A series approach to stochastic differential equations with infinite dimensional noise

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Abstract

This paper considers semilinear stochastic differential equations in Hilbert spaces with Lipschitz nonlinearities and with the noise terms driven by sequences of independent scalar Wiener processes (Brownian motions). The interpretation of such equations requires a stochastic integral. By means of a series of Itô integrals, an elementary and direct construction of a Hilbert space valued stochastic integral with respect to a sequence of independent scalar Wiener processes is given. As an application, existence and strong and weak uniqueness for the stochastic differential equation are shown by exploiting the series construction of the integral.

1 Introduction

In order to model noise in infinite dimensional dynamical systems, it might be necessary to consider noise terms that are driven by infinite dimensional Wiener processes. See for example [7] and references therein. A simple and direct way to formulate infinite dimensional noise is by means of a sequence of independent scalar Wiener processes. The proper formulation as an integral equation needs a corresponding stochastic integral. In general, the integral of [4] for Hilbert-Schmidt operator valued integrands and Hilbert space valued Wiener processes is used, and one needs to transform the model to this formulation by constructing the Wiener process and considering a spectral decomposition of its covariance operator (see, e.g., [1]). Instead of packing the sequence of scaler Wiener processes into one Hilbert space valued Wiener process and using the integral of [4], it is our aim to present a construction directly in terms of the sequence of independent scaler processes. The construction is a straightforward extension of Itô's integral with respect to a vector of independent Wiener processes as in [10], though we present it in a Hilbert space setting. In fact, we simply consider a series of Itô integrals with respect to the scalar processes.

*supported by an N.W.O. PIONIER-grant under 600-61-410

We will show that the construction yields an integral that is equivalent to the integral of [4], both for the trace class covariance and cylindrical case.

We will exploit the series construction of the stochastic integral to study the stochastic differential equation

$$X'(t) = AX(t) + F(t, X(t)) + B(t, X(t))W'(t), \quad t \geq 0, \tag{1}$$

where X will take values in a separable real Hilbert space H, A is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on H, and $F\colon [0,\infty) \times H \to H$ is a continuous function that is Lipschitz continuous in its second argument. In this equation W' denotes the infinite dimensional white noise corresponding to a sequence of independent scalar Wiener processes $((W_i(t))_{t\geq 0})_{i=0}^{\infty}$ and it is modified by the coefficient $B = (B_i)_{i=0}^{\infty}$, where each $B_i\colon [0,\infty) \times H \to H$ is continuous and Lipschitz continuous in its second argument. We suppose that each W_i contributes noise in a direction and with a weight given by B_i, depending on both t and $X(t)$, so that

$$B(t, X(t))W'(t) = \sum_{i=0}^{\infty} B_i(t, X(t))W_i'(t), \quad t \geq 0. \tag{2}$$

The corresponding stochastic integral will be defined by

$$\int_0^t \Phi(s)dW(s) = \sum_{i=0}^{\infty} \int_0^t \Phi_i(s)dW_i(s), \quad t \geq 0, \tag{3}$$

where $\Phi_i(s) = B_i(s, X(s))$, $s \geq 0$, and we interpret equation (1) as the integral equation

$$\begin{aligned} X(t) \quad &= \quad S(t)X(0) + \int_0^t S(t-s)F(s, X(s))ds \qquad (4) \\ &\qquad + \sum_{i=0}^{\infty} \int_0^t S(t-s)B_i(s, X(s))dW_i(s), \ t \geq 0. \end{aligned}$$

We will establish existence and strong uniqueness, and give a detailed proof of weak uniqueness, which is essential for certain coupling techniques (see [8]).

In the next section we present the construction of the stochastic integral and in Section 3 we discuss the equivalence with the integral of [4]. In Section 4 we prove the existence and uniqueness theorems for equation (4). As it is our aim to convince the reader that the series construction of the integral is simple and easy to deal with, we devised the paper to be self-contained.

2 The Construction of the stochastic integral

We will first give a construction of the Hilbert space valued Itô integral and then we use it to build the stochastic integral with respect to a sequence of independent scalar Wiener processes. We begin by introducing some terminology and fixing notations.

A *scalar Wiener process* is a family $(W(t))_{t\geq 0}$ of real valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that (i) $W(0) = 0$, (ii) for almost all $\omega \in \Omega$ the function $t \mapsto W(t)(\omega)$ is continuous, (iii) $W(t+s) - W(t)$ and $W(u)$ are independent for all $t \geq u \geq 0$, $s \geq 0$, and (iv) $W(t+s) - W(t)$ has a Gaussian distribution with mean 0 and variance $\sigma^2 s$ for all $s, t \geq 0$. If $\sigma = 1$, the Wiener process is called *normalized.*

The construction of the Itô integral with respect to the Wiener process $(W(t))_{t\geq 0}$ requires a certain kind of adaptedness of the integrand. For instance, one can assume that the integrand is at every t measurable with respect to the σ-algebra generated by $\{W(s)\colon\ 0 \leq s \leq t\}$. In view of other random variables involved, such as stochastic initial conditions or several other independent Wiener processes, it is often useful to allow families of larger σ-algebras called filtrations.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. An increasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ is called a *filtration* in $\mathcal{F}$ or, less precisely, in Ω. A Wiener process $(W(t))_{t\geq 0}$ on Ω is called a *Wiener process with respect to* $(\mathcal{F}_t)_{t\geq 0}$ if $W(t)$ is $\mathcal{F}_t$ measurable and $W(t+s) - W(t)$ is independent of $\mathcal{F}_t$, that is, for every Borel set $B \subset \mathbb{R}$ and $C \in \mathcal{F}_t$ one has $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$, where $A = \{\omega \in \Omega : (W(t+s) - W(t))(\omega) \in B\}$, for all $t, s \geq 0$. For instance, if $(X_i)_{i\in I}$ is a family of random variables on Ω with X_i independent of $W(t)$ for all i and t, and $\mathcal{F}_t$ is the sub-σ-algebra of $\mathcal{F}$ generated by $\{W(s)\colon\ 0 \leq s \leq t\}$ and $\{X_i\colon\ i \in I\}$, then $(\mathcal{F}_t)_{t\geq 0}$ is a filtration and W is a Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$. In particular, $(X_i)_{i\in I}$ may consist of one or several Wiener processes that are independent of W.

For an index set $I \subset [0, \infty)$, a family $(\Phi_t)_{t\in I}$ of random variables on Ω that take values in a separable real Hilbert space H is called *adapted* to $(\mathcal{F}_t)_{t\geq 0}$ if Φ_t is measurable with respect to $\mathcal{F}_t$ and the Borel σ-algebra of H for every $t \in I$. We will view $t \mapsto \Phi_t$ as a function and mainly consider the case that $\Phi : [0, \infty) \to L^2(\Omega; H)$, where $L^2(\Omega; H)$ denotes the Hilbert space of H-valued square Bochner integrable random variables.

For the Itô integral we will consider integrands $\Phi : [0, \infty) \to L^2(\Omega; H)$ that are adapted and either continuous or simple. In order to combine both cases we consider the following class. We call a function $f\colon\ [0, T] \to L^2(\Omega; H)$ *piecewise uniformly continuous* (PUC) if there are $0 = a_0 < a_1 < \cdots < a_n = T$ such that f is uniformly continuous on (a_{k-1}, a_k) for each k. Observe that each such function is bounded and that the class of PUC functions contains the continuous functions and the simple functions. A function $f\colon\ [0, \infty) \to L^2(\Omega; H)$ is called piecewise uniformly continuous if every restriction $f|_{[0,T]}$ $(T > 0)$ is PUC.

The setting for the construction will be as follows. Let H be a separable real Hilbert space with the inner product denoted by $\langle \cdot, \cdot \rangle$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W = (W_i)_{i\in\mathbb{N}}$ be a sequence of independent normalized scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. For instance, the probability space may be the product of countably many copies of a space belonging to one normalized scalar Wiener process and the filtration may consist of the σ-algebras $\mathcal{F}_t$ generated by $\{W_i(s)\colon\ 0 \leq s \leq t,\ i \in \mathbb{N}\}$, $t \geq 0$. As integrand we will consider a sequence $\Phi = (\Phi_i)_i$ of PUC functions from $[0,\infty)$ to $L^2(\Omega; H)$, with each Φ_i adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

The Hilbert space valued Itô integrals $\int_0^t \Phi_i(s)dW_i(s)$, $i \in \mathbb{N}$, will be defined as limits in $L^2(\Omega; H)$ of certain Riemann sums. They turn out to form an orthogonal sequence in the Hilbert space $L^2(\Omega; H)$ and their norms are given by the Itô isometry (6) below. If their norms are square summable we can define the stochastic integral of Φ with respect to W by (3).

Lemma 2.1. *Let H be a real separable Hilbert space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(W_i)_{i=0}^\infty$ be a sequence of independent normalized scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $T > 0$ and let $(\Phi_i)_{i=0}^\infty$ be a sequence of adapted piecewise uniformly continuous functions from $[0,T]$ to $L^2(\Omega; H)$. Then:*

(i) For each $i \in \mathbb{N}$, the integral $\int_0^T \Phi_i(s)dW_i(s)$ is well-defined as a limit in $L^2(\Omega; H)$ of Riemann sums of the form

$$\sum_{k=1}^{n} \Phi_i(s_{k-1})\Big(W_i(s_k) - W_i(s_{k-1})\Big), \tag{5}$$

where $0 = s_0 < s_1 < \cdots < s_n = T$. More precisely, there is an element in $L^2(\Omega; H)$, denoted by $\int_0^T \Phi_i(s)dW_i(s)$, such that for every $\varepsilon > 0$ there is a $\delta > 0$ with $\mathbb{E}\,\| \int_0^T \Phi_i(s)dW_i(s) - \sum_{k=1}^n \Phi_i(s_{k-1})(W_i(s_k) - W(s_{k-1}))\| < \varepsilon$ for any partition $0 = s_0 < s_1 < \cdots < s_n = T$ with $\max_k |s_k - s_{k-1}| < \delta$.

