I_v)$ be reals such that for some finite subset T of M_K we have

$$d_{iv} = 0 \quad \text{for } v \in M_K \backslash T,\ i \in I_v. \tag{7.1}$$

For $Q \geqslant 1$ and for $\mathbf{y} \in K^{n+1}$ we define

$$H_Q(\mathbf{y}) = \prod_{v \in M_K} \max_{i \in I_v} \left(|l_i(\mathbf{y})|_v \cdot Q^{-d_{iv}} \right). \tag{7.2}$$

We will refer to H_Q as a twisted (exponential) height. By the product formula we have $H_Q(\lambda \mathbf{y}) = H_Q(\mathbf{y})$ for $\lambda \in K^*$, therefore, H_Q may be viewed as a twisted height on $\mathbb{P}^n(K)$.

We extend H_Q to $\mathbb{P}^n(\overline{\mathbb{Q}})$ as follows. Let $\mathbf{y} \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Pick a finite extension L of K such that $\mathbf{y} \in \mathbb{P}^n(L)$. For a place $w \in M_L$ put

$$I_w = I_v, \qquad d_{iw} = \tfrac{[L_w : K_v]}{[L:K]} d_{iv}, \tag{7.3}$$

where $v \in M_K$ is the place lying below w. Then we put

$$H_Q(\mathbf{y}) = \prod_{w \in M_L} \max_{i \in I_w} \left(|l_i(\mathbf{y})|_w \cdot Q^{-d_{iw}} \right). \tag{7.4}$$

By (2.1) this is well-defined, i.e., independent of the choice of L.

7.2. Define

$$\Delta := \prod_{v \in M_K} |\det(l_i : i \in I_v)|_v, \tag{7.5}$$

where for any subset I of $\{0,\dots,N\}$ of cardinality $n+1$, $\det(l_i : i\in I)$ denotes the coefficient determinant of the linear forms l_i $(i\in I)$. Further, let

$$\mathcal{H}_{\mathcal{L}} := \prod_{v\in M_K}\Big(\max_I |\det(l_i : i\in I)|_v\Big), \tag{7.6}$$

where the maxima are taken over all subsets I of $\{0,\dots,N\}$ of cardinality $n+1$. We may view $\mathcal{H}_{\mathcal{L}}$ as a height of the family $\mathcal{L}=\{l_0,\dots,l_N\}$. We assume that the reals d_{iv} satisfy, apart from (7.1),

$$\sum_{v\in M_K}\sum_{i\in I_v} d_{iv} = 0, \tag{7.7}$$

$$\sum_{v\in M_K}\max_{i\in I_v} d_{iv} \leqslant 1. \tag{7.8}$$

Then Theorem 2.1 of [5] can be stated as follows:

Proposition 7.3. *Let $0<\varepsilon<1$. Let H_Q be defined by (7.2)–(7.4). Then there are proper linear subspaces $T_1,\dots,T_t$ of $\mathbb{P}^n$, defined over K, with*

$$t \leqslant 4^{(n+9)^2}\varepsilon^{-n-5}\log(3N)\log\log(3N) \tag{7.9}$$

for which the following holds:
For every real Q with

$$Q \geqslant \max\Big(\mathcal{H}_{\mathcal{L}}^{1/\binom{N+1}{n+1}}, (n+1)^{2/\varepsilon}\Big) \tag{7.10}$$

there is a space $T_i\in\{T_1,\dots,T_t\}$ such that

$$\Big\{\mathbf{y}\in\mathbb{P}^n(\overline{\mathbb{Q}}) :\, H_Q(\mathbf{y})\leqslant \Delta^{1/(n+1)}\cdot Q^{-\varepsilon}\Big\}\subset T_i. \tag{7.11}$$

In addition, we need the following estimate for Δ:

Lemma 7.4. $\Delta \geqslant \mathcal{H}_{\mathcal{L}}^{1-\binom{N+1}{n+1}}$.

Proof. Let $I_1, \ldots, I_R$ be the subsets I of cardinality $n+1$ of $\{0, \ldots, N\}$ such that $\{l_i : i \in I\}$ is linearly independent and put $a_k := \det(l_i : i \in I_k)$ for $k = 1, \ldots, R$. Then $\Delta = \prod_{v \in M_K} |a_{i_v}|_v$, where $i_v \in \{1, \ldots, R\}$ for $v \in M_K$. With the product formula and $R \leqslant \binom{N+1}{n+1}$ this gives

$$\Delta = \prod_{v \in M_K} \frac{|\prod_{k=1}^R a_k|_v}{\prod_{k \neq i_v} |a_k|_v} \geqslant \prod_{v \in M_K} \Big(\max_{1 \leqslant k \leqslant R} |a_i|_v \Big)^{1-R} = \mathcal{H}_{\mathcal{L}}^{1-R} \geqslant \mathcal{H}_{\mathcal{L}}^{1-\binom{N+1}{n+1}}.$$

