Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, Part II

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Abstract

We study the pointwise asymptotic behavior of the radially symmetric ground state solution of a quasilinear elliptic equation involving the m-Laplacian in $\mathbb{R}^n$ with two competing parameters. Roughly speaking, the first parameter ε measures the distance to a critical growth problem while the second parameter δ measures the weight of the "linear" term. We obtain the exact asymptotic behavior of the ground state, for any $1 < m < n$, both at the origin and outside the origin, when the equation tends to critical growth ($\varepsilon \to 0$). We also obtain the "equilibrium relation" between ε and δ so that when they both vanish according to this relation, ground states neither blow up nor vanish.

The results of this paper complete the description begun in [9].

Key words: m-Laplacian, ground states, shooting methods, asymptotic behavior.

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Résumé

Nous étudions le comportement asymptotique ponctuel de l'état fondamental à symétrie radiale d'une équation elliptique quasilinéaire contenant le m-Laplacien en $\mathbb{R}^n$ avec deux paramètres en compétition. En gros, le premier paramètre ε mesure la distance d'un problème à croissance critique tandis que le second paramètre δ mesure le poids du terme "linéaire". Nous obtenons le comportement asymptotique exact de l'état fondamental, pour tout $1 < m < n$, aussi bien à l'origine que ailleurs, lorsque l'équation tend à la croissance critique ($\varepsilon \to 0$). Nous obtenons également la "relation d'équilibre" entre ε et δ de façon que lorsqu'ils convergent vers 0 en respectant cette relation, l'état fondamental n'explose ni ne converge vers la solution nulle.

Les résultats de ce papier complètent la description commencée en [9].

1 Introduction

In this paper we study *ground states* of the quasilinear elliptic equation

$$-\Delta_m u = -\delta u^{m-1} + u^{p-1} \qquad \text{in } \mathbb{R}^n, \tag{P_p^δ}$$

in which $1 < m < n$, $m < p < m^*$, $\delta > 0$ and m^* is the critical Sobolev exponent

$$m^* = \frac{nm}{n-m},$$

and where the degenerate Laplace operator Δ_m is defined by

$$\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u).$$

Here, by a *ground state* we mean a *positive solution* of problem (P_p^δ) in $C^1(\mathbb{R}^n)$ – in the sense of distributions – which tends to zero as $|x| \to \infty$. In this paper we only deal with *radially symmetric solutions* of (P_p^δ). Thus, from now on we shall mean by a ground state a *radial* ground state.

We know from [10, 18] that (P_p^δ) admits a unique ground state for all values of $p \in (m, m^*)$ and $\delta > 0$. On the other hand, if either $p \geq m^*$ and $\delta > 0$ or $\delta = 0$ and $p \in (m, m^*)$, then equation (P_p^δ) admits no ground state (see [13, Theorem 5] and [12]). We observe as well that if *both* $\delta = 0$ and $p = m^*$, then $(P_{m^*}^0)$ possesses the one-parameter family of ground states

$$U_d(x) = d\left[1 + D\left(d^{m'/k}|x|\right)^{m'}\right]^{-k/m'} \qquad (d > 0), \tag{1.1}$$

in which

$$m' = \frac{m}{m-1}, \qquad k = \frac{n-m}{m-1}, \qquad D = D_{m,n} = 1/kn^{1/(m-1)}.$$

Here m' is the Hőlder conjugate of the exponent m, while k is the power decay rate of the fundamental solution of the equation $\Delta_m u = 0$.

These facts raise two questions. First, how does the ground state u of (P_p^δ) evolve and disappear as either $\delta \to 0$ or $p \to m^*$. Second, do the ground states u admit a limit whenever both $\delta \to 0$ and $p \to m^*$ at a suitable *equilibrium behavior.* To this purpose, in [9] it has been shown that

(A) if $\delta \to 0$, then $u \to 0$ uniformly in $\mathbb{R}^n$, see [9, Theorem 1],

(B) if $p \to m^*$, then u concentrates at the origin, i.e. $u(0) \to \infty$ and $u(x) \to 0$ at any point $x \neq 0$, see Figure 1. Moreover, $u(x) \to 0$ weakly in $W^{1,m}$. (The final statement is a direct consequence of the limit conditions (5), (6), (7), (8) in [9].)

(C) if $n > m^2$, if both $\delta \to 0$ and $p \to m^*$, and if δ and $m^* - p$ are *linearly related*, then u converges uniformly to a suitable function of the family (1.1). For positive solutions of the Dirichlet problem for (P_p^δ) in a bounded domain, similar but somewhat simpler phenomena have been found. For details we refer to [3, 5, 7, 8, 11, 16, 17] and references therein.

The purpose of the present paper is to make the statements (B) and (C) more precise, and in particular to supplement the results obtained in [9] so as to arrive at a complete asymptotic description of the ground state as $p \to m^*$ for all $m \in (1, n)$. Specifically, writing $p = m^* - \varepsilon$ $(\varepsilon > 0)$, and fixing $\delta > 0$, we shall prove the following behavior:

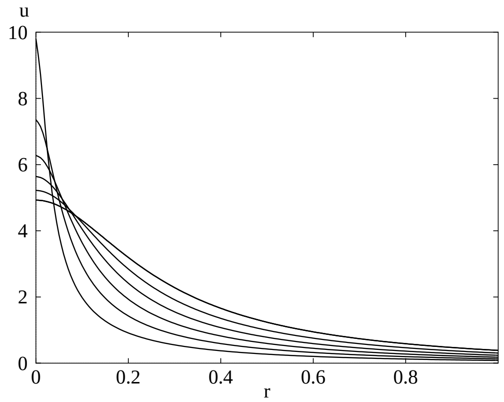
If $n > m^2$, then $u(0) \sim c_1 \varepsilon^{-k/mm'}$ as $\varepsilon \to 0$,

If $n = m^2$, then $u(0) \sim c_2 \varepsilon^{-1/m'}|\log \varepsilon|$ as $\varepsilon \to 0$,

If $n < m^2$, then $u(0) \sim c_3 \varepsilon^{-1/m'}$ as $\varepsilon \to 0$.

Here c_1, c_2, c_3 are constants which can be computed explicitly in terms of δ, m and n (see Theorem 1; note that the first result already appears in [9, Theorem 2]).

The difference between the cases $n > m^2$ and $n \leq m^2$ is ultimately due to the fact that the fundamental solution r^{-k}, which is never in $L^m(\mathbb{R}^n)$ whatever the dimension n of the space, nevertheless has its m-th power mass concentrated at infinity when $n < m^2$ though it is *integrable at the origin*; while when $n > m^2$ its m-th power mass is concentrated at the origin but is *integrable at infinity.* Therefore in turn one expects that the growth of $u(0)$ as



ε	$\sqrt{\varepsilon}\, u(0)$	$u(0)$
0	4.2003005	∞
0.2	4.4534773	9.958278
0.4	4.6797187	7.399285
0.6	4.8683761	6.285047
0.8	5.0476843	5.643483
1.0	5.2234980	5.223498
1.2	5.3988591	4.928462

Figure 1: Graphs of the function $u(r) = u_\varepsilon(r)$ for $\delta = 1$, $n = 3$, $m = 2$, $m^* = 6$ and for values of $\varepsilon = 6 - p = 0.2$, 0.4, 0.6, 0.8, 1.0 and 1.2. As ε decreases the values $u(0)$ of u at the origin increase monotonically to infinity as shown in the table; conversely, the values of u for large r decrease monotonically to zero.
The table also gives values of $\sqrt{\varepsilon}\, u(0)$, with the limiting case $\varepsilon \to 0$ included as the first entry (see Corollary 1 in Section 2 for $n = 3$). An asymptotic formula closely representing $u(0)$ is given by

$$u(0) = 4.20\varepsilon^{-1/2} + 1.25\varepsilon^{1/2} - 0.21\varepsilon^{3/2}.$$

$\varepsilon \to 0$ would be independent of n when $n < m^2$ and would have a higher rate when $n > m^2$. In consequence, a complete study of the asymptotic behavior of u, both at the origin and outside the origin, requires very fine estimates of the speed of transfer of mass.

In [9] the exact asymptotic behavior of $u(0)$ was determined in the case $n > m^2$ while for $n \le m^2$ only partial results were established. Moreover, the behavior of the ground state *outside* the origin (i.e. $u(x)$ for $x \ne 0$) was merely estimated in [9] by an inequality, see (3.11) below. It is our purpose in this paper to fill these gaps and to furnish a complete description of the asymptotic behavior of the ground state as $\varepsilon \to 0$ for all $1 < m < n$. Clearly, this description will also enable us to write down explicitly the above mentioned equilibrium behavior when $\varepsilon \to 0$ and $\delta \to 0$ simultaneously.

