Homoclinic orbits to a saddle-center in a fourth order differential equation

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1 Introduction

In recent years there has been a considerable interest in fourth order model equations such as the Extended Fisher-Kolmogorov equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad \gamma > 0, \tag{1.1}$$

proposed in 1988 by Dee & Van Saarloos [10] as a model for bi-stable systems, and the Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = \kappa u - \Big(1 + \frac{\partial}{\partial x^2}\Big)^2 u - u^3, \qquad \kappa \in \mathbb{R}, \tag{1.2}$$

proposed by Swift & Hohenberg [23] in studies of hydrodynamic instability. Both equations are used in the description and analysis of complex patterns [8, 12, 9, 13, 21] kinks, or heteroclinic orbits, which connect two different uniform states, pulses, or homoclinic orbits to one of the uniform states, and periodic patterns with a nontrivial structure. For an overview of recent results, specially about stationary patterns, we refer to [3] and [20] and the literature cited therein.

After an appropriate transformation, stationary solutions $u(x)$ of (1.1) and (1.2) are found to be solutions of the equation

$$u^{iv} + qu'' + f(u) = 0, \qquad q \in \mathbb{R}, \tag{1.3}$$

in which primes denote differentiation. For the Extended-Fisher Kolmogorov equation the transformation involved is

$$x = \gamma^{1/4} x^*, \qquad u(x) = u^*(x^*) \quad \text{and} \quad q = -\frac{1}{\sqrt{\gamma}}, \tag{1.4}$$

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and for the Swift-Hohenberg equation, when $\kappa > 1$, the scaling is

$$x = (\kappa - 1)^{-1/4} x^*, \qquad u(x) = \sqrt{\kappa - 1}\, u^*(x^*), \qquad q = \frac{2}{\sqrt{\kappa - 1}}. \tag{1.5}$$

In both cases the function $f(s)$ becomes the cubic

$$f(s) = s^3 - s. \tag{1.6}$$

In the present paper we analyse the existence of homoclinic orbits of equation (1.3) which tend to the origin $u = 0$ as $x \to \pm\infty$. Homoclinic orbits to the uniform states $u = \pm 1$ have been obtained for values of $q \in (-\sqrt{8}, \sqrt{8})$ in [20] and [22]. For these values of q the states $u = 1$ and $u = -1$ are *saddle-foci*, with two-dimensional stable and unstable manifolds. In contrast, $u = 0$ is a *saddle-center* for all values of q and the stable and unstable manifolds are both one-dimensional. Thus, the existence of homoclinic orbits to $u = 0$ is less likely. We shall have to carefully tune the parameter q in (1.3) in order for a pulse to exist.

Homoclinic orbits to saddle-centers have attracted a great deal of attention. We mention the paper by Mielke, Holmes and O'Reilly [19] and the surveys of Champneys [4] and [5] and the references quoted therein. Concerning equations such as (1.3), we mention a result of Amick & McLeod [1] , and Hammersley & Mazzarino [16] (see also Eckhaus [11]), which proves the nonexistence of such orbits for the function

$$f(s) = s^2 - s.$$

For the cubic function f of (1.6) Grotta-Ragazzo [14, 15] proved that once a homoclinic orbit is found for some value q^* there exist infinitely many such orbits for values of q in any small neighbourhood around q^*.

For a related equation

$$\frac{1}{12} u^{iv} + u'' - \lambda^2 u + u^3 + \frac{3}{4} u(uu'' - u'^2) = 0$$

Kivshar *et. al.* [17] established numerically the existence of homoclinic orbits for the equation for discrete values of λ, and for a similar equation, arising in the theory of water waves, with a second order nonlinear term, we mention the work of Champneys & Groves [6].

In the present paper we study equation (1.3) when f is given by the piecewise linear function

$$f(s) = \begin{cases} s + 1 & s \leqslant -\dfrac{1}{2}, \\ -s & |s| < \dfrac{1}{2}, \\ s - 1 & s \geqslant \dfrac{1}{2}. \end{cases} \tag{1.7}$$

Like the cubic function (1.6), this function is odd and has qualitatively the same shape. However, since it is piecewise linear, we can make explicit computations, and so obtain very precise results about existence and multiplicity of pulse-type solutions.

Since we seek homoclinic orbits to the constant solution $u = 0$, the local behaviour around $u = 0$ is important. Linearising around the origin we obtain the equation

$$u^{iv} + qu'' - u = 0,$$

with the associated characteristic equation

$$\lambda^4 + q\lambda^2 - 1 = 0.$$

It has two real roots $\lambda = \pm\mu$, and two imaginary roots $\lambda = \pm i\omega$. Here

$$\mu(q) = \frac{1}{\sqrt{2}}\sqrt{-q + \sqrt{q^2+4}} \qquad \text{and} \qquad \omega(q) = \frac{1}{\sqrt{2}}\sqrt{q + \sqrt{q^2+4}}. \tag{1.8}$$

For this reason, $u = 0$ is called a *saddle-center*. Thus, the stable and the unstable manifold are both one-dimensional, and it is not evident that they will intersect to form a homoclinic orbit. We shall find that this happens at an infinite sequence $\{q_k\}$ of values of q, which tend to infinity as $k \to \infty$.

Specifically we prove the following theorem:

Theorem 1.1 *There exists an infinite sequence* $\{q_k > 2 : k = 1, 2, \dots\}$ *which tends to infinity such that for* $q = q_k$ *equation* (1.3), *with* f *given by the nonlinearity* (1.7), *has an even homoclinic orbit* $u(x)$ *to the origin* $u = 0$. *It has the following properties:*

$$u > 0 \quad \textit{on} \quad \mathbb{R} \qquad \textit{and} \qquad u(0) > 1 + \frac{1}{\sqrt{2}}, \tag{1.9}$$

and

$$u'(x) < 0 \quad \textit{for} \quad 0 < x < \infty. \tag{1.10}$$

The graphs of the first ten homoclinic orbits are shown in Figure 1(a).

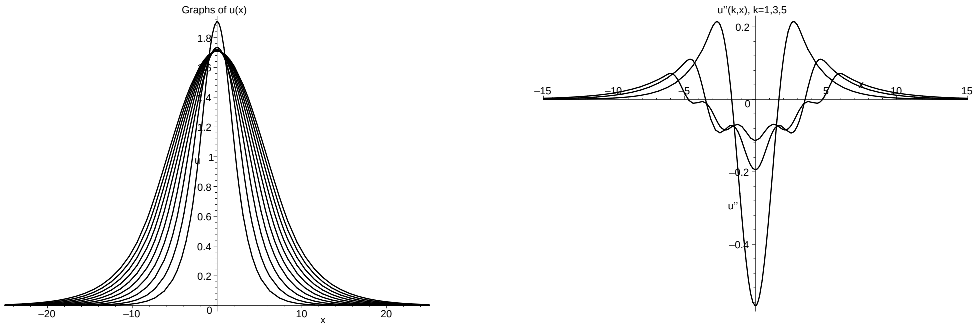
Whilst for every $k \geqslant 1$, u_k and u_k' have one sign on $\mathbb{R}^+$, the second derivative u_k'' exhibits more and more oscillations as k increases. In Figure 1(b), we show u_k'' for $k = 1,\ 3,\ 5$. In Appendix C we present a table of values of $q_k,\ \zeta_k$ and of u_k and u_k'' at the origin.

