# Asymptotic estimates for the periods of periodic points of nonexpansive maps

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**Abstract**

For each positive integer $n$ we use the concept of "admissible arrays on $n$ symbols" to define a set of positive integers $Q(n)$ which is determined solely by number theoretical and combinatorial constraints and whose computation reduces to a finite problem. In earlier joint work with M. Scheutzow, it was shown that the set $Q(n)$ is intimately connected to the set of periods of periodic points of classes of nonexpansive nonlinear maps defined on the positive cone in $\mathbb{R}^n$. In this paper we continue the characterization of $Q(n)$ and present precise asymptotic estimates for the largest element of $Q(n)$. For example, if $\gamma(n)$ denotes the largest element of $Q(n)$, then we show that $\lim_{n\to\infty}(n\log n)^{-1/2}\log\gamma(n) = 1$. We also discuss that further details about the fine structure of $Q(n)$ require some delicate number theoretical issues.

## 1 Introduction

Let $D \subset \mathbb{R}^n$ be a closed subset and $g : D \to D$ be a map from $D$ into $D$. The map $g^j : D \to D$ will denote the $j$-fold composition of $g$ with itself. If $\xi \in D$ and $g^p(\xi) = \xi$ for some $p \geq 1$, we call $\xi$ a periodic point of $g$ of period $p$. We call $p$ the minimal period if $g^j(\xi) \neq \xi$ for $1 \leq j < p$. The map $g : D \to D$ is called nonexpansive with respect to $\|\cdot\|$ if

$$\|g(x) - g(y)\| \leq \|x - y\| \quad \text{for all } x, y \in D. \tag{1.1}$$

In this paper we consider maps that are nonexpansive with respect to the $l_1$-norm, i.e., $\|x\|_1 := \sum_{j=1}^n |x_j|$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, but the questions we address are certainly relevant for other norms on $\mathbb{R}^n$, notable, the $l_\infty$-norm and, more generally, polyhedral norms on $\mathbb{R}^n$. See [6] for further information and partial results.

If $D \subset \mathbb{R}^n$ is closed and $f : D \to D$ is nonexpansive with respect to the $l_1$-norm and there exists an $\eta \in D$ such that $\sup_j \|f^j(\eta)\|_1 < \infty$, then it follows from results of Akcoglu and Krengel [1] that for every $x \in D$, there exists a positive integer $p = p(x)$ and a point $\xi = \xi(x) \in D$ such that $\xi$ is a periodic point of $f$ of minimal period $p$ and

$$\lim_{j\to\infty} f^{jp}(x) = \xi. \tag{1.2}$$

Let $K^n = \{x \in \mathbb{R}^n \mid x = (x_1, x_2, \ldots, x_n),\ x_i \geq 0\}$ denote the positive cone in $\mathbb{R}^n$. Linear maps that leave $K^n$ invariant and are nonexpansive with respect to the $l_1$-norm are given by $n \times n$-matrices $A = (a_{ij})$ with $a_{ij} \geq 0$ and $\sum_{i=1}^n a_{ij} \leq 1$ for $1 \leq j \leq n$. The Perron-Frobenius theory of nonnegative matrices implies that not only does (1.2) hold, but also the positive minimal periods $p$ which can arise are given by the least common multiple of sets of positive integers whose sum is less than or equal to $n$; see [14] and Section 9 of [12] for details and further generalizations.

Conversely, every permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ induces a linear map $f_\sigma : K^n \to K^n$ by $f_\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ which has a periodic point with minimal period equal to the order of the permutation. So, in the linear case, it is possible to describe exactly the set of possible minimal periods $p$.

Motivated by the linear case and the result of Akcoglu and Krengel, generalizations to classes of nonlinear maps have been studied. Recent joint work of the authors with M. Scheutzow, shows that for special classes of nonexpansive maps there exists an exact description of the set of possible minimal periods using number theoretical and combinatorial constraints; see [10, 11, 12].

This paper consists of five sections. In Section 2 we give an introduction to admissible arrays and explain the connection with periodic points of nonexpansive maps. In Section 3, we present some background information on the orders of the permutations on $n$ letters and prove that in dimensions $n \geq 8$, there are periods of periodic points of nonlinear nonexpansive maps that cannot be realized by linear maps in the same space. In Section 4 we prove our main result that will give an asymptotic estimate for the largest possible period of periodic points of nonexpansive maps. Finally, in Section 5 we discuss some further properties of the periods of the periodic points of nonexpansive maps.

# 2 Admissible arrays and periodic points of nonlinear maps

The cone $K^n = \{x \in \mathbb{R}^n \mid x_i \geq 0,\ 1 \leq i \leq n\}$ induces a partial ordering by $x \leq y$ if and only if $y - x \in K^n$. A map $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is *order-preserving* if $f(x) \leq f(y)$ for all $x, y \in D$ with $x \leq y$. If $f_j(x)$ denotes the $j^{th}$ coordinate of $f(x)$, then $f$ is called *integral-preserving* if

$$\sum_{j=1}^{n} f_j(x) = \sum_{j=1}^{n} x_j \qquad \text{for all } x \in D.$$

We begin by defining a class of maps which we denote below by $\mathcal{F}_3(n)$ and by giving some refinements of $\mathcal{F}_3(n)$.

**Definition 2.1** Define $u = (1, 1, \ldots, 1) \in \mathbb{R}^n$ and consider the following conditions on maps $f : K^n \to K^n$:

(1) $f(0) = 0$,

(2) $f$ is order-preserving,

(3) $f$ is integral-preserving,

(4) $f$ is nonexpansive with respect to the $l_1$-norm,

(5) $f(\lambda u) = \lambda u$ for all $\lambda > 0$,

We define sets of maps $\mathcal{F}_j(n)$, $1 \le j \le 3$, by

$$\begin{aligned}
\mathcal{F}_1(n) &= \{f : K^n \to K^n \mid f \text{ satisfies (1), (2), (3) and (5)}\}, \\
\mathcal{F}_2(n) &= \{f : K^n \to K^n \mid f \text{ satisfies (1), (2) and (3)}\}, \\
\mathcal{F}_3(n) &= \{f : K^n \to K^n \mid f \text{ satisfies (1) and (4)}\}.
\end{aligned}$$

A proposition of Crandall and Tartar [3] implies that if $f : K^n \to K^n$ is integral-preserving, then it is order-preserving if and only if it is $l_1$-norm nonexpansive. Thus we see that

$$\mathcal{F}_1(n) \subset \mathcal{F}_2(n) \subset \mathcal{F}_3(n).$$

If $f : K^n \to K^n$ is integral-preserving and order-preserving, one can easily check that $f$ satisfies (5) if and only if $f$ is sup-norm-decreasing, i.e., $\|f(x)\|_\infty \le \|x\|_\infty$ for all $x \in D$. Using this characterization of $\mathcal{F}_1(n)$ and a result of Lin and Krengel [4], we see that if $f \in \mathcal{F}_1(n)$ and $y \in K^n$ is a periodic point of $f$, then there is a permutation $\sigma$, depending on $f$ and $y$, such that

$$f(y) = (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}).$$

Examples of maps belonging to $\mathcal{F}_1(n)$ can be constructed as follows. Let $\sigma$ and $\tau$ be permutations of the set $\{1, 2, 3, \ldots, n\}$. Define the map $f : K^n \to K^n$ by

$$f(x)_j = \min\{x_{\sigma(j)}, 1\} + \max\{x_{\tau(j)}, 1\} - 1, \quad j = 1, 2, \ldots, n. \tag{2.1}$$

Even for such simple looking examples, it is not easy to determine the possible minimal periods of the periodic points of $f$.

In order to obtain more information about the possible periods, we define sets of positive integers $P_j(n)$, $1 \le j \le 3$, by

$$P_j(n) = \{p \ge 1 \mid \exists f \in \mathcal{F}_j(n) \text{ and a periodic point of } f \text{ of minimal period } p\}.$$

Our results describe the sets $P_2(n)$ and $P_3(n)$ precisely and provide considerable information about the set $P_1(n)$.

Because $\mathcal{F}_1(n) \subset \mathcal{F}_2(n) \subset \mathcal{F}_3(n)$ we have, by definition,

$$P_1(n) \subset P_2(n) \subset P_3(n). \tag{2.2}$$

If $S_n$ denotes the symmetric group on $n$ symbols and $\sigma$ an element of $S_n$ then, by permutation of the coordinates, $\sigma$ induces a linear map $\hat{\sigma}$ that belongs to $\mathcal{F}_1(n)$ and it is easy to see that $\xi = (1, 2, 3, \ldots, n) \in K^n$ is a periodic point of minimal period $p$ equal to the order of $\sigma$ as an element of symmetric group $S_n$. Thus $P_1(n)$ contains the set of all orders of elements of $S_n$. However, in general, $P_1(n)$ is larger than the set of orders of elements of $S_n$; see Theorem 3.1 for a precise result.

By constructing special maps, one can show (see [9] for $P_1(n)$ and Section 8 of [12] for $P_2(n)$) that the sets $P_1(n)$ and $P_2(n)$ have the following properties.

**Theorem 2.1** *Let $j = 1$ or $j = 2$. If $p_1 \in P_j(n_1)$ and $p_2 \in P_j(n_2)$, then*

$$lcm\,(p_1, p_2) \in P_j(n_1 + n_2).$$

*Furthermore, if $p_i \in P_j(m)$ for $1 \le i \le r$, then*

$$r\,lcm\,(p_1, p_2, \ldots, p_r) \in P_j(rm).$$

Claim (1) follows from concatenation of maps. If $p_i \in P_j(n_i)$, $i = 1, 2$, there exist maps $f_i \in \mathcal{F}_j(n_i)$ with, respectively, periodic points $\xi_i$ of minimal period $p_i$. The map $F : K^{n_1+n_2} \to K^{n_1+n_2}$ defined by $F(x, y) = (f_1(x), f_2(y))$ has a periodic point $\xi = (\xi_1, \xi_2)$ of minimal period $\mathrm{lcm}\,(p_1, p_2)$. To prove the second claim, we use the following nontrivial observation. If there are maps $f_i \in \mathcal{F}_j(m)$ with, respectively, periodic points $\xi_i$ of minimal period $p_i$, then there also exists a single map $F \in \mathcal{F}_j(m)$ with periodic points $\hat{\xi}_i$ of minimal period $p_i$, $i = 1, 2, \ldots, r$, simultaneously. Assuming the existence of such a map $F : K^m \to K^m$ we can construct a map $T : K^{rm} \to K^{rm}$ as follows

$$T(x_1, x_2, \ldots, x_r) = (F(x_r), x_1, \ldots, x_{r-1})$$

that has a periodic point $\xi = (\hat{\xi}_1, \ldots, \hat{\xi}_r)$ of minimal period $r\mathrm{lcm}\,(p_1, p_2, \ldots, p_r)$. For example, the map $f : K^4 \to K^4$ given by

$$f(y_1, y_2, y_3, y_4) = (\min\{y_3, 1\} + \max\{y_4, 1\} - 1, y_1, y_2, \max\{y_3, 1\} + \min\{y_4, 1\} - 1)$$

belongs to $\mathcal{F}_1(4)$ and has periodic points $(2, 1, 1, 1)$ and $(1, 0, 0, 1)$ of minimal period, respectively, 4 and 3. Note that $f$ is a special case of the map given by (2.1) with $n = 4$, $\sigma(1) = 3$, $\tau(1) = 4$, $\sigma(2) = \tau(2) = 1$, $\sigma(3) = \tau(3) = 2$, $\sigma(4) = 4$ and $\tau(4) = 3$. Consequently, $2 \times \mathrm{lcm}\,\{3, 4\} = 24 \in P_1(8)$. Since 24 is not the order of an element of the symmetric group on eight symbols, a nonlinear map is needed to have a periodic point of minimal period 24 in $K^8$.

Also note that, since $P_j(1) = \{1\}$, one has that $P_j(n) \subset P_j(n + 1)$ for all $n \ge 1$ and if $p \in P_j(n)$ and $d|p$, then $d \in P_j(n)$ $(j = 1, 2, 3)$.

To describe the set $P_3(n)$ precisely, we use the notion of admissible arrays introduced in [10].

**Definition 2.2** Suppose that $(L, \prec)$ is a finite, totally ordered set and that $\Sigma$ is a finite set with $n$ elements. Let $\mathbb{Z}$ denote the integers and for each $i \in L$, suppose that $\theta_i : \mathbb{Z} \to \Sigma$ is a map. We shall say that $\{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ is an *admissible array on $n$ symbols* if the maps $\theta_i$ satisfy the following conditions:

(i) For each $i \in L$, the map $\theta_i : \mathbb{Z} \to \Sigma$ is periodic of minimal period $p_i$, where $1 \le p_i \le n$. Furthermore, for $1 \le j < k \le p_i$ we have $\theta_i(j) \ne \theta_i(k)$.

