The group generated by the Weierstrass points of some plane quartics

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Abstract

The group generated by the Weierstrass points of a smooth curve in its Jacobian is an interesting intrinsic invariant of the curve. In this survey paper, we give the structure of this group for some particular smooth quartics and we review the techniques used to obtain these results. Since Weierstrass points are closely related to moduli spaces of curves, as an application, we get bounds on both the rank and the torsion part of this group for a generic quartic having a fixed number of hyperflexes in the moduli space $\mathcal{M}_3$ of curves of genus 3. In particular, we can conclude that for a generic quartic, it is a free abelian group of rank at least 11.

The Weierstrass points of an algebraic curve are a distinguished set of points of the curve, which have the unique property of being geometrically intrinsic, i.e. independent of isomorphism or of any embedding in projective space. Curves of genus 0 or 1 have no Weierstrass points, and in genus 2 the Weierstrass points are easily characterized as the six ramification points under the hyperelliptic involution, which generate the 2-torsion subgroup of the Jacobian. More generally, the same result holds for a hyperelliptic curve of any genus. Nonhyperelliptic curves of genus 3, which admit a plane quartic model, provide the first case in which the structure of the Weierstrass subgroup i.e. subgroup of the Jacobian generated by these points under the Abel-Jacobi map is a nontrivial object of study.

In this survey paper, we will state results concerning the structure of this group for particular curves having a fixed number of hyperflexes i.e. points at which the tangent line to the curve meets the curve with multiplicity four.

Weierstrass points are used to construct various stratifications of moduli spaces of curves. In particular, there exists a stratification of the moduli space $\mathcal{M}_3$ of genus 3 curves, due to Vermeulen [Ver], in terms of the number of hyperflexes. This stratification determines certain intrinsic families of quartics, corresponding to the strata of $\mathcal{M}_3$, characterized by the number of hyperflexes and their geometric configuration. Using specialization of curves in these families, we obtain bounds both on the rank and on the torsion part of the group generated by the Weierstrass points for a generic quartic in these strata. In particular, we show that for a generic quartic, it is a free abelian group of rank at least 11.

We will first recall the definitions of Weierstrass points and of the object of our study. We will then state some results concerning quartics with lots of automorphisms. In section 3, we will state two theorems fundamental to the present study. Finally, in section 4, we will deduce the above-mentioned results on the strata of $\mathcal{M}_3$.

1 Definitions and notations.

1.1 Weierstrass points.

Let $\mathcal{C}$ be a smooth projective curve of genus $g \geq 2$ and let P be any point on $\mathcal{C}$.

We will say that P is a *Weierstrass point* if and only if there exists a differential form $\omega \in H^\circ(C, \Omega_C)$, such that $\mathrm{ord}_P(\omega) \geq g$. Let $\mathcal{W}$ be the set of Weierstrass points on $\mathcal{C}$.

We can attach to a point a notion of weight, which will give an alternative characterization of Weierstrass points: $P \in \mathcal{W}$ if and only if $w(P) \geq 1$.

The weight is defined as follows: for any divisor D on $\mathcal{C}$, let $\mathcal{L}(D)$ be the Riemann-Roch space $\{f \in k(\mathcal{C}), \mathrm{div}\,(f) + D \geq 0\}$ and let $l(D)$ be its dimension. The gap sequence associated to P is then defined to be the set $G(P) = \{n \in \mathbb{Z}_{>0}, l(nP) = l((n-1)P)\}$. We can define the *weight* of a point to be $w(P) = \sum\limits_{n \in G(P)} n - g(g+1)/2$.

There is a finite number of Weierstrass points. More precisely, the number of Weierstrass points counted with multiplicities equal to their weights is $\sum_{P \in \mathcal{C}} w(P) = g(g^2-1)$.

1.2 The group W.

Choose $\infty \in \mathcal{W}$, the Abel-Jacobi map is given by

$$j: \begin{array}{ccc} \mathcal{C} & \to & J = \mathrm{Pic}^\circ(\mathcal{C}) \\ P & \mapsto & [P - \infty]. \end{array}$$

We define the Weierstrass subgroup W to be the group generated by the images of the Weierstrass points under this map. It is independent of the base point of the Abel-Jacobi map, hence it defines an interesting geometric invariant of the curve.