The asymptotic behaviour of the critical values q_k as $k \to \infty$ is given by

$$q_k = \frac{4}{3}k + \frac{2}{3} + O(k^{-1}) \qquad \text{as} \quad k \to \infty. \tag{1.11}$$

In view of (1.9) and (1.10) the graphs of the solutions $u_k(x)$ each intersect the level line $u = \frac{1}{2}$ precisely once, at a point which we shall denote by $x = \zeta_k = \zeta(q_k)$, i.e.

$$u_k(\zeta_k) = \frac{1}{2} \qquad \text{for} \quad k = 1, 2, 3, \dots. \tag{1.12}$$



(a) 10 solitons

(b) Orbits for $u_k''(x),\ k = 1, 3, 5$

Figure 1: Homoclinic orbits

We find that $\zeta_k \to \infty$ as $k \to \infty$, with the following asymptotic behaviour:

$$\zeta(q_k) = \frac{3\pi}{4}\sqrt{q_k} + O(q_k^{-3/2}) \qquad \text{as} \quad k \to \infty, \tag{1.13}$$

or

$$\zeta_k = \frac{\pi\sqrt{3}}{2}\sqrt{k} + o(1) \qquad \text{as} \quad k \to \infty. \tag{1.14}$$

On the other hand, as k increases, the maxima $u_k(0)$ decrease, and

$$u_k(0) \searrow 1 + \frac{1}{\sqrt{2}} \qquad \text{as} \quad k \to \infty. \tag{1.15}$$

Thus, the graphs of $u_k(x)$ become flatter and wider as $k \to \infty$. Scaling the independent variable x according to

$$y = \frac{x}{\sqrt{q}} \qquad \text{and} \qquad v_k(y) = u_k(x),$$

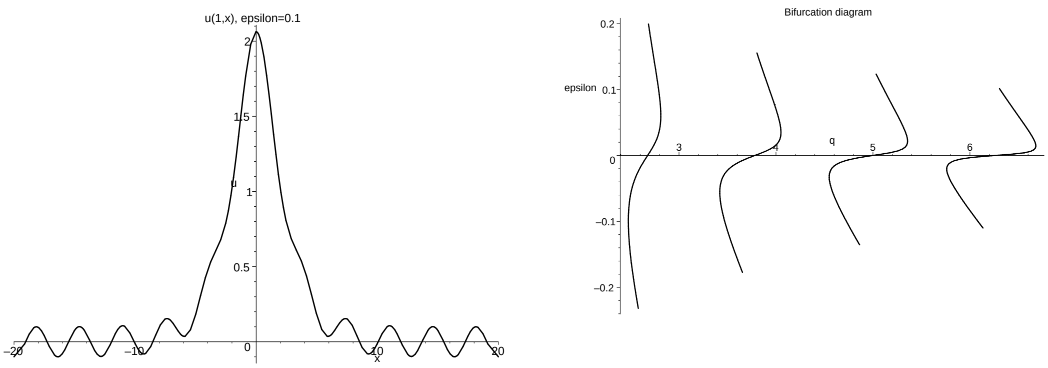
we find that

$$v_k(y) \to V(y) \qquad \text{as} \qquad k \to \infty$$

uniformly in $C^1(\mathbb{R})$, where V is the unique even positive solution of the problem

$$V''(y) + f(V(y)) = 0, \qquad V(\pm\infty) = 0.$$

As discussed in [18] and [7] (see also [4] and [5]) homoclinic orbits — or *solitons* — can be viewed as *embedded* in branches of homoclinic orbits to periodic orbits at $x = \pm\infty$. One such homoclinic orbit is shown in Figure 2(a).



(b) Bifurcation branches

(a) Homoclinic orbit to periodic solution

Figure 2: (a) A homoclinic orbit to a periodic orbit. (b) Bifurcation branches of homoclinic orbits.

In Section 5 we establish the existence of branches of such homoclinic orbits. They are shown in Figure 2(b), where the vertical axis gives the amplitude ε of the limiting periodic solution.

An interesting question concerns the phase shift 2ϕ between the limiting periodic solutions at $x = -\infty$ and $x = +\infty$. In Figure 3 we show 2ϕ versus the parameter q along the first four branches. The diamonds on the branches correspond to the solitons.

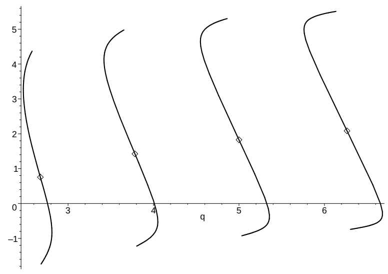


Figure 3: The phase shift 2ϕ versus q along the first four branches

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2 Preliminaries

We seek even homoclinic orbits to the origin, i.e. solutions $u(x)$ of equation (1.3) endowed with the symmetry property

$$u(-x) = u(x) \qquad \text{for} \quad x \in \mathbb{R}. \tag{2.1}$$

Thanks to the invariance of the equation with respect to the transformation $u \to -u$, we may assume without loss of generality that u is positive at the origin. In addition we assume that u has a local maximum at the origin. Thus

$$u(0) > 0 \qquad \text{and} \qquad u''(0) < 0. \tag{2.2}$$

As a first observation, we derive a lower bound for any critical value of such a solution. The essential ingredient here is the first integral of equation (1.3), which we shall often refer to as the *Energy Identity.* It is given by

$$\mathcal{E}(u) \stackrel{\text{def}}{=} u'(x)u'''(x) - \frac{1}{2}\{u''(x)\}^2 + \frac{q}{2}\{u'(x)\}^2 + F(u(x)) = E, \tag{2.3}$$

in which E is a constant, and F is the primitive of f, given by

$$F(u) = \int_0^u f(s)\,ds = \begin{cases} \frac{1}{2}u^2 + u + \frac{1}{4} & u \leqslant -\frac{1}{2}, \\ -\frac{1}{2}u^2 & |u| < \frac{1}{2}, \\ \frac{1}{2}u^2 - u + \frac{1}{4} & u \geqslant \frac{1}{2}, \end{cases} \tag{2.4}$$

which has been chosen so that $F(0) = 0$. Thus, a homoclinic orbit $u(x)$ which converges to the origin as $x \to \pm\infty$ has zero energy, i.e. $E = 0$.

Lemma 2.1 *Let $u(x)$ be a homoclinic orbit to $u = 0$, and let $u'(a) = 0$ for some $a \in \mathbb{R}$. Then either $u(a) = 0$ or*

$$|u(a)| \geqslant 1 + \frac{1}{\sqrt{2}}. \tag{2.5}$$

Proof. We evaluate the energy at $x = a$, and remember that $u'(a) = 0$. This yields

$$-\frac{1}{2}\{u''(a)\}^2 + F(u(a)) = 0, \tag{2.6}$$

so that

$$F(u(a)) = \frac{1}{2}\{u''(a)\}^2 \geqslant 0. \tag{2.7}$$

From the definition of F we see that $F(s) < 0$ for $s \in (-s_0, s_0) \setminus \{0\}$, where $s_0 = 1 + \frac{1}{\sqrt{2}}$. We therefore conclude that either $u(a) = 0$ or $|u(a)| \geqslant s_0$, as asserted.

An immediate consequence of Lemma 2.1 is that homoclinic orbits to $u = 0$ have monotone tails. In particular, we have

Lemma 2.2 *Let $u(x)$ be a nonnegative homoclinic orbit. Then there exist points x_1 and x_2 $(x_1 < x_2)$ such that*

$$u' > 0 \quad \textit{for} \quad x < x_1 \qquad \text{and} \qquad u' < 0 \quad \textit{for} \quad x > x_2,$$

and

$$u(x) > 1 + \frac{1}{\sqrt{2}} \quad \textit{for} \quad x_1 < x < x_2.$$

Plainly, if $u(x)$ is even, then $x_1 < 0$ and $x_2 > 0$ and $x_2 = -x_1$.

In the present paper we shall always assume that $q > 2$, and the question remains as to the existence of homoclinic orbits when $q \leqslant 2$, and the uniform states $u = \pm 1$ are no longer saddle-centers, but saddle-foci. For the cubic function (1.6):

$$f(s) = s^3 - s, \tag{1.6}$$

one can establish a lower bound for q [24].