(ii) For each i the Itô isometry *holds true:*

$$\mathbb{E}\,\Big\| \int_0^T \Phi_i(s)dW_i(s)\Big\|^2 = \int_0^T \mathbb{E}\,\|\Phi_i(s)\|^2 ds. \tag{6}$$

(iii) For each i and j with $i \neq j$ one has

$$\mathbb{E}\,\Big\langle \int_0^T \Phi_i(s)dW_i(s), \int_0^T \Phi_j(s)dW_j(s)\Big\rangle = 0. \tag{7}$$

The lemma can be proved by standard arguments. We have included a proof in the Appendix.

Remark 2.2. 1. In Lemma 2.1 we did not assume independence of Φ_i and Φ_j. Actually, independence of W_i and W_j $(i \neq j)$ and adaptedness of Φ_i and Φ_j already imply the orthogonality relation expressed by (7).

2. It follows from part (ii) of Lemma 2.1 that the value of the integral $\int_0^t \Phi_i(s)\,dW_i(s)$ is not affected by changing $s \mapsto \Phi_i(s)$ at finitely many points. Therefore the integral is well-defined on any interval $[a,b]$ with $0 \leq a \leq b$ by setting

$$\int_a^b \Phi_i(s)\,dW_i(s) := \int_0^b \mathbb{1}_{[a,b]}(s)\Phi_i(s)\,dW_i(s).$$

3. One can extend the definition of the Itô integral to all adapted functions in $L^2([0,T];L^2(\Omega;H))$. The proof of this observation comes down to showing that the adapted *simple functions* (i.e. piecewise uniformly continuous and taking only finitely many values) are dense in the subspace of $L^2([0,T];L^2(\Omega;H))$ consisting of adapted functions. Indeed, because of the Itô isometry, the extension then follows directly. In order to show that the adapted simple functions are dense, one can adapt one of several proofs that exist for the scalar valued case (see, e.g., [13, p.27–28]). We prefer, instead, an elegant translation argument presented by J. van Neerven [12]. First, note that by definition of square Bochner integrability (see [5, 6]), for every $f \in L^2([0,T];L^2(\Omega;H))$ and $\varepsilon > 0$, there are measurable sets $A_1,\dots,A_n \subset [0,T]$ and $a_1,\dots,a_n \in L^2(\Omega;H)$ such that

$$\|f-g\|_{L^2} = \left(\int_0^T \mathbb{E}\,\|f(s)-g(s)\|^2 ds\right)^{1/2} < \varepsilon,$$

where $g = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$. More than that, we may assume that the sets $A_1,\dots,A_n$ are disjoint intervals, as each A_i can be replaced by a union U of disjoint intervals such that $(A_i \setminus U) \cup (U \setminus A_i)$ has arbitrary small Lebesgue measure.

Let now $f \in L^2([0,T];L^2(\Omega;H))$ be adapted to $(\mathcal{F}_t)_{t\geq 0}$ and let $\varepsilon > 0$. Fix a $\tau > 0$ such that $\|f - f_\tau\|_{L^2} < \varepsilon/2$, where f_τ is defined by $f_\tau(t) = f(t-\tau)$, $\tau \leq t \leq T$, $f_\tau(t) = 0$, $0 \leq t < \tau$. If we define

$$E := \{f \in L^2([0,T];L^2(\Omega;H)) : f(t) \text{ is } \mathcal{F}_{n\tau}\text{-measurable} \\ \text{for each } n \text{ and } t \in [n\tau,(n+1)\tau] \cap [0,T]\},$$

then E is a closed subspace and every element of E is adapted. Let P denote the orthogonal projection on E. The P-image of a simple function is simple as well, and $Pf_\tau = f_\tau$. Next, take a simple function g with $\|f_\tau - g\|_{L^2} < \varepsilon/2$. Since

$$\|f_\tau - Pg\|_{L^2} = \|P(f_\tau - g)\|_{l^2} \leq \|f_\tau - g\|_{L^2} < \varepsilon/2,$$

it follows that $\|f - Pg\|_{L^2} < \varepsilon$. As Pg is an adapted simple function, the proof is complete.

The integrals $\int_0^T \Phi_i(s)dW_i(s)$, $i \in \mathbb{N}$, of Lemma 2.1 form an orthogonal sequence in the Hilbert space $L^2(\Omega; H)$. Hence their infinite sum exists if their norms are square summable, and their norms are given by (6). Thus we obtain the following.

Theorem 2.3 (construction of the stochastic integral). *Let H be a separable real Hilbert space. Let $(\Omega; \mathcal{F}, \mathbb{P})$ be a probability space and let $(W_i)_{i=0}^\infty$ be a sequence of independent normalized scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $T > 0$ and let $\Phi = (\Phi_i)_{i=0}^\infty$ be a sequence of adapted piecewise uniformly continuous functions from $[0,T]$ to $L^2(\Omega; H)$ such that*

$$\int_0^T \sum_{i=0}^\infty \mathbb{E}\,\|\Phi_i(s)\|^2 ds < \infty. \tag{8}$$

Then

$$\int_0^T \Phi(s)dW(s) := \sum_{i=0}^\infty \int_0^T \Phi_i(s)dW_i(s) \quad \textit{exists in } L^2(\Omega; H)$$

and

$$\mathbb{E}\,\|\int_0^T \Phi(s)dW(s)\|^2 = \int_0^T \mathbb{E}\sum_{i=0}^\infty \|\Phi_i(s)\|^2 ds.$$

Remark 2.4. 1. It is clear that the assumption in Theorem 2.3 that the W_i are normalized does not restrict the generality, as the scaling can be absorbed in the Φ_i for each $i \in \mathbb{N}$.

2. With aid of Remark 2.2.3, we may relax the condition of piecewise uniform continuity of the functions Φ_i in Theorem 2.3 to $\Phi_i \in L^2([0,T]; L^2(\Omega; H))$ for $i \in \mathbb{N}$.

3. The finite dimensional case is included in Theorem 2.3 by taking all but finitely many of the functions Φ_i identically zero.

4. Instead of viewing the integrand as a sequence of functions, it may be considered as one function with values in a sequence space. Indeed, if we define the Hilbert space

$$l^2(\mathbb{N}; H) := \{x\colon\ \mathbb{N} \to H\colon\ \textstyle\sum_{i=0}^\infty \|x_i\|^2 < \infty\}$$

with the inner product given by $\langle x, y\rangle_{l^2(\mathbb{N};H)} = \sum_{i=0}^\infty \langle x_i, y_i\rangle$, $x, y \in l^2(\mathbb{N}; H)$, then for every sequence $(X_i)_{i=0}^\infty$ in $L^2(\Omega; H)$ with $\sum_{i=0}^\infty \mathbb{E}\,\|X_i\|^2 < \infty$, the map $\omega \mapsto (X_i(\omega))_{i=0}^\infty$ is an element of $L^2(\Omega; l^2(\mathbb{N}; H))$. Moreover, a function Φ from $[0,T]$ to $L^2(\Omega; l^2(\mathbb{N}; H))$ is adapted and measurable if and only if $t \mapsto (\omega \mapsto$

$(\Phi(t)(\omega))_i)$ is adapted and measurable from $[0,T]$ to $L^2(\Omega;H)$ for each $i \in \mathbb{N}$, and

$$\|\Phi\|^2_{L^2([0,T];L^2(\Omega;l^2(\mathbb{N};H)))} = \int_0^T \int_\Omega \sum_{i=0}^\infty \|(\Phi(s)(\omega))_i\|^2 d\mathbb{P}(\omega)ds.$$

In this way, the stochastic integral is defined for all adapted elements of the space $L^2([0,T];L^2(\Omega;l^2(\mathbb{N};H)))$ and the Itô isometry can be written as

$$\mathbb{E}\,\|\int_0^T \Phi(s)dW(s)\|^2 = \|\Phi\|^2_{L^2([0,T];L^2(\Omega;l^2(\mathbb{N};H)))},$$

$\Phi \in L^2([0,T];L^2(\Omega;l^2(\mathbb{N};H)))$.

Proposition 2.5. *Let H be a separable real Hilbert space. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $(W_i)_{i=0}^\infty$ be a sequence of independent normalized scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $\Phi = (\Phi_i)_{i=0}^\infty$ be a sequence of adapted piecewise uniformly continuous functions from $[0,\infty)$ to $L^2(\Omega;H)$ such that $\int_0^t \sum_{i=0}^\infty \mathbb{E}\,\|\Phi_i(s)\|^2 ds < \infty$ for every $t \in [0,\infty)$. Then:*

(i) $t \mapsto \int_0^t \Phi(s)dW(s)\colon [0,\infty) \to L^2(\Omega;H)$ *is continuous and adapted.*

(ii) If $T\colon H \to H$ is a bounded linear operator, then for each $t \geq 0$

$$\int_0^t T\Phi(s)dW(s) = T\int_0^t \Phi(s)dW(s),$$

which means explicitly that

$$\sum_{i=0}^\infty \int_0^t T \circ (\Phi_i(s))dW_i(s) = T \circ \Big(\sum_{i=0}^\infty \int_0^t \Phi_i(s)dW_i(s)\Big).$$

Proof. (i): Note that, by construction, each $t \mapsto \int_0^t \Phi_i(s)dW_i(s)$ is adapted and therefore the infinite sum is adapted as well. The continuity of the integral with respect to t follows from the Itô isometry and Lebesgue's dominated convergence theorem. Indeed, we have that

$$\begin{aligned}
&\mathbb{E}\,\|\int_0^{t+h} \Phi(s)dW(s) - \int_0^t \Phi(s)dW(s)\|^2 \\
&= \mathbb{E}\,\|\sum_{i=0}^\infty \int_0^{t+h} \Phi_i(s)\mathbb{1}_{[t,t+h]}(s)dW_i(s)\|^2 \\
&= \sum_{i=0}^\infty \int_0^{t+h} \mathbb{E}\,\|\Phi_i(s)\mathbb{1}_{[t,t+h]}(s)\|^2 ds \\
&= \int_0^{t+1} \mathbb{1}_{[t,t+h]}(s)\mathbb{E}\sum_{i=0}^\infty \|\Phi_i(s)\|^2 ds \\
&\to 0 \text{ as } h \downarrow 0,
\end{aligned}$$

for every $t \geq 0$, and continuity follows.