The limiting form of the ground state $u = u_\varepsilon$ at fixed $\delta > 0$ can conveniently be characterized by two scalings. First, u_ε peaks in an *inner region* with radius of the order $O(\alpha^{-(p-m)/m})$, where $\alpha = u(0) = u_\varepsilon(0)$. Writing

$$\tilde{x} = \alpha^{(p-m)/m} x \qquad \text{and} \qquad \tilde{u}(\tilde{x}) = \alpha^{-1} u(x),$$

we obtain in the limit as $\varepsilon \to 0$ $(\alpha \to \infty)$ that $\tilde{u} \to U_1$ uniformly in $\mathbb{R}^n$, where U_1 is defined in (1.1), namely it is the unique radial solution of the equation

$$-\Delta_m U_1 = U_1^{m^*-1}, \qquad U_1(0) = 1; \tag{1.2}$$

see Lemma 2 in Section 3.

For fixed $x \ne 0$ it was shown in [9] that $u(x)$ tends to zero at the rate $O(\alpha^{-1/(m-1)})$. Thus in the *outer region* $|x| > 0$ a natural scaling is simply

$$w(x) = \alpha^{1/(m-1)} u(x).$$

Passing to the limit as $\alpha \to \infty$ we shall show that $w(x) \to W(x)$, where W is the solution of the homogeneous equation

$$\Delta_m W = \delta W^{m-1}, \qquad x \ne 0 \tag{1.3}$$

which behaves at the origin like the fundamental solution of the equation $\Delta_m u = 0$, namely

$$W(x) \sim Q|x|^{-k}, \tag{1.4}$$

where Q is a positive constant which can be determined explicitly in terms of m and n only (see Lemma 1 below).

For a detailed description of the inner and outer asymptotics, and especially their *matching asymptotic behavior*, we refer to Section 8 and Figures 2. The underlying heuristic arguments for our results can also be considered in these terms, as we indicate in that section.

The final ingredient in the analysis of equation (P_p^δ) is the Ni-Pucci-Serrin [12, 14] generalization of the Pohozaev identity for ground states u,

$$m\delta \int_{R^n} u^m(x)\,dx = \varepsilon \frac{n-m}{m^*-\varepsilon} \int_{R^n} u^p(x)\,dx, \qquad \varepsilon = m^* - p. \tag{1.5}$$

This identity allows us to relate the central value $\alpha = u_\varepsilon(0)$ with the principal parameter ε. Here it is interesting to note that $n = m^2$ is a critical dimension for the asymptotic evaluation of the integral on the left in this identity (1.5). That is, as we shall see, if $n > m^2$ one requires for this evaluation the *inner asymptotic solution*, while when $n < m^2$ it is the *outer solution* which is relevant for the evaluation. The integral on the right in (1.5) is asymptotically determined by the inner solution for *any* $m \in (1, n)$.

The paper is organized as follows. In the next section we state our main asymptotic results, Theorems 1–4. In Section 3 we set the problem (P_p^δ) in radial coordinates and recall some results from [9]. These will be the starting points for the proofs of Theorem 1 (in Section 5) and Theorem 2 (in Section 4). We first prove Theorem 2 because its results are used in the proof of Theorem 1. The proofs of Theorems 3 and 4 are also consequences of Theorem 2 and are given, respectively, in Sections 6 and 7.

2 Main results

We first keep $\delta > 0$ fixed and let $p \to m^*$. Thanks to the scaling

$$u_1(x) = \delta^{-1/(p-m)} u\Big(\frac{x}{\delta^{1/m}}\Big), \tag{2.1}$$

which transforms equation (P_p^δ) into (P_p^1), we may take $\delta = 1$ for the rest of this section (except Theorem 3).

In order to state the main asymptotic results for the case $\delta = 1$, we first introduce some notation. We recall that the beta function $B(\cdot,\cdot)$ is defined by

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,dt, \qquad a, b > 0.$$

It arises here in the limits for the three cases $n > m^2$, $n = m^2$ and $n < m^2$. In the first case we put

$$\beta_{m,n} = \left(n \left(\frac{m}{n-m}\right)^2 \frac{B\left(\frac{n}{m'}, \frac{n-m^2}{m}\right)}{B\left(\frac{n}{m'}, \frac{n}{m}\right)} \right)^{k/mm'} \qquad (n > m^2), \tag{2.2}$$

and in the second case

$$\omega_m = \left(\frac{m'^3}{mB\,(m(m-1), m)} \right)^{1/m'} \qquad (n = m^2). \tag{2.3}$$

To formulate the limit when $n < m^2$, we use the singular radial solution W of the related homogeneous equation

$$\Delta_m W = W^{m-1} \qquad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \tag{2.4}$$

with asymptotic behavior near the origin and at infinity given by

$$W(x) = W(r) \sim A_{m,n} r^{-k} \quad \text{as } r \to 0, \qquad W(x) = W(r) = o(r^{-k}) \quad \text{as } r \to \infty, \tag{2.5}$$

where

$$A_{m,n} = n^{k/m} k^{k/m'}, \qquad k = \frac{n-m}{m-1}. \tag{2.6}$$

Note that $A_{m,n} = D^{-k/m'}$, where $D = D_{m,n}$ is given in equation (1.1). In the special cases when $m = 2$ and when $n = m^2$ we have

$$A_{2,n} = [n(n-2)]^{(n-2)/2}, \qquad A_{m,m^2} = m^{m+1}.$$

In Section 4 we shall prove the following result about problem (2.4) – (2.5):

Lemma 1 *Problem* (2.4) – (2.5) *has a unique positive radial solution W. Moreover, the function*

$$V(r) = r^k W(r)$$

is decreasing, decays exponentially to zero as $r \to \infty$, and $V(r) \to A_{m,n}$ as $r \to 0$.

Note that for $n < m^2$ we have $-mk + n > 0$, so by Lemma 1 the integral

$$I_{m,n} = \int_0^\infty r^{n-1} W^m(r)\, dr$$

is finite (e.g., $I_{2,3} = 3/2$), and we can define the constant

$$\gamma_{m,n} = \left(\frac{nm'^3}{k^2} \frac{D^{n/m'}}{B\left(\frac{n}{m'}, \frac{n}{m}\right)} I_{m,n} \right)^{1/m'} \qquad (n < m^2). \tag{2.7}$$

We may now formulate our first main result, the **asymptotic behavior of u at the origin** as $p \to m^*$:

Theorem 1 *For all $p \in (m, m^*)$, let $u = u_\varepsilon$ be the unique ground state for equation (P_p^1) (i.e. $\delta = 1$). Then, writing $\varepsilon = m^* - p$, we have*

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \varepsilon^{k/mm'} u(0) = \beta_{m,n} && \textit{if} \quad n > m^2, \\
&\lim_{\varepsilon \to 0} \left(\frac{\varepsilon}{|\log \varepsilon|} \right)^{1/m'} u(0) = \omega_m && \textit{if} \quad n = m^2, \\
&\lim_{\varepsilon \to 0} \varepsilon^{1/m'} u(0) = \gamma_{m,n} && \textit{if} \quad n < m^2.
\end{aligned} \tag{2.8}$$

where $\beta_{m,n}$, ω_m and $\gamma_{m,n}$ have been defined in (2.2), (2.3) *and* (2.7) *respectively.*

Note that the constants $\beta_{m,n}$ defined in (2.2) and $\gamma_{m,n}$ defined in (2.7) have the properties

$$\beta_{m,n} \to \infty \quad \text{as} \quad m \uparrow \sqrt{n}, \qquad\qquad \gamma_{m,n} \to \infty \quad \text{as} \quad m \downarrow \sqrt{n}.$$

This fact gives a further explanation for the "logarithmic" behavior in (2.8) in the case $n = m^2$.

In the important case $m = 2$ we may restate Theorem 1 explicitly as

Corollary 1 *Let $n > 2$ and $0 < \varepsilon < \frac{4}{n-2}$, and let $u = u_\varepsilon$ be the unique ground state of the equation*

$$-\Delta u = -u + u^{\frac{n+2}{n-2}-\varepsilon} \qquad \text{in} \quad \mathbb{R}^n.$$

Then

$$\lim_{\varepsilon\to 0} \varepsilon^{(n-2)/4} u(0) = \left[\frac{16n(n-1)}{(n-4)(n-2)^2}\right]^{(n-2)/4} \qquad \text{if} \quad n > 4,$$

$$\lim_{\varepsilon\to 0} \sqrt{\frac{\varepsilon}{|\log \varepsilon|}}\, u(0) = 2\sqrt{6} \qquad \text{if} \quad n = 4,$$

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon}\, u(0) = 4\,\sqrt[4]{12/\pi^2} \qquad \text{if} \quad n = 3.$$

The proof of these limits follows immediately from Theorem 1 and from standard properties of the beta function B, see [1, Chapter 6].

With the help of the function W we may also describe the **exact asymptotic behavior of u away from the origin**:

Theorem 2 *For all $p \in (m, m^*)$, let $u = u_\varepsilon$ be the unique radial ground state for the equation (P_p^1) $(\delta = 1)$. Then, writing $\varepsilon = m^* - p$, we have for all $x \neq 0$,*

$$\lim_{\varepsilon\to 0} [u(0)]^{1/(m-1)} u(x) = W(x), \tag{2.9}$$

and

$$\lim_{\varepsilon\to 0} [u(0)]^{1/(m-1)} \nabla u(x) = \nabla W(x), \tag{2.10}$$

where W is the solution of problem (2.4) – (2.5). The convergences are uniform outside any neighborhood of the origin.