(ii) If $\prec$ denotes the ordering on $L$ and $m_1 \prec m_2 \prec \cdots \prec m_{r+1}$ is any given sequence of $(r + 1)$ elements of $L$ and if

$$\theta_{m_i}(s_i) = \theta_{m_{i+1}}(t_i)$$

for $1 \leq i \leq r$, then

$$\sum_{i=1}^{r}(t_i - s_i) \not\equiv 0 \mod \rho,$$

where $\rho = \gcd(\{p_{m_i} \mid 1 \leq i \leq r+1\})$.

The concept of an admissible array on $n$ symbols depends on the ordering $\prec$ on $L$, but it has been observed in [10] that if $|L| = m$, we can assume that $L = \{i \in \mathbb{Z} \mid 1 \leq i \leq m\}$ with the usual ordering and $\Sigma = \{j \in \mathbb{Z} \mid 1 \leq j \leq n\}$. An admissible array $\{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ can be identified with a semi-infinite matrix $(a_{ij})$, $i \in L$, $j \in \mathbb{Z}$, where $a_{ij} = \theta_i(j)$. For this reason, we shall sometimes talk about the "$i^{th}$ row of an array". We shall say that "an admissible array has $m$ rows" if $|L| = m$.

The period of an admissible array $\{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ is defined to be the least common multiple of the periods of the maps $\theta_i$, $i \in L$.

**Definition 2.3** Suppose that $S = \{q_i \mid 1 \leq i \leq m\}$ is a set of positive integers with $1 \leq q_i \leq n$ for $1 \leq i \leq m$ and $q_i \neq q_j$ for $1 \leq i < j \leq m$. We call $S$ an *array-admissible set for $n$* if there exists an admissible array on $n$ symbols $\{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ such that $\theta_i$ has minimal period $p_i$, and a one-to-one map $\sigma$ of $\{1, 2, \ldots, m\}$ onto $L$ such that $q_i = p_{\sigma(i)}$.

**Definition 2.4** $Q(n) = \{\mathrm{lcm}\,(S) \mid S \subset \{1, 2, \ldots, n\}$ is array-admissible for $n\}$.

To become more familiar with admissible arrays and the set $Q(n)$, we compute the sets $Q(n)$ for $1 \leq n \leq 6$ and refer to [12] for a systematic approach to compute $Q(n)$.

First observe that if $p$ is a prime and $p^\alpha \in Q(n)$ for some integers $\alpha \geq 0$ and $n \geq 1$, then $p^\alpha \leq n$. Furthermore, if an integer $q$ has prime factorization $q = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$ and $\sum_{j=1}^{m} p_j^{\alpha_j} \leq n$, then $q \in Q(n)$ (the maps $\theta_i$ in the definition of an admissible array can be positioned in such a way that the ranges of the maps $\theta_i$ do not intersect and this implies that the second condition in the definition of an admissible array is void). This last observation implies that the orders of the elements of the symmetric group on $n$ letters are contained in the set $Q(n)$. These observations yield $Q(1) = \{1\}$, $Q(2) = \{1, 2\}$ and $Q(3) \subset \{1, 2, 3, 6\}$. Can $6 \in Q(3)$? For this we need an admissible array with two maps $\theta_1$ and $\theta_2$ with periods 2 and 3. Since $n = 3$ the intersection of the ranges of $\theta_1$ and $\theta_2$ is nonempty. Hence there exist $t_1, s_1$ such that $\theta_1(s_1) = \theta_2(t_1)$ and the second condition in the definition of an admissible array yields $t_1 - s_1 \not\equiv 0 \mod 1$. A contradiction. Thus $Q(3) = \{1, 2, 3\}$. Similarly $Q(4) = \{1, 2, 3, 4\}$, $Q(5) = \{1, 2, 3, 4, 5, 6\}$ and $Q(6) \subset \{1, 2, 3, 4, 5, 6, 12\}$. Can $12 \in Q(6)$? One cannot take an admissible array $\{\theta_1, \theta_2\}$ with periods 3 and 4, but there exists an admissible array $\{\theta_1, \theta_2\}$ with periods 4 and 6; define $\theta_1(j) = j \mod 6$ and $\theta_2(j) = j + 1 \mod 4$. So $Q(6) = \{1, 2, 3, 4, 5, 6, 12\}$.

As an illustration of the use of admissible arrays, we prove the following lemma that will be used in the sequel; see Theorem 3.1 of [12] for a much more general result.

**Lemma 2.1** *Let $\theta = \{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ be an admissible array. If the periods of the maps $\theta_i$, $i \in L$ are relatively prime, then the period of the admissible array $\theta$ is the order of an permutation.*

**Proof.** Let $B_{q_i} = \{\theta_i(j) \mid j \in \mathbb{Z}\}$ denote the range of $\theta_i$, $i \in L$. If there exist integers $i_1$ and $i_2$ in $L$ such that $i_1 \neq i_2$ and $B_{q_{i_1}} \cap B_{q_{i_2}} \neq \emptyset$, then there exist integers $s$ and $t$ such that $\theta_{i_1}(s) = \theta_{i_2}(t)$, but then condition (ii) of Definition 2.2, implies that $t - s \not\equiv 0 \mod \rho$, where $\rho = \gcd(q_{i_1}, q_{i_2})$. From the assumption that the periods of the maps $\theta_i$ are relatively prime, it follows that $\rho = 1$ and hence $t - s \not\equiv 0 \mod 1$. A contradiction.

This shows that $B_{q_i} \cap B_{q_j} = \emptyset$ for $i, j \in L$ with $i \neq j$. From condition (ii) of Definition 2.2, it follows that the cardinality of the set $B_{q_i}$ equals $q_i$, $i \in L$. Thus $\sum_{i \in L} q_i \leq n$ and the disjoint cycle representation for permutations shows that $q = \operatorname{lcm}\{q_i \mid i \in L\}$ is the order of a permutation. $\square$

Earlier work by the first author and Scheutzow [10] showed that there is an intimate connection between the sets $P_i(n)$, $i = 1, 2, 3$ and $Q(n)$ which can be derived from the structure of the semilattice generated by a periodic orbit of a map in $\mathcal{F}_i(n)$, $i = 1, 2, 3$. To explain this connection we need some more definitions. If $x, y \in \mathbb{R}^n$, we define $x \wedge y$ and $x \vee y$ in the standard way:

$$\begin{aligned} x \wedge y &:= z, \quad z_i = \min\{x_i, y_i\} \quad \text{for } 1 \leq i \leq n \\ x \vee y &:= w, \quad w_i = \max\{x_i, y_i\} \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

If $V \subset \mathbb{R}^n$, $V$ is called a *lower semilattice* if $x \wedge y \in V$ whenever $x \in V$ and $y \in V$. If $A \subset \mathbb{R}^n$, there is a minimal (in the sense of set inclusion) lower semilattice $V \supset A$, the lower semilattice generated by $A$. If $|A| < \infty$, it follows that $|V| < \infty$. If $V$ is a lower semilattice, a map $h : V \to V$ is called a *lower semilattice homomorphism of* $V$ if

$$h(x \wedge y) = h(x) \wedge h(y) \quad \text{for all } x, y \in V.$$

If $W \subset \mathbb{R}^n$ is a lower semilattice, $h : W \to W$ is a lower semilattice homomorphism of $W$ and $\xi \in W$ is a periodic point of minimal period $p$ of $h$, we let $V$ denote the finite lower semilattice generated by

$$A = \{h^j(\xi) \mid 0 \leq j < p\}.$$

From the definitions it follows that $h(V) \subset V$ and $h^p(x) = x$ for all $x \in V$. In particular, $h|V$ is a lower semilattice homomorphism, $h|V$ is one-to-one, onto and

$$(h|V)^{-1} = h^{p-1}|V$$

is also a semilattice homomorphism of $V$.

The relevance of these ideas in our situation is indicated by the following theorem due to Scheutzow [14].

**Theorem 2.2** *Suppose that $f \in \mathcal{F}_3(n)$ and that $\xi \in K^n$ is a periodic point of $f$ of minimal period $p$. Let $A = \{f^j(\xi) \mid 0 \leq j < p\}$. If $V$ denotes the finite lower semilattice generated by $A$, then $f(V) \subset V$, $f|V$ is a lower semilattice homomorphism of $V$, $f^p(x) = x$ for all $x \in V$ and $(f|V)^{-1} = f^{p-1}|V$ is a lower semilattice homomorphism of $V$.*

**Definition 2.5** If $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$, we shall write $f \in \mathcal{G}_1(n)$ if and only if $D$ is a lower semilattice $f(D) \subset D$ and f is a lower semilattice homomorphism of $D$. We shall write $p \in Q_1(n)$ if and only if there exists a map $f \in \mathcal{G}_1(n)$ and a periodic point $\xi \in K^n$ of $f$ of minimal period $p$.

Note that from Theorem 2.2, it follows that $P_3(n) \subset Q_1(n)$. Our main theorem presented in [11] describes the situation precisely.

**Theorem 2.3** *For every positive integer $n$*

$$P_2(n) = P_3(n) = Q_1(n) = Q(n).$$

Note that if a map $f : K^n \to K^n$ has a periodic point of minimal period $p$ and $q$ is a divisor of $p$, then there exists a periodic point of $f$ of minimal period $q$ (if $\bar{x}$ is a periodic point of minimal period $p$ and $p = mq$, then $\bar{y} = f^m(\bar{x})$ is a periodic point of $f$ of minimal period $q$). This observation yields the following corollary of Theorem 2.3.

**Corollary 2.1** *If $p \in Q(n)$ and $q|p$ then $q \in Q(n)$.*

Given the very definition of an admissible array, this is indeed a nontrivial consequence of the main theorem. It allows us to introduce the notion of maximal elements of $Q(n)$. An integer $p \in Q(n)$ is called maximal if there does not exist a $q \in Q(n)$, $q \neq p$ and $p$ divides $q$. For example, the set of maximal elements of $Q(6)$ comprises the elements 5 and 12. The fact that one can restrict attention to maximal elements is crucial in any attempt to compute $Q(n)$ explicitly: see [12].

We end this section with the definition of an auxiliary set of integers that will play an essential role in the sequel.

**Definition 2.6** We define inductively, for each $n \geq 1$, a collection of positive integers $P(n)$ by $P(1) = \{1\}$ and, for $n > 1$, $p \in P(n)$ if and only if either

(A) $p = \mathrm{lcm}\,(p_1, p_2)$, where $p_1 \in P(n_1)$, $p_2 \in P(n_2)$ and $n_1$ and $n_2$ are positive integers with $n = n_1 + n_2$ or

(B) $n = rm$ for integers $r > 1$ and $m \geq 1$ and $p = r\mathrm{lcm}\,(p_1, p_2, \ldots, p_r)$, where $p_i \in P(m)$ for $1 \leq i \leq r$.

From Theorem 2.1, we obtain that

$$P(n) \subset P_1(n) \subset P_2(n).$$

Since Theorem 2.3 states that $P_2(n) = Q(n)$, we see that the set $P(n)$ provides a "lower bound" for $Q(n)$.

The set of maximal elements of $P(n)$ can easily be computed; and in order to compute $Q(n)$, it suffices to study the complement of $P(n)$ in $Q(n)$. This approach was used in [12] to compute $Q(n)$ explicitly for $1 \leq n \leq 50$. Actually, it turned out that $P(n) = Q(n)$ for $1 \leq n \leq 50$. However, in general $P(n)$ is not equal to $Q(n)$. It is proved in Section 7 of [12] that $P(78)$ is not equal to $Q(78)$, and $n = 78$ is the smallest known $n$ with $P(n)$ not equal to $Q(n)$.

In this paper we give a precise asymptotic estimate for the largest element of $Q(n)$ and $P(n)$, but first we have to collect some more results for linear maps.

# 3 The relation between linear and nonlinear maps

We have seen that a permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ induces a linear $l_1$-norm nonexpansive map $f_\sigma$ which, in fact, belongs to $\mathcal{F}_1(n)$. So the orders of the permutations on $n$ letters belong to the set $Q(n)$ and the largest order of a permutation on $n$ letters provides a lower bound for the largest element of $Q(n)$. Let $g(n)$ denote the largest order of a permutation on $n$ letters. The fundamental result, due to Landau [5], states that

$$\lim_{n\to\infty} \frac{\log g(n)}{\sqrt{n \log n}} = 1. \tag{3.1}$$

Since some basic properties of $g(n)$ and some arguments of the proof of (3.1) play a role in the analysis of the largest element of $Q(n)$, we summarize what is known about $g(n)$. A proof of (3.1) requires the Prime Number Theorem which states that if $\pi(x)$ denotes the number of primes not exceeding $x$, then

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1. \tag{3.2}$$

We refer to Miller [8] for a clear presentation of how to derive (3.1) from (3.2) and for historical notes about $g(n)$. One should note that an explicit upper bound for $\log g(n)$ was only recently obtained and that the function $n \mapsto g(n)$ has quite complicated behaviour: see Massias [7].