For hyperelliptic curves, the Weierstrass points are well-known: they are the ramification points of the curve. Moreover, the group they generate in the Jacobian is the whole 2-torsion group, that is $W = (\mathbb{Z}/2\mathbb{Z})^{2g}$, for a hyperelliptic curve of genus g.

1.3 Plane quartics.

We will henceforth restrict ourselves to the case of smooth plane quartics. Indeed, they are the non-hyperelliptic curves of smaller genus, i.e. genus 3. The Weierstrass points are the *flexes* of the curve, i.e. the points where the tangent line to the curve meets the curve with multiplicity at least 3. The *hyperflexes* are then the points for which this multiplicity equals 4; they correspond to Weierstrass points of weight 2. As we have seen before, there are 24 points counted with multiplicity equal to the weight, i.e. $r + 2s = 24$ if s is the number of hyperflexes and r is the number of ordinary flexes.

Choice of the base point: We will always take a hyperflex as base point whenever possible.

If ∞ the base point of j is a hyperflex, then the image of any hyperflex is a point of order 4 in the Jacobian. Indeed, if L_P (resp. L_∞) is the linear form defining the tangent line to the curve at P (resp. ∞), we have $\mathrm{div}\,(L_P/L_\infty) = 4(P) - 4(\infty)$, hence the order of $j(P)$ divides 4, and it cannot be 2 otherwise the curve would be hyperelliptic. Moreover, there is an extra relation between the images of the Weierstrass points, that is $\sum w(P)j(P) = 0$. We thus obtain that W is a quotient of $\mathbb{Z}^r \times (\mathbb{Z}/4\mathbb{Z})^{s-1}$. Hence, we get a "naïve" bound for the rank: $\mathrm{rank}(W) \leq 24 - 2s - 1$ if $s \neq 12$.

For $s = 12$, all points are hyperflexes, they are thus all of order 4 and the rank is 0.

2 Previous results

Prior to my study, the Weierstrass subgroup had been determined for curves with a large automorphism group. In all these cases, the group is of rank 0.

Theorem 1 (Prapavessi [Pra]). *The Klein quartic given by $X^3Y + Y^3Z + Z^3X = 0$ has a group of automorphisms of order 168, and has no hyperflexes. The group generated by the Weierstrass points is $W = (\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/7\,\mathbb{Z})^3$.*

Theorem 2 (Rohrlich [Roh]). *The Fermat quartic given by $X^4 + Y^4 = Z^4$ has a group of automorphisms of order 96 and has 12 hyperflexes The group generated by the Weierstrass points is $W = (\mathbb{Z}/4\,\mathbb{Z})^5 \times (\mathbb{Z}/2\,\mathbb{Z})$.*

Theorem 3 (Klassen-Schaefer [Kla-Sch]). *The Picard curve given by* $Y^3Z+Z^4=X^4$ *has a group of automorphisms of order 48 and 4 hyperflexes. The group generated by the Weierstrass points is* $W=(\mathbb{Z}/4\,\mathbb{Z})^2\times(\mathbb{Z}/3\,\mathbb{Z})^5$.

Remark: In those three cases, the Jacobian is isogenous to a product of three CM elliptic curves.

3 Specialization

In this section, we review the tools we use to obtain results both on particular curves and on families. In particular, we state two theorems concerning the behavior under specialization.

Let $\mathcal{C}\to\mathcal{S}$ be a family of smooth projective curves of genus g. Let W_η (resp. W_s) be the group generated by the Weierstrass points in the generic fiber (resp. a special fiber). The divisor of Weierstrass points form an algebraic family. More precisely, we have

Theorem 4.
(i) W_s *is a group quotient of* W_η.
(ii) The specialization is injective on the torsion part.

Proof: For the first part, see Hubbard [Hub] or Laksov-Thorup [Lak-Tho], the second part is classic (see [Hin-Sil]). ∎

Both these properties and the following theorem of Silverman are essential to deduce results for families of curves and also to obtain bounds on the rank and the torsion part of a generic quartic in the moduli space of curves of genus 3, as we will see in section 4.