When $m = 2$, equation (2.4) is linear and may be solved in terms of Bessel functions. We thus obtain

$$W(r) = \frac{2}{\Gamma(\sigma)} \left(\frac{n(n-2)}{2}\right)^{\sigma} r^{-\sigma} K_\sigma(r) \qquad \text{for} \quad r > 0, \qquad \sigma = \frac{n-2}{2}, \tag{2.11}$$

where we have used the fact that $K_\sigma(r) \sim \frac{1}{2}\Gamma(\sigma)(\frac{1}{2}r)^{-\sigma}$ as $r \to 0^+$ (see [1], p. 375). If in addition n is an odd integer, then (2.9) becomes:

$$n = 3, \qquad \lim_{\varepsilon\to 0} u(0)u(x) = 3^{1/2}\, \frac{e^{-|x|}}{|x|}$$

$$n = 5, \qquad \lim_{\varepsilon\to 0} u(0)u(x) = 15^{3/2}\,(1 + |x|)\frac{e^{-|x|}}{|x|^3}$$

$$n = 7, \qquad \lim_{\varepsilon\to 0} u(0)u(x) = 35^{5/2}\left(1 + |x| + \frac{|x|^2}{3}\right)\frac{e^{-|x|}}{|x|^5}$$

$$n = 9, \qquad \lim_{\varepsilon\to 0} u(0)u(x) = 63^{7/2}\left(1 + |x| + \frac{2|x|^2}{5} + \frac{|x|^3}{15}\right)\frac{e^{-|x|}}{|x|^7}.$$

These formulas are perhaps most easily obtained by the explicit solutions of the linear ordinary differential equation satisfied by V, namely

$$V'' - (n-3)\frac{V'}{r} - V = 0, \qquad V(0) = A_{2,n} = [n(n-2)]^{(n-2)/2}$$

(see equation (4.23) below with $m = 2$).

Improving [9, Theorem 4], we can now describe more explicitly the **equilibrium behavior** of u when both ε and δ tend to 0 simultaneously, in such a way that $u(0)$ tends *neither to infinity nor to zero.* The result is as follows.

Theorem 3 *Assume $m < p < m^*$, and let $u(x;\delta)$ be the unique radial ground state of problem (P_p^δ) with $\delta = \delta(\varepsilon)$, and*

$$\begin{aligned}
\delta(\varepsilon) &= \left(\frac{d}{\beta_{m,n}}\right)^{mm'/k}\varepsilon && \text{if } n > m^2\\
\delta(\varepsilon) &= \left(\frac{d}{\omega_m}\right)^{m'}\frac{\varepsilon}{|\log\varepsilon|} && \text{if } n = m^2 \qquad (2.12)\\
\delta(\varepsilon) &= \left[\left(\frac{d}{\gamma_{m,n}}\right)^{m'}\varepsilon\right]^{m/k} && \text{if } n < m^2,
\end{aligned}$$

where $\varepsilon = m^* - p$ and d is any given positive constant. Then

$$u(0;\delta(\varepsilon)) \to d, \quad u(\cdot\,;\delta(\varepsilon)) \to U_d \qquad \text{as} \qquad \varepsilon \to 0$$

uniformly on $\mathbb{R}^n$, where U_d is defined in (1.1).

We conclude with a **sharp estimation of the spike** of u at the origin, giving the rate at which it becomes thinner as $\alpha = u(0) \to \infty$.

Theorem 4 *Let M be any given positive constant, and let $\{r_\varepsilon\}$ be the family of radii defined by*

$$r_\varepsilon^k = \frac{A_{m,n}}{M\alpha^{1/(m-1)}},$$

where $\alpha = u(0)$ and $u = u_\varepsilon$ is the unique radial ground state of problem (P_p^1). Then

$$u(r_\varepsilon) \to M \qquad \text{as} \quad \varepsilon \to 0.$$

The Laplacian case $m = 2$ gives particularly simple formulas, that is

$$r_\varepsilon = \frac{n-2}{2}\sqrt[4]{\frac{n(n-4)}{n-1}}\,\frac{\sqrt[4]{\varepsilon}}{M^{1/(n-2)}} \qquad (n > 4),$$

$$r_\varepsilon = \frac{2}{\sqrt{M}}\sqrt[4]{\frac{\varepsilon}{6|\log\varepsilon|}} \qquad (n = 4),$$

$$r_\varepsilon = \frac{\sqrt[4]{12\pi^2}}{8}\,\frac{\sqrt{\varepsilon}}{M} \qquad (n = 3).$$

3 Preliminaries

The existence and uniqueness of radial ground states for equation (P_p^δ) is well known [2, 4, 10, 18]. We state this formally as

Proposition 1 *For all $n > m > 1$, $m < p < m^*$ and $\delta > 0$ the equation (P_p^δ) admits a unique radial ground state $u = u(r)$, $r = |x|$. Moreover $u'(r) < 0$ for $r > 0$.*

A radial ground state $u = u(r)$, $r = |x|$, of (P_p^δ) is in fact a C^1 solution of the ordinary differential equation

$$\begin{cases} (|u'|^{m-2}u')' + \dfrac{n-1}{r}|u'|^{m-2}u' = \delta u^{m-1} - u^{p-1}, & r > 0, \\ u(0) = \alpha > 0, \quad u'(0) = 0 \end{cases} \tag{3.1}$$

for some initial value $\alpha > 0$. For our purposes the dimension n may in fact be considered as any real number greater than m. From now on we refer indifferently to (3.1) or to (P_p^δ).

The ground state u of (3.1) has several important properties. First, we recall the Ni-Pucci-Serrin [12, 14] generalization of the Pohozaev identity given in (1.5):

$$m\delta\int_0^\infty r^{n-1}u^m(r)\,dr = \varepsilon\frac{n-m}{m^*-\varepsilon}\int_0^\infty r^{n-1}u^p(r)\,dr. \tag{3.2}$$

Next, from [9] we recall the exponential decay of the ground states of (P_p^δ):

Proposition 2 *Suppose that there exist positive constants λ, Λ, and ρ such that the function $f \in C^1[0,\infty)$ satisfies the inequalities*

$$-\lambda s^{m-1} \le f(s) \le -\Lambda s^{m-1} \quad \textit{for } 0 < s < \rho.$$

Let u be a (radial) ground state of

$$-\Delta_m u = f(u)$$

and let R be such that $u(r) \le \rho$ for all $r \ge R$. Let $\nu = [\Lambda/(m-1)]^{1/m}$. Then

$$u(r) \le \rho e^{\nu R}e^{-\nu r} \qquad \textit{for} \quad r \ge R.$$

Moreover, there exists a constant $\mu > 0$ (depending on m, n, λ and Λ) such that, for r suitably large,

$$|u'(r)| \le \mu e^{-\nu r}, \qquad |u''(r)| \le \mu e^{-\nu r}.$$

Proof. See [9, Theorem 8]. •

Using a Lyapunov function introduced in [11], the following result was proved in [9]:

Lemma 2 *For all* $r \geq 0$,

$$0 < z(\alpha^{(p-m)/m} r) - \frac{1}{\alpha} u(r) < c\varepsilon |\log \varepsilon| \tag{3.3}$$

where

$$z(s) = \frac{1}{[1 + (1-\eta)^{1/(m-1)} D s^{m'}]^{k/m'}}. \tag{3.4}$$

Here $c > 0$ *is a constant,* $D = D_{m,n}$ *is given in (1.1), and* $\eta = \alpha^{m-p} \to 0$ *as* $\varepsilon \to 0$.

Proof. These bounds follow directly from (30), (33), (49), and (59) in [9]. •

Remark. The function $z(s)$ in Lemma 2 is the solution of the problem

$$(s^{n-1}|z'|^{m-1}z')' + (1-\eta)s^{n-1}z^{m^*-1} = 0, \qquad z(0) = 1, \quad z'(0) = 0. \tag{3.5}$$

Note that

$$z(s) < \frac{A_{m,n}}{(1-\alpha^{m-p})^{k/m}}\, s^{-k} \qquad \text{for} \quad s > 0, \tag{3.6}$$

and

$$z(s) \sim \frac{A_{m,n}}{(1-\alpha^{m-p})^{k/m}}\, s^{-k} \qquad \text{as} \quad s \to \infty. \tag{3.7}$$

As already mentioned, part of the asymptotic results for (P_p^1) given in Section 2 were previously obtained by Gazzola and Serrin [9]. More precisely, there it was proved that

$$\lim_{\varepsilon \to 0} \varepsilon^{k/mm'} u(0) = \beta_{m,n} \quad \text{if } n > m^2. \tag{3.8}$$

Moreover, it was shown that if $n = m^2$, then

$$\left(\frac{\varepsilon}{|\log \varepsilon|}\right)^{1/m'} u(0) \approx 1, \tag{3.9}$$

while if $m < n < m^2$, then for appropriate positive constants

$$\text{Const.}\, |\log \varepsilon|^{(n-m^2)/m^2} \leq \varepsilon^{1/m'} u(0) \leq \text{Const.}\, |\log \varepsilon|^{(n-m)/m^2}. \tag{3.10}$$

Finally, for all $x \neq 0$

$$\lim_{\varepsilon \to 0} [u(0)]^{1/(m-1)} u(x) \leq A_{m,n}\, |x|^{-k} \tag{3.11}$$

and

$$\lim_{\varepsilon \to 0} [u(0)]^{1/(m-1)} |\nabla u(x)| \leq k\, A_{m,n} |x|^{-k-1}. \tag{3.12}$$

In (3.11) and (3.12) the convergence is uniform in closed sets which do not contain the origin.