Lemma 2.3 *Let $u(x)$ be a nontrivial homoclinic orbit to the origin that solves the differential equation* (1.3) *in which the nonlinearity f is given by the cubic polynomial in* (1.6). *Then*

$$q \geqslant \frac{2\sqrt{7}}{3}. \tag{2.8}$$

For completeness we give the proof in Appendix A.

3 Existence of homoclinic orbits

In this section we turn to the question of the existence of homoclinic orbits of equation (1.3) which tend to the origin as $x \to \pm\infty$. We consider *even* solutions, and hence it will suffice to find solutions of the following Initial Value Problem:

$$\begin{cases} u^{iv} + qu'' + f(u) = 0 & \text{for} \quad x > 0, \\ u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0, \\ u(x) \to 0 \quad \text{as} \quad x \to \infty, \end{cases} \tag{3.1}$$

where α and β are so chosen that the Energy Identity (2.3) is satisfied at the origin, i.e.

$$-\frac{1}{2}\beta^2 + F(\alpha) = 0. \tag{3.2}$$

We assume that $u(x)$ attains its maximum at the origin, and hence, that $\beta = u''(0) < 0$. We therefore conclude from (3.1) that $F(\alpha) \geqslant 0$, and hence that

$$\alpha \geqslant 1 + \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \beta = -\sqrt{2F(\alpha)}. \tag{3.3}$$

We establish the existence of nonnegative solutions $u(x)$ of Problem (3.1) by explicitly constructing them. By Lemma 2.2 the graph of $u(x)$ intersects the line $u = \frac{1}{2}$ only once on $\mathbb{R}^+$, and as before (see (1.12)), we denote this point of intersection by ζ. We then put

$$u(x) = \begin{cases} u_+(x) & \text{for} \quad 0 \leqslant x \leqslant \zeta, \\ u_-(x) & \text{for} \quad \zeta \leqslant x < \infty. \end{cases} \tag{3.4}$$

By the definition of $f(s)$ in (1.7), the function u_- solves the equation

$$u^{iv} + qu'' - u = 0.$$

Thus, since $u_-(\zeta) = \frac{1}{2}$ it follows that

$$u_-(x) = \frac{1}{2}e^{-\mu(x-\zeta)} \qquad \text{for} \qquad \zeta < x < \infty, \tag{3.5}$$

where μ has been defined in (1.8).

We write $u_+(x) = 1 + v(x)$. Then the function $v(x)$ solves the equation

$$v^{iv} + qv'' + v = 0.$$

Its characteristic polynomial is given by

$$\lambda^4 + q\lambda^2 + 1 = 0. \tag{3.6}$$

For $q > 2$ the roots of (3.6) are purely imaginary. They are given by $\lambda = \pm ia$ and $\lambda = \pm ib$, where $a = a(q)$ and $b = b(q)$ are positive constants given by

$$a^2(q) = \frac{1}{2}(q + \sqrt{q^2 - 4}) \tag{3.7}$$

and

$$b^2(q) = \frac{1}{2}(q - \sqrt{q^2 - 4}) = \frac{1}{a^2(q)}. \tag{3.8}$$

Therefore, if $q > 2$ then in view of the symmetry with respect to the origin, $u_+(x)$ can be written as

$$u_+(x) = 1 + A\cos(ax) + B\cos(bx), \tag{3.9}$$

where A and B are constants.

The function $u(x)$, composed of $u_+(x)$ for $0 \leqslant x \leqslant \zeta$ and $u_-(x)$ for $x \geqslant \zeta$, will be a solution of Problem (3.1) if we can find values for the parameters q, ζ, A and B such that u_+ and u_-, as well as their first three derivatives, match at $x = \zeta$. It then follows that $u \in C^4(\mathbb{R})$, and that u satisfies the Energy Identity

(2.3). This gives us the following set of conditions:

$$
\begin{aligned}
1 + A\cos(a\zeta) + B\cos(b\zeta) &= \frac{1}{2}, &(3.10)\\
aA\sin(a\zeta) + bB\sin(b\zeta) &= \frac{1}{2}\,\mu, &(3.11)\\
a^2A\cos(a\zeta) + b^2B\cos(b\zeta) &= -\frac{1}{2}\,\mu^2, &(3.12)\\
a^3A\sin(a\zeta) + b^3B\sin(b\zeta) &= -\frac{1}{2}\,\mu^3. &(3.13)
\end{aligned}
$$

If we eliminate B from (3.11) and (3.13), we obtain

$$A\sin(a\,\zeta) = -\frac{\mu}{2a}\frac{\mu^2+b^2}{a^2-b^2}, \tag{3.14}$$

and if we eliminate A from (3.11) and (3.13), we obtain

$$B\sin(b\,\zeta) = +\frac{\mu}{2b}\frac{\mu^2+b^2}{a^2-b^2}. \tag{3.15}$$

Similarly, when we eliminate A and B from (3.10) and (3.12), we find that

$$A\cos(a\,\zeta) = -\frac{1}{2}\frac{\mu^2-b^2}{a^2-b^2} \tag{3.16}$$

and

$$B\cos(b\,\zeta) = +\frac{1}{2}\frac{\mu^2-a^2}{a^2-b^2}. \tag{3.17}$$

We finally eliminate the constants A and B by dividing (3.14) by (3.16) and (3.15) by (3.17). This yields the expressions

$$\begin{cases} \tan(a\zeta) = -\dfrac{\mu}{a}\dfrac{b^2+\mu^2}{b^2-\mu^2} \stackrel{\text{def}}{=} -P(q), \\[2ex] \tan(b\zeta) = -\dfrac{\mu}{b}\dfrac{a^2+\mu^2}{a^2-\mu^2} \stackrel{\text{def}}{=} -Q(q). \end{cases} \tag{3.18}$$

Lemma 3.1 *The functions $a(q)$, $b(q)$ and $\mu(q)$ have the following properties:*

(a) *$a(q)$ is strictly increasing, $b(q)$ is strictly decreasing, and $\mu(q)$ is strictly decreasing.*

(b)
$$a(q) \to 1, \quad b(q) \to 1, \quad \mu(q) \to \sqrt{\sqrt{2}-1} \qquad \textit{as} \qquad q \to 2^+.$$

(c)
$$a(q) = \sqrt{q}\Big(1 - \frac{1}{2q^2} + O(q^{-4})\Big) \qquad \textit{as} \quad q \to \infty.$$

(d)
$$b(q) = \frac{1}{\sqrt{q}}\Big(1 + \frac{1}{2q^2} + O(q^{-4})\Big) \qquad as \quad q \to \infty.$$

(e)
$$\mu(q) = \frac{1}{\sqrt{q}}\Big(1 - \frac{1}{2q^2} + O(q^{-4})\Big) \qquad as \quad q \to \infty.$$

Lemma 3.1 is readily proved by inspecting the expressions for $\mu(q)$, $a(q)$ and $b(q)$ given by respectively (1.8), (3.7) and (3.8).

Inspection of the functions $P(q)$ and $Q(q)$ reveals the following:

Lemma 3.2 *The functions $P(q)$ and $Q(q)$ have the following properties:*

(a) *$P(q)$ is strictly increasing and $Q(q)$ is strictly decreasing.*

(b)
$$P(q) \to \frac{1}{\sqrt{\sqrt{2}-1}}, \qquad Q(q) \to \frac{1}{\sqrt{\sqrt{2}-1}} \qquad as \qquad q \to 2^+.$$

(c)
$$P(q) = q\Big\{1 - \frac{3}{2q^4} + O(q^{-8})\Big\} \qquad as \qquad q \to \infty.$$

(d)
$$Q(q) = 1 + \frac{1}{q^2} + O(q^{-4}) \qquad as \qquad q \to \infty.$$

Lemma 3.2 is proved by an elementary explicit computation.