7.5. *Proof of Theorem 3.2.* Let $X \subset \mathbb{P}^N$ be the linear variety from Theorem 3.2, defined over a number field K. Choose a basis $\mathbf{a}_0 = (a_{00}, \ldots, a_{0N}), \ldots, \mathbf{a}_n = (a_{n0}, \ldots, a_{nN})$ of $X(\overline{\mathbb{Q}})$ (considered as a vector space) with $\mathbf{a}_0, \ldots, \mathbf{a}_n \in K^{N+1}$. Define the family of linear forms

(7.12) $\mathcal{L} = \{l_0, \ldots, l_N\}$ with $l_j = a_{0j}x_0 + \cdots + a_{Nj}x_N \quad (j = 0, \ldots, N)$.

For $v \in S$, let I_v be a subset of cardinality $n+1$ of $\{0, \ldots, N\}$ which is independent with respect to X such that $\sum_{i \in I_v} c_{iv}$ is maximal. For $v \in M_K \backslash S$, let I_v be any independent subset of cardinality $n+1$ of $\{0, \ldots, N\}$. Thus, for $v \in M_K$, $\{l_i : i \in I_v\}$ is a set of $n+1$ linearly independent linear forms. Notice that the quantity $\mathcal{H}_{\mathcal{L}}$ defined by (7.6) satisfies

$$\mathcal{H}_{\mathcal{L}} = \exp(h(X)), \tag{7.13}$$

where $h(X)$ is the logarithmic height of X. Further, by (7.13) and Lemma 7.4 we have for the quantity Δ defined by (7.5):

$$\Delta \geqslant \exp\Big(- \Big\{ \tbinom{N+1}{n+1} - 1 \Big\} h(X) \Big). \tag{7.14}$$

Put

$$\begin{cases} d_{iv} := E^{-1} \cdot \big(E_v - c_{iv} \big) & (v \in S,\ i \in I_v), \\ \\ d_{iv} := 0 & (v \in M_K \backslash S,\ i \in I_v), \end{cases} \tag{7.15}$$

where

$$E_v := \frac{1}{n+1} \sum_{i \in I_v} c_{iv} \quad (v \in S), \quad E := \sum_{v \in S} E_v. \tag{7.16}$$

It is clear that the numbers d_{iv} satisfy (7.1), (7.7). Further, using that the numbers c_{iv} are $\geqslant 0$, it follows easily that the numbers d_{iv} satisfy (7.8).

Let $\varphi : \mathbb{P}^n \to X$ be the bijective linear map given by $\mathbf{y} = (y_0 : \cdots : y_n) \mapsto \sum_{i=0}^n y_i \mathbf{a}_i$. Let $\mathbf{x} = (x_0 : \cdots : x_N) \in \mathcal{S}_X(\overline{\mathbb{Q}})$ be a point with (3.6). This means that $\mathbf{x} \in X(L)$ and $\mathbf{x}$ satisfies (3.2) for some finite extension L of K. Let $\mathbf{y} = \varphi^{-1}(\mathbf{x})$. Then $\mathbf{y} \in \mathbb{P}^n(L)$ and by (7.12),

$$x_i = l_i(\mathbf{y}) \quad \text{for } i = 0, \ldots, N. \tag{7.17}$$

Put

$$Q := \exp\big(E \cdot h(\mathbf{x})\big). \tag{7.18}$$

We estimate from above $H_Q(\mathbf{y})$, where H_Q is defined by (7.2)–(7.4).

Put $I_w = I_v$, $d_{iw} := d_{iv} \cdot \frac{[L_w:K_v]}{[L:K]}$ for $w \in M_L$, $i \in I_w$. Further, let S_L be the set of places of L lying above the places in S, and put $E_w := \frac{1}{n+1}\sum_{i \in I_w} c_{iw}$ for $w \in S_L$. Then by (3.3) and (7.15) we have

$$\begin{cases} d_{iw} := E^{-1} \cdot \big(E_w - c_{iw}\big) & (w \in S_L,\ i \in I_w), \\ \\ d_{iw} := 0 & (w \in M_L \backslash S_L,\ i \in I_w), \end{cases} \tag{7.19}$$

Further, by (3.3), (7.16), (3.4) and the choices of the sets I_v we have

$$\sum_{w \in S_L} E_w = E \geqslant 1 + \delta. \tag{7.20}$$

For $w \in S_L$ we have by (7.17), (7.18), (7.19), (3.2),

$$\begin{aligned} \max_{i \in I_w} \big(|l_i(\mathbf{y})|_w Q^{-d_{iw}}\big) &= \max_{i \in I_w} \big(|x_i|_w \exp((c_{iw} - E_w)h(\mathbf{x}))\big) \\ &\leqslant \|\mathbf{x}\|_w \exp(-E_w h(\mathbf{x})), \end{aligned}$$

while for $w \in M_L \backslash S_L$ we have by (7.17), (7.19),

$$\max_{i \in I_w} \big(|l_i(\mathbf{y})|_w Q^{-d_{iw}}\big) = \max_{i \in I_w} |x_i|_w \leqslant \|x\|_w.$$