Theorem 5 (Silverman [Sil]). *Let* $A\to C$ *be a (flat) family of abelian varieties, all defined over a global field* K*, where* C *is a smooth projective curve. At a point* $t\in C(\overline{K})$ *for which the fibre* A_t *is non-singular, the specialization map is defined by* $\sigma_t: A(C)\to A_t(\overline{K}),\quad P\mapsto P_t$.

Assume that A *has no constant part, then the set* $\{t\in C(\overline{K})\mid \sigma_t$ *is not injective* $\}$ *is a set of bounded height in* $C(\overline{K})$.

In particular, if K *is a number field and* $d\geq 1$ *is an integer, then* σ_t *is injective for all but finitely many* $t\in \bigcup_{[L:K]\leq d} C(L)$.

We will now give a quick sketch of how the proof goes for families of curves.

First, we use elementary geometric arguments to restrict the number of generators of W. By elementary, I mean that the relations we are considering are those arising from 4 Weierstrass points lying on a line (which will give a sum of 4 points equal to 0 in the Jacobian) or a conic tangent to the curve at 4 Weierstrass points (which will give a 2-torsion point in the Jacobian). Thus we obtain that W is a quotient of some W_0, i.e. $\boxed{W \twoheadrightarrow W_0}$.

It often turns out that by these geometric arguments we have already obtained all the existing relations between the Weierstrass points. We show that for a well-chosen specialization, we have already determined the structure of the group, i.e. $\boxed{W_{C_0} \simeq W_0}$. In order to compute the group for a particular curve, two methods prove to be useful. If the Jacobian is isogenous to a product of three elliptic curves, we use reduction on these elliptic curves to prove independence of points. Otherwise, we carry out an explicit descent via an isogeny (following Schaefer [Sch]). Since, W_{C_0} is a group quotient of W_η, we obtain that $\boxed{W_\eta \simeq W_0}$.

We then use the specialization theorem of Silverman 5 to conclude that W is isomorphic to W_0 for almost all curves.

4 Moduli space of curves

In this section, we will use Vermeulen's stratification of the moduli space $\mathcal{M}_3$ of curves of genus 3 ([Ver]) and the determination of the group W for particular curves to deduce some bounds on the rank and on the torsion part of a generic quartic having a fixed number of hyperflexes in a particular geometric configuration.

Let $\mathcal{M}_3^\circ$ be $\{[C] \in \mathcal{M}_3,\ C \text{ non-hyperelliptic}\}$, $M_s = \{[C] \in \mathcal{M}_3^\circ,\ C$ has at least s hyperflexes$\}$, and $M_s^\circ = \{[C] \in \mathcal{M}_3^\circ,\ C$ has exactly s hyperflexes$\}$.

The reference for all statements concerning the strata is Vermeulen [Ver]. I'm following his notations. In particular, the X_i have dimension 3, and the Y_i's dimension 2. In all the statements, groups called F are always finite.

There are two quartics with 12 hyperflexes, the Fermat quartic and the curve Ψ.

Theorem 6 (G-Tzermias [Gir-Tze]). *The Weierstrass subgroup of the curve Ψ given by the equation $X^4+Y^4+Z^4+3\,(X^2Y^2+X^2Z^2+Y^2Z^2)=0$ is $W \cong (\mathbb{Z}/4\mathbb{Z})^5$.*

There are no quartics with either 10 or 11 hyperflexes. There are two curves with exactly 9 hyperflexes:

Theorem 7 ([Gir3]). *The Weierstrass subgroup of each of the two curves Ω_i given by $(21 \pm 8\sqrt{7})x^4 - 6(2 \pm \sqrt{7})yx^2 + (3 \pm \sqrt{7})(y^3+1)x - 3y^2 = 0$ is $W = \mathbb{Z}^5 \times (\mathbb{Z}/4\mathbb{Z})^5$.*

The set of curves with exactly eight hyperflexes M_8° consists of a curve Σ, and of a 1-dimensional family $Z_{1,t}$:

Theorem 8 ([Gir3]). *The Weierstrass subgroup of the curve Σ, given by the equation $8(x^4+y^4) + 48i\sqrt{7}x^2y^2 + 24(1-i\sqrt{7})xyz^2 - (7-3i\sqrt{7})z^4 = 0$, is $W = \mathbb{Z}^5 \times (\mathbb{Z}/4\mathbb{Z})^4$.*

Theorem 9 ([Gir3]). *Let $Z_{1,t}$ be the smooth projective curve birational to the affine curve $(t^2+1)(x^2-y)^2 - y(2x-y-1)(2tx-y-t^2) = 0$ where $t \notin \{0, 1, -1, 3, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$. For any number field K, there is a finite set $S_K \subset K$ such that for each $t \in K \setminus S_K$ we have $W_{Z_{1,t}} \cong \mathbb{Z}^5 \times (\mathbb{Z}/4\mathbb{Z})^4$.*

From these results, using the specialization theorem 4 we can deduce than the rank is at least 5 and the torsion part is at most $(\mathbb{Z}/4\mathbb{Z})^4$ or $(\mathbb{Z}/4\mathbb{Z})^5$ for all strata containing one of the above curves. For curves with 4 hyperflexes or less, we obtain sharper results. To begin with, the following family has 4 hyperflexes on a line.

Theorem 10 ([Gir2]). *Let $\mathcal{C}_t$ be the smooth projective curve birational to the affine curve $y^4 = x(x-1)(x-t)$ where $t \notin \{0,1\}$. For any number field K, there is a finite set $S_K \subset K$, for each $t \in K \setminus S_K$, we have $W_{\mathcal{C}_t} \cong \mathbb{Z}^9 \times (\mathbb{Z}/4\mathbb{Z})^2$.*

A straightforward study of the Riemann-Roch spaces corresponding to multiples of the base point of the Abel-Jacobi map yields the following lemma

Lemma 11. *Let $\mathcal{C}$ be a smooth quartic having exactly 4 hyperflexes. If 3 of these points lie on a line, the fourth point lies also on this line and the group they generate is $(\mathbb{Z}/4\mathbb{Z})^2$. Otherwise, the group generated by these 4 points is $(\mathbb{Z}/4\mathbb{Z})^3$ or $(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})$. More precisely, this group is $(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})$ if and only if there exists a conic curve tangent to the curve at these 4 points.*

The set of curves with 4 hyperflexes M_4 consists of 5 irreducible components X_1, Y_2, Y_3, Y_4 and Y_5, for which we state the following results:

Proposition 12.
(i) The Weierstrass subgroup of a generic quartic of Y_3 or Y_4 is of the form $W_\eta = \mathbb{Z}^r \times (\mathbb{Z}/4\mathbb{Z})^3 \times F$ with $5 \leq r \leq 15$ and $F \subset (\mathbb{Z}/4\mathbb{Z})$.
(ii) The Weierstrass subgroup of a generic quartic in Y_5 is of the form $W_\eta = \mathbb{Z}^r \times (\mathbb{Z}/4\mathbb{Z})^3 \times F$ with $5 \leq r \leq 15$ and $F \subset (\mathbb{Z}/4\mathbb{Z})^2$.

Proof: Indeed, according to the above lemma, the group generated by the hyperflexes is $(\mathbb{Z}/4\mathbb{Z})^3$. The curve Σ of theorem 8 is in both Y_3 and Y_4. The curves Ω_i's of theorem 7 are in Y_5. The specialization theorem 4 implies that the torsion part is at most $(\mathbb{Z}/4\mathbb{Z})^4$ or $(\mathbb{Z}/4\mathbb{Z})^5$ respectively, and gives bounds on the rank. ∎

Proposition 13. *The Weierstrass subgroup of a generic quartic of Y_2 is of the form $W_\eta = \mathbb{Z}^r \times F$ with $5 \leq r \leq 15$ and $(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z}) \subset F \subset (\mathbb{Z}/4\mathbb{Z})^4$.*

Proof: A curve in Y_2 has the property that there is a conic curve which is tangent at 4 hyperflexes. Moreover, the curve Σ of theorem 8 is in Y_2. The above lemma and the specialization theorem 4 give the result. ∎

Most of the remaining results were proved in [Gir2], but since the proofs are straightforward, we give them here as well.