From (3.18) we obtain two expressions for $\zeta(q)$:

$$\zeta(q) = X_m(q) \stackrel{\text{def}}{=} \frac{1}{a(q)}\{m\pi - \arctan(P(q))\}, \tag{3.19}$$

and

$$\zeta(q) = Y_n(q) \stackrel{\text{def}}{=} \frac{1}{b(q)}\{n\pi - \arctan(Q(q))\}, \tag{3.20}$$

where m and n are integers. Since $\zeta > 0$, they need to be positive. Plainly, the values of q for which a homoclinic orbit may exist must solve the equation

$$X_m(q) = Y_n(q) \tag{3.21}$$

for some positive values of m and n.

Lemma 3.3 *The functions X_m are decreasing and the functions Y_n are increasing.*

Proof. According to Lemma 3.1, $a(q)$ is increasing, and according to Lemma 3.2, $P(q)$ is increasing, so that $m\pi - \arctan(P(q))$ is decreasing. Therefore the quotient is decreasing.

Similarly, we know from Lemma 3.1 that $b(q)$ is decreasing and from Lemma 3.2, that $Q(q)$ is decreasing. Therefore, $n\pi - \arctan(Q(q))$ is increasing, and the quotient is increasing.

Note that at $q = 2$, we have

$$X_m(2) = m\pi - \arctan\left(\frac{1}{\sqrt{\sqrt{2}-1}}\right),$$
$$Y_n(2) = n\pi \ - \arctan\left(\frac{1}{\sqrt{\sqrt{2}-1}}\right),$$

and for large values of q the behaviour of $X_m(q)$ and $Y_n(q)$ is given by:

$$X_m(q) \sim \left(m - \frac{1}{2}\right)\pi q^{-1/2} \qquad \text{as} \quad q \to \infty,$$
$$Y_n(q) \sim \left(n - \frac{1}{4}\right)\pi q^{1/2} \qquad \text{as} \quad q \to \infty.$$

Hence, for every pair (m, n) such that $m > n$, there exists precisely one point where the graphs of X_m and Y_n intersect, and therefore one root $q_{m,n}$ of equation (3.21). Graphs of $X_m(q)$ $(m = 2, 3, 4, \dots)$ and $Y_1(q)$ are shown in Figure 4.

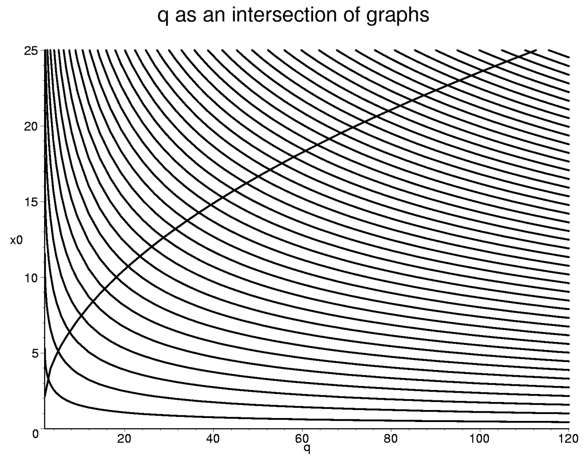


Figure 4: Plots of $X_m(q), m = 2, 3, 4, \dots$ and $Y_1(q)$

For q to admit a homoclinic orbit to $u = 0$, it is necessary that it is a solution of (3.21). However, it is not sufficient. We also require that

$$u(x) > \frac{1}{2} \qquad \text{for} \qquad x \in (-\zeta, \zeta). \tag{3.22}$$

In the following lemma we show that this condition is satisfied if and only if $n = 1$ and $m \geqslant 2$.

Lemma 3.4 *For the pair (m, n), the root $q_{m,n}$ of equation (3.21) corresponds to a solution of Problem (3.1) of the form (3.4) if and only if $n = 1$ and $m \geqslant 2$.*

Proof. Suppose that $n \geqslant 2$. Then, by (3.20),

$$\zeta(q) \geqslant \frac{1}{b(q)}\{2\pi - \arctan(Q(q))\} > \frac{3\pi}{2b(q)}.$$

Thus, the point $y = \frac{\pi}{b(q)}$ lies in the interval $(-\zeta, \zeta)$. We compute u at the points $x = 0$ and $y = \frac{\pi}{b(q)}$. We have

$$\begin{aligned} u(0) &= 1 + B + A, \\ u(y) &= 1 - B + A\cos(a^2\pi). \end{aligned}$$

¿From (3.16), (3.17) and (3.18) we deduce that

$$|A| = \frac{1}{2}\sqrt{1 + P^2(q)} \cdot \frac{b^2(q) - \mu^2(q)}{a^2(q) - b^2(q)}, \tag{3.23}$$

$$|B| = \frac{1}{2}\sqrt{1 + Q^2(q)} \cdot \frac{a^2(q) - \mu^2(q)}{a^2(q) - b^2(q)}. \tag{3.24}$$

This means in particular that

$$|B| - |A| > \frac{1}{\sqrt{2}} \qquad \text{for every} \quad q > 2.$$

Thus

$$\min\{u(0), u(y)\} \leqslant 1 - |B| + |A| \leqslant 1 - \frac{1}{\sqrt{2}} < \frac{1}{2}.$$

Because we know that $u(0) > 1 + \frac{1}{\sqrt{2}}$ we conclude that $u(y) < \frac{1}{2}$, and we have shown that for $n \geqslant 2$ the graph of $u(x)$ crosses the line $u = \frac{1}{2}$ at some point in $(-\zeta, \zeta)$. Therefore $u(x)$ cannot be a solution of (3.1).

It remains to show that for $n = 1$ and $m \geqslant 2$ the point $q_{m,n}$ corresponds to a solution of Problem (3.1). We shall show that $u' < 0$ on the interval $(0, \zeta)$. We observe from (3.20) that $\zeta < \frac{3\pi}{4}\frac{1}{b\sqrt{q}}$.

For convenience, we rescale the variables and define

$$y = x\sqrt{q}, \qquad v(y) = u(x) \qquad \text{and} \qquad \tilde{\zeta} = \frac{\zeta}{\sqrt{q}}. \tag{3.25}$$

We consider the intervals $I_1 = (0, \frac{\pi}{4}\frac{1}{b(q)\sqrt{q}})$ and $I_2 = (\frac{\pi}{4}\frac{1}{b(q)\sqrt{q}}, \frac{3\pi}{4})$ separately: we show that $v'' < 0$ on I_1 and $v' < 0$ on I_2. Since $v'(0) = 0$ and $\tilde{\zeta} < \frac{3\pi}{4}$, this implies that $v' < 0$ on $(0, \tilde{\zeta}]$, as asserted.

For $y \in I_1$ we have

$$\begin{aligned} v''(y) &= -Aqa^2\cos(a\sqrt{q}y) - Bqb^2\cos(b\sqrt{q}y) \\ &< |A|qa^2 - Bqb^2\frac{1}{\sqrt{2}} \leqslant |A|qa^2 - \frac{1}{2} < 0, \end{aligned}$$

and for $y \in I_2$ we have

$$\begin{aligned} v'(y) &= -A\sqrt{q}a\sin(a\sqrt{q}y) - B\sqrt{q}b\sin(b\sqrt{q}y) \\ &< |A|\sqrt{q}a - B\sqrt{q}b\frac{1}{\sqrt{2}} \leqslant |A|\sqrt{q}a(q) - \frac{1}{2} < 0. \end{aligned}$$

The last two inequalities follow from a simple Maple calculation.