(ii): As the map $f \mapsto T \circ f$ on $L^2(\Omega; H)$ is linear, bounded, and preserves adaptedness, the assertion follows from the very construction of the integral. $\square$

3 Equivalence with Da Prato and Zabczyk's approach

In this section we will show that the stochastic integral constructed in Section 2 contains the same information as the integrals given by Da Prato and Zabczyk in [4], both for trace class covariance and cylindrical Wiener processes. A translation scheme between the two definitions is provided by making series expansions of the Wiener processes and considering elementary functions. For earlier work on the integral of [4] we refer to [2] and [3, Chapter 5] and references therein.

We will need the following lemma on Bochner spaces. A proof is given in the Appendix. If $(S, \mathcal{F}, \mu)$ is a σ-finite measure space, X is a Banach space, and $1 \leq p < \infty$, then $\mathcal{L}^p(S; X)$ denotes the space of strongly measurable functions $f : S \to X$ with $\int_S \|f\|^p d\mu < \infty$. By $L^p(S; X)$ we denote the Banach space of equivalence classes with respect to the relation of equality almost everywhere on S (see [5]). For $f \in \mathcal{L}^p(S; X)$ we denote by $[f]$ the equivalence class of $L^p(S; X)$ that contains f.

Lemma 3.1. *Let $(s, \mathcal{F}, \mu)$ and $(T, \mathcal{G}, \nu)$ be σ-finite measure spaces, let X be a Banach space, and let $1 \leq p < \infty$. Then $L^p(S \times T; X)$ and $L^p(S; L^p(T; X))$ are naturally isometrically isomorphic. More specifically, let*

$$D = \{f \in \mathcal{L}^p(S \times T; X) : f(s, \cdot) \in \mathcal{L}^p(T; X) \text{ for all } s \in S\}$$

and define for $f \in D$ the function $I(f) : S \to L^p(T; X)$ by $I(f) := s \mapsto [f(s, \cdot)]$. Then I is a linear map from D to $\mathcal{L}^p(S; L^p(T; X))$, $\|I(f)\|_{\mathcal{L}^p(S;L^p(T;X))} = \|f\|_{\mathcal{L}^p(S \times T;X)}$ for all $f \in D$, and for every $f, g \in \mathcal{L}^p(S \times T; X)$ one has $f = g$ a.e. on $S \times T$ if and only if $I(f) = I(g)$ a.e. on S. Moreoever, for every $F \in \mathcal{L}^p(S; L^p(T; X))$ there is an $f \in D$ with $I(f) = F$ a.e. on S, each equivalence class of $L^p(S \times T; X)$ contains an element of D, and therefore I induces a linear, isometrical bijection from $L^p(S \times T; X)$ onto $L^p(S; L^p(T; X))$.

Let H be a separable real Hilbert space. For $I \subset \mathbb{N}$, define the Hilbert space

$$l^2(I; H) := \{x\colon I \to H\colon \sum_{i \in I} \|x_i\|^2 < \infty\},$$

with the inner product $\langle x, y \rangle_{l^2} = \sum_{i \in I} \langle x_i, y_i \rangle$, $x, y \in l^2(I; H)$. Let U be a separable real Hilbert space and consider a Wiener process W in U with covariance operator Q that is of trace class. Since Q is self-adjoint, positive definite, and of trace class, there is an orthonormal basis of eigenvectors $(e_i)_i$ of Q with corresponding eigenvalues $(\lambda_i)_i$ with $\lambda_i \geq 0$ for all i and $\sum_i \lambda_i < \infty$. Let $I := \{i \in \mathbb{N}\colon\ \lambda_i \neq 0\}$ and define

$$W_i(t) = \lambda_i^{-1/2} \langle W(t), e_i \rangle, \quad t \geq 0,\ i \in I.$$

We have that $(W_i)_{i\in I}$ is a sequence of independent normalized scalar Wiener processes. (In fact, the process W is constructed by $W(t) = \sum_i \sqrt{\lambda_i} W_i(t) e_i$, $t \geq 0$.) The space $U_0 := Q^{1/2}(U)$ endowed with the inner product induced by $Q^{1/2}$, that is,

$$\langle x, y\rangle_{U_0} = \left\langle Q^{-1/2}x, Q^{-1/2}y \right\rangle_U, \quad x, y \in U_0,$$

contains e_i for all $i \in I$ and has $(\sqrt{\lambda_i} e_i)_{i\in I}$ as an orthonormal basis. Therefore a linear operator Ψ from U_0 to H is Hilbert-Schmidt if and only if

$$\sum_{i\in I} \|\Psi(\sqrt{\lambda_i} e_i)\|^2 = \sum_{i\in I} \lambda_i \|\Psi e_i\|^2 < \infty,$$

which means that $(\sqrt{\lambda_i}\Psi e_i)_i \in l^2(I;H)$. Thus, the Hilbert space L_2^0 of [4] consisting of the Hilbert-Schmidt operators from U_0 to H is isometrically isomorphic to $l^2(I;H)$, where the isomorphism $\mathcal{I}$ is given by

$$\mathcal{I}(\Psi) = (\sqrt{\lambda_i}\Psi e_i)_{i\in I}, \qquad \Psi \in L_2^0. \tag{9}$$

In particular,

$$\|\Psi\|^2_{L_2^0} = \sum_i \lambda_i \|\Psi e_i\|^2 = \|(\sqrt{\lambda_i}\Psi e_i)_i\|^2_{l^2(I;H)} = \|\mathcal{I}(\Psi)\|^2_{l^2(I;H)}.$$

Since the integrands that Da Prato and Zabczyk consider in [4] are functions with values in L_2^0, we can use the isomorphism $\mathcal{I}$ of (9) to identify them with functions with values in $l^2(I;H)$. More precisely, we have that $L^2([0,T]\times\Omega; L_2^0)$ is isometrically isomorphic to $L^2([0,T]\times\Omega; L^2(I;H))$, which is on its turn naturally isometrically isomorphic to $L^2([0,T]; L^2(\Omega; l^2(I;H)))$, by Lemma 3.1. Moreover, the 'predictable' elements of $L^2([0,T]\times\Omega; L_2^0)$ considered in [4] correspond to the adapted elements of $L^2([0,T]; L^2(\Omega; l^2(I;H)))$. Therefore the integrands of [4] coincide with the integrands in our approach (see Remark 2.4.4).

Let us now establish that the integrals coincide as well. A function Φ from $[0,T]$ to $L^2(\Omega; \mathcal{L}(U,H))$ is called *elementary* in [4] if

$$\Phi(s) = \sum_{k=0}^{n-1} \Phi_k \mathbb{1}_{(t_k,t_{k+1}]}(s), \quad 0 \leq s \leq T, \tag{10}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $\Phi_0, \ldots, \Phi_{n-1}$ are random variables, each taking only a finite number of values in the space $\mathcal{L}(U,H)$ of bounded linear operators from U to H and such that each Φ_k is $\mathcal{F}_{t_k}$-measurable. For a function Φ given by (10) the integral in [4] is defined by

$$\int_0^T \Phi(s)dW(s) := \sum_{k=0}^{n-1} \Phi_k\big(W(t_{k+1}) - W(t_k)\big). \tag{11}$$

It is then observed in [4] that every elementary function Φ may be viewed as a predictable element of $L^2([0,T] \times \Omega; L_2^0)$ and that the elementary functions are dense in the subspace of $L^2([0,T] \times \Omega; L_2^0)$ consisting of predictable functions. This allows one to extend the stochastic integral to an isometry from the predictable functions of $L^2([0,T] \times \Omega; L_2^0)$ to $L^2(\Omega; H)$.

As $L^2([0,T]\times\Omega; L_2^0)$ is isometrically isomorphic to $L^2([0,T]; L^2(\Omega; l^2(I;H)))$, it suffices to show that our definition of $\int_0^T \Phi(s)dW(s)$ coincides with the definition given above for elementary processes. (For general processes the equivalence then follows by extension.) So, let Φ be elementary as in (10) and write out (11),

$$\begin{aligned}
\sum_{k=0}^{n-1} \Phi_k\big(W(s_{k+1}) - W(s_k)\big) &= \sum_{k=0}^{n-1} \Phi_k \sum_{i\in I} \sqrt{\lambda_i}\big(W_i(s_{k+1}) - W_i(s_k)\big)e_i \\
&= \sum_{i\in I} \sqrt{\lambda_i} \int_0^T \Phi(s)e_i dW_i(s) \\
&= \int_0^T (\sqrt{\lambda_i}\Phi(s)e_i)_{i\in I} d(W_i)_{i\in I}(s),
\end{aligned}$$

where the latter integral is as in Section 2. The sequence $(\sqrt{\lambda_i}\Phi(s)e_i)_{i\in I}$ is exactly the image of $\Phi(s)$ under the isomorphism $\mathcal{I}$ of (9) between L_2^0 and $l^2(I;H)$.

We have a similar equivalence with the integral of [4] for cylindrical Wiener processes. A cylindrical Wiener process W in U is a Wiener process with a covariance operator Q on U that is bounded linear, self-adjoint, and positive definite, but not of trace class. Such a process can be constructed in the same fashion as in the trace class case, but one can no longer use the space U itself. Instead, let U_1 be a separable real Hilbert space such that U is continuously embedded in U_1 and such that there is an embedding J from $U_0 = Q^{1/2}(U)$ into U_1 that is Hilbert-Schmidt. So, given an orthonormal basis $(g_i)_i$ of U_0 one has that $\sum_i \|Jg_i\|^2 < \infty$, and

$$W(t) := \sum_i W_i(t)Jg_i, \quad t \geq 0, \tag{12}$$

defines a Wiener process in U_1. Furthermore, it turns out that the covariance operator Q_1 of W equals JJ^* which is a trace class operator on U_1. The process W defined by (12) is called a cylindrical Wiener process on U. Notice that the space U_1 is not unique. However, it can be shown that $\tilde{U}_0 := Q_1^{1/2}(U_1)$ is isometrically isomorphic to U_0 and therefore independent of the choice of U_1. Accordingly, the space $\tilde{L}_2^0$ of Hilbert-Schmidt operators from $\tilde{U}_0$ to H is unambiguous and the stochastic integral of $\tilde{L}_2^0$-valued integrands with respect to W is therefore well-defined by the above construction with U and Q replaced by U_1 and Q_1. This is the definition given in [4] and the correspondence with the series definition of Section 2 proceeds along the same lines as above, with Q replaced by Q_1.