Let $L(n)$ denote the set of orders of the permutations on $n$ letters. From the disjoint cycle representation for permutations, it follows that

$$\begin{aligned} L(n) \;\; = \;\; \big\{ p \in \mathbb{N} \mid p = \operatorname{lcm}\{m_1, m_2, \ldots, m_s), \; m_i \geq 0, \; i = 1, 2, \ldots, s, \\ \sum_{i=1}^{m} m_i \leq n, \; \text{for some } s > 0 \big\} \end{aligned} \tag{3.3}$$

This representation for the orders of the permutations implies that $L(n) \subset P(n)$, so that $g(n) = \max\{p \mid p \in L(n)\}$ is actually a lower bound for the largest element of $P(n)$. In fact, the set $L(n)$ is the smallest set of positive integers such that $n \in L(n)$ and $L(n)$ is closed under Rule A of Definition 2.6. This fact together with the observation, made at the end of Section 2, that it suffices to compute the maximal elements of $L(n)$, yields a simple procedure to compute $g(n)$ up to $n = 100$.

The basic idea of the proof of (3.1) is the fact that the prime factorization of a given integer tells us whether the integer belongs to $L(n)$. Since this idea also plays a role in the analysis of the largest element of $Q(n)$, we recall the definition and the basic properties of the so-called $S$-function (see also Miller [8]).

**Definition 3.1** Let the function $S : \mathbb{N} \to \mathbb{N}$ be defined by $S(1) = 1$ and $S(p) = \sum_{j=1}^{s} p_j^{\alpha_j}$ for $p > 1$, where $p = \prod_{j=1}^{s} p_j^{\alpha_j}$ is the prime factorization of $p$.

Lemmas 3.1 and 3.2 below give standard properties of the $S$-function and are presented for the reader's convenience.

**Lemma 3.1** *If $m \geq 1$ is an integer, then $S(m) \leq m$.*

**Proof.** The lemma is trivially true if $m = p_1^{\alpha}$, where $p_1$ is a prime and $\alpha$ a nonnegative integer. Now we use mathematical induction with respect to the number of distinct primes in the prime factorization of $m$. We can assume that $m = p_1^{\alpha_1} m'$, where $m' > p_1$ and $p_1 \geq 2$ is a prime. By induction, $S(m') \leq m'$ and therefore

$$S(m) = p_1^{\alpha_1} + S(m') \leq p_1^{\alpha_1} + m'$$

and it remains to prove $p_1^{\alpha_1} + m' \leq p_1^{\alpha_1} m'$ or, equivalently,

$$p_1^{\alpha_1} \geq \frac{m'}{m'-1}. \tag{3.4}$$

Since $p_1^{\alpha_1} \geq 2$, $m' > 3$ and $m'/(m'-1) \leq 3/2$ for $m' \geq 3$, the inequality follows. □

**Lemma 3.2** *If $m_1, m_2, \ldots, m_t$ are positive integers, then*

$$S\big(lcm\,(m_1, m_2, \ldots, m_t)\big) \leq \sum_{i=1}^{t} S(m_i). \tag{3.5}$$

*Furthermore, if the integers $m_1, m_2, \ldots, m_t$ are relatively prime, so $gcd\,(m_i, m_j) = 1$, $1 \leq i < j \leq t$, then equality holds in* (3.5).

**Proof.** Let for $1 \leq j \leq t$, the prime factorization of $m_j$ be given by

$$m_j = \prod_{i=1}^{s} p_i^{\alpha_i(j)}, \qquad \alpha_i(j) \geq 0,\ p_i \text{ is prime },1 \leq i \leq s.$$

If $m = \operatorname{lcm}(m_1, m_2, \ldots, m_t)$, then the prime factorization of $m$ becomes, by definition,

$$m = \prod_{i=1}^{s} p_i^{\beta_i}, \qquad \beta_i = \max_{1 \leq j \leq t} \alpha_i(j), \quad 1 \leq i \leq s.$$

So, by removing possible common factors in $m_j$, $1 \leq j \leq t$, that do not effect $m$, we can write $m = \tilde{m}_1 \tilde{m}_2 \cdots \tilde{m}_t$, where $\tilde{m}_i | m_i$, $\gcd(\tilde{m}_i, \tilde{m}_j) = 1$, $1 \leq i < j \leq t'$, $\tilde{m}_i = 1$ and $t' \leq t$. In other words,

$$\tilde{m}_j = \prod_{i=1}^{s} p_i^{\beta_i(j)}, \qquad 1 \leq j \leq t',$$

where

$$\beta_i(j) = \begin{cases} \alpha_i(j) & \text{if } \alpha_i(j) = \beta_i \text{ and no } j' < j \text{ exists with } \alpha_i(j') = \beta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by construction,

$$S(m) = \sum_{j=1}^{t'} S(\tilde{m}_j) \quad \text{and} \quad S(\tilde{m}_j) \leq S(m_j).$$

This proves the lemma. $\square$

Using the $S$-function and Lemma 3.1 and 3.2, we can write

$$L(n) = \{m \in \mathbb{N} \mid S(m) \leq n\}.$$

From the definitions, it follows that $L(n) = P(n) = Q(n)$ for $1 \leq n \leq 5$ and the set $Q(6) = P(6) = L(6) \cup \{12\}$. For $n = 7$ one again has $L(n) = P(n) = Q(n)$, but actually 7 is the largest integer $n$ for which this equality holds.

**Theorem 3.1** *For $n \geq 8$, the set $P(n)$ is strictly bigger than the set $L(n)$.*

**Proof.** The proof of the theorem is based on the characterization of $L(n)$ using the $S$-function. We shall construct elements $q_n \in P(n)$ with the property that $S(q_n) > n$. The construction is based on the fact that for any integer $n \geq 1$, there exists a prime between $n$ and $2n$.

Define $p_k$, $k \geq 2$, to be the largest prime between $2^{k-1}$ and $2^k$. We claim that

$$q_{2^k} = 2^k \times 3 \times 7 \times \cdots \times p_{k-1} \in P(2^k), \qquad k \geq 3. \tag{3.6}$$

The proof of the claim uses mathematical induction. For $k = 3$, we have that $3 \in P(4)$ and $4 \in P(4)$. Therefore, it follows, by Rule B of Definition 2.6 with $r = 2$, that

$$q_8 = 2\mathrm{lcm}\,(4, 3) = 2^3 \times p_2 \in P(8).$$

Suppose that the claim holds for all $k$ with $3 \leq k < l$ and that

$$q_{2^{l-1}} = 2^{l-1} \times 3 \times 7 \times \cdots \times p_{l-2} \in P(2^{l-1}).$$

Since, by construction, $p_{l-1} \in P(2^{l-1})$, it follows from Rule B of Definition 2.6 with $r = 2$ that

$$q_{2^l} = 2\mathrm{lcm}\left(q_{2^{l-1}}, p_{l-1}\right) \in P(2^l).$$

This proves the claim. To define $q_n$ for $2^k < n < 2^{k+1}$, we define the elements $q_n$ for $8 < n < 16$ explicitly and again proceed by induction. Define

$$q_8 = q_9 = q_{10} = 8 \times 3, \;\; q_{11} = 4 \times 3 \times 5$$

and

$$q_{12} = q_{13} = q_{14} = q_{15} = 8 \times 3 \times 5.$$

Then, by construction, $q_n \in P(n)$ and $S(q_n) > n$ for $8 < n < 16$. In general, we define for $2^k < n < 2^k + \sum_{l=2}^{k-1} p_l$,

$$q_n = q_{2^k}.$$

Since $S(q_{2^k}) = 2^k + \sum_{l=2}^{k-1} p_l$, it follows that for $2^k < n < 2^k + \sum_{l=2}^{k-1} p_l$,

$$q_n \in P(2^k) \subset P(n) \quad \text{and } S(q_n) > n.$$

Therefore, $q_n \in P(n)$ but $q_n \notin L(n)$ for $2^k < n < 2^k + \sum_{l=2}^{k-1} p_l$.

By our construction, for $k \geq 4$, we have

$$\sum_{l=2}^{k-1} p_l > 2 + \sum_{l=2}^{k-1} 2^{l-1} = 2^{k-1},$$

we can define $q_n$ for $2^k + \sum_{l=2}^{k-1} p_l \leq n < 2^{k+1}$, $k \geq 4$, as follows. First write $n = n' + p_k$, and note that we have

$$n' = n - p_k < 2^k + 2^{k-1} + 2^{k-1} - 2^{k-1} = 2^k + 2^{k-1} < 2^k + \sum_{l=2}^{k-1} p_l$$

and

$$n' = n - p_k \geq 2^k + \sum_{l=2}^{k-1} p_l - 2^k > 2^{k-1}.$$

Therefore for $k \geq 4$ and for integers $n$ with $2^k + \sum_{l=2}^{k-1} p_l \leq n < 2^{k+1}$, it follows that

$$2^{k-1} < n' < 2^k + \sum_{l=2}^{k-1} p_l.$$

Now take $q_{n'} \in P(n')$ and $p_k \in P(p_k)$, so, by rule A, $\operatorname{lcm}(q_{n'}, p_k) \in P(n' + p_k) = P(n)$. Thus, for integers $n$ with $2^k + \sum_{l=2}^{k-1} p_l \leq n < 2^{k+1}$, we can define

$$q_n = \operatorname{lcm}(q_{n'}, p_k) \in P(n), \qquad n' = n - p_k.$$

It follows by mathematical induction and by our construction that $S(q_{n'}) > n'$. Furthermore, by the definition of $q_9, \ldots, q_{15}$ and by the construction of $q_n$ in general, we see that the largest prime in the prime factorization of $q_{n'}$ is less than or equal to $p_{k-1}$. We conclude that the integers $q_{n'}$ and $p_k$ are relatively prime, so

$$S(q_n) = S(q_{n'}) + S(p_k) > n' + p_k = n.$$

Thus $q_n \notin L(n)$ also for $n$ with $2^k + \sum_{l=2}^{k-1} p_l \leq n < 2^{k+1}$. This completes the proof. □

The theorem implies that for $n \geq 8$, the set $Q(n)$ is strictly bigger than $L(n)$. In the next section we discuss the asymptotic behaviour of the largest element of $Q(n)$.

# 4 An asymptotic estimate for the largest element

We are now ready to prove an asymptotic estimate for the largest element of $Q(n)$.

**Theorem 4.1** *If $\gamma(n)$ denotes the largest element of $Q(n)$, then*

$$\lim_{n\to\infty} \frac{\log \gamma(n)}{\sqrt{n \log n}} = 1. \tag{4.7}$$

**Proof.** Let the prime factorization of $\gamma(n)$ be given by

$$\gamma(n) = \prod_{i=1}^{t} p_i^{\alpha_i} \quad \text{with } p_1 < p_2 < p_3 < \ldots < p_t.$$

Since $\gamma(n)$ belongs to $Q(n)$, we have that $\gamma(n) = \mathrm{lcm}\,(q_1, q_2, \ldots, q_s)$, where $q_i$ are the periods of maps $\theta_i$ of an admissible array $\theta = (\theta_1, \theta_2, \ldots, \theta_s)$. Therefore, basic properties of the least common multiple imply that for every factor $p_i^{\alpha_i}$ of $\gamma(n)$, there exists a $q_{j_i}$, $1 \le j_i \le s$, such that $p^{\alpha_i} | q_{j_i}$. From the definition of an admissible array one has that the periods of the maps are less than or equal to $n$, $q_i \le n$ for $1 \le i \le s$. So we have

$$p_i^{\alpha_i} \le n, \qquad 1 \le i \le t. \tag{4.8}$$

Define the integer $l$, $1 \le l \le t$, such that $p_l \le \sqrt{n}$ and $p_{l+1} > \sqrt{n}$. Set

$$\gamma_1(n) = \prod_{i=1}^{l} p_i^{\alpha_i} \quad \text{and} \quad \gamma_2(n) = \prod_{i=l+1}^{t} p_i^{\alpha_i}. \tag{4.9}$$

From Corollary 2.1, it follows that $\gamma_1(n) \in Q(n)$ and $\gamma_2(n) \in Q(n)$.