For later use, we state the following elementary calculus lemma for the function

$$g(y) = (1-y)^{\vartheta} - 1, \qquad y < 1, \tag{3.13}$$

in which ϑ is a given positive constant.

Lemma 3 *We have*

(i) *If* $y < 0$, *then* $0 \leq g(y) \leq \max\{1, 2^{\vartheta-1}\}(1 + |y|^{\vartheta})$;

(ii) *If* $-1 < y < 0$, *then* $0 \leq g(y) \leq \max\{\vartheta, 2^{\vartheta} - 1\}|y|$;

(iii) *If* $0 < y < 1$, *then* $0 \leq -g(y) \leq \max\{1, \vartheta\} y$.

4 Proof of Lemma 1 and Theorem 2

In light of the scaling law (2.1) we may put $\delta = 1$.

In order to improve (3.11) and (3.12) to the form (2.9) and (2.10), we make the substitutions

$$v(r) = \alpha^{1/(m-1)}\, r^k\, u(r), \qquad k = \frac{n-m}{m-1}, \qquad \alpha = u(0). \tag{4.1}$$

It is also convenient to introduce the simpler rescaling

$$w(r) = \alpha^{1/(m-1)} u(r) = r^{-k} v(r). \tag{4.2}$$

The function w now obeys the equation

$$\Delta_m w = w^{m-1}(1 - u^{p-m}),$$

or in radial terms with primes denoting differentiation with respect to r,

$$\left(r^{n-1}|w'|^{m-2}w'\right)' = r^{n-1}w^{m-1}(1 - u^{p-m}). \tag{4.3}$$

Note in particular that, like u, the functions v and w depend on ε, though in general this will not be explicitly emphasized in the notation.

The first main step in the proof is to show that the family of functions $v = v_\varepsilon$ defined by (4.1) converges as $\varepsilon \to 0$. To this end, we prove a crucial uniform upper bound.

Lemma 4 *When ε is small enough, we have*

$$0 < v(r) \le 2A_{m,n} \qquad \text{for} \quad r > 0, \tag{4.4}$$

and also,

$$v(r) \le 2A_{m,n}e^{\nu}\, r^k\, e^{-\nu r} \quad \text{for} \quad r > 1, \qquad \nu = \{2(m-1)\}^{-1/m}. \tag{4.5}$$

Proof. Using Lemma 2, (3.6) and (3.7) and recalling the substitution (4.1) above, we obtain

$$v(r) < \alpha^{m'} r^k z(\alpha^{(p-m)/m} r) < \frac{A_{m,n}}{(1-\alpha^{m-p})^{k/m}}\, \alpha^{\varepsilon(k/m)} \qquad \text{for} \quad r > 0. \tag{4.6}$$

Since $\alpha \to \infty$ and $\alpha^\varepsilon \to 1$ as $\varepsilon \to 0$, the first assertion follows.

To obtain the second bound, we apply Proposition 2 and the fact that $w(1) = v(1) \le 2A_{m,n}$, to get

$$w(r) \le 2A_{m,n}e^{\nu}\, e^{-\nu r} \quad \text{for} \quad r > 1, \qquad \nu = \{2(m-1)\}^{-1/m}.$$

Since $v = r^k w$ the assertion is proved. •

We can now prove the convergence of the family $v = v_\varepsilon$ as $\varepsilon \to 0$.

Lemma 5 *Let v be the function defined by (4.1). Then, there exists a function $V \in C^{0,1}(0,\infty)$ such that (along an appropriate subsequence of values ε)*

$$\lim_{\varepsilon\to 0} v(r) = V(r) \qquad \text{pointwise on } (0,\infty) \tag{4.7}$$

and also uniformly outside any neighborhood of the origin. Moreover,

$$0 \le V(r) \le 2A_{m,n} \qquad \text{for} \quad r > 0,$$

and $V(r)$ decays exponentially to 0 as $r \to \infty$.

Proof. Observe that
$$v' = \alpha^{1/(m-1)} \left[r^k u' + k r^{k-1} u \right].$$
Hence by (3.11) and (3.12) we obtain, for r in any compact interval I of $(0,\infty)$ and for all sufficiently small ε,
$$|v'(r)| \le 4k \frac{A_{m,n}^{1/(m-1)}}{r}.$$
In turn, by the Ascoli-Arzelà Theorem, there exists a function $V \in C(I)$ such that $v \to V$ along an appropriate subsequence of values ε going to 0, the convergence being uniform on I. A standard diagonal process then proves (4.7), uniformly on any compact subset of $(0,\infty)$. The first required result then follows using the uniform exponential decay (4.5) of $v(r)$ as $r \to \infty$.

The final part of the lemma is now an immediate consequence of Lemma 4. •

The rest of the proof of Theorem 2 relies on various further estimates for v_ε as $\varepsilon \to 0$, the principal goal being (4.19) and (4.20) below. We first introduce the family of radii $\{\rho_\varepsilon\}$ according to the condition
$$\alpha^{m'} \rho_\varepsilon^k = \varepsilon^{-\theta}, \qquad 0 < \theta < 1. \tag{4.8}$$
The following lemma then holds.

Lemma 6 *One has $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover*
$$\lim_{\varepsilon \to 0} v(\rho_\varepsilon) = A_{m,n}.$$

Proof. As $\varepsilon \to 0$, we see from (3.8) that
$$\alpha \approx \varepsilon^{-k/mm'} \qquad \text{if } n > m^2,$$
and from (3.9), (3.10) that
$$\alpha \approx \varepsilon^{-1/m'} \qquad \text{if } n \le m^2,$$
where in the second case the approximation is up to logarithmic terms which do not affect the following argument. That $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ now follows at once.

To prove the main assertion, we observe to begin with that, by (4.6),
$$\limsup_{\varepsilon \to 0} v(\rho_\varepsilon) \le A_{m,n}.$$
Thus it remains to show
$$\liminf_{\varepsilon \to 0} v(\rho_\varepsilon) \ge A_{m,n}. \tag{4.9}$$
By Lemma 2, we have
$$v(r) > -C\alpha^{m'} r^k \varepsilon |\log \varepsilon| + \alpha^{m'} r^k z(\alpha^{(p-m)/m} r) \qquad \text{for} \quad r > 0. \tag{4.10}$$
An easy computation shows that $\alpha^{(p-m)/m} \rho_\varepsilon \to \infty$ as $\varepsilon \to 0$, so that by (3.7),
$$\alpha^{m'} \rho_\varepsilon^k z(\alpha^{(p-m)/m} \rho_\varepsilon) \to A_{m,n} \qquad \text{as} \quad \varepsilon \to 0. \tag{4.11}$$
By definition, however, we have
$$\alpha^{m'} \rho_\varepsilon^k \, \varepsilon |\log \varepsilon| = \varepsilon^{1-\theta} |\log \varepsilon|. \tag{4.12}$$

Thus by (4.10) – (4.12) and the fact that $0 < \theta < 1$, we conclude that

$$v(\rho_\varepsilon) > A_{m,n} - o(1) \quad \text{as} \quad \varepsilon \to 0,$$

and (4.9) follows. This completes the proof. •

We continue with two useful integral identities.

Lemma 7 *The functions v and w defined in (4.1) and (4.2) satisfy the following identities:*

$$r^{n-1}|w'(r)|^{m-1} = a - \int_{\rho_\varepsilon}^{r} s^{n-1} w^{m-1}(1 - u^{p-m})ds, \qquad r > 0,$$

and

$$v(r) = \frac{a^{1/(m-1)}}{k} - br^k + a^{1/(m-1)} r^k \int_r^1 g\Big(\frac{y}{a}\Big)\frac{dt}{t^{k+1}}, \qquad r > 0, \tag{4.13}$$

where

$$g(z) = (1-z)^{1/(m-1)} - 1 \qquad\qquad (z < 1),$$

$$y = y_\varepsilon(t) = \int_{\rho_\varepsilon}^{t} (s\,v)^{m-1}(1 - u^{p-m})ds \qquad (y < a).$$

Here $a = a_\varepsilon > 0$ and $b = b_\varepsilon$ are constants which depend on ε.