Thus we arrive at the following equivalence.

Theorem 3.2 (equivalence of the stochastic integrals). *Let W be a Q-Wiener process in a separable real Hilbert space U with Q of trace class and let Φ be a predictable process with values in L_2^0 such that*

$$|||\Phi|||_T := (\mathbb{E} \int_0^T \|\Phi(s)\|_{L_2^0}^2 ds)^{1/2} < \infty$$

(see [4]). Take an orthonormal basis $(e_i)_i$ in U and numbers $\lambda_i \geq 0$, $i \in \mathbb{N}$, with $\sum_i \lambda_i < \infty$ so that $W(t) = \sum_i W_i(t)\sqrt{\lambda_i} e_i$, $t \geq 0$, where $(W_i)_i$ is a sequence of independent normalized scalar Wiener processes. Define

$$\Phi_i(t)(\omega) := \sqrt{\lambda_i}\Phi(t,\omega)e_i, \qquad \omega \in \Omega,\ t \geq 0,\ i \in \mathbb{N}.$$

Then one may view Φ_i as an element of $L^2([0,T]; L^2(\Omega; H))$ and Φ_i is adapted for all $i \in \mathbb{N}$. Moreover, $(\Phi_i)_i$ satisfies (8) and one has for the integral as defined in [4]

$$\int_0^T \Phi(s)dW(s) = \sum_{i=0}^{\infty} \int_0^T \Phi_i(s)dW_i(s) = \int_0^T (\Phi_i)_i(s)d(W_i)_i(s), \tag{13}$$

where the latter integral is in the sense of Section 2. If the Wiener process W is cylindrical, then it is actually a Q_1-Wiener process on some separable real Hilbert space U_1 with Q_1 of trace class and one has the same conclusion as above with U replaced by U_1.

Conversely, let $(W_i)_{i\in\mathbb{N}}$ be a sequence of independent normalized scalar Wiener processes and let $(\Phi_i)_{i=0}^{\infty}$ be a sequence of adapted functions in the space $L^2([0,T]; L^2(\Omega; H))$ satisfying (8). Choose an arbitrary separable real Hilbert space U with an orthonormal basis $(e_i)_{i=0}^{\infty}$ and numbers $\lambda_i > 0$ with $\sum_i \lambda_i < \infty$, put $Qx := \sum_i \lambda_i \langle x, e_i \rangle$, $x \in U$, and define a Q-Wiener process in U by $W(t) := \sum_i W_i(t)\sqrt{\lambda_i} e_i$. Use the natural isomorphism of $L^2([0,T]; L^2(\Omega; H))$ and $L^2([0,T] \times \Omega; H)$ to define

$$\Phi(t,\omega)x := \sum_{i\in I} \lambda_i^{-1/2} \langle x, e_i \rangle \Phi_i(t)(\omega), \qquad x \in U_0,\ t \geq 0,\ \omega \in \Omega,$$

where $U_0 = Q^{1/2}(U)$. Then Φ is a predictable process with values in L_2^0 satisfying $|||\Phi|||_T < \infty$, and one has again relation (13).

Remark 3.3. The only information on W that is used for the definition of Da Prato and Zabczyk's integral in [4]—both in the trace class and cylindrical case—is the information contained in the covariance operator Q. More than that, the integrands still leave the freedom to change the directions and amplitudes of the noise components. In the construction of Section 2, we have absorbed all such information into the integrands and undressed the Wiener process to only a sequence of independent normalized scalar Wiener processes. We believe that the remaining construction is simpler and from a modelling point of view maybe even more natural.

4 Stochastic differential equations

In this section we consider semilinear stochastic differential equations of the form (1) with infinite dimensional noise and Lipschitz nonlinearities. We interpret (1) as the integral equation (4) with the stochastic integral defined as in Section 2. We will prove existence and strong and weak uniqueness for the initial value problem associated to (4). Existence and strong uniqueness results based on the stochastic integral developed in [4] are readily available in the literature. However, weak uniqueness results are often less explicit and therefore we treat it in detail.

Let H be a separable real Hilbert space and let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup in H with generator A. Let $F\colon [0,\infty) \times H \to H$ be a continuous function and let $B = (B_i)_{i=0}^{\infty}$ be a sequence of continuous functions $B_i\colon [0,\infty) \times H \to H$, $i \in \mathbb{N}$, such that there exist constants $L_F, L_B \geq 0$ with for all $t \geq 0$ and $x, y \in H$:

$$\begin{aligned} \|F(t,x) - F(t,y)\| &\leq L_F\|x-y\|, \\ \sum_{i=0}^{\infty} \|B_i(t,x) - B_i(t,y)\|^2 &\leq L_B^2\|x-y\|^2, \\ \int_0^t \sum_{i=0}^{\infty} \|B_i(s,0)\|^2\,ds &< \infty. \end{aligned} \tag{14}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W = (W_i)_{i=0}^{\infty}$ be a sequence of independent normalized scalar Wiener processes on Ω with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $X_0 \in L^2(\Omega; H)$ be $\mathcal{F}_0$-measurable and consider the initial value problem

$$\begin{cases} dX(t) = AX(t)dt + F(t,X(t))dt + B(t,X(t))dW(t), \quad t \geq 0, \\ X(0) = X_0. \end{cases} \tag{15}$$

We interpret (15) as the stochastic integral equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s))dW(s), \quad t \geq 0, \tag{16}$$

where the latter integral is defined as in Section 2:

$$\int_0^t S(t-s)B(s,X(s))dW(s) = \sum_{i=0}^{\infty} \int_0^t S(t-s)B_i(s,X(s))dW_i(s).$$

Observe that the integrals are well-defined whenever X is a continuous and adapted function from $[0,t]$ to $L^2(\Omega; H)$. Indeed, in that case the functions $s \mapsto F(s,X(s))$ and $s \mapsto B_i(s,X(s))$ are continous and adapted for every i. Hence $s \mapsto S(t-s)F(s,X(s))$ and $s \mapsto S(t-s)B_i(s,X(s))$ are continous from $[0,t]$ to $L^2(\Omega; H)$ and adapted. Furthermore, let $M_t \geq 0$ be such that $\|S(s)\| \leq M_t$ for all $0 \leq s \leq t$, $t \geq 0$, and observe that

$$\begin{aligned}
&\int_0^t \mathbb{E} \sum_{i=0}^{\infty} \|S(t-s)B_i(s,X(s))\|^2 ds \\
&\leq M_t^2 \int_0^t \mathbb{E} \sum_{i=0}^{\infty} \left(2\|B_i(s,X(s)) - B_i(s,0)\|^2 + 2\|B_i(s,0)\|^2\right) ds \\
&\leq 2M_t^2 L_B^2 \int_0^t \mathbb{E}\, \|X(s)\|^2 ds + 2M_t^2 \int_0^t \sum_{i=0}^{\infty} \|B_i(s,0)\|^2 ds \\
&< \infty \qquad \text{for } t \geq 0.
\end{aligned}$$

A continuous and adapted function $X\colon [0,\infty) \to L^2(\Omega; H)$ that satisfies equation (16) for each $t \geq 0$ is called a *mild solution* of (15) and (16).

Our aim in this section is to prove existence and uniqueness of a mild solution of (16). Uniqueness in this sense is usually called *strong uniqueness*. However, in many questions concerning the stochastic behavior of the solution, it is not the function $X\colon [0,\infty) \times \Omega \to H$ that is important, but rather its family of distributions $\{\mu_{X(t)} : t \geq 0\}$. Typical questions are usually: with what probability is $X(3)$ in a set $V \subset H$? with what probability is $X(t)$ less than 0 for $t \leq 1$ and more than 10 for $t \geq 5$? is there for every $\varepsilon > 0$ a compact set $K \subset H$ such that $X(t)$ is in K with probability at least $1-\varepsilon$ for all $t \geq 0$? For such questions the underlying space Ω is irrelevant. More than that, when (15) is viewed as a model equation for a certain stochastic phenomenon, the choice of the probability space Ω is arbitrary. What matters is that the noise is generated by Wiener processes and not on what probability space these processes are constructed. If one would use another probability space, the essential behavior of the solution should not be affected, or rather its distributions should be the same. This kind of uniqueness is called *weak uniqueness* (or *uniqueness in law* [11]). Its definition is exactly the first statement of Theorem 4.2.

Theorem 4.1 (existence and strong uniqueness of mild solutions). *There exists one and only one function $X : [0,\infty) \to L^2(\Omega; H)$ that is continuous, adapted to $(\mathcal{F}_t)_{t\geq 0}$, and that satisfies (16).*

Theorem 4.2 (weak uniqueness of mild solutions). *Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space and let $W' = (W_i')_{i=0}^{\infty}$ be a sequence of independent normalized scalar Wiener processes on Ω' with respect to some filtration $(\mathcal{F}_t')_{t\geq 0}$ in $\mathcal{F}'$. Let $X'\colon [0,\infty) \to L^2(\Omega'; H)$ be continuous, adapted, and such that*

$$X'(t) = S(t)X_0' + \int_0^t S(t-s)F(s, X'(s))ds + \int_0^t S(t-s)B(s, X'(s))dW'(s),$$

for all $t \geq 0$, where $X_0' \in L^2(\Omega'; H)$ is $\mathcal{F}_0'$-measurable. If X_0' has the same distribution as X_0, then $X(t)$ (with X as in Theorem 4.1) and $X'(t)$ have the same distribution for all $t \geq 0$. More than that, for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ and $m \in \mathbb{N}$, the random vectors

$$\big(X(t_1), \ldots, X(t_n), W_1(t_1), \ldots, W_1(t_n), \ldots, W_m(t_1), \ldots, W_m(t_n)\big)$$

and

$$\big(X'(t_1), \ldots, X'(t_n), W_1'(t_1), \ldots, W_1'(t_n), \ldots, W_m'(t_1), \ldots, W_m'(t_n)\big)$$

have the same distribution.