By taking the product over $w \in M_L$, invoking (7.18), (7.20), we obtain

$$H_Q(\mathbf{y}) \leqslant \exp\big(-(E-1)h(\mathbf{x})\big) = Q^{-(E-1)/E} \leqslant Q^{-(1+\delta^{-1})^{-1}}.$$

From (7.18) and (7.20), our assumption (3.6) and (7.14) it follows

$$\begin{aligned} \log Q &\geqslant h(\mathbf{x}) \geqslant (1+\delta^{-1})(N+1)^{n+1}(1+h(X)) \\ &\geqslant 2(1+\delta^{-1})\binom{N+1}{n+1}h(X) \geqslant 2(1+\delta^{-1})\log\Delta^{-1/(n+1)}, \end{aligned}$$

and so

$$H_Q(\mathbf{y}) \leqslant \Delta^{1/(n+1)} Q^{-(2(1+\delta^{-1}))^{-1}}.$$

Thus we are in a position to apply Proposition 7.3 with $\varepsilon = (2(1+\delta^{-1}))^{-1}$. Our assumption (3.6), in combination with (7.18), (7.20), (7.13), implies that

$$\log Q \geqslant \log\max\big(\mathcal{H}_{\mathcal{L}}^{1/\binom{N+1}{n+1}}, (n+1)^{4(1+\delta^{-1})}\big),$$

i.e., that condition (7.10) of Proposition 7.3 is satisfied with our choice of ε. It follows that there are proper linear subspaces $T_1, \dots, T_t$ of $\mathbb{P}^n$ defined over K, with

$$\begin{aligned} t &\leqslant 4^{(n+9)^2}\big(2(1+\delta^{-1})\big)^{n+5}\log(3N)\log\log(3N) \\ &\leqslant 4^{(n+10)^2}(1+\delta^{-1})^{n+5}\log(3N)\log\log(3N), \end{aligned}$$

such that $\mathbf{y} \in T_1 \cup \dots \cup T_t$. Then $\mathbf{x} \in Y_1 \cup \dots \cup Y_t$, where $Y_i = \varphi(T_i)$ $(i = 1, \dots, t)$ are proper linear subspaces of X defined over K which do not depend on $\mathbf{x}$. This completes the proof of Theorem 3.2. $\square$

8. Heights

8.1. Let K be a number field. Denote by M_K^∞ the set of archimedean places and by M_K^0 the set of non-archimedean places of K. For each $v \in M_K^\infty$, there is an isomorphic embedding $\sigma_v : K \hookrightarrow \mathbb{C}$ such that $|x|_v = |\sigma_v(x)|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ for $x \in K$. For $\mathbf{x} = (x_0, \dots, x_N) \in K^{N+1}$, $v \in M_K^\infty$ we put

$$\|\mathbf{x}\|_{v,1} = \Big(\sum_{i=0}^{N}|\sigma_v(x_i)|\Big)^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}, \quad \|\mathbf{x}\|_{v,2} = \Big(\sum_{i=0}^{N}|\sigma_v(x_i)|^2\Big)^{[K_v:\mathbb{R}]/2[K:\mathbb{Q}]}.$$

We then define heights $h_1(\mathbf{x})$, $h_2(\mathbf{x})$ for $\mathbf{x} \in \overline{\mathbb{Q}}^{N+1}\backslash\{\mathbf{0}\}$ by choosing a number field K with $\mathbf{x} \in K^{N+1}$ and putting

$$\begin{aligned} h_1(\mathbf{x}) &= \sum_{v \in M_K^\infty} \log \|\mathbf{x}\|_{v,1} + \sum_{v \in M_K^0} \log \|\mathbf{x}\|_v, \\ h_2(\mathbf{x}) &= \sum_{v \in M_K^\infty} \log \|\mathbf{x}\|_{v,2} + \sum_{v \in M_K^0} \log \|\mathbf{x}\|_v; \end{aligned}$$

these quantities are independent of the choice of K. By the product formula, h_1, h_2 define heights on $\mathbb{P}^N(\overline{\mathbb{Q}})$. We have

$$\left\{ \begin{array}{l} h(\mathbf{x}) \leqslant h_2(\mathbf{x}) \leqslant h_1(\mathbf{x}), \\ h_1(\mathbf{x}) \leqslant h(\mathbf{x}) + \log(N+1), \quad h_2(\mathbf{x}) \leqslant h(\mathbf{x}) + \frac{1}{2}\log(N+1) \end{array} \right. \tag{8.1}$$

for $\mathbf{x} \in \overline{\mathbb{Q}}^{N+1}$ (or $\mathbf{x} \in \mathbb{P}^N(\overline{\mathbb{Q}})$) and

$$h_2(\mathbf{x}_0 \wedge \cdots \wedge \mathbf{x}_n) \leqslant \sum_{i=0}^n h_2(\mathbf{x}_i) \quad \text{(Hadamard's inequality)} \tag{8.2}$$

for $\mathbf{x}_0, \ldots, \mathbf{x}_n \in \overline{\mathbb{Q}}^{N+1}$. Given a polynomial P with coefficients in $\overline{\mathbb{Q}}$, we define $h_1(P)$, $h_2(P)$ to be the respective heights of the coefficient vector of P.