Proof. Integration of the (radial) equation (4.3) for $w(r)$ yields the first identity, with $a = a_\varepsilon = \rho_\varepsilon^{n-1}|w'(\rho_\varepsilon)|^{m-1}$. Note that $a > 0$ and $y < a$, since $u'(r)$, and hence also $w'(r)$, is negative for all $r > 0$.

Rewriting the first identity as

$$w'(r) = -r^{-k-1}\left(a - \int_{\rho_\varepsilon}^{r} s^{n-1}w^{m-1}(1 - u^{p-m})ds\right)^{1/(m-1)},$$

integrating over $(r, 1)$ and using the relation $w = r^{-k}v$, we obtain

$$v(r) = r^k v(1) + r^k \int_r^1 \left(a - \int_{\rho_\varepsilon}^{t} s^{m-1}v^{m-1}(1 - u^{p-m})ds\right)^{1/(m-1)} \frac{dt}{t^{k+1}}.$$

Then from the fact that

$$r^k \int_r^1 \frac{dt}{t^{k+1}} = \frac{1 - r^k}{k},$$

we arrive at the second identity, with $b = a^{1/(m-1)}/k - v(1)$. •

The second identity is in fact arranged so that the function

$$R_\varepsilon(\rho_\varepsilon) = a^{1/(m-1)}\rho_\varepsilon^k \int_{\rho_\varepsilon}^1 g\Big(\frac{y}{a}\Big)\frac{dt}{t^{k+1}} \tag{4.14}$$

tends to zero as $\varepsilon \to 0$. Before proving this delicate fact, we first show that the constants $a = a_\varepsilon$ and $b = b_\varepsilon$ are uniformly bounded as $\varepsilon \to 0$.

Lemma 8 *There are postive constants $M_\pm$ and M^* such that, for ε sufficiently small,*

$$M_- < a_\varepsilon < M_+ \quad \text{and} \quad |b_\varepsilon| < M^*.$$

Proof. Letting $r \to \infty$ in the first identity of Lemma 7, and recalling that both v and w decay exponentially (see Lemma 4), gives

$$a = \int_{\rho_\varepsilon}^{\infty} s^{n-1} w^{m-1}(1-u^{p-m})ds \le \int_{\rho_\varepsilon}^{1} s^{m-1} v^{m-1} ds + \int_{1}^{\infty} s^{n-1} w^{m-1} ds.$$

Estimating v by means of Lemma 4 and w by (4.2) and (4.5), we conclude that $a \le C A_{m,n}^{m-1}$ for some constant $C > 0$. Next, putting $r = 1$ in the second identity, we find that

$$0 < v(1) = \frac{a^{1/(m-1)}}{k} - b \le 2A_{m,n},$$

so that b is also bounded.

It remains to show that a is bounded away from 0. To this end, we first assert that, for $t \in (\rho_\varepsilon, 1)$,

$$\int_{\rho_\varepsilon}^{t} (s\,v)^{m-1} u^{p-m} ds \le \text{Const. } \varepsilon^{\theta m/(n-m)} = o(1) \qquad \text{as} \quad \varepsilon \to 0. \tag{4.15}$$

Granting this assertion for the moment, it then follows from Lemma 3 (i) with $\vartheta = 1/(m-1)$ that, when $y < 0$, for all $t \in (\rho_\varepsilon, 1)$ we have

$$\begin{aligned} a^{1/(m-1)} g\Big(\frac{y}{a}\Big) &\le \text{Const.} \left[a^{1/(m-1)} + \Big(\int_{\rho_\varepsilon}^{t} (s\,v)^{m-1} u^{p-m} ds \Big)^{1/(m-1)} \right] \\ &\le \text{Const.}\, a^{1/(m-1)} + o(1) \qquad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

On the other hand, when $y > 0$, then obviously $a^{1/(m-1)} g(y/a) < 0$. Hence after an easy integration,

$$R_\varepsilon(\rho_\varepsilon) \le \text{Const. } a^{1/(m-1)} + o(1) \qquad \text{as} \quad \varepsilon \to 0.$$

Now putting $r = \rho_\varepsilon$ in (4.13), there results

$$v(\rho_\varepsilon) = \frac{a^{1/(m-1)}}{k} - b\,\rho_\varepsilon^{k} + R_\varepsilon(\rho_\varepsilon) \le \text{Const.}\, a^{1/(m-1)} + o(1) \qquad \text{as} \quad \varepsilon \to 0.$$

But $v(\rho_\varepsilon) \to A_{m,n}$ as $\varepsilon \to 0$ by Lemma 6, so that

$$\liminf_{\varepsilon \to 0} a^{1/(m-1)} \ge \text{Const. } A_{m,n},$$

as required.

To prove the assertion (4.15), we first use the fact that $u(r) = \alpha^{-1/(m-1)} r^{-k} v(r)$. Then, since $v \le 2A_{m,n}$ and $p - m = m^2/(n-m) - \varepsilon$, we find easily that, for $t \in (\rho_\varepsilon, 1)$,

$$\int_{\rho_\varepsilon}^{t} (s\,v)^{m-1} u^{p-m} ds \le \alpha^{-m^2/(m-1)(n-m) + \varepsilon/(m-1)} (2A_{m,n})^{p-1} \int_{\rho_\varepsilon}^{t} s^{(1-2m)/(m-1)+\varepsilon k} ds,$$

so that finally (recall $\alpha^\varepsilon \le$ Const.)

$$\int_{\rho_\varepsilon}^{t} (s\,v)^{m-1} u^{p-m} ds \le \text{Const.}\, (\alpha^{m'/k} \rho_\varepsilon)^{-m'} = \text{Const.}\, \varepsilon^{\theta m/(n-m)},$$

where in the last equality we have taken into account the definition (4.8) of ρ_ε. This proves the assertion. •

Thanks to Lemma 8 we may prove that the important quantity $R_\varepsilon(\rho_\varepsilon)$, defined in (4.14), vanishes in the limit:

$$\lim_{\varepsilon\to 0} R_\varepsilon(\rho_\varepsilon) = 0. \tag{4.16}$$

From the definition of R_ε we see that the integration variable t is restricted to the interval $(\rho_\varepsilon, 1)$. Therefore, the variable $y = y_\varepsilon(t)$ in (4.14) satisfies

$$-o(1) \le y(t) \le \frac{(2A_{m,n})^{m-1}}{m}\, t^m,$$

the left hand inequality being due to (4.15), and the right hand following from an easy integration.

Accordingly, by Lemma 3 (ii), and Lemma 8 we get

$$-\mathrm{Const.}\, t^m \le g\Big(\frac{y(t)}{a}\Big) \le o(1). \tag{4.17}$$

Combining the preceding lines then gives

$$|R_\varepsilon(\rho_\varepsilon)| \le a^{1/(m-1)}\rho_\varepsilon^k \int_{\rho_\varepsilon}^1 [\mathrm{Const.}\ t^{m-k-1} + o(1)t^{-k-1}]dt = o(1) \qquad \text{as} \quad \varepsilon \to 0,$$

which is (4.16), as required.

Now insert $r = \rho_\varepsilon$ in (4.13), and let $\varepsilon \to 0$. By (4.16) and Lemmas 6 and 8 there results

$$\lim_{\varepsilon\to 0} \frac{a^{1/(m-1)}}{k} = A_{m,n}. \tag{4.18}$$

Next, for fixed $r > 0$, we let $\varepsilon \to 0$ through an appropriate subsequence, so that both $v \to V$ (by Lemma 5) and $b \to B$ (for some constant B with $|B| \le M^*$, see Lemma 8). Using (4.13), (4.18), the estimate (4.15), and an easy application of dominated convergence, then yields the principal integral equation for V:

$$V(r) = A_{m,n} - Br^k + kA_{m,n}R(r) \qquad \text{for } r > 0, \tag{4.19}$$

where

$$R(r) = r^k \int_r^1 g\left((kA_{m,n})^{1-m}\int_0^t (s\,V)^{m-1}ds\right)\frac{dt}{t^{k+1}},$$

and the function g was defined in Lemma 7.

It is immediate from the preceding discussion that $R(r) \to 0$ as $r \to 0$. Hence by (4.19) we get

$$V(r) \to A_{m,n} \qquad \text{as } r \to 0. \tag{4.20}$$

Moreover, by reversing the steps which were used to derive the integral equation (4.13), one finds that the function $W = r^{-k}V$ satisfies

$$(r^{n-1}|W'|^{m-2}W')' = r^{n-1}W^{m-1}, \tag{4.21}$$

that is, the radial form of (1.3) with $\delta = 1$; also by (4.20) and Lemma 5

$$W(r) \sim A_{m,n}r^{-k} \quad \text{as } r \to 0, \qquad W(r) = o(r^{-k}) \quad \text{as } r \to \infty.$$

In other words, we have shown that the function W given by

$$W(r) = r^{-k}V(r), \tag{4.22}$$

where V is the limit function introduced in Lemma 5, satisfies (2.4) – (2.5). Furthermore, multiplying both sides of (4.7) by r^{-k} then gives (2.9) on a subsequence.