First we analyse $\gamma_2(n)$. Note that

$$\gamma_2(n) = \prod_{l+1}^{t} p_i^{\alpha_i} \quad \text{with } p_i > \sqrt{n},\ l+1 \le i \le t.$$

Together with (4.8) this implies that $\alpha_i = 1$ for $l+1 \le i \le t$. Hence the prime factorization of $\gamma_2(n)$ is given by

$$\gamma_2(n) = \prod_{l+1}^{t} p_i \quad \text{with } p_i > \sqrt{n},\ l+1 \le i \le t. \tag{4.10}$$

Since $\gamma_2(n)$ belongs to $Q(n)$, we have that $\gamma_2(n) = \mathrm{lcm}\,(p_1, p_2, \ldots, p_s)$, where $p_i$ are the periods of maps $\psi_i$ of an admissible array $\psi = (\psi_1, \psi_2, \ldots, \psi_s)$. Therefore, for every prime factor $p_i$, $l+1 \le i \le t$ of $\gamma_2(n)$, there exists a $p_{k(i)} \in \{p_1, p_2, \ldots, p_s\}$. Since $p_{k(i)} \le n$ and $p_i > \sqrt{n}$ for $l+1 \le i \le t$, it follows that there cannot be another $p_j$, $j \ne i$, such that $p_j | p_{k(i)}$. Thus $p_i = p_{k(i)}$ for $l+1 \le i \le t$ and the periods of the admissible array $\psi$ corresponding to $\gamma_2(n)$ are just a permutation of the prime factors in the prime factorization (4.10) of $\gamma_2(n)$. This implies, in particular, that $\gcd\,(q_i, q_j) = 1$ for $1 \le i < j \le s$. So an application of Lemma 2.1 yields that $\gamma_2(n) \in L(n)$. Thus,

$$\gamma_2(n) \le g(n). \tag{4.11}$$

In order to estimate $\gamma_1(n)$, we use (4.8) and the prime factorization (4.9) of $\gamma_1(n)$. If $\pi(\sqrt{n})$ denotes the number of primes less than $\sqrt{n}$, then

$$\gamma_1(n) = \prod_{i=1}^{l} p_i^{\alpha_i} \leq n^{\pi(\sqrt{n})}. \tag{4.12}$$

Thus, using (4.11) and (4.12) we arrive at the following basic estimate for $\gamma(n)$, the largest element of $Q(n)$,

$$g(n) \leq \gamma(n) \leq g(n) n^{\pi(\sqrt{n})}. \tag{4.13}$$

The prime number theorem, see (3.2), states that

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

Furthermore, there exist effective bounds that improve this estimate: see Theorem 1 in [13],

$$\pi(\sqrt{n}) \leq 2\frac{\sqrt{n}}{\log n}\big(1 + \frac{3}{\log n}\big) \quad \text{for } n > 1. \tag{4.14}$$

Taking the logarithm in (4.13) and using (4.14), we have

$$\begin{aligned} \log g(n) \leq \log\gamma(n) \quad &\leq \quad \log g(n) + \pi(\sqrt{n})\log n \\ &\leq \quad \log g(n) + 2\sqrt{n}\big(1 + \frac{3}{\log n}\big), \quad n > 1. \end{aligned} \tag{4.15}$$

From (3.1) and the squeezing lemma, it follows that

$$\lim_{n\to\infty} \frac{\log\gamma(n)}{\sqrt{n\log n}} = 1.$$

This completes the proof of the theorem. □

The estimate (4.15) actually gives an effective upper bound for $\log\gamma(n)$ in terms of $g(n)$. Regarding $g(n)$, Massias [7] derived an explicit upper bound for $g(n)$ and determined the value of $n$ at which $(\log g(n) - \sqrt{n\log n})$ attains it maximum. Thus, we actually have an effective upper bound for $\log\gamma(n)$.

The known upper bound for the largest element of $Q(n)$ is equal to $\mathrm{lcm}\,(1, 2, \ldots, n)$: see [14, 15]. This bound has been slightly improved in [9] and it was shown that the largest element of $Q(n)$ is less than $2^n$: see page 366 and 367 of [9]. These estimates shows that our bound for the logarithm of the largest element is rather sharp. However, one has to realize that a sharp bound that directly applies to the largest element in $Q(n)$ is still lacking.

Since $L(n) \subset P(n)$ and Theorem 2.3 implies that $P(n) \subset Q(n)$, the same estimate (4.7) also holds for the largest element of $P(n)$.

**Corollary 4.1** *If $\gamma_0(n)$ denotes the largest element of $P(n)$, then*

$$\lim_{n\to\infty} \frac{\log\gamma_0(n)}{\sqrt{n\log n}} = 1. \tag{4.16}$$

## 5 Further analysis of the $S$-function

We have observed in Section 3 that a positive integer $q$ is an element of $L(n)$ if and only if $S(q)/n \leq 1$. Thus one approach to understanding the difference between $P(n)$ and $Q(n)$ or between $P(n)$ and $L(n)$ is to study $\{S(q)/n \mid q \in Q(n)\}$ or $\{S(q)/n \mid q \in P(n)\}$. In particular, it is of interest to study sequences $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ defined by

$$c_n = \max\{\frac{S(q)}{n} \mid q \in Q(n)\} \quad \text{and} \quad d_n = \max\{\frac{S(q)}{n} \mid q \in P(n)\}. \tag{5.1}$$

It is proved in [12] that $P(n) = Q(n)$ for $1 \leq n \leq 50$, so $c_n = d_n$ for $1 \leq n \leq 50$; and since $P(n) \subset Q(n)$ for all $n$, we always have $d_n \leq c_n$. With the aid of a computer (see [12] for the case $1 \leq n \leq 50$) one can show that, at least for moderate values of $n$, the sequence $(d_n)_{n\geq 1}$ is irregular and takes relatively small values. Actually, the maximum value of $d_n$ for $1 \leq n \leq 80$ arises for $n = 68$, see Table 1 below.

In this section we shall present some preliminary results concerning the numbers $\big(S(q)/n\big)$ for $q \in Q(n)$ or $q \in P(n)$. As we shall discuss below, it is very likely that a deeper understanding will involve some delicate number theoretical issues.

**Proposition 5.1** *If $q$ is the period of an admissible array on $n$ symbols with two rows, then $S(q) \leq (3/2)n$. If $q$ is the period of an admissible array on $n$ symbols with three rows, then $S(q) \leq 2n$. If $q \in Q(n)$ and $q = p_1^{\alpha_1} p_2^{\alpha_2}$, where $p_1$ and $p_2$ are primes, then $S(q) \leq (3/2)n$. If $q \in Q(n)$ and $q = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, where $p_1, p_2$ and $p_3$ are primes, then $S(q) \leq 2n$.*

**Proof.** Let $L = \{1, 2\}$ with the usual ordering, $\Sigma$ denote a set with $n$ elements and $\{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ denote an admissible array (see Definition 2.2) whose period is $q$. If $q_i \leq n$ denotes the period of $\theta_i$, then $q = \text{lcm}\,(q_1, q_2)$. Let $B_{q_i}$ denote the range of $\theta_i$, so $|B_{q_i}| = q_i$. If $\gcd\,(q_1, q_2) = 1$, then condition (ii) of Definition 2.2 implies that $B_{q_1} \bigcap B_{q_2} = \emptyset$, so $q_1 + q_2 \leq n$. It follows in this case that

$$S\big(\text{lcm}\,(q_1, q_2)\big) \leq S(q_1) + S(q_2) \leq q_1 + q_2 \leq n.$$

If $B_{q_1} \bigcap B_{q_2} \neq \emptyset$, condition (ii) of Definition 2.2 implies that $\gcd\,(q_1, q_2) = \rho \geq 2$. Thus there exists a prime factor $p$ of $\rho$ such that either (a) $\text{lcm}\,(q_1, q_2) = \text{lcm}\,(q_1/p, q_2)$ or (b) $\text{lcm}\,(q_1, q_2) = \text{lcm}\,(q_1, q_2/p)$, depending on which term has the higher power of $p$ as a factor. In case (a)

$$\begin{aligned} S\big(\text{lcm}\,(q_1, q_2)\big) = S\big(\text{lcm}\,(q_1/p, q_2)\big) &\leq S(q_1/p) + S(q_2) \\ &\leq \frac{q_1}{p} + q_2 \leq n(1 + \frac{1}{p}) \leq \frac{3}{2}n. \end{aligned}$$

The proof that $S(q) \leq (3/2)n$ in case (b) is the same.

Next suppose that $q$ is the period of an admissible array $\{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ on $n$ symbols with three rows. We can assume that $L = \{1, 2, 3\}$ has the standard ordering,

| $n$ | $d_n$ | | $n$ | $d_n$ | | $n$ | $d_n$ | | $n$ | $d_n$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | 21 | $\frac{31}{21}$ | | 41 | $\frac{66}{41}$ | | 61 | $\frac{88}{61}$ |
| 2 | 1 | | 22 | $\frac{16}{11}$ | | 42 | $\frac{11}{7}$ | | 62 | $\frac{89}{62}$ |
| 3 | 1 | | 23 | $\frac{32}{23}$ | | 43 | $\frac{66}{43}$ | | 63 | $\frac{10}{7}$ |
| 4 | 1 | | 24 | $\frac{35}{24}$ | | 44 | $\frac{35}{22}$ | | 64 | $\frac{61}{32}$ |
| 5 | 1 | | 25 | $\frac{7}{5}$ | | 45 | $\frac{71}{45}$ | | 65 | $\frac{122}{65}$ |
| 6 | $\frac{7}{6}$ | | 26 | $\frac{37}{26}$ | | 46 | $\frac{71}{46}$ | | 66 | $\frac{127}{66}$ |
| 7 | 1 | | 27 | $\frac{43}{27}$ | | 47 | $\frac{72}{47}$ | | 67 | $\frac{127}{67}$ |
| 8 | $\frac{11}{8}$ | | 28 | $\frac{43}{28}$ | | 48 | $\frac{25}{16}$ | | 68 | $\frac{131}{68}$ |
| 9 | $\frac{11}{9}$ | | 29 | $\frac{43}{29}$ | | 49 | $\frac{75}{49}$ | | 69 | $\frac{131}{69}$ |
| 10 | $\frac{13}{10}$ | | 30 | $\frac{47}{30}$ | | 50 | $\frac{3}{2}$ | | 70 | $\frac{131}{70}$ |
| 11 | $\frac{13}{11}$ | | 31 | $\frac{47}{31}$ | | 51 | $\frac{77}{51}$ | | 71 | $\frac{131}{71}$ |
| 12 | $\frac{4}{3}$ | | 32 | $\frac{29}{16}$ | | 52 | $\frac{79}{52}$ | | 72 | $\frac{131}{72}$ |
| 13 | $\frac{16}{13}$ | | 33 | $\frac{58}{33}$ | | 53 | $\frac{79}{53}$ | | 73 | $\frac{131}{73}$ |
| 14 | $\frac{9}{7}$ | | 34 | $\frac{59}{34}$ | | 54 | $\frac{14}{9}$ | | 74 | $\frac{131}{74}$ |
| 15 | $\frac{6}{5}$ | | 35 | $\frac{59}{35}$ | | 55 | $\frac{84}{55}$ | | 75 | $\frac{131}{75}$ |
| 16 | $\frac{13}{8}$ | | 36 | $\frac{5}{3}$ | | 56 | $\frac{3}{2}$ | | 76 | $\frac{7}{4}$ |
| 17 | $\frac{26}{17}$ | | 37 | $\frac{60}{37}$ | | 57 | $\frac{28}{19}$ | | 77 | $\frac{134}{77}$ |
| 18 | $\frac{13}{9}$ | | 38 | $\frac{61}{38}$ | | 58 | $\frac{42}{29}$ | | 78 | $\frac{24}{13}$ |
| 19 | $\frac{26}{19}$ | | 39 | $\frac{64}{39}$ | | 59 | $\frac{85}{59}$ | | 79 | $\frac{144}{79}$ |
| 20 | $\frac{31}{20}$ | | 40 | $\frac{33}{20}$ | | 60 | $\frac{29}{20}$ | | 80 | $\frac{73}{40}$ |

Table 1: This table tabulates $d_n = \max\{S(q)/n : q \in P(n)\}$ for $1 \le n \le 80$.

and for convenience we take $\Sigma = \{j \in \mathbb{N} \mid 1 \le j \le n\}$. As before, let $q_i \le n$ denote the period of $\theta_i$ and let $B_{q_i}$ denote the range of $\theta_i$. If $q_1 + q_2 + q_3 \le 2n$, we see that

$$S(q) = S\big(\operatorname{lcm}(q_1, q_2, q_3)\big) \le \sum_{i=1}^{3} S(q_i) \le \sum_{i=1}^{3} q_i \le 2n.$$