In what follows, X is a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d, defined over $\overline{\mathbb{Q}}$. Let $P_X \subset \overline{\mathbb{Q}}[x_0, \ldots, x_N]$ denote the prime ideal of X. Given any number field K such that X is defined over K, denote by $\mathcal{X}$ its Zariski closure over $\mathrm{Spec}(O_K)$, i.e. $\mathcal{X} = \mathrm{Proj}(R/P_X \cap R)$ where $R = O_K[x_0, \ldots, x_N]$. Let $\tilde{h}(\mathcal{X})$ be the logarithmic height of $\mathcal{X}$ as defined by Faltings [6], pp. 552, 553. We then define the absolute Faltings height of X by $h_{\mathrm{Falt}}(X) := \frac{1}{[K:\mathbb{Q}]}\tilde{h}(\mathcal{X})$. By [1], p. 948 this is independent of the choice of K.

Lemma 8.2. $h_{\mathrm{Falt}}(X) \leqslant h(X) + d(n+1)(1 + 2\log(N+1))$.

Proof. From [1], Theorem 4.3.8, pp. 989, 990, formulas (4.3.31), (4.3.32), it follows that

$$h_{\mathrm{Falt}}(X) \leqslant h_1(F_X) + d(n+1)\log(N+1). \tag{8.3}$$

Since the Chow form F_X is homogeneous of degree d in each of the $n+1$ blocks of $N+1$ variables, its number of coefficients is at most $\binom{N+d}{N}^{n+1} \leqslant (e(N+1))^{d(n+1)}$

with $e = 2.71\ldots$, where the latter inequality follows from

$$\binom{x+y}{x} \leqslant \frac{(x+y)^{x+y}}{x^x y^y} = (1+x/y)^y(1+y/x)^x \leqslant (e(1+y/x))^x \tag{8.4}$$

for positive integers x, y. So by (8.1) we have

$$h_1(F_X) \leqslant h(F_X) + \log\left((e(N+1))^{d(n+1)}\right) = h(X) + d(n+1)(1+\log(N+1)).$$

By combining this with (8.3) we obtain the lemma. □

Lemma 8.3. *For every $\varepsilon > 0$, the set*

$$X(\varepsilon) := \{\mathbf{x} \in X(\overline{\mathbb{Q}}) : h_2(\mathbf{x}) \leqslant d^{-1}h_{\mathrm{Falt}}(X) + \varepsilon\}$$

is Zariski dense in X.

Proof. This follows from Zhang [20], p. 208, Theorem 5.2. □

Let m be a positive integer and put $R := \binom{N+m}{N} - 1$. Choose homogeneous coordinates $(y_0 : \ldots : y_R)$ on $\mathbb{P}^R$. Let $\mathbf{x}^{\mathbf{a}_0}, \ldots, \mathbf{x}^{\mathbf{a}_R}$ be the monomials of degree m. Consider the Veronese embedding

$$\varphi_m : \mathbb{P}^N \hookrightarrow \mathbb{P}^R : \mathbf{x} \mapsto (\mathbf{x}^{\mathbf{a}_0} : \cdots : \mathbf{x}^{\mathbf{a}_R}). \tag{8.5}$$

Denote by X_m the smallest linear subvariety of $\mathbb{P}^R$ containing $\varphi_m(X)$. Then clearly, a linear form $\sum_{i=0}^R \gamma_i y_i$ vanishes identically on X_m if and only if the polynomial of degree m $\sum_{i=0}^R \gamma_i \mathbf{x}^{\mathbf{a}_i}$ vanishes identically on X. In other words, there is an isomorphism

$$\overline{\mathbb{Q}}[x_0, \ldots, x_N]_m/(P_X)_m \xrightarrow{\sim} X_m^\vee : \quad \mathbf{x}^{\mathbf{a}_i} \mapsto y_i \ \ (i = 0, \ldots, R), \tag{8.6}$$

where $(P_X)_m$ is the vector space of homogeneous polynomials of degree m in P_X and $X_m^\vee$ is the vector space of linear forms in $\overline{\mathbb{Q}}[y_0, \ldots, y_R]$ modulo the linear forms vanishing identically on X_m.