We next assert that V, that is $r^k W$, is such that $V(r) > 0$ and $V'(r) < 0$ for all $r > 0$, and that V is unique (and so also W). This will first of all prove Lemma 1 of Section 2, and moreover, since both V and W are unique, that it is in fact unnecessary to use subsequences in (4.7) and (2.9).

To prove the assertion, we first note that the strict positivity of V is equivalent to the strict positivity of W. This last result however can be shown exactly as in the proof of [6, Proposition 1.3.2].

Next, we see from (4.21) and (4.22) that V satisfies the equation

$$(m-1)\left|V' - k\frac{V}{r}\right|^{m-2}\left[V'' - (k-1)\frac{V'}{r}\right] - V^{m-1} = 0. \tag{4.23}$$

To show that $V'(r) < 0$ for $r > 0$, let us suppose for contradiction that there is some point $r_0 > 0$ where $V'(r_0) > 0$. Then, by Lemma 5, there must be some local maximum point $r_1 > r_0$ of V. Clearly $V'(r_1) = 0$ so from the equation one then has $V''(r_1) > 0$, contradicting the fact that r_1 is a local maximum. Similarly if $V'(r_0) = 0$, then $V''(r_0) > 0$, and again there would be a local maximum $r_1 > r_0$ of V, giving once more a contradiction.

In order to prove the uniqueness, we write (4.23) as

$$V'' = (k-1)\frac{V'}{r} + \frac{1}{m-1}\left(\frac{k}{r} - \frac{V'}{V}\right)^{2-m} V, \tag{4.24}$$

where we have used the fact that $V(r) > 0$, $V'(r) < 0$ for all $r > 0$.

Now for contradiction, assume that (4.24) admits two different solutions V_1 and V_2 such that

$$V_1(0) = V_2(0) = A_{m,n}. \tag{4.25}$$

Of course also both V_1 and V_2 decay exponentially to 0 as $r \to \infty$. From these conditions it is easy to see that there exists $R \in (0,\infty)$ such that the function $V_1(r) - V_2(r)$ attains either a positive global maximum or a negative global minimum at R. By switching V_1 and V_2 we may in fact assume that R is a global maximum, that is

$$V_1(R) - V_2(R) = d > 0, \qquad V_1'(R) - V_2'(R) = 0, \qquad V_1''(R) - V_2''(R) \le 0. \tag{4.26}$$

Let $h = -V_2'(R) = -V_1'(R)$ so that $h > 0$ by what has been shown above. By subtracting the equations (4.24) relative to V_1 and to V_2 at the point $r = R$, we get

$$V_1''(R) - V_2''(R) = \Phi(V_2(R) + d) - \Phi(V_2(R)), \tag{4.27}$$

where

$$\Phi(s) = \frac{s}{m-1}\left(\frac{k}{R} + \frac{h}{s}\right)^{2-m}.$$

Also

$$\Phi'(s) = \frac{1}{m-1}\left(\frac{k}{R} + \frac{h}{s}\right)^{1-m}\left[\frac{k}{R} + (m-1)\frac{h}{s}\right] > 0.$$

Hence, by (4.27), we get $V_1''(R) - V_2''(R) > 0$, which contradicts (4.26). This completes the proof of the assertion.

It remains to prove (2.10). Of course, it is enough to consider its radial version, namely

$$\lim_{\varepsilon \to 0} \alpha^{1/(m-1)} u'(r) = W'(r), \qquad \alpha = u(0). \tag{4.28}$$

Let w be as in (4.2). Since $w' < 0$ for $r > 0$, we may rewrite (4.3) as

$$-\left(r^{n-1}|w'|^{m-1}\right)' = r^{n-1}\left(w^{m-1} - \frac{w^{p-1}}{\alpha^{(p-m)/(m-1)}}\right). \tag{4.29}$$

Now fix $r > 0$. By Proposition 2, integration of (4.29) over $[r, \infty)$ yields

$$r^{n-1}|w'(r)|^{m-1} = \int_r^\infty t^{n-1}\left(w^{m-1}(t) - \frac{w^{p-1}(t)}{\alpha^{(p-m)/(m-1)}}\right) dt. \tag{4.30}$$

Thanks to an obvious modification of Lemma 4, we may apply Lebesgue's Theorem to the right hand side of (4.30), so that by (2.9) we have

$$r^{n-1}|w'(r)|^{m-1} \;\to\; \int_r^\infty t^{n-1} W^{m-1}(t)\, dt = r^{n-1}|W'(r)|^{m-1},$$

where the last equality follows by integrating (4.21). Returning to the function u by means of (4.2), this proves (4.28).

The uniform convergence of w' outside any neighborhood of the origin follows at once, since for all $d > 0$ we have, from the last two displayed equations,

$$\begin{aligned}\sup_{r \ge d} \Big| |w'(r)|^{m-1} - |W'(r)|^{m-1} \Big| \;\le\; & d^{1-n} \left(\int_d^\infty t^{n-1} |w^{m-1}(t) - W^{m-1}(t)|\, dt \right. \\ & \left. + \alpha^{-(p-m)/(m-1)} \int_d^\infty t^{n-1} w^{p-1}(t)\, dt \right).\end{aligned}$$

The right side can be made arbitrarily small as $\alpha \to \infty$ when we use (2.9), completing the proof of Theorem 2.

5 Proof of Theorem 1

The asymptotic behavior of $u(0)$ given in (2.8) for $n > m^2$ was proved in [9], see also (3.8) above. In this section we establish this behavior for $n \le m^2$. We treat the cases $n < m^2$ and $n = m^2$ separately. As in the previous section, we put $\delta = 1$.

5.1 The case $n < m^2$

We start from the generalized Pohozaev identity (3.2). Multiplying by $\alpha^{m/(m-1)}/m$ and making the substitution (4.2) in the integral on the left hand side, we obtain the identity

$$m \int_0^\infty r^{n-1} w^m(r)\, dr = \varepsilon \frac{n-m}{m^* - \varepsilon}\, \alpha^{m/(m-1)} \int_0^\infty r^{n-1} u^p(r)\, dr. \tag{5.1}$$

From Lemma 2 it follows that (see (52) in [9])

$$\int_0^\infty r^{n-1} u^p(r)\, dr \to \frac{1}{m'} D^{-n/m'} B\left(\frac{n}{m'}, \frac{n}{m}\right) \qquad \text{as} \quad \varepsilon \to 0, \tag{5.2}$$

this being valid for any $n > m$.

Estimating the integral on the left hand side of (5.1) is more delicate. From Lemma 5 we know that if $\varepsilon \to 0$, then $v(r) \to V(r)$ and $w(r) \to W(r)$ for all $r > 0$, and by Lemma 4,

$$r^{n-1} w^m(r) = r^{n-1-mk} v^m(r) \le (2A_{m,n})^m \; r^{(m^2-n)/(m-1)-1} \qquad \text{for} \quad r > 0.$$

Hence, if $n < m^2$ the integrand of the integral on the left in (5.1) is bounded uniformly by a function which is integrable near the origin. By Proposition 2 it is also uniformly bounded by a function which is integrable at infinity. Therefore, it follows from the dominated convergence theorem that

$$\int_0^\infty r^{n-1} w^m(r)\, dr \to \int_0^\infty r^{n-1} W^m(r)\, dr = I_{m,n} \qquad \text{as} \quad \varepsilon \to 0. \tag{5.3}$$

Putting (5.3) and (5.2) into (5.1) yields the desired limit for $n < m^2$ in Theorem 1.

5.2 The case $n = m^2$

First of all we note that (3.9) yields

$$\frac{\log \alpha}{|\log \varepsilon|} \to \frac{1}{m'} \qquad \text{and} \qquad \alpha^\varepsilon = 1 + O\Big(\frac{\log^2 \alpha}{\alpha^{m'}}\Big) \qquad \text{as} \quad \varepsilon \to 0. \tag{5.4}$$

When $n = m^2$ the limit (5.2) of the right hand integral in (5.1) still holds. However the argument used to obtain the limit of the left hand integral when $n < m^2$, now breaks down at the origin. We therefore split this integral into two parts at the radius

$$R_0 = R_0(\varepsilon) = |\log \varepsilon|^{-2/m}. \tag{5.5}$$

With the help of Lemma 4 we can prove

Lemma 9 *Let $n = m^2$, and let R_0 be as in* (5.5)*. Then*

$$\int_{R_0}^\infty r^{m^2-1} w^m(r)\, dr = O(\log|\log \varepsilon|) \qquad \textit{as} \quad \varepsilon \to 0.$$

Proof. By the substitution (4.2) and the fact that $k = m$ in the present case, the statement of the lemma is equivalent to

$$\int_{R_0}^\infty \frac{v^m(r)}{r}\, dr = O(\log|\log \varepsilon|) \qquad \text{as} \quad \varepsilon \to 0. \tag{5.6}$$

Since $R_0(\varepsilon) \to 0$ as $\varepsilon \to 0$, we may write $[R_0, \infty) = [R_0, 1] \cup [1, \infty)$. By Lemma 4 we know that there exists a constant $c > 0$ such that

$$\int_1^\infty \frac{v^m(r)}{r}\, dr \le c \tag{5.7}$$

for all suitably small ε. On the other hand, using Lemma 4 again, we infer that

$$\int_{R_0}^1 \frac{v^m(r)}{r}\, dr \le (2A_{m^2,m})^m |\log R_0|$$

for ε sufficiently small. Therefore

$$\int_{R_0}^{1} \frac{v^m(r)}{r}\,dr = O(\log|\log\varepsilon|) \qquad \text{as} \qquad \varepsilon \to 0,$$

which, together with (5.7), proves (5.6). •

From now on, we argue mostly in terms of α instead of ε ($\alpha \to \infty$ if and only if $\varepsilon \to 0$); then by (5.4) and Lemma 9, we can write (5.6) in the form

$$\int_{R_0}^{\infty} r^{m^2-1} w^m(r)\,dr = O(\log\log\alpha) \qquad \text{as} \qquad \alpha \to \infty. \tag{5.8}$$

Next we estimate the integral over the interval $(0, R_0)$.