The basic idea is to use a Banach fixed point argument, similar to the proofs in a finite dimensional setting in [14, Chapter 5].

Proof of Theorem 4.1. Let $T > 0$. It suffices to prove existence and strong uniqueness on $[0,T]$. Let C_{ad} denote the subspace of $C([0,T]; L^2(\Omega; H))$ consisting of functions that are adapted to $(\mathcal{F}_t)_t$. The subspace C_{ad} is closed with respect to the norm $\|.\|_\infty$ given by

$$\|X\|_\infty := \max_{0\leq t\leq T} \|X(t)\|_{L^2(\Omega;H)} = \max_{0\leq t\leq T} (\mathbb{E}\,\|X(t)\|^2)^{1/2},$$

$X \in C([0,T]; L^2(\Omega; H))$. Define for $X \in C_{\mathrm{ad}}$ the function

$$R(X)(t) := S(t)X_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)B(s, X(s))dW(s), \tag{17}$$

$t \in [0,T]$. Since $R(X)(t) \in L^2(\Omega; H)$ for all t, $R(X)$ is continuous from $[0,T]$ to $L^2(\Omega; H)$ (use Itô isometry), and $R(X)$ is adapted, it follows that R maps C_{ad} into itself.

Let $X, Y \in C_{\mathrm{ad}}$. For $t \in [0,T]$ we have

$$
\begin{aligned}
&\mathbb{E}\,\|R(X)(t) - R(Y)(t)\|^2 \\
&\le \mathbb{E}\,\Bigg\{ 2\left(\int_0^t \|S(t-s)\big(F(s,X(s)) - F(s,Y(s))\big)\|ds\right)^2 \\
&\qquad\qquad + 2\|\sum_{i=0}^{\infty}\int_0^t S(t-s)\big(B_i(s,X(s)) - B_i(s,Y(s))\big)dW_i(s)\|^2 \Bigg\} \\
&\le 2\mathbb{E}\left(\int_0^t M_t L_F \|X(s) - Y(s)\|ds\right)^2 \\
&\qquad\qquad + 2\sum_{i=0}^{\infty}\int_0^t \mathbb{E}\,\|S(t-s)\big(B_i(s,X(s)) - B_i(s,Y(s))\big)\|^2 ds \\
&\le 2M_t^2 L_F^2 t\,\mathbb{E}\int_0^t \|X(s) - Y(s)\|^2 ds \\
&\qquad\qquad + 2M_t^2 \int_0^t \mathbb{E}\sum_{i=0}^{\infty} \|B_i(s,X(s)) - B_i(s,Y(s))\|^2 ds \\
&\le 2M_t^2 L_F^2 t \int_0^t \mathbb{E}\,\|X(s) - Y(s)\|^2 ds + 2M_t^2 L_B^2 \int_0^t \mathbb{E}\,\|X(s) - Y(s)\|^2 ds \\
&\le 2(L_F^2 T + L_B^2) M_T^2 \int_0^t \mathbb{E}\,\|X(s) - Y(s)\|^2 ds.
\end{aligned}
$$

Hence for any $a > 0$ and $t \in [0,T]$ we find

$$
\begin{aligned}
&e^{-at}\mathbb{E}\,\|R(X)(t) - R(Y)(t)\|^2 \\
&\le 2(L_F^2 T + L_B^2) M_T^2 \int_0^t e^{-a(t-s)} e^{-as}\mathbb{E}\,\|X(s) - Y(s)\|^2 ds \\
&\le 2(L_F^2 T + L_B^2) M_T^2 \int_0^t e^{-a(t-s)} ds \max_{0\le s\le t} e^{-as}\mathbb{E}\,\|X(s) - Y(s)\|^2 \\
&\le 2a^{-1}(L_F^2 T + L_B^2) M_T^2 \max_{0\le s\le t} e^{-as}\mathbb{E}\,\|X(s) - Y(s)\|^2,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&\max_{0\le t\le T} e^{-at}\mathbb{E}\,\|R(X)(t) - R(Y)(t)\|^2 \\
&\qquad\qquad \le 2a^{-1}(L_F^2 T + L_B^2) M_T^2 \max_{0\le s\le T} e^{-as}\mathbb{E}\,\|X(s) - Y(s)\|^2.
\end{aligned}
$$

Now choose $a > 2(L_F^2 T + L_B^2) M_T^2$, then R is a strict contraction on C_{ad} with respect to the norm $|||.|||$ given by

$$
|||X||| = \max_{0\le t\le T}\left(e^{-at}\mathbb{E}\,\|X(t)\|^2\right)^{1/2}, \quad X \in C_{\mathrm{ad}}.
$$

The norm $|||.|||$ is equivalent to $\|.\|_\infty$ and therefore $(C_{\mathrm{ad}}, |||.|||)$ is a Banach space. Banach's fixed point theorem therefore yields that R has a unique fixed point in C_{ad}. It is clear that this fixed point is the unique solution of (16) on $[0, T]$. $\square$

To prove the weak uniqueness theorem, we will consider the sequence of iterates $X_1 = R(X_0), X_2 = R(X_1), \ldots$ (with the notations from the proof of Theorem 4.1) and show that the distribution of each $X_n(t)$ depends only on the distribution of X_0 and not on the underlying probability spaces of X_0 and W. By the series construction of the stochastic integral, the main point is to understand when the distributions of two Itô integrals $\int_0^t g(s)dB(s)$ and $\int_0^t g'(s)dB'(s)$ are equal, where B and B' are scalar normalized Wiener processes. A naive guess (and a naive interpretation of the text in the proof of Corollary 5.13 on p. 127 of [14]) is that it suffices that $(g(t_1), \ldots, g(t_n))$ and $(g'(t_1), \ldots, g'(t_n))$ have the same distribution for every $0 \le t_1 \le t_2 \le \cdots \le t_n \le t$. However, one has to be more careful.

Example 4.3. Let B be a normalized scalar Wiener process and let N_1 and N_2 be two independent standard Gaussian random variables both independent of B. Consider the functions

$$\begin{aligned} g(t) &= B(1)\mathbb{1}_{[1,2)}(t) + B(2)\mathbb{1}_{[2,3]}(t) \\ &= B(1)\mathbb{1}_{[1,3)}(t) + (B(2) - B(1))\mathbb{1}_{[2,3]}(t), \quad t \ge 0, \end{aligned}$$

and

$$g'(t) = N_1\mathbb{1}_{[1,3)}(t) + N_2\mathbb{1}_{[2,3]}(t), \quad t \ge 0.$$

If $(\mathcal{F}_t)_{t\ge 0}$ is the filtration generated by $(B(t))_{t\ge 0}$, N_1, and N_2, then B is a Wiener proces with respect to $(\mathcal{F}_t)_{t\ge 0}$ and both g and g' are adapted simple functions from $[0,\infty)$ to $L^2(\Omega; H)$. For every $0 \le t_1 \le t_2 \le \cdots \le t_n$ the random vectors $(g(t_1), \ldots, g(t_n))$ and $(g'(t_1), \ldots, g'(t_n))$ have the same distribution. However, $\int_0^3 g(s)dB(s)$ and $\int_0^3 g'(s)dB(s)$ do not have the same distribution. Indeed,

$$\begin{aligned} &\mathbb{E}\Big(\int_0^3 g(s)dB(s)\Big)^3 = \mathbb{E}\Big(B(1)\big(B(2)-B(1)\big) + B(2)\big(B(3)-B(2)\big)\Big)^3 \\ &= \mathbb{E}\,B(1)^3\big(B(2)-B(1)\big)^3 + 3\mathbb{E}\,B(1)^2\big(B(2)-B(1)\big)^2 B(2)\big(B(3)-B(2)\big) \\ &\quad +3\mathbb{E}\,B(1)\big(B(2)-B(1)\big)B(2)^2\big(B(3)-B(2)\big)^2 + \mathbb{E}\,B(2)^3\big(B(3)-B(2)\big)^3 \\ &= 3\mathbb{E}\,B(1)\big(B(2)-B(1)\big)B(2)^2\mathbb{E}\,\big(B(3)-B(2)\big)^2 \end{aligned}$$

$$
\begin{aligned}
&= 3\mathbb{E}\, B(1)\big(B(2)-B(1)\big)\big(B(1)+B(2)-B(1)\big)^2 \\
&= 3\mathbb{E}\, B(1)\big(B(2)-B(1)\big)\Big(B(1)^2+2B(1)\big(B(2)-B(1)\big)+\big(B(2)-B(1)\big)^2\Big) \\
&= 6\mathbb{E}\, B(1)^2\big(B(2)-B(1)\big)^2 \\
&= 6\mathbb{E}\, B(1)^2\mathbb{E}\,\big(B(2)-B(1)\big)^2 = 6,
\end{aligned}
$$

whereas

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^3 g'(s)dB(s)\Big)^3 = \mathbb{E}\,\Big(N_1\big(B(3)-B(1)\big)+N_2\big(B(3)-B(2)\big)\Big)^3 \\
= \;& \mathbb{E}\, N_1^3\big(B(3)-B(1)\big)^3+3\mathbb{E}\, N_1^2\big(B(3)-B(1)\big)^2N_2\big(B(3)-B(2)\big) \\
& +3\mathbb{E}\, N_1\big(B(3)-B(1)\big)N_2^2\big(B(3)-B(2)\big)^2+\mathbb{E}\, N_2^3\big(B(3)-B(2)\big)^3 \\
= \;& 0,
\end{aligned}
$$

so that $\int_0^3 g(s)dB(s)$ and $\int_0^3 g'(s)dB(s)$ do not have the same distribution.