We conclude that when $q = q_{m,1}$, where m is an arbitrary integer greater than one, then Problem (3.1) possesses a solution $u(x)$ of the form (3.4). In what follows we shall put

$$q_k = q_{k+1,1}, \qquad k \geqslant 1,$$

and we denote the corresponding solutions by $u_k(x)$.

4 Asymptotics

The asymptotic estimates obtained in Section 3 enable us to establish the asymptotic behaviour of the eigenvalues q_k, the points of intersection ζ_k and the homoclinic orbits u_k as $k \to \infty$.

Lemma 4.1 *The constants q_k and ζ_k have the following asymptotic behaviour:*

$$q_k = \frac{4}{3}k + \frac{2}{3} + O(k^{-1}) \qquad as \qquad k \to \infty,$$

and

$$\begin{aligned} \zeta_k &= \frac{3\pi}{4}\sqrt{q_k}\left\{1 - \frac{1}{2q_k^2}\left(\frac{4}{3\pi} + 1\right) + O(q_k^{-4})\right\} \\ &= \frac{\pi\sqrt{3}}{2}\sqrt{k} + o(1) \qquad as \quad k \to \infty. \end{aligned}$$

Proof. From the two expressions (3.19) and (3.20) for ζ we obtain a relation between q_k and k. By Lemma 3.2, $P(q) \to \infty$ and $Q(q) \to 1$ as $q \to \infty$. Hence, by the asymptotic expansions of $\arctan(x)$ for $x \to \infty$ and for $x \to 1$, we find that

$$m\pi - \frac{\pi}{2} + \frac{1}{q} + O(q^{-3}) = \frac{a(q)}{b(q)}\left\{\pi - \left(\frac{\pi}{4} + \frac{1}{2q^2} + O(q^{-6})\right)\right\}.$$

Using the asymptotic expansions of $a(q)$ and $b(q)$ for $q \to \infty$ from Lemma 3.1, we obtain

$$q_{m,1} = \frac{4}{3}m - \frac{2}{3} + O(q_{m,1}^{-1}) \qquad \text{as} \quad m \to \infty.$$

The desired estimate follows when we put $m = k + 1$.

Remark. We have

$$\arctan(1+x) = \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + O(x^3) \qquad \text{as} \quad x \to 0.$$

We use equation (3.20) to obtain a relation between ζ_k and q_k:

$$\zeta(q) = Y_1(q) \stackrel{\text{def}}{=} \frac{1}{b(q)}\{\pi - \arctan(Q(q))\}.$$

Estimating $b(q_k)$ by means of Lemma 3.1 and $Q(q_k)$ by means of Lemma 3.2 we obtain the estimate for ζ_k as $k \to \infty$.

In order to find the asymptotic behaviour of $u_k(x)$ we first derive the behaviour of the constants A_k and B_k as $k \to \infty$.

Lemma 4.2 *The coefficients A_k and B_k have the following asymptotic behaviour:*

$$A(q_k) = (-1)^{k+1}\frac{1}{q_k^3} + O(q_k^{-5}) \qquad \textit{as} \qquad k \to \infty,$$

and

$$B(q_k) = \frac{1}{\sqrt{2}}\Big\{1 + \frac{1}{2q_k^2} + \frac{19}{8q_k^4} + O(q_k^{-6})\Big\} \qquad \textit{as} \qquad k \to \infty.$$

Proof. It follows from Lemma 3.1 and Lemma 4.1 that

$$b(q_k)\,\zeta_k = \frac{3\pi}{4} - \frac{1}{2q_k^2} + O(q_k^{-4}) \qquad \text{as} \qquad k \to \infty.$$

Using Lemma 3.1 to estimate the right hand side of (3.17) we find the desired limit for $B(q_k)$.

To obtain the limiting behaviour for $A(q_k)$ we use the energy identity. Evaluating it at the origin we find that

$$\frac{1}{2}(a^2A + b^2B)^2 = F(1 + A + B).$$

Because $F(1+\frac{1}{\sqrt{2}}) = 0$, we can write this as

$$\frac{1}{2}a^4\Big\{A + \Big(\frac{b}{a}\Big)^2 B\Big\}^2 = F\Big(1 + \frac{1}{\sqrt{2}} + A + \Big[B - \frac{1}{\sqrt{2}}\Big]\Big).$$

Expanding the left hand side and the right hand side, and using the previous results, we obtain the desired estimate for $A(q_k)$.

We now turn to the asymptotic behaviour of the solution $u_k(x)$. The limiting behaviour of ζ_k established in Lemma 4.1 suggests to rescale the equations as in (3.25). When we thus transform the differential equation we find that $v_k(y)$ is a solution of the equation

$$\frac{1}{q^2}v^{iv} + v'' + f(v) = 0, \tag{4.1}$$

and that $\tilde{\zeta}_k \to \frac{3\pi}{4}$ as $k \to \infty$. We prove the following limit:

Lemma 4.3 *We have*

$$v_k(y) \to V(y) \qquad \textit{as} \qquad k \to \infty$$

uniformly in $C^2(\mathbb{R})$, where V is the unique even positive solution of the problem

$$V''(y) + f(V(y)) = 0, \qquad V(\pm\infty) = 0.$$

Remark. Note that

$$V(y) = \begin{cases} 1 + \dfrac{1}{\sqrt{2}}\cos(y) & \text{for} \qquad |y| \leqslant \dfrac{3\pi}{4}, \\ \dfrac{1}{2}e^{\frac{3\pi}{4} - |y|} & \text{for} \qquad |y| \geqslant \dfrac{3\pi}{4}. \end{cases}$$

Proof. For $y \geqslant \tilde{\zeta}_k$ we have

$$v_k(y) = \frac{1}{2}e^{-\mu_k\sqrt{q}(y-\tilde{\zeta}_k)},$$

so that we deduce from the asymptotic behaviour of $\mu(q)$, as given in Lemma 3.1, that

$$v_k(y) \to \frac{1}{2}e^{\frac{3\pi}{4} - y} \qquad \text{for} \qquad y > \frac{3\pi}{4}. \tag{4.2}$$

For $y \leqslant \tilde{\zeta}_k$ we have

$$v_k(y) = 1 + A_k\cos(a(q_k)\sqrt{q}_k y) + B_k\cos(b(q_k)\sqrt{q}_k y). \tag{4.3}$$

We recall from Lemma 3.1 that $a(q)\sqrt{q} \sim q$ and that $b(q)\sqrt{q} \to 1$ as $q \to \infty$ and from Lemma 4.2 that $A_k = O(k^{-3})$ and $B_k \to \frac{1}{\sqrt{2}}$ as $k \to \infty$, we find that

$$v_k(y) \to 1 + \frac{1}{\sqrt{2}}\cos(y) \qquad \text{for} \qquad |y| \leqslant \frac{3\pi}{4}.$$

Differentiation of (4.3) gives us that $v_k^{(n)}(y) \to V^{(n)}(y)$ for $n = 1, 2$, as asserted.

Remark. Note that the convergence is uniform in $C^2(\mathbb{R})$, but that $v_k'''(y)$ does not converge, even pointwise. In Appendix B we show that derivatives $v_k^{(n)}$ of order $n \geqslant 3$, converge to $V^{(n)}$ in some *weak* sense.