If $q_1 + q_2 + q_3 > 2n$, one can easily verify that $B_{q_1} \cap B_{q_2} \cap B_{q_3} \ne \emptyset$. (To see this, let $\chi_i(t)$, $t \in \{1, 2, \ldots, n\}$, denote the characteristic function of $B_{q_i}$, so $\chi_i(t) = 1$ if and only if $t \in B_{q_i}$. If $\bigcap_{i=1}^{3} B_{q_i} = \emptyset$, then $\sum_{i=1}^{3} \chi_i(t) \le 2$ for all $t$ and

$$2n \ge \sum_{t=1}^{n}\Big(\sum_{i=1}^{3} \chi_i(t)\Big) = \sum_{i=1}^{3}\Big(\sum_{t=1}^{n} \chi_i(t)\Big) \ge \sum_{i=1}^{3} q_i > 2n,$$

a contradiction.) It follows that there exist $s_1$, $t_1 = s_2$ and $t_2$ such that

$$\theta_1(s_1) = \theta_2(t_1) = \theta_2(s_2) = \theta_3(t_2).$$

If $\rho = \gcd(q_1, q_2, q_3)$, condition (ii) of Definition 2.2 implies that

$$(s_1 - t_1) + (s_2 - t_2) \not\equiv 0 \mod \rho,$$

so we must have $\rho > 1$. If $p$ is a prime factor of $\rho$, there exists an integer $\alpha \geq 1$ and an integer $j \in L$ such that $p^\alpha | q_j$ but $p^{\alpha+1} \not| q_i$ for $i \in L$ and $i \neq j$. If we denote by $i$ and $k$ the elements of $L$ which are unequal to $j$, we find that

$$\begin{aligned} S(q) &= S\big(\operatorname{lcm}(q_j, q_i, q_k)\big) = S\big(\operatorname{lcm}(q_j, q_i/p, q_j/p)\big) \\ &\leq S(q_j) + S(q_i/p) + S(q_k/p) \\ &\leq q_j + q_i/p + q_k/p \leq n\big(1 + \frac{2}{p}\big) \leq 2n. \end{aligned}$$

To obtain the final statement of Proposition 5.1, note that if $q \in Q(n)$ and $q = \prod_{i=1}^k p_i^{\alpha_i}$, where $\alpha_i \in \mathbb{N}$ and $p_i$ is a prime for $1 \leq i \leq k$, then $q$ is the period of an admissible array on $n$ symbols and $m$ rows, $m \leq k$. A proof of this for the case $k = 3$ is given in the proof of Corollary 5.5 on p. 28 of [12], and the general argument is the same. □

**Remark 5.1** If $q \in Q(n)$ and $q$ is the period of an admissible array on $n$ symbols and $k \leq 3$ rows, then it is proved in [12] that $q \in P(n)$. See Proposition 5.1 and Theorem 5.2 on p. 24 in [12].

**Remark 5.2** If $q$ is the period of an admissible array on $n$ symbols and three rows, one can prove that $S(q) \leq (7/4)n$, so the estimate in Proposition 5.1 is not optimal. The proof that $S(q) \leq (7/4)n$ depends on first considering several cases depending on the intersection pattern of $B_{q_i} \cap B_{q_j}$ for $i, j \in L$ and then carefully using condition (ii) of Definition 2.2. For the sake of brevity, we omit the proof. We shall prove below (see Remark 5.3) that the constants 3/2 and 7/4 are optimal.

Given an interval of real numbers $J$, a real number $\epsilon$ and a set of real numbers $T$, we shall say that "$T$ is $\epsilon$-dense in $J$" if, for each $x \in J$, there exists $t \in T$ with $|t - x| < \epsilon$.

**Lemma 5.1** *Suppose that $0 < \epsilon < 1$, that $p$ is a prime number and that $m$ is a positive integer such that for all $\nu \geq m$, the interval $\big((1 - \epsilon/2)p^\nu, p^\nu\big)$ contains at least $p - 1$ distinct prime numbers. Then if $\mu$ is a positive integer and $\delta = \delta(\mu) := \epsilon/2 + p^{-\mu}$, the set $\{S(q)/p^{m+\mu} \mid q \in P(p^{m+\mu})\}$ is $\delta$-dense in the interval $(0, 2)$.*

**Proof.** By assumption, for each integer $\nu \geq m$, there exist at least $p-1$ distinct primes $p_{\nu,j}$, $1 \leq j \leq p - 1$, such that

$$(1 - \frac{\epsilon}{2})p^\nu < p_{\nu,j} < p^\nu.$$

For $\nu = m$, we have that $p^m \in P(p^m)$ and $p_{m,j} \in P(p^m)$ for $1 \leq j \leq p - 1$, so the properties of the sets $P(k)$, $k \geq 1$, imply that

$$\operatorname{plcm}\big(p^m, p_{m,j} \mid 1 \leq j \leq p - 1\big) = p^{m+1} \prod_{j=1}^{p-1} p_{m,j} \in P(p^{m+1}).$$

Arguing by mathematical induction, assume for some $\mu \geq 1$ that we have proved that

$$p^{m+\mu} \prod_{\nu=m}^{m+\mu-1} \prod_{j=1}^{p-1} p_{\nu,j} \in P(p^{m+\mu}). \tag{5.2}$$

It follows that

$$p\mathrm{lcm}\left(p^{m+\mu} \prod_{\nu=m}^{m+\mu-1} \prod_{j=1}^{p-1} p_{\nu,j}, p_{m+\mu,i} \mid 1 \leq i \leq p-1\right) = p^{m+\mu+1} \prod_{\nu=m}^{m+\mu} \prod_{j=1}^{p-1} p_{\nu,j} \in P(p^{m+\mu+1}).$$

By mathematical induction we conclude that (5.2) holds for all $\mu \geq 1$.

Recall (see [12]) that if $q \in P(k)$, then any divisor of $q$ is also an element of $P(k)$. For each $\nu \geq m$, let $A_\nu$ be a subset of $\{j \in \mathbb{N} \mid 1 \leq j \leq p-1\}$ and let $a_\nu = |A_\nu|$, the cardinality of $A_\nu$. If $A_\nu$ is empty, define $\prod_{j \in A_\nu} p_{\nu,j} = 1$ and $\sum_{j \in A_\nu} p_{\nu,j} = 0$. Select a fixed integer $\mu \geq 1$, select a nonnegative integer $\alpha \leq m+\mu$ and observe that

$$q := p^{\alpha} \prod_{\nu=m}^{m+\mu-1} \prod_{j \in A_\nu} p_{\nu,j} \in P(p^{m+\mu}), \tag{5.3}$$

because it is a divisor of the left hand side of (5.2). If $\alpha > 0$, we see that

$$\begin{aligned} p^{-(m+\mu)} S(q) &= p^{\alpha-m-\mu} + \sum_{\nu=m}^{m+\mu-1} \sum_{j \in A_\nu} \frac{p_{\nu,j}}{p^{m+\mu}} \\ &> p^{\alpha-m-\mu} + \sum_{\nu=m}^{m+\mu-1} (1-\frac{\epsilon}{2}) a_\nu p^{\nu-m-\mu}, \end{aligned}$$

and if $\alpha = 0$ and $q > 1$,

$$p^{-(m+\mu)} S(q) > \sum_{\nu=m}^{m+\mu-1} (1-\frac{\epsilon}{2}) a_\nu p^{\nu-m-\mu}.$$

Note that an upper bound on $p^{-(m+\mu)} S(q)$ follows by replacing $(1-\epsilon/2)$ by 1 in these two inequalities. Recall that any real number $x$, $0 \leq x \leq 1$, can be written in the form

$$x = \sum_{t=1}^{\infty} b_t p^{-t},$$

where $0 \leq b_t \leq p-1$ is an integer for $t \geq 1$. Given such a number $x$, define $a_\nu = b_{m+\mu-\nu}$ for $m \leq \nu \leq m+\mu-1$, $\alpha = m+\mu$ and let $q$ be as in (5.3), with $|A_\nu| = a_\nu$. Then we find that

$$\begin{aligned}
0 \le (1+x) - p^{-(m+\mu)}S(q) &\le (1+x) - 1 - (1-\frac{\epsilon}{2})\sum_{t=1}^{\mu} b_t p^{-t} \\
&= \sum_{t=1}^{\infty} b_t p^{-t} - (1-\frac{\epsilon}{2})\sum_{t=1}^{\mu} b_t p^{-t} \\
&\le \frac{\epsilon}{2}\sum_{t=1}^{\mu} b_t p^{-t} + \sum_{t=\mu+1}^{\infty} b_t p^{-t} \\
&< \frac{\epsilon}{2} + p^{-\mu}.
\end{aligned}$$

It follows that $P(p^{m+\mu})$ is $\delta$-dense in $[1,2]$ with $\delta := \epsilon/2 + p^{-\mu}$.

Set $\delta := \epsilon/2 + p^{-\mu}$. If $x$ is as above, and $0 \le x \le \delta$, select $q = 1$ and note that $|x - p^{-(m+\mu)}S(q)| < \delta$, because $S(1) = 1$. If $x > \delta$, define $a_\nu = b_{m+\mu-\nu}$ for $m \le \nu \le m+\mu-1$ and define $\alpha = 0$. Note that $a_\nu > 0$ for some $\nu$ with $m \le \nu \le m+\mu-1$, since otherwise $x \le p^{-\mu} < \delta$. Arguing as above, we see that

$$0 \le x - p^{-(m+\mu)}S(q) \le \sum_{t=1}^{\infty} b_t p^{-t} - (1-\epsilon/2)\sum_{t=1}^{\mu} b_t p^{-t} < \frac{\epsilon}{2} + p^{-\mu} = \delta.$$

It follows that $P(p^{m+\mu})$ is $\delta$-dense in $[0,1]$, which completes the proof. $\square$

**Remark 5.3** Take $p = 2$ and $0 < \epsilon < 1$ in Lemma 5.1. The prime number theorem implies that there exists an integer $m \ge 1$ such that for all integers $\nu \ge m$, the interval $\big((1-\epsilon/2)2^\nu, 2^\nu\big)$ contains a prime $p_{\nu,1}$. Thus the hypotheses of Lemma 5.1 are satisfied. If we take $\mu = 1$ in (5.2), we see that $2^{m+1}p_{m,1} \in P(2^{m+1})$ and

$$S(2^{m+1}p_{m,1}) = 2^{m+1} + p_{m,1} \ge 2^{m+1} + (1-\frac{\epsilon}{2})2^m = 2^{m+1}(\frac{3}{2} - \frac{\epsilon}{4}).$$

Since $\epsilon > 0$ was arbitrary, this shows the constant $3/2$ in Proposition 5.1 is optimal. If we take $\mu = 2$ in (5.2), we also see that the constant $7/4$ in Remark 5.2 is optimal.

**Theorem 5.1** *Given $\epsilon > 0$, there exists a positive integer $m = m(\epsilon)$ such that the set $\big\{S(q)/n \mid q \in P(n)\big\}$ is $\epsilon$-dense in the interval $(0,2)$ for all $n \ge m$.*

**Proof.** Select $\epsilon$, $0 < \epsilon < 1$, and let $p_k$ denote the $k^{th}$ prime. By using the prime number theorem, we see that there exists an integer $N \ge 1$ such that the interval $\big((1-\epsilon/2)p_n^j, p_n^j\big)$ contains at least $p_n - 1$ primes for all $n \ge N$ and all $j \ge 2$. For $n \ge N$, if, in the notation of Lemma 5.1, we write $p = p_n$, $m = 2$ and $\mu = 1$, Lemma 5.1 implies that $\big\{S(q)/p_n^3 \mid q \in P(p_n^3)\big\}$ is $\delta$-dense in $(0,2)$ for $\delta = \epsilon/2 + p_n^{-1}$. Furthermore, the argument in Lemma 5.1 actually showed that for each $x \in [0,2]$, there exists $q \in P(p_n^3)$ such that

$$0 \le \frac{S(q)}{p_n^3} \le 2 \quad \text{and} \quad \big|x - \frac{S(q)}{p_n^3}\big| \le \delta.$$

By using the prime number theorem again, we see that there exists an integer $N_1 \geq N$ such that for any $n \geq N_1$, $p_n^{-1} < \epsilon/6$ and $(1-\epsilon/6)p_{n+1}^3 \leq p_n^3$. If $m \geq p_{N_1}$, select $n \geq N_1$ such that $p_n^3 \leq m < p_{n+1}^3$. If $x \in [0,2]$, we have seen that there exists $q \in P(p_n^3) \subset P(m)$ with $0 \leq S(q)/p_n^3 \leq 2$ and

$$|x - \frac{S(q)}{p_n^3}| \leq \delta = \epsilon/2 + p_n^{-1}.$$

For this $q$, it follows that

$$\begin{aligned} |x - \frac{S(q)}{m}| &\leq |x - \frac{S(q)}{p^3}| + |\frac{S(q)}{p^3} - \frac{S(q)}{m}| \\ &\leq \delta + \frac{S(q)}{p_n^3}[1 - \frac{p_n^3}{m}] \\ &\leq \delta + 2[1 - \frac{p_n^3}{p_{n+1}^3}] < \epsilon, \end{aligned}$$

which completes the proof. □

Theorem 5.1 implies that $\liminf_{n\to\infty} d_n \geq 2$, where $d_n$ is as in (5.1); but we believe this estimate is not representative for large $n$. In fact we make the following conjecture.