Lemma 8.4. *(i) If X is defined over a number field K then X_m is defined over K.*
(ii) $\dim X_m = H_X(m) - 1 \leqslant d\binom{m+n}{n} - 1$.
(iii) $h(X_m) \leqslant dm\binom{m+n}{n} \cdot \left(d^{-1}h(X) + (3n+4)\log(N+1)\right)$.

Proof. If X is defined over K then $(P_X)_m$ is generated by polynomials with coefficients in K, therefore, X_m is defined by linear forms with coefficients in K. This implies (i). By (8.6), we have $\dim X_m = \dim X_m^\vee - 1 = H_X(m) - 1$ and together with Lemma 6.6 this implies (ii).

In order to prove (iii), let $\varepsilon > 0$ and let X'_m be the smallest linear subspace of $\mathbb{P}^R$ containing $\varphi_m(X(\varepsilon))$. We claim that $X'_m = X_m$. For assume the contrary: then there is a non-zero linear form vanishing identically on X'_m but not on X_m. Hence there is a non-zero polynomial of degree m vanishing identically on $X(\varepsilon)$ but not on X, which contradicts Lemma 8.3.

Therefore, $X_m(\overline{\mathbb{Q}})$ (considered as a vector space) has a basis of the shape $\{\varphi_m(\mathbf{x}_i) : i = 1, \dots, H\}$, with $H = \dim X_m + 1 = H_X(m)$ and $\mathbf{x}_i \in X(\varepsilon)$ for $i = 1, \dots, H$. By (2.4), (8.1), (8.2) we have

$$h(X_m) \leqslant h_2(\varphi_m(\mathbf{x}_1) \wedge \cdots \wedge \varphi_m(\mathbf{x}_H)) \leqslant \sum_{i=1}^{H} h_2(\varphi_m(\mathbf{x}_i)).$$

Further, by (8.1), (8.4) we have for $i = 1, \dots, H$,

$$\begin{aligned} h_2(\varphi_m(\mathbf{x}_i)) &\leqslant \tfrac{1}{2}\log\tbinom{m+N}{N} + h(\varphi_m(\mathbf{x}_i)) \\ &\leqslant \tfrac{1}{2}m(1+\log(N+1)) + mh(\mathbf{x}_i) \leqslant m\big(\tfrac{1}{2}(1+\log(N+1)) + h_2(\mathbf{x}_i)\big) \\ &\leqslant m\Big(\tfrac{1}{2}(1+\log(N+1)) + d^{-1}h_{\mathrm{Falt}}(X) + \varepsilon\Big). \end{aligned}$$

Hence

$$h(X_m) \leqslant mH \cdot \Big(\tfrac{1}{2}(1+\log(N+1)) + d^{-1}h_{\mathrm{Falt}}(X) + \varepsilon\Big).$$

By inserting Lemma 6.6, Lemma 8.2 and using $N \geqslant 2$, we obtain

$$\begin{aligned} h(X_m) &\leqslant dm\binom{m+n}{n} \cdot \Big(\tfrac{1}{2}(1+\log(N+1)) + \\ &\qquad\qquad + d^{-1}h(X) + (n+1)(1+2\log(N+1)) + \varepsilon\Big) \\ &\leqslant dm\binom{m+n}{n} \cdot \Big(d^{-1}h(X) + (3n+4)\log(N+1) + \varepsilon\Big). \end{aligned}$$

Since we may choose ε arbitrarily small, this implies (iii). $\square$

9. Proof of Theorem 3.4 (the general case)

9.1. We keep the notation from Sections 2,3. In particular, X is a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d defined over a number field K, where $1 \leqslant n < N$. We assume that none of the coordinates x_j ($j = 0, \ldots, N$) vanishes identically on X which is no loss of generality. Indeed, suppose for instance that $x_{M+1}, \ldots, x_N$ vanish identically on X whereas $x_0, \ldots, x_M$ do not vanish identically on X. Let $X' = \pi(X)$ where π is the projection $(x_0 : \cdots : x_N) \mapsto (x_0 : \cdots : x_M)$. We construct from (3.2) a new system of inequalities with solutions in X' by removing all inequalities involving x_i ($i = M+1, \ldots, N$). For the Chow forms of X, X' we have that $F_X = F_{X'} \in \overline{\mathbb{Q}}[h_{00}, \ldots, h_{0M}; \ldots; h_{n0}, \ldots, h_{nM}]$ and this implies that for the Chow weights we have $e_X(\mathbf{c}_v) = e_{X'}(\mathbf{c}'_v)$ for $v \in S$, where $\mathbf{c}'_v = (c_{0v}, \ldots, c_{Mv})$. Therefore, the new system satisfies (3.10) with $\mathbf{c}'_v$ in place of $\mathbf{c}_v$ for $v \in S$. So it suffices to prove Theorem 3.4 for the new system in place of (3.2).

In the remainder of the proof we distinguish two cases.