Let us now consider conditions that provide that the random variables $\int_0^t g(s)dB(s)$ and $\int_0^t g'(s)dB'(s)$ are equally distributed for all t. It turns out that it suffices that the joint distributions of g and B equal the joint distributions of g' and B'. In view of later application, we will prove a more general statement. We first prove the following lemma on convergence. In the sequel, the distribution of a random variable $X \in L^2(\Omega;H)$ will be denoted by μ_X.

Lemma 4.4. *Let H be a separable real Hilbert space and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space.*

(i) If a sequence $(X_n)_n$ in $L^2(\Omega;H)$ and $X \in L^2(\Omega;H)$ are such that $X_n \to X$ in $L^2(\Omega;H)$, then there is a subsequence, given by $(n_k)_k$, such that $\mu_{X_{n_k}} \to \mu_X$ in the sense of weak convergence of measures.

(ii) If $(X_n)_n$ and $(Y_n)_n$ are sequences in $L^2(\Omega;H)$, $X,Y \in L^2(\Omega;H)$, $X_n \to X$, $Y_n \to Y$ in $L^2(\Omega;H)$, and $\mu_{X_n} = \mu_{Y_n}$ for all n, then $\mu_X = \mu_Y$.

Proof. (i): There is a subsequence such that $X_{n_k} \to X$ a.e. on Ω. (Indeed, $\int_\Omega \|X_n - X\|^2 d\mathbb{P} \to 0$ so that $\|X_{n_k} - X\|^2 \to 0$ a.e. for a certain subsequence.) Then $f(X_{n_k}) \to f(X)$ a.e. for every bounded continuous $f\colon H \to H$ so that, by dominated convergence, $\mathbb{E}\, f(X_{n_k}) \to f(X)$ for every bounded continuous $f\colon H \to H$. Hence $\mu_{X_{n_k}} \to \mu_X$.

(ii): Direct from (i). □

By a *family of random variables* on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ we mean a family $(X_i)_{i\in I}$ where each X_i is a measurable mapping from Ω to some Hilbert space H_i. For a finite choice $i_1,\dots,i_n \in I$ the random vector $(X_{i_1},\dots,X_{i_n})$

may be viewed as a random variable and its distribution is a measure on the product Hilbert space $H_{i_1} \times \cdots \times H_{i_n}$. Given two families $(X_i)_{i\in I}$ and $(X'_i)_{i\in I}$ of random variables on Ω and another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, respectively, we say that they are *equally distributed* if for any $i_1, \ldots, i_n \in I$ the vectors $(X_{i_1}, \ldots, X_{i_n})$ and $(X'_{i_1}, \ldots, X'_{i_n})$ map into the same Hilbert space and have the same distribution.

Lemma 4.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ be probability spaces and let $(Z_i^{(n)})_{i\in I}$ and $(Z_i^{\infty})_{i\in I}$, $n \in \mathbb{N}$, and $(Z_i'^{(n)})_{i\in I}$ and $(Z_i'^{\infty})_{i\in I}$, $n \in \mathbb{N}$, be families of random variables on Ω and Ω', respectively. Suppose that for each i the random variables $Z_i^{(n)}$, $Z_i'^{(n)}$, Z_i^{∞}, and $Z_i'^{\infty}$ take values in a separable real Hilbert space H_i for all n. Suppose also that $Z_i^{(n)} \to Z_i^{\infty}$ in $L^2(\Omega; H_i)$ and $Z_i'^{(n)} \to Z_i'^{\infty}$ in $L^2(\Omega'; H_i)$. If $(Z_i^{(n)})_{i\in I}$ and $(Z_i'^{(n)})_{i\in I}$ are equally distributed for every n, then $(Z_i^{\infty})_{i\in I}$ and $(Z_i'^{\infty})_{i\in I}$ are equally distributed.*

Proof. For any $i_1, \ldots, i_m \in I$ we have $(Z_{i_1}^{(n)}, \ldots, Z_{i_m}^{(n)}) \to (Z_{i_1}^{\infty}, \ldots, Z_{i_m}^{\infty})$ and $(Z_{i_1}'^{(n)}, \ldots, Z_{i_m}'^{(n)}) \to (Z_{i_1}'^{\infty}, \ldots, Z_{i_m}'^{\infty})$ in $L^2(\Omega; H_{i_1} \times \cdots \times H_{i_m})$ and $L^2(\Omega'; H_{i_1} \times \cdots \times H_{i_m})$, respectively, so that Lemma 4.4 yields that $(Z_{i_1}^{\infty}, \ldots, Z_{i_m}^{\infty})$ and $(Z_{i_1}'^{\infty}, \ldots, Z_{i_m}'^{\infty})$ have the same distribution. $\square$

Lemma 4.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ be probability spaces and let $(Z_i)_{i\in I}$ and $(Z_i')_{i\in I}$ be families of random variables on Ω and Ω', respectively. Let H be a separable real Hilbert space.*

(i) Let $g\colon [0,\infty) \to L^2(\Omega; H)$ and $g'\colon [0,\infty) \to L^2(\Omega'; H)$ be piecewise uniformly continuous functions. If

$$\big(g(t), Z_i\colon\ t \geq 0,\ i \in I\big)$$

and $\big(g'(t), Z'_i\colon\ t \geq 0,\ i \in I\big)$ are equally distributed, then

$$\big(\textstyle\int_0^t g(s)\, ds, Z_i\colon\ t \geq 0,\ i \in I\big)$$

and $\big(\int_0^t g'(s)\, ds, Z'_i\colon\ t \geq 0,\ i \in I\big)$ are equally distributed.

(ii) Let $W = (W_k)_{k\in\mathbb{N}}$ and $W' = (W_k')_{k\in\mathbb{N}}$ be sequences of independent scalar Wiener processes on Ω and Ω', respectively, and let $g = (g_k)_{k\in\mathbb{N}}$ and $g' = (g_k')_{k\in\mathbb{N}}$ be sequences of adapted piecewise uniformly continuous functions from $[0,\infty)$ to $L^2(\Omega; H)$ and $L^2(\Omega'; H)$, respectively. Assume that $\int_0^t \mathbb{E} \sum_i \|g_i(s)\|^2\, ds < \infty$ and $\int_0^t \mathbb{E} \sum_i \|g_i'(s)\|^2\, ds < \infty$ for all $t \geq 0$. If

$$\big(g_k(t), W_k(t), Z_i\colon\ t \geq 0,\ k \in \mathbb{N},\ i \in I\big)$$

and $\left(g_k'(t), W_k'(t), Z_i' \colon\ t \geq 0,\ k \in \mathbb{N},\ i \in I\right)$ *are equally distributed, then*

$$\left(\int_0^t g(s)\, dW(s), Z_i \colon\ t \geq 0,\ i \in I\right)$$

and $\left(\int_0^t g'(s)\, dW'(s), Z_i' \colon\ t \geq 0,\ i \in I\right)$ *are equally distributed.*

Proof. (i): Fix $t_1, \ldots, t_n \geq 0$ and $i_1, \ldots, i_m \in I$. For any partitions $0 = s_0^k \leq \cdots \leq s_{p_k}^k = t_k$, $k = 1, \ldots, n$, we have that

$$\left(g(s_1^1), \ldots, g(s_{p_1}^1), \ldots, g(s_1^n), \ldots, g(s_{p_n}^n), Z_{i_1}, \ldots, Z_{i_m}\right)$$

and $\left(g'(s_1^1), \ldots, g'(s_{p_1}^1), \ldots, g'(s_1^n), \ldots, g'(s_{p_n}^n), Z_{i_1}', \ldots, Z_{i_m}'\right)$ have the same distribution. Therefore

$$\left(\sum_l g(s_l^1)(s_{l+1}^1 - s_l^1), \ldots, \sum_l g(s_l^n)(s_{l+1}^n - s_l^n), Z_{i_1}, \ldots, Z_{i_m}\right)$$

and $\left(\sum_l g'(s_l^1)(s_{l+1}^1 - s_l^1), \ldots, \sum_l g'(s_l^n)(s_{l+1}^n - s_l^n), Z_{i_1}', \ldots, Z_{i_m}'\right)$ have the same distribution as well. With aid of Lemma 4.4(ii), it follows that

$$\left(\int_0^{t_1} g(s)\, ds, \ldots, \int_0^{t_n} g(s)\, ds, Z_{i_1}, \ldots, Z_{i_m}\right)$$

and $\left(\int_0^{t_1} g'(s)\, ds, \ldots, \int_0^{t_n} g'(s)\, ds, Z_{i_1}', \ldots, Z_{i_m}'\right)$ have the same distribution.

(ii): Fix $t_1, \ldots, t_n \geq 0$ and $i_1, \ldots, i_m \in I$. Similar to the proof of (i), we can consider partitions $0 = s_0^k \leq \cdots \leq s_{p_k}^k = t_k$, $k = 1, \ldots, n$, and the random vector composed of $g_j(s_l^k)$, $W_j(s_l^k)$, with $l = 0, \ldots, p_k$, $k = 1, \ldots, n$, $j = 0, \ldots, J$, and the corresponding vector with primes, in order to find that

$$\left(\sum_{j=0}^J \int_0^{t_1} g_j(s)\, dW_j(s), \ldots, \sum_{j=0}^J \int_0^{t_n} g_j(s)\, dW_j(s), Z_{i_1}, \ldots, Z_{i_m}\right)$$

and $\left(\sum_{j=0}^J \int_0^{t_1} g_j'(s)\, dW_j'(s), \ldots, \sum_{j=0}^J \int_0^{t_n} g_j'(s)\, dW_j'(s), Z_{i_1}', \ldots, Z_{i_m}'\right)$ have the same distribution. Now let $J \to \infty$ and use Lemma 4.5 to conclude that

$$\left(\int_0^{t_1} g(s)\, dW(s), \ldots, \int_0^{t_n} g(s)\, dW(s), Z_{i_1}, \ldots, Z_{i_m}\right)$$

and $\left(\int_0^{t_1} g'(s)\, dW'(s), \ldots, \int_0^{t_n} g'(s)\, dW'(s), Z_{i_1}', \ldots, Z_{i_m}'\right)$ have the same distribution. $\square$