Proposition 14 ([Gir2]). *The Weierstrass subgroup of a generic quartic of X_1 is of the form $W_\eta = \mathbb{Z}^r \times (\mathbb{Z}/4\mathbb{Z})^2$ with $9 \leq r \leq 15$.*

Proof: A curve in X_1 has 4 hyperflexes lying on a line, hence, the group they generate is $(\mathbb{Z}/4\mathbb{Z})^2$. Moreover, the family $\mathcal{C}_t$ of theorem 10 is in X_1, hence the torsion part is at most $(\mathbb{Z}/4\mathbb{Z})^2$ and the bound on the rank. ∎

The set M_3° of curves with exactly 3 hyperflexes consists of 2 irreducible components X_2 and X_3. The set X_1 is in M_3, hence if a curve has 3 hyperflexes on a line, it lies in X_1 and has at least 4 hyperflexes.

Proposition 15. *The Weierstrass subgroup of a generic quartic of either X_2 or X_3 is of the form $W_\eta = \mathbb{Z}^r \times (\mathbb{Z}/4\mathbb{Z})^2 \times F$ with $5 \leq r \leq 17$ and $F \subset (\mathbb{Z}/4\mathbb{Z})^2$.*

Proof: Both components contain Y_3 and the group generated by 3 hyperflexes is $(\mathbb{Z}/4\mathbb{Z})^2$, according to the above lemma. Proposition 12((i)) and the specialization theorem 4 give the result. ∎

Since M_1 et M_2 are irreducible, we deduce the following results.

Proposition 16 ([Gir2]). *The Weierstrass subgroup of a generic quartic of M_2 is of the form $W_\eta = \mathbb{Z}^r \times (\mathbb{Z}/4\mathbb{Z}) \times F$ with $9 \leq r \leq 19$ and $F \subset (\mathbb{Z}/4\mathbb{Z})$.*

Proof: Let $W_\eta = \mathbb{Z}^r \times F$ be the group generated by the Weierstrass points in the generic fiber. One has $F \subset (\mathbb{Z}/4\mathbb{Z})^2$ and $F \supset (\mathbb{Z}/4\mathbb{Z})$, since the image of a hyperflex is of order 4 and the family $\mathcal{C}_t$ of theorem 10 is in M_2. For the last reason, the rank is at least 9. ∎

Finally, the knowledge of the group W for the following particular family of curves implies that W_η is a free abelian group of rank at least 11 for a generic quartic (also for a generic quartic with one hyperflex).

Theorem 17 ([Gir1]). *Let $\mathcal{C}_\beta$ be the smooth projective curve birational to the affine curve $y^3 = x(x-1)(x^2 - 2\beta x + \beta)$ where $\beta \notin \{0,1\}$. For any number field K, there is a finite set $S_K \subset K$ such that for each $\beta \in K \setminus S_K$, we have $W_{\mathcal{C}_\beta} \cong \mathbb{Z}^4 \times (\mathbb{Z}/3\mathbb{Z})^5$.*

Theorem 18 ([Gir2]).
(i) The Weierstrass subgroup of a generic quartic is of the form $W_\eta = \mathbb{Z}^r$ with $11 \leq r \leq 23$.
(ii) The Weierstrass subgroup of a generic quartic with at least 1 hyperflex is of the form $W_\eta = \mathbb{Z}^r$ with $11 \leq r \leq 21$.

Proof: Let $W_\eta = \mathbb{Z}^r \times F$ be the group generated by the Weierstrass points in the generic fiber. Both families C_β of thm 17 and C_t of thm 10 are in M_0 and M_1. Hence, the finite part F satisfies $F \subset (\mathbb{Z}/3\mathbb{Z})^5$ and $F \subset (\mathbb{Z}/4\mathbb{Z})^2$ thus $F = \emptyset$; moreover $\mathbb{Z}^4 \times (\mathbb{Z}/3\mathbb{Z})^5$ and $\mathbb{Z}^9 \times (\mathbb{Z}/4\mathbb{Z})^2$ are quotients of $\mathbb{Z}^r$, hence $r \geq 11$. ∎

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