9.2. First assume that

$$\sum_{v \in S} \max_{0 \leqslant j \leqslant N} c_{jv} \geqslant 2 \min\big((n+1)d, N+1\big). \tag{9.1}$$

For $v \in S$, choose $j_v \in \{0, \ldots, N\}$ such that $c_{j_v,v} = \max_{0 \leqslant j \leqslant N} c_{jv}$ and put

$$d_{j_v,v} = c_{j_v,v}, \quad d_{jv} = 0 \text{ for } j = 0, \ldots, N,\, j \neq j_v. \tag{9.2}$$

Let X_1 be the smallest linear subspace of $\mathbb{P}^N$ which contains X. Put $H := \dim X_1$. By Lemma 8.4, (i) with $m = 1$, X_1 is defined over K. For $v \in S$, let I_v be a subset of $\{0, \ldots, N\}$ of cardinality $H+1$ containing j_v which is independent with respect to X_1, i.e., no non-trivial linear combination of the variables x_j ($j \in I_v$) vanishes identically on X_1; such a set exists since x_{j_v} does not vanish identically on X, hence not on X_1. By Lemma 8.4, (ii) with $m = 1$, we have $H \leqslant \min((n+1)d - 1, N)$. Together with (9.2), (9.1) this implies

$$\frac{1}{H+1} \sum_{v \in S} \sum_{j \in I_v} d_{jv} \geqslant 2. \tag{9.3}$$

For any finite extension L of K we put $j_w = j_v$, $d_{jw} = \frac{[L_w:K_v]}{[L:K]} d_{jv}$ for $w \in S_L$, $j = 0, \ldots, N$, where $v \in S$ is the place lying below w. Then by (9.2), (3.3) we have $d_{j_w,w} = c_{j_w,w}$, $d_{jw} = 0$ if $j \neq j_w$.

Let $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$. Then for some finite extension L of K, $\mathbf{x} \in X(L)$, $\mathbf{x}$ satisfies (3.2) for some finite extension L of K and, by what we just observed,

$$\log\left(\frac{|x_j|_w}{\|\mathbf{x}\|_w}\right) \leqslant -d_{jw}h(\mathbf{x}) \quad \text{for } w \in S_L,\ j = 0, \dots, N. \tag{9.4}$$

We apply Theorem 3.2 with X_1, H, 1, $\{d_{jv}\}$ in place of X, n, δ, $\{c_{jv}\}$. Notice that condition (3.4) is satisfied in view of (9.3). It follows that the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x}) \geqslant 2(N+1)^{H+1}(1 + h(X_1)) \tag{9.5}$$

is contained in the union of at most

$$t_0 = 4^{(H+10)^2} 2^{H+5} \log(3N) \log\log(3N) \tag{9.6}$$

proper linear subspaces of X_1 which are all defined over K.
Note that by Lemma 8.4,(ii),(iii) with $m = 1$ the right-hand side of (9.5) is at most

$$\begin{aligned} 2(N+1)^{d(n+1)}\Big(1 + (n+1)h(X) + d(3n+4)\log(N+1)\Big) \\ \leqslant c_3(N, n, d, \delta)(1 + h(X)), \end{aligned}$$

hence (9.5) is implied by (3.14). The intersection of X with a proper linear subspace of X_1 defined over K is a proper Zariski closed subset of X, and by Bézout's theorem, it is the union of at most d proper K-subvarieties of X, each of degree $\leqslant d$. Hence the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ with (3.14) is contained in the union of at most $t = dt_0$ proper K-subvarieties of X of degree at most d. Inserting Lemma 8.4, (ii) with $m = 1$ into (9.6) we obtain

$$t \leqslant d \cdot 4^{((n+1)d+9)^2} 2^{(n+1)d+4} \log 3N \log\log 3N \leqslant c_1(N, n, d, \delta).$$

Further, $d \leqslant c_2(N, n, d, \delta)$. This shows (3.12) and (3.13). Thus under assumption (9.1), Theorem 3.4 follows.

9.3. Now assume that

$$\sum_{v \in S} \max_{0 \leqslant j \leqslant N} c_{jv} < 2 \min\big((n+1)d, N+1\big). \tag{9.7}$$

Choose

$$m = 1 + \big[(8n+4)(1+\delta^{-1})d \min\big((n+1)d, N+1\big)\big]\ . \tag{9.8}$$

Put $R := \binom{N+m}{N} - 1$. Let $\varphi_m : \mathbb{P}^N \hookrightarrow \mathbb{P}^R$ be the Veronese embedding defined by (8.5), and let X_m be the smallest linear subvariety of $\mathbb{P}^R$ containing $\varphi_m(X)$. Recall that by Lemma 8.4, X_m is defined over K and $\dim X_m = H_X(m) - 1$.