Corollary 4.7. *Let H be a real separable Hilbert space, $(S(t))_{t \geq 0}$ a strongly continuous semugroup on H, $F : [0, \infty) \times H \to H$ a continuous function, and $B = (B_i)_{i=0}^\infty$ a sequence of continuous functions $B_i : [0, \infty) \times H \to H$ satisfying (14). Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ be probability spaces and $W = (W_i)_{i=0}^\infty$ and $W' = (W_i')_{i=0}^\infty$ be sequences of independent Wiener processes on Ω and Ω', respectively. Let $X_0 \in L^2(\Omega; H)$ and $X_0' \in L^2(\Omega'; H)$. Let $X \colon\ [0, \infty) \to L^2(\Omega; H)$*

and $X'\colon\ [0,\infty) \to L^2(\Omega';H)$ *be adapted piecewise uniformly continuous functions and let*

$$\begin{aligned} Y(t) &:= S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s) \\ Y'(t) &:= S(t)X_0' + \int_0^t S(t-s)F(X'(s))ds + \int_0^t S(t-s)B(X'(s))dW'(s) \end{aligned}$$

If

$$\big(X(t),\ X_0,\ W_i(t)\colon\ t \geq 0,\ i \in \mathbb{N}\big)$$

and $\big(X'(t),\ X_0',\ W_i'(t)\colon\ t \geq 0,\ i \in \mathbb{N}\big)$ *are equally distributed, then*

$$\big(Y(t),\ X_0,\ W_i(t)\colon\ t \geq 0,\ i \in \mathbb{N}\big)$$

and $\big(Y'(t),\ X_0',\ W_i'(t)\colon\ t \geq 0,\ i \in \mathbb{N}\big)$ *are equally distributed.*

Proof. The assumptions yield that

$$\big(S(t-s)F(X(s)),\ S(t-s)B_k(X(s)),\ S(t)X_0,\ W_k(t)\colon\ 0 \leq s \leq t,\ t \geq 0,\ k \in \mathbb{N}\big)$$

and

$$\big(S(t-s)F(X'(s)),\ S(t-s)B_k(X'(s)),\ S(t)X_0',\ W_k'(t)\colon\ 0 \leq s \leq t,\ t \geq 0,\ k \in \mathbb{N}\big)$$

are equally distributed. By applying Lemma 4.6 twice and with appropriate choices of the families $(Z_i)_{i\in I}$ and $(Z_i')_{i\in I}$, we find that

$$\big(\textstyle\int_0^t S(t-s)F(X(s))ds,\ \int_0^t S(t-s)B(X(s))dW(s),\ S(t)X_0,\ W_k(t)\colon\ t \geq 0,\ k \in \mathbb{N}\big)$$

and

$$\big(\textstyle\int_0^t S(t-s)F(X'(s))ds,\ \int_0^t S(t-s)B(X'(s))dW'(s),\ S(t)X_0',\ W_k'(t)\colon\ t \geq 0,\ k \in \mathbb{N}\big)$$

are equally distributed, and the assertion follows. $\square$

Proof of Theorem 4.2. Let $X_0(t) := X_0$, $X_0'(t) := X_0'$, $t \geq 0$. Define

$$\begin{aligned} X_{n+1}(t) &:= S(t)X_0 + \int_0^t S(t-s)F(X_n(s))ds + \int_0^t S(t-s)B(X_n(s))dW(s), \\ X_{n+1}'(t) &:= S(t)X_0' + \int_0^t S(t-s)F(X_n'(s))ds + \int_0^t S(t-s)B(X_n'(s))dW'(s), \end{aligned}$$

$t \geq 0$, $n = 0, 1, \ldots$. Fix a $T > 0$. With the operator R defined by (17) we have that $X_{n+1} = R(X_n)$ and since R is a contraction (with respect to the suitably scaled norm $|||.|||$), we obtain that $|||X_n - X||| \to 0$, where X is the continuous and adapted function from $[0,\infty)$ to $L^2(\Omega;H)$ satisfying

$$X(t) := S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s), \quad t \geq 0.$$

In particular, $X_n(t) \to X(t)$ in $L^2(\Omega; H)$ for every $t \in [0,T]$, and since T was arbitrary, for every $t \geq 0$. Similarly, $X_n'(t) \to X'(t)$ in $L^2(\Omega'; H)$ for every $t \geq 0$.

The families $(X_0(t),\ X_0,\ W_i(t)\colon\ t \geq 0,\ i \in \mathbb{N})$ and $(X_0'(t),\ X_0',\ W_i'(t)\colon\ t \geq 0,\ i \in \mathbb{N})$ are equally distributed and it follows from Corollary 4.7 that then

$$(X_n(t),\ X_0,\ W_i(t)\colon\ t \geq 0,\ i \in \mathbb{N}) \text{ and } (X_n'(t),\ X_0',\ W_i'(t)\colon\ t \geq 0,\ i \in \mathbb{N})$$

are equally distributed. Lemma 4.5 now yields that

$$(X(t),\ X_0,\ W_i(t)\colon\ t \geq 0,\ i \in \mathbb{N}) \text{ and } (X'(t),\ X_0',\ W_i'(t)\colon\ t \geq 0,\ i \in \mathbb{N})$$

are equally distributed, and the proof is complete. □

To conclude this paper we would like to point out by means of an example that the definition of the stochastic integral as a series of stochastic integrals with respect to scalar Wiener processes may have advantages for numerical simulations. Indeed, in certain cases this definition allows us to straightforwardly reduce the infinite dimensional equation to a numerically tractable finite dimensional equation (see also [1]).

Example 4.8. Let us assume that the linear operator A in (15) has a complete orthonormal system of eigenvectors $(e_i)_{i=0}^{\infty}$ with corresponding eigenvalues $(\lambda_i)_{i=0}^{\infty}$. If each Wiener process W_i generates noise in the direction of e_i, that is,

$$B_i(t,x) = \varphi_i(t,x)e_i, \qquad t \geq 0,\ x \in H,\ i \in \mathbb{N},$$

where each φ_i is a scalar function, then the infinite dimensional integral equation (16) reduces to a system of scalar equations. Indeed, we have that $S(t)e_i = e^{\lambda_i t}$, and if we decompose

$$F(t,x) = \sum_{i=0}^{\infty} f_i(t,x)e_i,$$

where $f_i(t,x) = \langle F(t,x), e_i\rangle$, $t \geq 0$, $x \in H$, and let $X_i(t) = \langle X(t), e_i\rangle$, $t \geq 0$, $i \in \mathbb{N}$, then (16) can be written as

$$X_i(t) = e^{\lambda_i t}X_i(0) + \int_0^t e^{\lambda_i(t-s)} f_i(s, X(s))ds + \int_0^t e^{\lambda_i(t-s)}\varphi_i(s, X(s))\,dW_i(s), \tag{18}$$

$t \geq 0$, $i \in \mathbb{N}$. If the infinite dimensional function X is approximated by $X(t) = \sum_{i=1}^m X_i(t)e_i$, $t \geq 0$, then (18) reduces to a genuinely finite dimensional system of coupled equations.

Appendix

Proof of Lemma 2.1. (i): For convenience, we suppress the index i in the proof. Let $0 = a_0 < a_1 < \cdots < a_N = T$ be such that Φ is uniformly continuous on each interval (a_{k-1}, a_k). Let $M := \sup_{0\le s\le T} \mathbb{E}\,\|\Phi(s)\|^2$ and let $\varepsilon > 0$. Take $\delta > 0$ such that $8MN\delta < \varepsilon$ and such that for each k one has $\mathbb{E}\,\|\Phi(r)-\Phi(s)\|^2 < \varepsilon/2T$ whenever $r, s \in (a_{k-1}, a_k)$. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0, t]$ and let $0 = s_0 < s_1 < \cdots < s_m = T$ be a refinement. Define the simple function

$$\Psi := \sum_{k=1}^{n} \Phi(t_{k-1}) \mathbb{1}_{[t_{k-1},t_k)},$$

then the Riemann sum of Φ corresponding to the partition $t_1, \ldots, t_n$ is

$$\sum_{k=1}^{n} \Phi(t_{k-1})\big(W(t_k) - W(t_{k-1})\big) = \sum_{k=1}^{m} \Psi(s_{k-1})\big(W(s_k) - W(s_{k-1})\big).$$

By adaptedness one has

$$\begin{aligned}
&\mathbb{E}\,\Big\|\sum_{k=0}^{n-1} \Phi(t_k)\big(W(t_{k+1}) - W(t_k)\big) - \sum_{k=0}^{m-1} \Phi(s_k)\big(W(s_{k+1}) - W(s_k)\big)\Big\|^2 \\
&= \mathbb{E} \sum_{k=0}^{m-1}\sum_{l=0}^{m-1} \Big\langle \Psi(s_k) - \Phi(s_k), \Psi(s_l) - \Phi(s_l)\Big\rangle \\
&\qquad\qquad \cdot\big(W(s_{k+1}) - W(s_k)\big)\big(W(s_{l+1}) - W(s_l)\big) \\
&= \sum_{k=0}^{m-1} \mathbb{E}\,\|\Psi(s_k) - \Phi(s_k)\|^2 \mathbb{E}\,\big(W(s_{k+1}) - W(s_k)\big)^2 \\
&\qquad +2\sum_{k=0}^{m-1}\sum_{l=0}^{k-1} \mathbb{E}\,\Big\langle \Psi(s_k) - \Phi(s_k), \Psi(s_l) - \Phi(s_l)\Big\rangle \\
&\qquad\qquad \cdot\mathbb{E}\,\big(W(s_{k+1}) - W(s_k)\big)\mathbb{E}\,\big(W(s_{l+1}) - W(s_l)\big) \\
&= \sum_{k=0}^{m-1} \mathbb{E}\,\|\Psi(s_k) - \Phi(s_k)\|^2 (s_{k+1} - s_k).
\end{aligned}$$

Denote by J the set of indices j for which the interval $[t_{j-1}, t_j)$ contains one or more of the jump points a_i. Note that J has at most N elements. Hence

$$\begin{aligned}
&\sum_{k=0}^{m-1} \mathbb{E}\,\|\Psi(s_k) - \Phi(s_k)\|^2 (s_{k+1} - s_k) \\
&\quad = \Big(\sum_{j\in J}\ \sum_{k:\ s_k\in[t_{j-1},t_j)} + \sum_{j\notin J}\ \sum_{k:\ s_k\in[t_{j-1},t_j)}\Big) \mathbb{E}\,\|\Psi(s_k) - \Phi(s_k)\|^2 (s_{k+1} - s_k)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j\in J} \sum_{k:\ s_k\in[t_{j-1},t_j)} \mathbb{E}\left(\|\Psi(s_k)\|+\|\Phi(s_k)\|\right)^2 (s_{k+1}-s_k)\\
&\qquad\qquad + \sum_{j\notin J} \sum_{k:\ s_k\in[t_{j-1},t_j)} (\varepsilon/2T)(s_{k+1}-s_k)\\
&\le \sum_{j\in J} 4M(t_j-t_{j-1}) + (\varepsilon/2T)T\\
&\le 4MN\delta + \varepsilon/2 < \varepsilon.
\end{aligned}
$$

Therefore

$$
\mathbb{E}\,\|\sum_{k=1}^{n} \Phi(t_{k-1})(W(t_k)-W(t_{k-1})) - \sum_{k=1}^{m} \Phi(s_{k-1})(W(s_k)-W(s_{k-1}))\|^2 < \varepsilon.
$$

Since $L^2(\Omega;H)$ is complete, it follows that the limit in (5) exists.