**Conjecture 5.1** *For $d_n$ as in* (5.1) *one has* $\limsup_{n\to\infty} d_n = \infty$.

Conjecture 5.1 would imply (since $d_n \leq c_n$) that $\limsup_{n\to\infty} c_n = \infty$, but the converse is not known to be true, so we also propose a weaker conjecture.

**Conjecture 5.2** *For $c_n$ as in* (5.1) *one has* $\limsup_{n\to\infty} c_n = \infty$.

It seems that proving Conjecture 5.1 may be closely related to a difficult question in the theory of transcendental numbers. Indeed, we suspect that if one knew the truth of the following purely number theoretical conjecture, one might be able to prove Conjecture 5.1.

**Conjecture 5.3** *There exists a strictly increasing sequence of prime numbers $(p_k)_{k\geq 1}$ such that $\{(\log p_j)^{-1} \mid 1 \leq j \leq N\}$ is linearly independent over the rational numbers $\mathbb{Q}$ for every $N \geq 1$.*

Indeed, as we shall show below, a slightly weaker version of Conjecture 5.3 implies that $\limsup_{n\to\infty} d_n \geq 3$; and at present, the only way we know to prove this result is essentially to assume Conjecture 5.3.

We are indebted to our colleague at Rutgers, Professor Jozsef Beck, who has pointed out to us that the truth of Conjecture 5.3 would follow from the so-called Schanuel conjecture: see p. 120 of [2]. Recall that complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$ are algebraically independent over $\mathbb{Q}$ if they do not satisfy a polynomial equation $p(\alpha_1, \alpha_2, \ldots, \alpha_N) = 0$, where $p$ is a nonzero polynomial in $N$ variables with coefficients in $\mathbb{Q}$. Schanuel's conjecture asserts that if the complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$ are linearly independent over $\mathbb{Q}$, then at least $N$ of the $2N$ numbers $\alpha_j, e^{\alpha_j}$, $1 \le j \le N$, are algebraically independent over $\mathbb{Q}$. Notice that if $p_1, p_2, \ldots, p_N$ are distinct prime numbers, then the prime factorization theorem implies that $\log p_1, \log p_2, \ldots, \log p_N$ are linearly independent over $\mathbb{Q}$. Taking $\alpha_j = \log p_j$, Schanuel's conjecture would imply that the numbers $\log p_j$, $1 \le j \le N$, are algebraically independent over $\mathbb{Q}$, so the numbers $(\log p_j)^{-1}$, $1 \le j \le N$, would be linearly independent over $\mathbb{Q}$.

It seems that a deeper analysis of the sets $P(n)$ and $Q(n)$ involves a theorem of Kronecker which can be formulated as follows (see p. 80 of [16]). Given a real $\nu_1 \times \nu_2$ matrix $A$ and a real $\nu_2$-vector $b$, the following statements are equivalent:

(i) For every $\epsilon > 0$ there exists $x \in \mathbb{Z}^{\nu_2}$ such that $\|Ax - b\| < \epsilon$.

(ii) If $y \in \mathbb{R}^{\nu_1}$ and $y^T A \in \mathbb{Z}^{\nu_2}$, then $y^T b \in \mathbb{Z}$.

As an application of Kronecker's theorem and an initial indication of the importance of Conjecture 5.3 we obtain

**Lemma 5.2** *Assume that $p_1, p_2, \ldots, p_N$ are primes such that $\{(\log p_i)^{-1} \mid 1 \le i \le N\}$ is linearly independent over $\mathbb{Q}$. For every $\epsilon > 0$ and every positive integer $n_*$, there exist positive integers $n \ge n_*$ and $\alpha_j$, $1 \le j \le N$, such that*

$$(1-\epsilon)n < p_j^{\alpha_j} \le n. \tag{5.4}$$

**Proof.** By relabelling we can assume that $p_j < p_{j+1}$ for $1 \le j < N$. Let $q > p_N$ be a prime so large that $q^{-N}(q-1)^N > 1 - \epsilon/2$. The idea of the proof is to use Kronecker's theorem to find positive integral solutions $\alpha_j$, $1 \le j \le N$, of the approximate equations

$$p_j^{\alpha_j} p_{j+1}^{-\alpha_{j+1}} \approx \frac{q-1}{q}, \qquad 1 \le j < N. \tag{5.5}$$

Notice that the prime factorization theorem implies that none of the equations (5.5) can have an exact solution. Taking logarithms gives

$$\alpha_j \log p_j - \alpha_{j+1} \log p_{j+1} \approx \log \frac{q-1}{q}, \qquad 1 \le j < N. \tag{5.6}$$

If $x$ and $b$ are $N$-column vectors, with $x_j = \alpha_j$ and $b_j = \log\big((q-1)/q\big)$, $1 \le j \le N$, and if $A = (a_{ij})$ is the $(N-1) \times N$ matrix with $a_{ii} = \log p_i$, $a_{i,i+1} = -\log p_{i+1}$ for $1 \le i \le N-1$ and $a_{ij} = 0$ otherwise, then we wish to find solutions $x \in \mathbb{Z}^N$ of $Ax \approx b$. None of the equations (5.6) can have exact integral solutions, but Kronecker's theorem implies that for every $\delta > 0$, there exists $x \in \mathbb{Z}^N$ with $0 < \|Ax - b\| < \delta$, if, whenever $y \in \mathbb{R}^{N-1}$ and $y^T A \in \mathbb{Z}^N$, it follows that $y^T b \in \mathbb{Z}$. If the components of $y$ are $y_j$, $1 \le j \le N-1$, the equation $y^T A \in \mathbb{Z}^N$ implies that $y_1 \log p_1 \in \mathbb{Z}$, $-y_{j-1} \log p_{j-1} + y_j \log p_j \in \mathbb{Z}$ for $2 \le j \le N-1$ and $-y_{N-1} \log p_N \in \mathbb{Z}$. One derives from this that there are integers $m_j$, $1 \le j \le N$, such that

$$y_k = \sum_{j=1}^{k} \frac{m_j}{\log p_j}, \quad 1 \le k \le N-1, \text{ and } y_{N-1} = -\frac{m_N}{\log p_N}. \tag{5.7}$$

Using the equation for $y_{N-1}$ in (5.7) we find that

$$\sum_{j=1}^{N} \frac{m_j}{\log p_j} = 0, \tag{5.8}$$

which implies that $m_j = 0$ and $y_j = 0$ for $1 \leq j \leq N$ and $y^T b = 0$. By Kronecker's theorem, for every $\delta > 0$, there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{Z}^N$ such that, for $1 \leq j \leq N-1$

$$\log \frac{q-1}{q} - \delta < \alpha_j \log p_j - \alpha_{j+1} \log p_{j+1} < \log \frac{q-1}{q} + \delta. \tag{5.9}$$

Since none of the equations (5.9) has an exact integral solution, for any $a > 0$ we can arrange, by taking $\delta > 0$ sufficiently small, that there is a solution $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{Z}^N$ of (5.9) such that $|\alpha_j| \geq a$ for $1 \leq j \leq N$. If

$$\delta < \log\bigl(q/(q-1)\bigr) \quad \text{and} \quad \delta + \log\bigl(q/(q-1)\bigr) < \log 2$$

(both of which are true for $\delta > 0$ sufficiently small), one can easily check that all the integers $\alpha_j$, $1 \leq j \leq N$, which solve (5.9) are strictly positive or all are strictly negative. If the $\alpha_j$ solving (5.9) are strictly positive and $\delta < \log\bigl(q/(q-1)\bigr)$, we obtain for $1 \leq j < N$

$$p_{j+1}^{\alpha_{j+1}} \exp\bigl[\log\bigl((q-1)/q\bigr) - \delta\bigr] < p_j^{\alpha_j} < p_{j+1}^{\alpha_{j+1}} \exp\bigl[\log\bigl((q-1)/q\bigr) + \delta\bigr]. \tag{5.10}$$

If the $\alpha_j$ solving (5.9) are strictly negative and $\delta < \log\bigl(q/(q-1)\bigr)$, we write $\beta_j = -\alpha_j > 0$ and observe that for $1 \leq j < N$

$$p_j^{\beta_j} \exp\bigl[\log\bigl((q-1)/q\bigr) - \delta\bigr] < p_{j+1}^{\beta_{j+1}} < p_j^{\beta_j} \exp\bigl[\log\bigl((q-1)/q\bigr) + \delta\bigr]. \tag{5.11}$$

In the case that the $\alpha_j$ are positive, we deduce from (5.11) that, since $\delta < \log\bigl(q/(q-1)\bigr)$, $p_j^{\alpha_j} < p_{j+1}^{\alpha_{j+1}}$ for $1 \leq j < N$ and

$$p_N^{\alpha_N} \bigl((q-1)/q\bigr)^N \exp(-N\delta) < p_j^{\alpha_j} < p_N^{\alpha_N}, \qquad 1 \leq j < N. \tag{5.12}$$

If we choose $\delta > 0$ so small that $\exp(-N\delta) > (1 - \epsilon/2)$, define $n = p_N^{\alpha_N}$ and recall that $\bigl((q-1)/q\bigr)^N > 1 - \epsilon/2$, we see that (5.12) implies that

$$(1 - \epsilon)n < p_j^{\alpha_j} \leq n, \qquad 1 \leq j \leq N.$$

Note that by taking $\delta > 0$ sufficiently small, we can insure that $\alpha_N$ is as large as desired and guarantee that $p_N^{\alpha_N} = n \geq n_*$.

In the case that the $\alpha_j$ are negative, we replace $\alpha_j$ by $\beta_j = -\alpha_j$ and use (5.10). In this case we have $n = p_1^{\beta_1}$ and a similar argument completes the proof. $\square$

Our next theorem generalizes aspects of Theorem 5.1.

**Theorem 5.2** *Assume that $p_1, p_2, \ldots, p_N$ are prime numbers such that the following numbers $\bigl(\log p_j\bigr)^{-1}$, $1 \leq j \leq N$, are linearly independent over $\mathbb{Q}$. Then, for each $\epsilon$, $0 < \epsilon < 1/2$, and each positive integer $n_*$, there exists $n \geq n_*$ and elements $q_j \in P(n)$, $1 \leq j \leq N$, such that*

(a) *gcd* $(q_j, q_k) = 1$ *for* $1 \le j < k \le N$;

(b) $S(q_j) \ge (2-\epsilon)n$ *for* $1 \le j \le N$ *and*

(c) $q_j$ *has a factor of the form* $p_j^{\alpha_j}$, $\alpha_j \in \mathbb{N}$, *where* $(1-\epsilon/4)n < p_j^{\alpha_j} \le n$.

*In general, without any assumptions on the primes* $p_j$, $1 \le j \le N$, *if* $p_1 = N$ *and if there exist an integer* $n \ge 1$ *and elements* $q_j \in P(n)$ *which satisfy (a), (b) and (c) for some* $\epsilon$, $0 < \epsilon < 1/2$, *then* $p_1 lcm(q_1, \dots, q_N) \in P(p_1 n)$ *and*

$$S\big(p_1 lcm(q_1, q_2, \dots, q_N)\big) \ge [3 - \frac{5\epsilon}{4} - \frac{1}{p_1}](p_1 n).$$

**Proof.** Select $\epsilon$, $0 < \epsilon < 1/2$, let $\theta := (1-\epsilon/2)$ and $r := \sum_{j=1}^N p_j$. The prime number theorem implies that there exists $m \ge 2$ such that for all integers $\nu \ge m$ and for each $j$, $1 \le j \le N$, the interval $[\theta p_j^\nu, p_j^\nu]$ contains at least $r$ distinct primes. For given integers $i \ne j$, $1 \le i, j \le N$ and a given integer $\nu \ge 1$, notice that the interval $[\theta p_j^\nu, p_j^\nu]$ has nonempty intersection with at most one interval of the form $[\theta p_i^k, p_i^k]$, $k \ge 1$. For suppose that $[\theta p_i^k, p_i^k]$ and $[\theta p_i^l, p_i^l]$, $1 \le k < l$, both intersect $[\theta p_j^\nu, p_j^\nu]$. Then we must have

$$\theta p_j^\nu \le p_i^k < \theta p_i^l \le p_j^\nu,$$

which implies that

$$\frac{p_j^\nu}{\theta p_j^\nu} = \frac{1}{\theta} > \frac{\theta p_i^l}{p_i^k} \ge \theta p_i \ge 2\theta.$$

However, the inequality $1 > 2\theta^2$ is impossible for $\theta = 1 - \epsilon/2$ and $0 < \epsilon < 1/2$.