Lemma 10 *Let $n = m^2$, and let R_0 be as in* (5.5). *Then*

$$\int_{0}^{R_0} r^{m^2-1} w^m(r)\,dr = \frac{\log\alpha}{(m-1)D^{m(m-1)}} + O(\log\log\alpha) \qquad \textit{as} \qquad \alpha \to \infty. \tag{5.9}$$

Proof. For convenience we write

$$\varphi(r) = \frac{1}{[1 + D\alpha^{(p-m)/(m-1)} r^{m'}]^{k/m'}}$$

and formulate an upper and a lower bound for w^m. More precisely, we claim that there exist constants c_1, $c_2 > 0$ such that

$$w^m(r) \le \alpha^{mm'}\{\varphi^m(r) + c_1\,\eta\,\varphi^{m-1}(r)\}, \qquad \eta = \alpha^{m-p}, \tag{5.10}$$

and

$$w^m(r) \ge \alpha^{mm'}\{\varphi^m(r) - c_2\,\varepsilon|\log\varepsilon|\varphi^{m-1}(r)\}. \tag{5.11}$$

In order to prove these bounds we remark that Lemma 2 yields

$$u(r) \le \alpha\{\varphi(r) + C\,\eta\}$$

for some constant $C > 0$. The upper bound (5.10) then follows by taking the m-th power and transforming to w. On the other hand, Lemma 2 also gives

$$u(r) \ge \alpha\{\varphi(r) - C\,\varepsilon|\log\varepsilon|\}, \tag{5.12}$$

where C is a different positive constant. If the right hand side of (5.11) is negative, there is nothing to prove. If it is positive, then the right hand side of (5.12) is also positive and (5.11) follows by taking the m-th power and transforming to w.

The bounds (5.10)–(5.11) suggest to write

$$\int_0^{R_0} r^{m^2-1} w^m(r)\,dr = I + J,$$

where the principal term I is given by

$$I = \alpha^{mm'} \int_0^{R_0} r^{m^2-1}\,\varphi^m(r)\,dr.$$

We first estimate I. With the substitutions

$$t = D\alpha^{(p-m)/(m-1)} r^{m'} \qquad \text{and} \qquad T = D\alpha^{(p-m)/(m-1)} R_0^{m'}, \tag{5.13}$$

we obtain

$$I = \alpha^{mm'} \int_0^{R_0} r^{m^2-1} \varphi^m(r)\, dr = \frac{\alpha^{\varepsilon m}}{m' D^{m(m-1)}} \int_0^T \frac{t^{m(m-1)-1} dt}{(1+t)^{m(m-1)}}.$$

Since $T \to \infty$ as $\alpha \to \infty$ and $\varepsilon \to 0$, we find that[1]

$$I = \frac{\log T}{m' D^{m(m-1)}} + O(1) \quad \text{as} \quad \alpha \to \infty.$$

However, from (5.13) and the definition (5.5) of R_0 it is not hard to see that

$$\log T = \frac{m}{(m-1)^2} \log \alpha + O(\log\log \alpha) \quad \text{as} \quad \alpha \to \infty.$$

Thus, finally

$$I = \frac{\log \alpha}{(m-1) D^{m(m-1)}} + O(\log\log \alpha) \qquad \text{as} \quad \alpha \to \infty.$$

Next, we estimate the remainder term J, which we write as

$$J = \int_0^{R_0} r^{m^2-1} [w^m(r) - \alpha^{mm'} \varphi^m(r)]\, dr. \tag{5.14}$$

To do this, we use the upper and lower bound (5.10)–(5.11). Both bounds involve the integral

$$J_0 \equiv \alpha^{mm'} \int_0^{R_0} r^{m^2-1}\, \varphi^{m-1}(r)\, dr, \tag{5.15}$$

which we can write as

$$J_0 = \frac{\alpha^{\varepsilon m}}{m' D^{m(m-1)}} \int_0^T \frac{t^{m(m-1)-1} dt}{(1+t)^{(m-1)^2}}.$$

However $t^{m(m-1)-1}/(1+t)^{(m-1)^2} \le t^{m-2}$, so plainly

$$J_0 = O(T^{m-1}) \qquad \text{as} \quad \alpha \to \infty.$$

[1] The calculation is as follows. Put $a = m(m-1)$ and write

$$\frac{t^{a-1}}{(1+t)^a} = \frac{1}{1+t} - \frac{1}{1+t}\left(1 - \left(\frac{t}{1+t}\right)^a\right) + \frac{t^{a-1}}{(1+t)^{a+1}}.$$

But

$$\int_0^T \frac{t^{a-1}}{(1+t)^{a+1}}\, dt = \frac{1}{a}\left(\frac{T}{1+T}\right)^a \le \frac{1}{a}$$

and by Lemma 3 (iii) with $\vartheta = a$,

$$1 - \left(\frac{t}{1+t}\right)^a = -g\left(\frac{1}{1+t}\right) \le \max\{1,\, a\}\, \frac{1}{1+t}.$$

Now from the three preceding lines and the fact that $\int_0^T (1+t)^{-2} dt = 1 - (1+T)^{-1} \le 1$ we thus get

$$\int_0^T \frac{t^{a-1}}{(1+t)^a}\, dt = \log(1+T) + \kappa,$$

where $\max\{1,\, a\} \le \kappa \le 1/a$, and the result now follows.

From (5.14) and use of (5.10), (5.11) and (5.15), we get

$$-c_2\,\varepsilon|\log\varepsilon|\,J_0\ \le J \le\ c_1\eta\,J_0.$$

Moreover, from the definitions (5.13) and (5.5) of T and R_0,

$$\eta\,T^{m-1} = D^{m-1}R_0^m = \frac{D^{m-1}}{|\log\varepsilon|^2} \to 0,$$

whence

$$\varepsilon|\log\varepsilon|\,T^{m-1} = D^{m-1}\frac{\varepsilon}{|\log\varepsilon|}\,\alpha^{m'-\varepsilon} \le \text{Const},$$

where (3.9) was used at the last step. Thus, finally, $J = O(1)$ as $\alpha\to\infty$, so that

$$\int_0^{R_0} r^{m^2-1}w^m(r)\,dr = \frac{\log\alpha}{(m-1)D^{m(m-1)}} + O(\log\log\alpha) \qquad \text{as} \quad \alpha\to\infty,$$

as asserted. This completes the proof of Lemma 10. •

When we substitute (5.2), (5.8) and (5.9) into (5.1) we arrive at the limit

$$\frac{\varepsilon}{|\log\varepsilon|}\,\alpha^{m'} \to \frac{(m')^3}{m}\cdot\frac{1}{B(m(m-1),m)} \qquad \text{as} \quad \alpha\to\infty,$$

which is equivalent to the second limit in Theorem 1.

6 Proof of Theorem 3

In this section we translate the results we obtained for equation (P_p^1), in which the coefficient δ has been chosen equal to 1, to the solution $u(x;\delta)$ of equation (P_p^δ) in which δ is arbitrary positive. To this end we use the scaling invariance

$$u(x;\delta) = \delta^{1/(p-m)}u(\delta^{1/m}x, 1). \tag{6.1}$$

From Theorem 1 we know that as $\varepsilon\to 0$,

$$\begin{aligned}
&u(0;1) \sim \beta_{m,n}\,\varepsilon^{-k/mm'} && \text{if } n > m^2\\
&u(0;1) \sim \omega_m\,(|\log\varepsilon|/\varepsilon)^{1/m'} && \text{if } n = m^2\\
&u(0;1) \sim \gamma_{m,n}\,\varepsilon^{-1/m'} && \text{if } n < m^2.
\end{aligned}$$

Therefore, by rescaling back according to (6.1), we obtain

$$\begin{aligned}
&u(0;\delta) \sim \beta_{m,n}(\delta/\varepsilon)^{k/mm'} && \text{if } n > m^2\\
&u(0;\delta) \sim \omega_m(|\log\varepsilon|\delta/\varepsilon)^{1/m'} && \text{if } n = m^2\\
&u(0;\delta) \sim \gamma_{m,n}(\delta^{k/m}/\varepsilon)^{1/m'} && \text{if } n < m^2.
\end{aligned}$$

If $\delta = \delta(\varepsilon)$, where $\delta(\varepsilon)$ is defined in (2.12), then these limits imply that $u(0;\delta(\varepsilon))\to d$ in all the three cases above. The proof of Theorem 3 may now be completed as in [9, Theorem 4].