(ii): As in the proof of (i), we find for any partition $0 = t_0 < t_1 < \cdots < t_n = T$:

$$
\mathbb{E}\,\|\sum_{k=1}^{n} \Phi_i(t_{k-1})\big(W(t_k)-W(t_{k-1})\big)\|^2 = \sum_{k=1}^{n} \mathbb{E}\,\|\Phi_i(t_{k-1})\|^2 (t_k - t_{k-1}),
$$

and by considering partitions with mesh sizes tending to zero the equality follows.

(iii): Because of (i) and continuity of the inner product in $L^2(\Omega;H)$, it suffices to show that

$$
\mathbb{E}\left\langle \sum_{k=0}^{n-1} \Phi_i(s_k)(W_i(s_{k+1})-W_i(s_k)), \sum_{l=0}^{n-1} \Phi_j(s_{l-1})(W_j(s_l)-W_j(s_l)) \right\rangle = 0
$$

for each partition $0 = s_0 < s_1 < \cdots < s_n = T$.

Using that $W_i(s_k) - W_i(s_{k-1})$ is independent of both $W_j(s_l) - W_j(s_{l-1})$ and $\langle \Phi_i(s_{k-1}), \Phi_j(s_{l-1})\rangle$ for all i, j and $l \ge k$, we obtain that

$$
\begin{aligned}
&\mathbb{E}\ \left\langle \sum_{k=0}^{n-1} \Phi_i(s_k)\big(W_i(s_{k+1})-W_i(s_k)\big), \sum_{l=0}^{n-1} \Phi_j(s_l)\big(W_j(s_{l+1})-W_j(s_l)\big) \right\rangle =\\
&2\sum_{k=0}^{n-1}\sum_{l=0}^{k-1} \mathbb{E}\left(W_i(s_{k+1})-W_i(s_k)\right)\left(W_j(s_{l+1})-W_j(s_l)\right)\langle \Phi_i(s_k), \Phi_j(s_l)\rangle +\\
&\sum_{k=0}^{n-1} \mathbb{E}\left(W_i(s_{k+1})-W_i(s_k)\right)\left(W_j(s_{k+1})-W_j(s_k)\right)\langle \Phi_i(s_k), \Phi_j(s_k)\rangle = 0.
\end{aligned}
$$

□

Proof of Lemma 3.1. Without loss of generality we may assume that $\mathcal{F}$ and $\mathcal{G}$ are complete. Let us call a set of the form $A \times B$ with $A \in \mathcal{F}$ and $B \in \mathcal{G}$ a *rectangle* in $S \times T$ and consider

$$\mathcal{R} := \{\bigcup_{i=1}^{n} R_i : R_1, \ldots, R_n \text{ mutually disjoint rectangles in } S \times T,\ n \in \mathbb{N}\}.$$

We call $f = \sum_{i=1}^n a_i \mathbb{1}_{R_i}$ an $\mathcal{R}$-*step function* if $R_i \in \mathcal{R}$ and $a_i \in X$ for all i. The proof follows from the following sequence of claims.

(a) If $E \in \mathcal{F} \otimes \mathcal{G}$, then $\{t : (s,t) \in E\} \in \mathcal{G}$ for almost every $s \in S$, and E is a null set if and only if $\{t : (s,t) \in E\}$ is a null set for almost every $s \in S$.

(b) For every $E \in \mathcal{F} \otimes \mathcal{G}$ and $\varepsilon > 0$ there is an $R \in \mathcal{R}$ such that $(\mu \otimes \nu)((E \setminus R) \cup (R \setminus E)) < \varepsilon$. For a proof, combine [9, VII, §33, Theorem E] and [6, Lemma III.8.3].

(c) For every stepfunction $f \in \mathcal{L}^p(S \times T; X)$ and $\varepsilon > 0$, there is an $\mathcal{R}$-stepfunction g with $\|f - g\|_{\mathcal{L}^p(S \times T; X)} < \varepsilon$. This claim follows from (b).

(d) If f is an $\mathcal{R}$-stepfunction, then $t \mapsto f(s,t)$ is a stepfunction in $\mathcal{L}^p(T; X)$ for every $s \in S$, and $s \mapsto [f(s, \cdot)]$ is a stepfunction in $\mathcal{L}^p(S; L^p(T; X))$.

(e) If $f \in \mathcal{L}^p(S \times T; X)$, then $\int_{S \times T} \|f\|^p d(\mu \otimes \nu) = \int_S \left(\int_T \|f(s,t)\|^p d\nu(t)\right) d\mu(s)$. This is a consequence of Fubini's theorem, since $(s,t) \mapsto \|f(s,t)\|^p$ is measurable.

(f) If $f \in \mathcal{L}^p(S \times T; X)$, then $t \mapsto f(s,t) \in \mathcal{L}^p(T; X)$ for almost every $s \in S$. To prove this, let $f \in \mathcal{L}^p(S \times T; X)$ and take, with aid of (c), a sequence of $\mathcal{R}$-stepfunctions $(g_n)_n$ such that $\|f - g_n\|_{\mathcal{L}^p(S \times T; X)} \to 0$ as $n \to \infty$. By means of (e) we can find a subsequence $(g_{n_k})_k$ such that $\int_T \|f(s,t) - g_{n_k}(s,t)\|^p d\nu(t) \to 0$ as $n \to \infty$ for almost every $s \in S$. The claim now follows from (d).

(g) For every $f \in D$ we have $I(f) \in \mathcal{L}^p(S; L^p(T; X))$ and $\|I(f)\|_{\mathcal{L}^p(S; L^p(T; X))} = \|f\|_{\mathcal{L}^p(S \times T; X)}$. Indeed, let $f \in \mathcal{L}^p(S \times T; X)$ and let $(g_n)_n$ be the sequence from the proof of claim (f). As $f \in D$, it follows from (d) and (e) that $\int_S \|[f(s, \cdot)] - [g_n(s, \cdot)]\|^p_{L^p(T; X)} d\mu(s) \to 0$ as $n \to \infty$, and therefore $I(f) \in \mathcal{L}^p(S; L^p(T; X))$. The equality of the norms of $I(f)$ and f follows from (e).

(h) If $f : S \to L^p(T; X)$ is a step function and $\varepsilon > 0$, then there is an $\mathcal{R}$-stepfunction g such that $\|f - I(g)\|^p_{\mathcal{L}^p(S; L^p(T; x))} < \varepsilon$. For a proof, let $f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$ with $a_i \in L^p(T; X)$, $A_i \in \mathcal{F}$, and $\mu(A_i) < \infty$ for all i, and let $\varepsilon > 0$. We can approximate every a_i by a stepfunction b_i from T to X in such a way that $\sum_{i=1}^n \|a_i - [b_i]\|^p_{L^p(T; X)} \mu(A_i) < \varepsilon$. If we set $g(s,t) := \sum_{i=1}^n b_i(t) \mathbb{1}_{A_i}(s)$, $s \in S$, $t \in T$, then g is an $\mathcal{R}$-stepfunction and $\|f - I(g)\|^p_{\mathcal{L}^p(S; L^p(T; X))} = \int_S \| \sum_{i=1}^n (a_i - [b_i]) \mathbb{1}_{A_i}(s)\|^p_{L^p(T; X)} d\mu(s) < \varepsilon$.

(i) For every $F \in \mathcal{L}^p(S; L^p(T; X))$ there is an $f \in D$ with $I(f) = F$. In order to prove this claim, let $F \in \mathcal{L}^p(S; L^p(T; X))$ and let $(F_n)_n$ be a sequence of stepfunctions from S to $L^p(T; X)$ such that $\|F_n - F\|_{\mathcal{L}^p(S; L^p(T; X))} \to 0$ as $n \to \infty$. Because of (h), we can modify $(F_n)_n$ to a sequence of $\mathcal{R}$-stepfunctions

$(f_n)_n$ with $\|F - I(f_n)\|_{\mathcal{L}^p(S;L^p(S;X))} \to 0$ as $n \to \infty$. From (g) it follows that $(f_n)_n$ is a Cauchy sequence in $\mathcal{L}^p(S \times T; X)$, and there is an $f \in \mathcal{L}^p(S \times T; X)$ such that $\|f_n - f\|_{\mathcal{L}^p(S\times T;X)} \to 0$, $n \to \infty$. By (g), again, we obtain $I(f) = F$ a.e. on S.

To complete the proof of the lemma, combine (g) and (i), and observe that I is linear. Furthermore, (a) yields that $f = g$ a.e. on $S \times T$ if and only if $I(f) = I(g)$ a.e. on S, for every $f, g \in D$, and that it follows from (f) that each equivalence class of $L^p(S \times T; X)$ contains an elements of D. $\square$

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