5 Homoclinic orbits to periodic solutions

In this section we show how the solitons found in the previous sections are embedded in families of homoclinic orbits with oscillatory tails. These tails are associated with the imaginary eigenvalues $\lambda = \pm i\omega$ given in (1.8), and take the form

$$w(x) = \varepsilon \cos(\omega x + \phi), \tag{5.1}$$

where ε is the amplitude and ϕ a phase shift. Since $f(s) = -s$ for $s \in [-\frac{1}{2}, \frac{1}{2}]$ only, we need to restrict ε to the interval $[-\frac{1}{2}, \frac{1}{2}]$. Without loss of generality we shall always assume that $\varepsilon \geqslant 0$. The energy $\mathcal{E}$ of w defined in (2.3) is found to be

$$\mathcal{E}(w) = -\frac{1}{4}\sqrt{q^2+4}\left(q + \sqrt{q^2+4}\right)\varepsilon^2. \tag{5.2}$$

As before we look for *even* solutions, so that we consider the initial-boundary value problem

$$\begin{cases} u^{iv} + qu'' + f(u) = 0 & \text{for} \quad x > 0, \\ u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0, \\ u(x) \to w(x) \quad \text{as} \quad x \to \infty, \end{cases} \tag{5.3}$$

where ε has been fixed and ϕ will be chosen appropriately. We look for a solution $u(x)$ of Problem (5.3) which lies entirely above the line $u = -\frac{1}{2}$, and intersects the line $u = +\frac{1}{2}$ exactly at one point $\zeta \in \mathbb{R}^+$. We write

$$u(x) = \begin{cases} u_+(x) & \text{for} \quad 0 \leqslant x \leqslant \zeta, \\ u_-(x) & \text{for} \quad \zeta \leqslant x < \infty. \end{cases} \tag{5.4}$$

Solving Problem (5.3) for $u > \frac{1}{2}$ and for $|u| < \frac{1}{2}$ we find that

$$u_+(x) = 1 + A\cos(ax) + B\cos(bx), \tag{5.5}$$

where A and B are constants, and

$$u_-(x) = Ce^{-\mu x} + \varepsilon\cos(\omega x + \phi), \tag{5.6}$$

in which C is a constant. The amplitude ε will be chosen sufficiently small, so that $u(x)$ intersects $u = \frac{1}{2}$ only once on $\mathbb{R}^+$.

At ζ we need to match u_+ and u_- as well as their first three derivatives. This yields the following conditions on the constants A, B, C, ω and ϕ. Because $u_+(\zeta) = \frac{1}{2}$ and $u_-(\zeta) = \frac{1}{2}$ we obtain

$$Ce^{-\mu\zeta} + \varepsilon\cos(\omega\zeta + \phi) = \frac{1}{2}, \tag{5.7}$$

$$A\cos(a\zeta) + B\cos(b\zeta) = -\frac{1}{2}. \tag{5.8}$$

Matching the derivatives yields:

$$\begin{aligned} -a\,A\sin(a\zeta)-b\,B\sin(b\zeta) &= -\mu\,Ce^{-\mu\zeta}-\varepsilon\omega\sin(\omega\zeta+\phi), && (5.9)\\ -a^2A\cos(a\zeta)-b^2B\cos(b\zeta) &= +\mu^2Ce^{-\mu\zeta}-\varepsilon\omega^2\cos(\omega\zeta+\phi), && (5.10)\\ +a^3A\sin(a\zeta)+b^3B\sin(b\zeta) &= -\mu^3Ce^{-\mu\zeta}+\varepsilon\omega^3\sin(\omega\zeta+\phi). && (5.11) \end{aligned}$$

The system (5.7) – (5.11) can be simplified. First, the four last equations are linear in A and B and hence these variables can be expressed in terms of the others. In this way, we have a system with two fewer equations and unknowns. Second, by division, it can be expressed in terms of $\tan(a\zeta)$ and $\tan(b\zeta)$. Introducing the variable $\theta=\omega\zeta+\phi$, we can write the resulting system as

$$Ce^{-\mu\zeta}+\varepsilon\cos(\theta)=\frac{1}{2}, \tag{5.12}$$

and

$$\begin{cases} \tan(a\zeta)=-\dfrac{\mu(b^2+\mu^2)[1-2\varepsilon\cos(\theta)]+2\varepsilon\omega(\omega^2-b^2)\sin(\theta)}{a[b^2-\mu^2+2(\mu^2+\omega^2)\varepsilon\cos(\theta)]}, \\ \tan(b\zeta)=-\dfrac{\mu(a^2+\mu^2)[1-2\varepsilon\cos(\theta)]+2\varepsilon\omega(a^2-\omega^2)\sin(\theta)}{b[a^2-\mu^2+2(\mu^2+\omega^2)\varepsilon\cos(\theta)]}. \end{cases} \tag{5.13}$$

Thus, the original system is reduced to a set of two equations with four parameters: ε, q, ζ and θ.

For $\varepsilon=0$ this system is the same as in (3.18), and for $\varepsilon\neq 0$ we expect to find a branch of solutions which passes through q_k when $\varepsilon=0$.

Before exhibiting this branch we first simplify the system above still further. This can be done thanks to the following lemma, which enables us to eliminate the variable θ.

Lemma 5.1 *We have*

$$\sin(\theta)=0.$$

Proof. The proof relies on following the energy $\mathcal{E}(u)$ along the solution. Since it is constant, we must have

$$\mathcal{E}(u_+)=\mathcal{E}(u_-)=\mathcal{E}(w), \tag{5.14}$$

where $\mathcal{E}(w)$ is given in (5.2). An elementary computation shows that

$$\mathcal{E}(u_+)=-\frac{1}{4}\{\sqrt{q^2-4}(q+\sqrt{q^2-4})A^2-\sqrt{q^2-4}(-q+\sqrt{q^2-4})B^2+1\} \tag{5.15}$$

From (5.7) - (5.11) we obtain the following expressions for A and B in terms of q, ζ, ε and θ:

$$A=\frac{b^2-\mu^2+2\varepsilon(\mu^2+\omega^2)\cos\theta}{2(a^2-b^2)\cos(a\zeta)}$$

and

$$B = -\frac{a^2 - \mu^2 + 2\varepsilon(\mu^2+\omega^2)\cos\theta}{2(a^2-b^2)\cos(b\zeta)}.$$

When we substitute them into (5.15), we obtain the much simpler expression

$$\begin{aligned}\mathcal{E}(u_+) &= -\frac{1}{4}\sqrt{q^2+4}(q+\sqrt{q^2+4})\varepsilon^2 - \frac{2\varepsilon(1-2\varepsilon\cos\theta)\sin\theta}{\sqrt{q^2-4}} = \\ &= \mathcal{E}(u_-) - \frac{2\varepsilon(1-2\varepsilon\cos\theta)\sin\theta}{\sqrt{q^2-4}}.\end{aligned}$$

Thus, the equality $\mathcal{E}(u_+) = \mathcal{E}(u_-)$ implies that either $\varepsilon = 0$ or, since $|\varepsilon| < \frac{1}{2}$, that $\sin\theta = 0$.

Lemma 5.1 allows us to considerably simplify the equality (5.13). Since the term $\cos\theta$ is multiplied by ε, either we consider $\varepsilon > 0$ and $\cos\theta = \pm 1$ or else, admit negative values of ε, so that $\varepsilon \in \mathbb{R}$ and set $\cos\theta = 1$. Thus, the bifurcation diagram is described by

$$\begin{cases} \tan(a\zeta) = -\dfrac{\mu(b^2+\mu^2)(1-2\varepsilon)}{a[b^2-\mu^2+2(\mu^2+\omega^2)\varepsilon]} \overset{\text{def}}{=} -\overline{P}(q), \\ \tan(b\zeta) = -\dfrac{\mu(a^2+\mu^2)(1-2\varepsilon)}{b[a^2-\mu^2+2(\mu^2+\omega^2)\varepsilon]} \overset{\text{def}}{=} -\overline{Q}(q). \end{cases} \tag{5.16}$$

Note that $\overline{P}(q,0) = P(q)$ and $\overline{Q}(q,0) = Q(q)$. In what follows we shall drop the bar over P and Q.