Let $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$. Then $\mathbf{x} \in X(L)$ and $\mathbf{x}$ satisfies (3.2) for some finite extension L of K. Put $y_i = \mathbf{x}^{\mathbf{a}_i}$ $(i = 0, \dots, R)$, $\mathbf{y} = (y_0 : \cdots : y_R) = \varphi_m(\mathbf{x})$. Further, put

$$d_{iv} := \frac{1}{m}\mathbf{a}_i \cdot \mathbf{c}_v \quad (v \in S,\ i = 0, \dots, R) \tag{9.9}$$

and $d_{iw} := \frac{[L_w:K_v]}{[L:K]} d_{iv}$ $(w \in S_L,\ i = 0, \dots, R)$ where $v \in S$ is the place below w. Write $\mathbf{a}_i = (a_{i0}, \dots, a_{iN})$ for $i = 0, \dots, R$. Then using $\|\mathbf{y}\|_w = \|\mathbf{x}\|_w^m$, $h(\mathbf{y}) = mh(\mathbf{x})$, (3.3), we obtain

$$\begin{aligned} \log\Big(\frac{|y_i|_w}{\|\mathbf{y}\|_w}\Big) &= \sum_{k=0}^{N} a_{ik} \log\Big(\frac{|x_k|_w}{\|\mathbf{x}\|_w}\Big) \leqslant -\big(\sum_{k=0}^{N} a_{ik} c_{kw}\big) h(\mathbf{x}) \\ &\leqslant -d_{iw} h(\mathbf{y}) \quad \text{for } w \in S_L,\ j = 0, \dots, R, \end{aligned} \tag{9.10}$$

We consider system (9.10) with solutions $\mathbf{y} \in X_m$. We show that the analogue of (3.4) for this system is satisfied.

Denote by $\mathcal{I}_{X_m}$ the collection of subsets of $\{0, \dots, R\}$ of cardinality $\dim X_m + 1 = H_X(m)$ which are independent with respect to X_m. Recall that a subset I of $\{0, \dots, R\}$ is independent with respect to X_m if no non-trivial linear combination of the variables y_i $(i \in I)$ vanishes identically on X_m. According to (8.6), this means precisely that $\{\mathbf{x}^{\mathbf{a}_i} : i \in I\}$ is linearly independent in $\overline{\mathbb{Q}}[x_0, \dots, x_N]_m/(P_X)_m$. Hence

$$I \in \mathcal{I}_{X_m} \iff \{\mathbf{x}^{\mathbf{a}_i} : i \in I\} \text{ is a basis of } \overline{\mathbb{Q}}[x_0, \dots, x_N]_m/(P_X)_m.$$

In combination with (9.9) this implies

$$\tfrac{1}{\dim X_m + 1} \cdot \max_{I \in \mathcal{I}_{X_m}} \sum_{i \in I} d_{iv} = \tfrac{1}{mH_X(m)} \cdot s_X(m, \mathbf{c}_v),$$

where $s_X(m, \mathbf{c}_v)$ is given by (4.4). Further, from Theorem 4.6, (3.10), (9.7), (9.8), we infer

$$\begin{aligned} \tfrac{1}{mH_X(m)} \cdot \sum_{v \in S} s_X(m, \mathbf{c}_v) &\geqslant \tfrac{1}{(n+1)d} \cdot \sum_{v \in S} e_X(\mathbf{c}_v) - \tfrac{(2n+1)d}{m} \cdot \sum_{v \in S} \max_{0 \leqslant j \leqslant N} c_{jv} \\ &\geqslant 1 + \delta - \tfrac{(2n+1)d \cdot 2\min((n+1)d, N+1))}{m} \\ &\geqslant 1 + \delta/2\,. \end{aligned}$$

Thus we arrive at

$$\frac{1}{\dim X_m+1}\sum_{v\in S}\left(\max_{I\in\mathcal{I}_{X_m}}\Big(\sum_{i\in I}d_{iv}\Big)\right)\geqslant 1+\delta/2, \tag{9.11}$$

which is the analogue of (3.4) for system (9.10) with $\delta/2$ replacing δ.

Thus the conditions of Theorem 3.2 are satisfied with X_m, $R=\binom{N+m}{N}-1$, $H_X(m)-1$, $\delta/2$, $\{d_{jv}\}$ in place of X, N, n, δ, $\{c_{jv}\}$. It follows that there are proper linear subspaces $Z_1,\dots,Z_{t_0}$ of X_m, all defined over K, with

$$t_0=4^{(H_X(m)+9)^2}(1+2\delta^{-1})^{H_X(m)+4}\log\left(3\tbinom{N+m}{N}\right)\log\log\left(3\tbinom{N+m}{N}\right)$$

such that for every finite extension L of K the set of solutions $\mathbf{y}\in X_m(L)$ of (9.10) with

$$h(\mathbf{y})\geqslant h_0=\binom{N+m}{N}^{H_X(m)}(1+2\delta^{-1})(1+h(X_m))$$

is contained in $Z_1\cup\cdots\cup Z_{t_0}$.