7 Proof of Theorem 4

As $\varepsilon \to 0$, the solution u of problem (P_p^1) develops a spike at the origin, which becomes progressively taller and thinner. In this section we give an estimate for the rate at which level curves shrink to a point as $\varepsilon \to 0$.

Let u be the unique radial ground state of problem (P_p^1), and let M be a postive number. We then define the radius r_ε through

$$r_\varepsilon^k = \frac{A_{m,n}}{M}\alpha^{-1/(m-1)}, \tag{7.1}$$

where $\alpha = u(0)$. We shall show that

$$u(r_\varepsilon) \to M \qquad \text{as} \quad \varepsilon \to 0. \tag{7.2}$$

We first prove that $r_\varepsilon > \rho_\varepsilon$ for ε small enough. By (4.8),

$$\rho_\varepsilon^k = \alpha^{-m'} \cdot \varepsilon^{-\theta}, \qquad 0 < \theta < 1.$$

Hence, by (7.1) and the asymptotic estimates (3.8), (3.9) and (3.10),

$$\Big(\frac{r_\varepsilon}{\rho_\varepsilon}\Big)^k \sim \frac{A_{m,n}}{M}\alpha\,\varepsilon^\theta \to \infty \qquad \text{as} \quad \varepsilon \to 0,$$

provided θ is restricted to $0 < \theta < 1/m'$.[2]

We now use the integral equation (4.13) for v in Lemma 7 to determine the limit of $u(r_\varepsilon)$ as $\varepsilon \to 0$. By (4.1) and (7.1), we have

$$\lim_{\varepsilon\to 0} u(r_\varepsilon) = \frac{M}{A_{m,n}} \lim_{\varepsilon\to 0} v(r_\varepsilon),$$

and by (4.13) and (4.18) we have

$$\lim_{\varepsilon\to 0} v(r_\varepsilon) = A_{m,n} + \lim_{\varepsilon\to 0} R_\varepsilon(r_\varepsilon),$$

where

$$R_\varepsilon(r) = a^{1/(m-1)} r^k \int_r^1 g\Big(\frac{y(t)}{a}\Big)\frac{dt}{t^{k+1}}.$$

Therefore, the assertion (7.2) is proved once we have shown that $R_\varepsilon(r_\varepsilon) \to 0$ as $\varepsilon \to 0$.

In (4.17) we have established that for some positive constant C,

$$-Ct^m < g\Big(\frac{y(t)}{a}\Big) < o(1) \qquad \text{as} \quad \varepsilon \to 0$$

for any $t \in (\rho_\varepsilon, 1)$. Hence, since $r_\varepsilon > \rho_\varepsilon$,

$$R_\varepsilon(r_\varepsilon) \le a^{1/(m-1)} r_\varepsilon^k \int_{r_\varepsilon}^1 t^{-k-1} dt \cdot o(1) \to 0$$

and

$$R_\varepsilon(r_\varepsilon) \ge -Ca^{1/(m-1)} r_\varepsilon^k \int_{r_\varepsilon}^1 t^{m-k-1} dt \to 0$$

as $\varepsilon \to 0$. Therefore, $R_\varepsilon(r_\varepsilon) \to 0$ as $\varepsilon \to 0$, which we set out to prove.

[2] Up to this point, any value $\theta \in (0,1)$ would have sufficed, say $\theta = 1/2$; it is only here that further care in the choice of θ is required.

8 Matching

In the Introduction we noted that to describe the limiting form of the ground state $u = u_\varepsilon$ one can distinguish an inner and an outer region, each being associated with a particular scaling. In this section we return to this observation and provide more details. For simplicity one can take $\delta = 1$ here.

We begin with the *inner region*. It is associated with the scaling

$$s = \alpha^{(p-m)/m} r \qquad \text{and} \qquad \tilde{u}(s) = \alpha^{-1} u(r). \tag{8.1}$$

By Lemma 2, it is not hard to see that for all $s \geq 0$,

$$\tilde{u}(s) \to U_1(s) \qquad \text{as} \quad \varepsilon \to 0, \tag{8.2}$$

where U_1 is defined by (1.1). That is, in the inner region, whose radius shrinks to zero as $\varepsilon \to 0$, the solution, when normalised, converges to U_1.

In the *outer region* we use the scaling (see (4.1), (4.2))

$$v(r) = \alpha^{1/(m-1)} r^k u(r). \tag{8.3}$$

It is proved in Lemma 5 that (no subsequence being needed)

$$v(r) \to V(r) \qquad \text{as} \quad \varepsilon \to 0, \qquad r > 0; \tag{8.4}$$

see Figure 2.1.

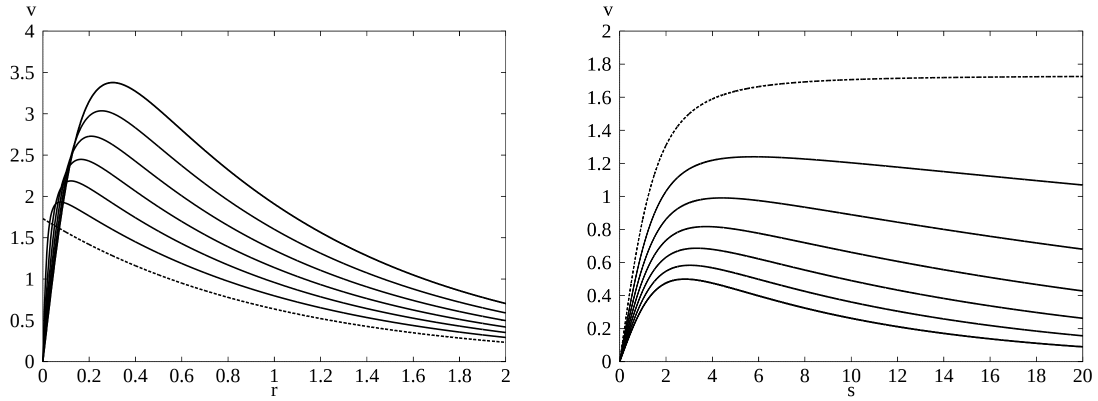


Figure 2: **2.1** Graphs of the function $v(r) = v_\varepsilon(r)$ for $n = 3$, $m = 2$, $\delta = 1$ and for $\varepsilon = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$, and 1.2. For large r the graphs decrease as ε decreases to zero. The lowest curve (dashed) is the graph of the limiting function $V(r) = \sqrt{3}e^{-r}$; note that $V(0) = \sqrt{3}$.
2.2 Graphs of the function $\tilde{v}(s) = \tilde{v}_\varepsilon(s)$ for $n = 3$, $m = 2$, $\delta = 1$ and for $\varepsilon = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$, and 1.2. For any fixed s the graphs eventually decrease as ε decreases to zero. The dashed curve is the graph of the limiting function $\tilde{V}(s) = sU_1(s) = s/(1 + \frac{1}{3}s^2)^{1/2}$; note that $\tilde{V}(\infty) = \sqrt{3}$.

To show that the two limiting solutions $U_1(s)$ and $V(r)$ match, we first transform the function $v(r)$ to the variable s, that is, we set $\tilde{v}(s) = v(r)$ and $r = \alpha^{-(p-m)/m} s$, so that

$$\tilde{v}(s) = \alpha^{m/(m-1)} \alpha^{-(p-m)k/m} s^k \tilde{u}(s) = \alpha^{\varepsilon k/m} s^k \tilde{u}(s).$$

Then from (8.2),

$$\tilde{v}(s) \to \tilde{V}(s) = s^k U_1(s) \qquad \text{as} \quad \varepsilon \to 0,$$

see Figure 2.2.

Plainly, by the definition (1.1) of U_1 together with (2.6), we have

$$\tilde{V}(s) \to A_{m,n} \qquad \text{as} \quad s \to \infty. \tag{8.5}$$

On the other hand, by (4.20),

$$V(r) \to A_{m,n} \qquad \text{as} \quad r \to 0. \tag{8.6}$$

From (8.5) and (8.6) one concludes that the limiting profiles in the inner and the outer region match at the *outer* boundary of the inner region and the *inner* boundary of the outer region.

The limits $\tilde{v} \to \tilde{V}$ and $v \to V$ can be expected intuitively, by virtue of the related limiting differential equations (1.2) and (1.3), except that one would then only get $V(r) \to Q$ as $r \to 0$ for some $Q \geq 0$, see (1.4). Recalling the limit (8.5), it then follows by heuristic matching, as above, that $Q = A_{m,n}$. While certainly suggestive, this approach should still be clearly understood only as a heuristic procedure, requiring the full apparatus here for a rigorous and convincing proof.

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