Thus, formally, in a small neighborhood of q_k, the bifurcation diagram around q_k is described by the equation

$$G(q_k,\varepsilon) = 0, \tag{5.17}$$

where

$$G(q,\varepsilon) = a^2(q)\{\pi - \arctan Q(q,\varepsilon)\} - (k+1)\pi + \arctan P(q,\varepsilon). \qquad k = 1,2\ldots \tag{5.18}$$

A straightforward computation made with the help of the program Maple shows that

$$\frac{\partial G}{\partial q}(q,0) \searrow \frac{3\pi}{4}, \qquad q^{-3}\frac{\partial G}{\partial \varepsilon}(q,0) \searrow -1 \qquad \text{as} \quad q \to \infty. \tag{5.19}$$

In particular, $\frac{\partial G}{\partial q}(q_k,0) \neq 0$. Thus, by the Implicit Function Theorem the following result holds:

Theorem 5.1

1. *For every $k \geqslant 1$ there exists an interval $[\varepsilon_k^-, \varepsilon_k^+]$ around $\varepsilon = 0$ and an analytic function $\overline{q_k} : [\varepsilon_k^-, \varepsilon_k^+] \to \mathbb{R}$ which solves (5.17) and satisfies $\overline{q_k}(0) = q_k$.*

2. We have

$$q_k^{-3}\frac{d\overline{q_k}}{d\varepsilon}(0)\nearrow\frac{4}{3\pi}\qquad as\quad k\to\infty.$$

Part 2 of Theorem 5.1 follows at once from (5.19).

This is a local result. The next question is to find the maximal interval $[\varepsilon_k^-, \varepsilon_k^+]$ along which we can extend the branches. This issue is closely related to the behaviour of $u(x)$ and the existence and position of its local maxima and minima.

First, let us make a remark about the following detail: the implicit function theorem can be applied as long as the function (5.17) is smooth on (q_k, ε). However, for $\varepsilon < 0$, the value of $P(q_k, \varepsilon)$ defined in (5.16) blows up to ∞ but it never vanishes. Aware of this fact, to obtain a global result it can be more convenient to consider the following system which is equivalent to (5.16):

$$\begin{cases}\cot(a\zeta) = -\dfrac{a[b^2-\mu^2+2(\mu^2+\omega^2)\varepsilon]}{\mu(b^2+\mu^2)(1-2\varepsilon)} = -\dfrac{1}{P(q,\varepsilon)},\\[2ex] \cot(b\zeta) = -\dfrac{b[a^2-\mu^2+2(\mu^2+\omega^2)\varepsilon]}{\mu(a^2+\mu^2)(1-2\varepsilon)} = -\dfrac{1}{Q(q,\varepsilon)}.\end{cases}\tag{5.20}$$

It can be verified explicitly (using Maple) that the functions $1/P(q, \varepsilon)$ and $1/Q(q, \varepsilon)$ are analytic for $q > 2$ and $|\varepsilon| < \frac{1}{2}$. Thus, the curve $\overline{q_k}(\varepsilon)$ is analytic in ε, and in the maximal interval of existence, the bifurcation diagram is described by

$$H(q_k, \varepsilon) = 0\tag{5.21}$$

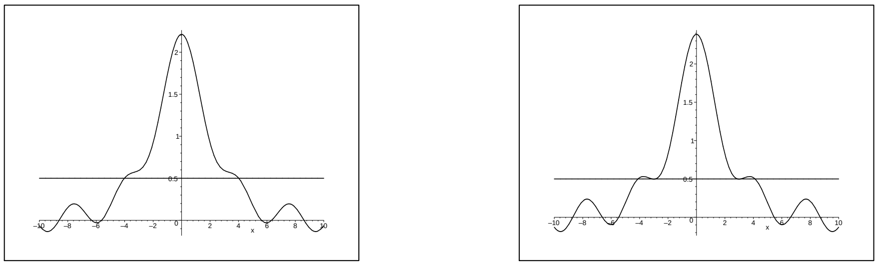
where

$$H(q,\varepsilon) = a^2(q)\Big\{\pi - \operatorname{arccot}\Big(\frac{1}{Q(q,\varepsilon)}\Big)\Big\} - (k+1)\pi + \operatorname{arccot}\Big(\frac{1}{P(q,\varepsilon)}\Big)\qquad k = 1, 2\ldots\tag{5.22}$$

Note that around $\varepsilon = 0$, this is equivalent to the expression for G given by (5.18); thus, the local results of Theorem 5.1 remain unchanged in the whole interval $[\varepsilon_k^-, \varepsilon_k^+]$ in which the function $q_k(\varepsilon)$, defined in Theorem 5.1, exists.

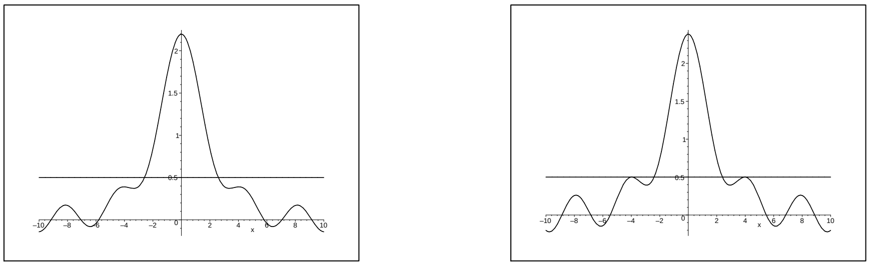
6 The extent of the bifurcation branches in the (q, ε) plane

In this section we follow the Embedded Solitons along the bifurcation branches, both for $\varepsilon > 0$ and for $\varepsilon < 0$. We first focus on the branch passing through q_1, i.e. we set $k = 1$, and then we briefly describe comparable results for the branches through the higher eigenvalues, i.e. for $k > 1$. We want to analyse the dependence of the orbits on ε. For this purpose, in this section we will adopt an approach partially analytical, partially numerical.



(a) $0 < \varepsilon < \varepsilon^+$ (b) $\varepsilon = \varepsilon^+$

Figure 5: The upper branch.



(a) $\varepsilon^- < \varepsilon < 0$ (b) $\varepsilon = \varepsilon^-$

Figure 6: The lower branch.

To emphasize the dependence of the orbit on ε, we denote it by $u(x, \varepsilon)$. Let us first describe what happens when ε becomes positive and when it becomes negative. For small $\varepsilon > 0$ we see that the graph of $u(x, \varepsilon)$ develops an inflection point – and then a local maximum and a local minimum – in the region in $\mathbb{R}^+$ where $u > \frac{1}{2}$, that is in the interval $(0, \zeta)$. Eventually, at $\varepsilon = \varepsilon^+$, the local minimum touches the line $u = \frac{1}{2}$ and the branch terminates. In Figure 5 we show two graphs of $u(x, \varepsilon)$, one for $\varepsilon \in (0, \varepsilon^+)$ and one for $\varepsilon = \varepsilon^+$.

For small $\varepsilon < 0$ the graph of $u(x, \varepsilon)$ develops an inflection point – and then a local maximum and a local minimum – in the region in $\mathbb{R}^+$ where $u < \frac{1}{2}$, that is in the interval (ζ, ∞). As ε drops further the branch terminates at $\varepsilon = \varepsilon^-$ when the local maximum touches the line $u = \frac{1}{2}$. In Figure 6 we show two graphs of $u(x, \varepsilon)$, one for $\varepsilon \in (\varepsilon^-, 0)$ and one for $\varepsilon = \varepsilon^-$.

Summarising we find that as ε *increases* from zero, the graph of $u_k(x)$ first intersects the line $u = \frac{1}{2}$ at a point in the interval $(0, \zeta)$ and as ε *decreases* from zero, $u_k(x)$ first intersects the line $u = \frac{1}{2}$ at a point in the interval on (ζ, ∞).