We now refine the argument in Lemma 5.1. For $\nu \ge m$, select $p_1 - 1$ primes $p_{1,\nu,j}$, $1 \le j \le p_1 - 1$ such that $\theta p_1^\nu < p_{1,\nu,j} < p_1^\nu$. The argument in the proof of Lemma 5.1 (see eq. (5.2)) shows that for $\mu_1 \ge 1$,

$$q_{1,\mu_1} := p_1^{m+\mu_1} \prod_{\nu=m}^{m+\mu_1} \prod_{j=1}^{p_1-1} p_{1,\nu,j} \in P(p_1^{m+\mu_1}).$$

For a given $\nu \ge m$, at most one of the intervals $[\theta p_1^k, p_1^k]$ intersects the interval $[\theta p_2^\nu, p_2^\nu]$. Thus for each $\nu \ge m$, there exist primes $p_{2,\nu,j}$, $1 \le j \le p_2 - 1$, such that $\theta p_2^\nu < p_{2,\nu,j} < p_2^\nu$ and none of the primes $p_{2,\nu,j}$ lies in the set

$$\{p_{1,k,i} \mid k \ge m, 1 \le i \le p_1 - 1\} \cup \{p_1\}.$$

As in Lemma 5.1, it follows that for $\mu_2 \ge 1$

$$q_{2,\mu_2} := p_2^{m+\mu_2} \prod_{\nu=m}^{m+\mu_2} \prod_{j=1}^{p_2-1} p_{2,\nu,j} \in P(p_2^{m+\mu_2}).$$

Our construction insures that gcd $(q_{1,\mu_1}, q_{2,\mu_2}) = 1$ for all $\mu_1 \ge 1$ and $\mu_2 \ge 1$.

Continuing in this way we see that for each $t$, $1 \le t \le N$, and each $\nu \ge m$, there exist $p_t - 1$ primes $p_{t,\nu,j}$, $1 \le j \le p_t - 1$, such that $\theta p_t^\nu < p_{t,\nu,j} < p_t^\nu$ and

$$p_{t,\nu,j} \notin \bigcup_{s=1}^{t-1} \big( \{ p_{s,\nu,j} \mid \nu \ge m, 1 \le j \le p_s - 1 \} \cup \{p_s\} \big).$$

It follows as in Lemma 5.1 that for $\mu_t \ge 1$,

$$q_{t,\mu_t} := p_t^{m+\mu_t} \prod_{\nu=m}^{m+\mu_t} \prod_{j=1}^{p_t-1} p_{t,\nu,j} \in P\big(p_t^{m+\mu_t}\big).$$

Our construction insures that for $1 \le s < t \le N$ and for all positive integers $\mu_s$ and $\mu_t$, $\gcd(q_{s,\mu_s}, q_{t,\mu_t}) = 1$.

By Lemma 5.2, there exist arbitrarily large positive integers $\alpha_1, \alpha_2, \ldots, \alpha_N$ and an integer $n \ge n_*$ such that $(1 - \epsilon/4)n < p_t^{\alpha_t} \le n$ for $1 \le t \le n$. We define $\mu_t = \alpha_t - m$ and $q_t := q_{t,\mu_t}$, so $q_t \in P(p_t^{\alpha_t}) \subset P(n)$. Our previous remarks show that condition (a) and (c) of Theorem 5.2 are satisfied. By Lemma 5.1, there exists $a \ge 1$ such that if $\alpha_t \ge a$ for $1 \le t \le N$, then

$$S(q_{t,\mu_t}) \ge \big(2 - \frac{\epsilon}{2}\big) p_t^{\alpha_t}, \qquad 1 \le t \le N,$$

and by Lemma 5.2 we can assume that $\alpha_t \ge a$ for $1 \le t \le N$. It follows that

$$S(q_{t,\mu_t}) \ge \big(2 - \frac{\epsilon}{2}\big)\big(1 - \frac{\epsilon}{4}\big) n > \big(2 - \epsilon\big) n,$$

so condition (b) is satisfied.

If $q_j$, $1 \le j \le N := p_1$ satisfy conditions (a),(b) and (c) of the theorem, then Rule B of Definition 2.6 implies that $p_1 \mathrm{lcm}\,(q_1, q_2, \ldots, q_N) \in P(p_1 n)$. Using conditions (a),(b) and (c) we obtain

$$\begin{aligned} S\big(p_1 \mathrm{lcm}\,(q_1, q_2, \ldots, q_N)\big) &= S\big(\mathrm{lcm}\,(p_1 q_1, q_2, \ldots, q_N)\big) \\ &= S(p_1 q_1) + \sum_{j=2}^{N} S(q_j) \\ &\ge p_1^{\alpha_1+1} + \big[(2-\epsilon)n - p_1^{\alpha_1}\big] + (p_1 - 1)(2-\epsilon)n \\ &= p_1 n\big[(2-\epsilon) + \frac{p_1^{\alpha_1}}{n} - \frac{p_1^{\alpha_1-1}}{n}\big] \\ &\ge p_1 n\big[(2-\epsilon) + (1 - \epsilon/4) - \frac{1}{p_1}\big] \\ &= p_1 n\big[3 - \frac{5\epsilon}{4} - \frac{1}{p_1}\big]. \end{aligned}$$

This completes the proof of Theorem 5.2. $\square$

**Corollary 5.1** *Assume that $p_1$ and $p_2$ are prime numbers. Then for each $\epsilon$, $0 < \epsilon < 1/2$, and each positive integer $n_*$, there exist $n \geq n_*$ and elements $q_j \in P(n)$, $1 \leq j \leq 2$, such that (a) $gcd(q_1, q_2) = 1$, (b) $S(q_j) \geq (2-\epsilon)n$ for $1 \leq j \leq 2$ and (c) $q_j$ has a prime factor $p_j^\alpha$, $\alpha \in \mathbb{N}$, where $(1-(\epsilon/4))n < p_j^{\alpha_j} \leq n$. If $p_1 = 2$, then $2lcm(q_1, q_2) \in P(2n)$ and*

$$S(2lcm(q_1, q_2)) \geq \left[\frac{5}{2} - \frac{5\epsilon}{4}\right]2n.$$

*This implies that* $\limsup_{n\to\infty} d_n \geq 5/2$.

**Proof.** For any two prime numbers $p_1$ and $p_2$, the numbers $(\log p_1)^{-1}$ and $(\log p_2)^{-1}$ are linearly independent over $\mathbb{Q}$. Therefore, the statement immediately follows from Theorem 5.2. □

The hypotheses of the next corollary would be satisfied if we knew the truth of Conjecture 5.3.

**Corollary 5.2** *Assume that $T$ is an infinite collection of primes such that for every $p \in T$, there exist prime numbers (depending on $p$) $r_1 = p, r_2, r_3, \ldots, r_p$ such that $\{(\log r_j)^{-1} : 1 \leq j \leq p\}$ is linearly independent over $\mathbb{Q}$. Then* $\limsup_{n\to\infty} d_n \geq 3$.

**Proof.** For every prime $p \in T$, Theorem 5.2 implies that $\limsup_{n\to\infty} d_n \geq (3 - p^{-1})$. Since $T$ is infinite, the prime $p$ can be made as large as desired and $\limsup_{n\to\infty} d_n \geq 3$. □

**Remark 5.4** Actually we would be able to show that, given the assumptions of Corollary 5.2, $\limsup_{n\to\infty} d_n = \infty$, provided we could show that a certain inhomogeneous linear systems of equations has a *positive* integer valued solution. While Kronecker's theorem asserts that there is an integer valued solution to this inhomogeneous system, it seems to be a nontrivial problem to conclude that the solution is actually positive.

Although we conjecture that $\limsup_{n\to\infty} c_n = \infty$, we shall now show that the growth rate of $c_n$ is necessarily very slow.

**Proposition 5.2** *Let $q$ be the period of an admissible array on $n$ symbols with four rows, $\theta = \{\theta_i : \mathbb{Z} \to \Sigma \mid i \in L = \{1,2,3,4\}\}$. It then follows that $S(q) \leq (5/2)n$.*

**Proof.** Let $q_i$ denote the period of $\theta_i$, $1 \leq i \leq 4$, so $q_i \leq 4$, and let $B_{q_i} \subset \Sigma$ denote the range of $\theta_i$, so $|B_{q_i}| = q_i$. We can assume that the admissible array $\theta$ is "minimal", in the sense that any proper subarray of $\theta$ has a strictly smaller period; for if the array were not minimal, we could replace it by an array with three or fewer rows and use Proposition 5.1 and Remark 5.2.

We consider two cases.

Case (a). Assume that there exist $i, j \in L$ such that $B_{q_i} \cap B_{q_j} = \emptyset$. It follows that $n \geq |B_{q_i} \cup B_{q_j}| = q_i + q_j$. Let $k$ and $l$ denote the remaining two elements of $L$. Then we have

$$\begin{aligned} S\big(\mathrm{lcm}\,(q_1, q_2, q_3, q_4)\big) &= S\big(\mathrm{lcm}\,(\mathrm{lcm}\,(q_i, q_j), \mathrm{lcm}\,(q_k, q_l))\big) \\ &\leq S\big(\mathrm{lcm}\,(q_i, q_j)\big) + S\big(\mathrm{lcm}\,(q_k, q_l)\big). \end{aligned} \tag{5.13}$$

Because $\mathrm{lcm}\,(q_k, q_l)$ is the period of a two-row admissible array on $n$ symbols, Proposition 5.1 implies that

$$S\big(\mathrm{lcm}\,(q_k, q_l)\big) \le \frac{3}{2}n. \tag{5.14}$$

We also have that

$$S\big(\mathrm{lcm}\,(q_i, q_j)\big) \le S(q_i) + S(q_j) \le q_i + q_j \le n. \tag{5.15}$$

Combining (5.13), (5.14) and (5.15), we see that

$$S\big(\mathrm{lcm}\,(q_1, q_2, q_3, q_4)\big) \le \frac{5}{2}n.$$

Case (b). Assume that $B_{q_i} \cap B_{q_j} \neq \emptyset$ for all $i, j \in L$. Condition (ii) in Definition (2.2) of admissible arrays implies $\rho = \gcd\,(q_1, q_2, q_3, q_4) \ge 2$. If $p$ is the largest prime factor of $\rho$, define $\alpha$ to be the largest integer such that $p^\alpha | \rho$. So $p^\alpha \ge 2$ and $p^\alpha \ge 3$ if $\rho \ge 3$. Let $\gamma$ be the largest positive integer such that $p^\gamma | \mathrm{lcm}\,(q_1, q_2, q_3, q_4)$ and select $i \in L$ such that $p^\gamma | q_i$. If $j, k, l$ denote the remaining elements of $L$, we obtain that

$$\begin{aligned} S\big(\mathrm{lcm}\,(q_1, q_2, q_3, q_4)\big) &= S\big(\mathrm{lcm}\,(q_i, p^{-\alpha}q_j, p^{-\alpha}q_k, p^{-\alpha}q_l)\big) \\ &\le q_i + \frac{1}{2}\big(q_j + q_l + q_k\big) \le n + \frac{3}{2}n = \frac{5}{2}n. \end{aligned}$$

This completes the proof of Proposition 5.2. □

**Remark 5.5** It is natural to ask for the optimal constant $\lambda$ such that $S(q) \le \lambda n$ whenever $n \ge 1$ and $q$ is the period of a four-row admissible array on $n$ symbols. Proposition 5.2 proves that $\lambda \le 5/2$; and if one uses (5.2) with $p = 2$ and $\mu = 3$, one can see that $\lambda \ge 1 + 1/2 + 1/4 + 1/8 = 15/8$. It seems likely that a better estimate on $\lambda$ than that in Proposition 5.2 is true.