For $i=1,\dots,t_0$, the intersection $X\cap\varphi_m^{-1}(Z_i)$ is contained in $X\cap Z(f_i)$, where $Z(f_i)$ is the zero locus of a homogeneous polynomial $f_i\in K[x_0,\dots,x_N]$ of degree m not vanishing identically on X. By Bézout's Theorem, $X\cap Z(f_i)$ is equal to the union of at most dm K-subvarieties, each of degree $\leqslant dm$. Using that $h(\varphi_m(\mathbf{x}))=mh(\mathbf{x})$, it follows that the set of $\mathbf{x}\in\mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x})\geqslant m^{-1}h_0=m^{-1}\binom{N+m}{N}^{H_X(m)}(1+2\delta^{-1})(1+h(X_m)) \tag{9.12}$$

is contained in the union of at most

$$t=dmt_0=dm\cdot 4^{(H_X(m)+9)^2}(1+2\delta^{-1})^{H_X(m)+4}\log\left(3\tbinom{N+m}{N}\right)\log\log\left(3\tbinom{N+m}{N}\right) \tag{9.13}$$

proper K-subvarieties of X, each of degree $\leqslant dm$.

Using Lemma 6.6, (8.4), (9.8), $n\geqslant 1$, $N\geqslant 2$, we obtain

$$\begin{aligned}
H_X(m) &\leqslant d\tbinom{m+n}{n}\leqslant d\big(e(m+1)\big)^n\\
&\leqslant d\Big(e(8n+5)(n+1)d^2(1+\delta^{-1})\Big)^n\leqslant d\Big(71n^2d^2(1+\delta^{-1})\Big)^n,\\
\binom{N+m}{N} &\leqslant \big(e(N+1)\big)^m\leqslant\big(e(N+1)\big)^{26n^2d^2(1+\delta^{-1})}.
\end{aligned}$$

Together with Lemma 8.4, (iii), this implies that the right-hand side of (9.12) is at most

$$\begin{aligned}
&m^{-1}\big(e(N+1)\big)^{dm(e(m+1))^n}\big(1+2\delta^{-1}\big)\cdot \\
&\qquad\cdot\Big(1+m\tbinom{n+m}{n}\big(h(X)+d(3n+4)\log(N+1)\big)\Big) \\
&\leqslant \big(e(N+1)\big)^{dm(e(m+1))^n}\big(1+2\delta^{-1}\big)\cdot \\
&\qquad\cdot m\tbinom{n+m}{n}\cdot\big(1+d(3n+4)\log(N+1)\big)\cdot\big(1+h(X)\big) \\
&\leqslant \big(e(N+1)\big)^{d(e(m+1))^{n+1}}\cdot(1+h(X)) \\
&\leqslant \big(e(N+1)\big)^{d(71n^2d^2(1+\delta^{-1}))^{n+1}}\cdot(1+h(X)) \\
&\leqslant (3N)^{(10n)^{2n+2}d^{2n+3}(1+\delta^{-1})^{n+1}}\cdot(1+h(X)) = c_3(N,n,d,\delta)\cdot(1+h(X)),
\end{aligned}$$

hence (9.12) is implied by (3.14).

In order to estimate from above the upper bound t for the number of subvarieties from (9.13), we first observe that

$$\begin{aligned}
\log\big(3\tbinom{N+m}{N}\big)\log\log\big(3\tbinom{N+m}{N}\big) &\leqslant \log\big(3(e(N+1))^m\big)\log\log\big(3(e(N+1))^m\big) \\
&\leqslant \log\big((3N)^{2m}\big)\log\log\big((3N)^{2m}\big) \\
&\leqslant 2m^2\log(3N)\log\log(3N).
\end{aligned}$$

Therefore,

$$\begin{aligned}
t &\leqslant dm\cdot 4^{(d\binom{m+n}{n}+9)^2}\cdot(1+2\delta^{-1})^{d\binom{m+n}{n}+4}\cdot 2m^2\log(3N)\log\log(3N) \\
&\leqslant \big(4e^{1/71}\big)^{\left((71n^2)^n d^{2n+1}(1+\delta^{-1})^n+10\right)^2}\log(3N)\log\log(3N) \\
&\leqslant \exp\Big((10n)^{4n}d^{4n+2}(1+\delta^{-1})^{2n}\Big)\cdot\log(3N)\log\log(3N) = c_1(N,n,d,\delta).
\end{aligned}$$

Finally, by (9.8), we have $md \leqslant (8n+5)(1+\delta^{-1})d^2\min((n+1)d, N+1) = c_2(N,n,d,\delta)$. Hence (3.12), (3.13) hold true. This completes the proof of Theorem 3.4. $\square$

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Universiteit Leiden, Mathematisch Instituut, Postbus 9512, NL-2300 RA Leiden

Departement Mathematik, ETH Zentrum, CH-8092 Zürich

E-mail address: evertse@math.leidenuniv.nl, ferretti@math.ethz.ch