**Lemma 5.3** *Suppose that $q \in Q(n)$ has a prime factorization $q = \prod_{i=1}^m p_i^{\alpha_i}$, where $p_i$, $1 \le i \le m$, are distinct primes and $\alpha_i \ge 1$. Let $k$ be a positive integer such that $n^{k^{-1}} \ge 2$ and assume that $n^{(k+1)^{-1}} < p_i \le n^{k^{-1}}$ for $1 \le i \le m$. If $\pi(x)$ denotes the number of primes less than or equal to $x$ and $[x]$ denotes the greatest integer less than or equal to $x$, we have $S(q) \le n$ for $k = 1$ and*

$$S(q) \le n + R(k, n) \quad \textit{for } k \ge 2, \tag{5.16}$$

*where*

$$R(k, n) := \big(\pi(n^{k^{-1}}) - \pi(n^{(k+1)^{-1}})\big)[n^{k^{-1}}]^{k-1}, \quad \textit{for } k \ge 1. \tag{5.17}$$

*Furthermore,*

$$\lim_{n\to\infty} \frac{R(k, n)}{n} = 0. \tag{5.18}$$

**Proof.** As was noted in Section 2, $p_i^{\alpha_i} \leq n$ for $1 \leq i \leq m$, so our assumptions imply that $\alpha_i \leq k$ for $1 \leq i \leq m$. Let $T = \{i \mid \alpha_i = k\}$ and $\Gamma = \{i \mid \alpha_i < k\}$ and write

$$q_1 = \prod_{i \in T} p_i^k \quad \text{and} \quad q_2 = \prod_{i \in \Gamma} p_i^{\alpha_i}.$$

We have that $S(q) = S(q_1) + S(q_2)$. There are at most $\pi(n^{k^{-1}} - \pi(n^{(k+1)^{-1}})$ prime factors in $q_2$ and each factor $p_i^{\alpha_i}$ satisfies

$$p_1^{\alpha_i} \leq [n^{k^{-1}}]^{k-1}.$$

Thus we obtain

$$S(q_2) \leq \sum_{i \in \Gamma} p_i^{\alpha_i} \leq R(k, n), \tag{5.19}$$

where $R(k, n)$ is given by (5.17). Because $q_1$ is a factor of $q$, we know that $q_1 \in Q(n)$. Thus $q_1$ is the period of an admissible array on $n$ symbols $\{\theta_j : \mathbb{Z} \to \Sigma : j \in L\}$. We can assume that the array is "minimal", as in the proof of Proposition 5.2. Let $q_{1,j}$ denote the period of $\theta_j$. For each $i \in T$, there exists $\sigma(i) = j \in L$ such that $p_i^k | q_{1,j}$. If $p_t | q_{1,\sigma(i)}$ for some $t \in T$, $t \neq i$, we find that

$$q_{1,\sigma(i)} \geq p_t p_i^k > n,$$

which is a contradiction. It follows that $q_{1,\sigma(i)} = p_i^k$, and the same argument shows that $\sigma$ is one-to-one. The minimality of the array implies that $\sigma$ is onto. It follows that for $j, t \in L$ with $j \neq t$, $\gcd(q_{1,j}, q_{1,t}) = 1$, so the properties of admissible arrays imply that $B_{q_{1,j}} \cap B_{q_{1,t}} = \emptyset$ for all $j, t \in L$, $j \neq t$, and

$$n \geq \Big| \bigcup_{j \in L} B_{q_{1,j}} \Big| = \sum_{j \in L} q_{1,j} = \sum_{t \in T} p_i^{\alpha_i} = S(q_1).$$

This proves (5.16). The limit (5.18) follows directly from the prime number theorem. □

With the aid of Propositions 5.1 and 5.2, we can obtain an upper bound on $c_n$ in (5.2). Recall that Theorem 1 on page 69 of [13] gives "effective" bounds in the prime number theorem:

$$\frac{x}{\log x}\big(1 + (2 \log x)^{-1}\big) < \pi(x) \quad \text{for } x \geq 59$$

and

$$\pi(x) < \frac{x}{\log x}\big(1 + 3(2 \log x)^{-1}\big) \quad \text{for } x > 1.$$

**Theorem 5.3** *Assume that $n \geq 11$ and let $\nu$ be a positive integer such that $11^\nu \leq n$. If $q \in Q(n)$ and $q = \prod_{i=1}^m p_i^{\alpha_i}$, where $p_i$ are distinct primes and $\alpha_i$ are positive integers for $1 \leq i \leq m$, define $T = \{i \mid p_i^{\nu+1} \leq n\}$ and $\tilde{q} = \prod_{i \in T} p_i^{\alpha_i}$. If $T$ is empty, define $S(\tilde{q}) = 0$. Then we have*

$$S(q) \leq S(\tilde{q}) + \nu n + \sum_{k=2}^{\nu} R(k, n), \tag{5.20}$$

*where $R(k,n)$ is given by (5.17) and the summation equals zero if $\nu = 1$. If $\nu \geq 2$,*

$$\begin{aligned} S(q) &\leq S(\tilde{q}) + \nu n + \pi(\sqrt{n})\sqrt{n} - \pi\big(n^{(\nu+1)^{-1}}\big)n^{(\nu-1)\nu^{-1}} \\ &\qquad + \sum_{k=3}^{\nu} \pi(n^{1/k}) n^{(k-1)k^{-1}} \frac{\log n}{k(k-1)}, \end{aligned} \tag{5.21}$$

*where the summation equals zero if $\nu = 2$. If $\nu \geq 2$, we also obtain that*

$$\begin{aligned} S(q) &\leq S(\tilde{q}) + \nu n + \frac{2n}{\log n}\big(1 + 3(\log n)^{-1}\big) - \pi\big(n^{(\nu+1)^{-1}}\big)n^{(\nu-1)\nu^{-1}} \\ &\qquad + n(1 + 3(2\log n)^{-1}) \log(\nu - 1) + \frac{3n}{2\log n}(\nu - 2). \end{aligned} \tag{5.22}$$

**Proof.** Let $T_k = \big\{i \mid n < p_i^{k+1} \text{ and } p_i^k \leq n\big\}$ for $1 \leq k \leq \nu$ and define $q_k = \prod_{i \in T_k} p_i^{\alpha_i}$. We define $S(q_k) = 0$ if $T_k = \emptyset$. Since $q_k$ is a factor of $q$ and $q \in Q(n)$, we know that $q_k \in Q(n)$. Lemma 5.3 implies that $S(q_1) \leq n$ and for $k \geq 2$

$$S(q_k) \leq n + R(k,n),$$

where $R(k,n)$ is given by (5.17). We also have that

$$S(q) = S\Big(\tilde{q}\prod_{k=1}^{\nu} q_k\Big) = S(\tilde{q}) + \sum_{k=1}^{\nu} S(q_k),$$

and by combining these equations we obtain (5.20). If $\nu \geq 2$ and if we replace $[n^{k^{-1}}]^{k-1}$ in (5.20) by $n^{(k-1)k^{-1}}$ and rearrange the terms, we obtain

$$\begin{aligned} S(q) &\leq S(\tilde{q}) + \pi(\sqrt{n})\sqrt{n} - \pi\big(n^{(\nu+1)^{-1}}\big)n^{(\nu-1)\nu^{-1}} \\ &\qquad + \sum_{k=3}^{\nu} \pi(n^{1/k}) n^{(k-1)k^{-1}} \big(1 - n^{-k^{-1}(k-1)^{-1}}\big). \end{aligned} \tag{5.23}$$

Writing $\xi = -(k(k-1))^{-1} \log n$ and using the mean value theorem, we obtain

$$1 - n^{-k^{-1}(k-1)^{-1}} = \exp(0) - \exp(\xi) \leq -\xi$$

and substituting this estimate in (5.23) yields (5.21). If we use the upper bound for $\pi(x)$ given just before Theorem 5.3, we find that, for $k \geq 3$,

$$n^{(k-1)k^{-1}} \pi\big(n^{k^{-1}}\big)\frac{\log n}{k(k-1)} < \frac{n}{k-1}\big(1 + 3k(2\log n)^{-1}\big)$$

and

$$\pi(\sqrt{n})\sqrt{n} < \frac{2n}{\log n}\big(1 + 3(\log n)^{-1}\big).$$

It follows that (for $\nu \geq 3$)

$$\begin{aligned}
\sum_{k=3}^{\nu} \pi\left(n^{k^{-1}}\right) n^{(k-1)k^{-1}} \frac{\log n}{k(k-1)} &< n \sum_{k=3}^{\nu} \frac{1}{k-1} + \frac{3n}{2\log n} \sum_{k=3}^{\nu} \left(1 + \frac{1}{k-1}\right) \\
&< n \log(\nu - 1) + \frac{3n}{2\log n}\big((\nu - 2) + \log(\nu - 1)\big),
\end{aligned}$$

where we have used the standard estimate $\sum_{j=2}^{\nu-1} j^{-1} < \log(\nu - 1)$. Substituting these estimates in (5.21) gives (5.22). □

**Corollary 5.3** *Assume that $n \geq 11$ and let $\nu$ be a positive integer such that*

$$11^{\nu} \leq n < 11^{\nu+1}. \tag{5.24}$$

*If $q \in Q(n)$ and $q = \prod_{i=1}^{m} p_i^{\alpha_i}$, where $p_i$ are distinct primes and $\alpha_i$ are positive integers for $1 \leq i \leq m$, define $T = \{i \mid p_i^{\nu+1} \leq n\}$ and $\tilde{q} = \prod_{i \in T} p_i^{\alpha_i}$. The set $T$ contains at most four elements; and if $T \neq \emptyset$, then $\tilde{q} = 2^{\alpha}3^{\beta}5^{\gamma}7^{\delta}$ for some nonnegative integers $\alpha, \beta, \gamma$ and $\delta$ and $S(\tilde{q}) \leq (5/2)n$. If $\nu = 1$, then $c_n \leq (7/2)$. If $\nu > 1$, then*

$$\begin{aligned}
c_n \;\leq\; & \frac{5}{2} + \nu + \frac{3(\nu-2)}{2\log n} + (1 + 3(2\log n)^{-1})\log(\nu - 1) + \frac{2}{\log n}(1 + 3(\log n)^{-1}) \\
& - \pi\left(n^{(\nu+1)^{-1}}\right) n^{-\nu^{-1}}.
\end{aligned} \tag{5.25}$$

*If $\tilde{c}_n$ denotes the right hand side of (5.25), then*

$$\lim_{n \to \infty} \tilde{c}_n - \left(\frac{5}{2} + \nu + \frac{3}{2\log 11} + \log(\nu - 1) - \frac{4}{11}\right) = 0.$$

**Proof.** If $\nu$ satisfies (5.24), then if $p$ is a prime and $p^{\nu+1} < n$, we must have $p < 11$, so $p = 2, 3, 5$ or $7$. Thus the set $T$ contains at most four elements, and if $T \neq \emptyset$, then $\tilde{q} = 2^{\alpha}3^{\beta}5^{\gamma}7^{\delta}$. It follows from Propositions 5.1 and 5.2 (since $\tilde{q} \in Q(n)$ and $\tilde{q}$ is the period of an admissible array on $n$ symbols with at most four rows), that $S(\tilde{q}) \leq (5/2)n$. Substituting this estimate in (5.22) (when $\nu \geq 2$) or in (5.20) (when $\nu = 1$) gives $c_n \leq 7/2$ for $11 \leq n < 11^2$ and (5.25). The remainder of the proof is straightforward. □

**Remark 5.6** If $\nu \geq 2$ in Corollary 5.3, then

$$\pi\left(n^{(\nu+1)^{-1}}\right) = \begin{cases} 2 & \text{for } \nu = 2 \\ 3 & \text{for } \nu = 3, 4 \\ 4 & \text{for } \nu \geq 5 \end{cases}$$

and, for all $\nu \geq 1$,

$$\frac{1}{11} 11^{-\nu^{-1}} < n^{-\nu^{-1}} \leq \frac{1}{11}.$$

These estimates can be used to further improve (5.25) in specific cases.

**Remark 5.7** In [12], the sets $Q(n)$ have been explicitly computed for $1 \le n \le 50$, so $c_n$ can be explicitly computed for $n \le 50$. In particular, $c_n \le 11/8$ for $n \le 11$ and $c_n \le 29/16$ for $n \le 50$. Computer programs are available to compute the sets $P(n) \subset Q(n)$ for reasonable large values of $n$ (see [12]), so $d_n$ can also be computed for reasonable large $n$ (see Table 1). However, $Q(n)$ and $c_n$ resist computation. It is known that in general $P(n) \neq Q(n)$ (see [12], Section 7), but more precise information is lacking. For instance is $P(n)$ always a "good approximation" to $Q(n)$? If $d_n$ always a good approximation to $c_n$?

**Remark 5.8** Equation (5.20) or the less precise estimate (5.22) can be used to give upper bounds for $c_n$ for large $n$. For example, if $11^7 \le n < 11^8$ and $q \in Q(n)$, then (5.20) implies that

$$\begin{aligned} S(q) \quad \le \quad & \frac{19}{2}n - 4[n^{1/7}]^6 + \pi(\sqrt{n})[\sqrt{n}] \\ & + \sum_{j=3}^{7} \pi\left(n^{j^{-1}}\right)\left([n^{j^{-1}}]^{j-1} - [n^{(j-1)^{-1}}]^{j-2}\right). \end{aligned} \tag{5.26}$$

Using this formula, one obtains after some tedious calculations that for $n = 10^8$ and $q \in Q(n)$,

$$S(q) \le \frac{19}{2}n + \frac{2}{3}n.$$

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