Thus, as we increase ε we follow the graph of $u_+(x, \varepsilon)$ on $(0, \zeta)$ and as we decrease ε we follow $u_-(x, \varepsilon)$ on the interval (ζ, ∞) respectively. We begin by proving an introductory lemma which states that it suffices to consider $u_+(x, \varepsilon)$

and $u_-(x,\varepsilon)$ on the intervals

$$I_+ = \Big(\zeta(q) - \frac{2\pi}{a(q)},\, \zeta(q)\Big) \qquad \textit{and} \qquad I_- = \Big(\zeta(q),\, \zeta(q) + \frac{\pi}{\omega(q)}\Big). \tag{6.1}$$

Lemma 6.1 *Let $\varepsilon \in (-\frac{1}{2}, \frac{1}{2})$. Then the following statements hold:*

1. *If $u(x,\varepsilon) < \frac{1}{2}$ for every $x \in I_-$, then $u(x,\varepsilon) < \frac{1}{2}$ for $|x| > \zeta$.*
2. *Suppose that $q \geqslant 2.47$ and $-\frac{1}{4} \leqslant \varepsilon \leqslant \frac{1}{2}$. If $u(x,\varepsilon) > \frac{1}{2}$ for every $x \in I_+$, then $u(x,\varepsilon) > \frac{1}{2}$ for $|x| < \zeta$.*
3. *Let $\varepsilon \in (0, \frac{1}{2})$. Then $u(x,\varepsilon) < \frac{1}{2}$ for $|x| > \zeta$.*

Remark. Note that Part 2 of Lemma 6.1 is a statement about $u(x,\varepsilon)$ under the condition that $q \geqslant 2.47$ and $-\frac{1}{4} \leqslant \varepsilon \leqslant \frac{1}{2}$. For $\varepsilon = 0$, all the eigenvalues q_k verify this condition. If we follow the branches along $\varepsilon \in (-\frac{1}{2}, \frac{1}{2})$, all of them continue to have the property that $q_k(\varepsilon) \geqslant 2.47$. Thus, the only restriction that remains involves the lower bound for ε. Numerical computations show that if we *decrease* ε from zero, then, before reaching the value $-\frac{1}{4}$, the bifurcation branch is already broken. Summarising, we can state that the condition of Part 2 is always satisfied along the branches of embedded solitons.

Proof. For $x > \zeta$, $u(x,\varepsilon)$ is defined by (5.6). By Lemma 5.1 we can set $\phi = -\omega\zeta$ and (5.12) gives us the value for C. Substitution on (5.6) gives us

$$u_-(x,\varepsilon) = \Big(\frac{1}{2} - \varepsilon\Big)e^{-\mu(x-\zeta)} + \varepsilon\cos\big(\omega(x-\zeta)\big). \tag{6.2}$$

Thus $u_-(x,\varepsilon)$ is the sum of a decreasing function and a periodic one (it corresponds to the function w defined in (5.1)) that reaches its extremum at $x = \zeta$, which is a maximum for $\varepsilon > 0$ and a minimum for $\varepsilon < 0$. This implies that over half a period, this is, for $\zeta \leqslant x \leqslant \zeta + \frac{\pi}{\omega}$, u_- reaches its global maximum in the region $[\zeta, \infty)$. This proves Part 1 of Lemma 6.1.

If $\varepsilon > 0$, then ζ is indeed the global maximum. This proves Part 3 of Lemma 6.1.

In order to prove Part 2 of the lemma, we need expressions for the coefficients A and B in $u_+(x,\varepsilon)$. They will also be useful for explicit computations. Using the formulae obtained in Section 5, we find

$$A(q,\varepsilon) = \frac{\sqrt{1+P^2(q,\varepsilon)}}{a^2(q)-b^2(q)}\Big(\frac{b^2(q)-\mu^2(q)}{2} + \varepsilon\big(\omega^2(q)+\mu^2(q)\big)\Big), \tag{6.3}$$

$$B(q,\varepsilon) = \frac{\sqrt{1+Q^2(q,\varepsilon)}}{a^2(q)-b^2(q)}\Big(\frac{a^2(q)-\mu^2(q)}{2} + \varepsilon\big(\omega^2(q)+\mu^2(q)\big)\Big). \tag{6.4}$$

By means of an explicit computation we can check that $B(q,\varepsilon) > 0$ for $\varepsilon \in [-\frac{1}{4}, \frac{1}{2}]$ and $q \geqslant 2.47$ (in fact, it is trivial from (6.4) that B is positive for $\varepsilon \geqslant 0$). As it was mentioned in the Remark above, we know *a posteriori* that along the branches, ε and q lie in the assigned intervals.

We note from (5.16) that

$$0 < b(q(\varepsilon)\zeta(q,\varepsilon)) = \pi - \arctan(Q(q,\varepsilon)) < \pi,$$

so that the term $\cos(b(q)x)$ is decreasing for $0 < x < \zeta$. Recall that $u_+(x,\varepsilon)$ is defined on $x \in (0,\zeta)$ by (5.5) and consider $y = x - \frac{2\pi}{a(q)}$. Then we have

$$u_+(y,\varepsilon) = 1 + A\cos(ax) + B\cos(by) < 1 + A\cos(ax) + B\cos(bx) = u_+(x,\varepsilon).$$

Thus, the maximum of u_+ is reached in the interval I_+. This completes the proof of Lemma 6.1.

By Lemma 6.1, we are only interested in the behaviour of $u(x,\varepsilon)$ for $x \in I_+$ since for $x > \zeta$ we already know that $u(x,\varepsilon) < u(\zeta,\varepsilon) = \frac{1}{2}$. We will carefully analyse the case $k = 1$, the first branch, for $\varepsilon > 0$ by studying the behaviour of $u'(x,\varepsilon)$ on I_+.

For $\varepsilon = 0$, the function $u_+(x,0)$ is decreasing. Thus, $u'_+(x,0)$ is negative for $x \in (0,\zeta)$ and at the boundaries $u'_+(0,0) = 0$ and $u'_+(\zeta,0) < 0$. As $u''_+(0,0) < 0$, by continuity, we can conclude that for $\varepsilon > 0$ sufficiently small, $u'_+(x,\varepsilon) < 0$ for $x \in (0,\zeta)$.

Recall that u' is given by

$$u'(x,\varepsilon) = -Aa\sin(ax) - Bb\sin(bx). \tag{6.5}$$

With this expression we can simplify our analysis. As $0 < x < \zeta < \frac{\pi}{b}$ and $B(q,\varepsilon) > 0$, the second term is always negative. By a similar argument, as $0 < x < \zeta < \frac{2\pi}{a}$, we know that the first term is negative for $0 < x < \frac{\pi}{a}$ and positive for $\frac{\pi}{a} < x < \zeta$. Thus, we are only concerned about this second interval $\frac{\pi}{a} < x < \zeta$.

Consider now the point where the first term has an extremum, this is, the point $x = \frac{3\pi}{2a}$. Then

$$u'(\frac{3\pi}{2a},\varepsilon) = Aa - Bb\sin(\frac{3\pi b}{2a}). \tag{6.6}$$

With Maple we check that the right hand side of (6.6) is positive for $\varepsilon > 0.18$. The choice of x is not optimal, but it is sufficient to show that for $\varepsilon > 0$ large enough, u_+ is not monotone. This motivates the following definition:

$$\begin{aligned} \varepsilon^* &= \sup\{\varepsilon \in \left(0,\frac{1}{2}\right) : u'(x,\varepsilon) < 0 \quad \text{for} \quad 0 < x < \zeta\}, & (6.7)\\ x^* &= \sup\{x \in (0,\zeta) : u'(x,\varepsilon^*) < 0\}. & (6.8) \end{aligned}$$

From the argument above we can conclude that $0 < \varepsilon^* < 0.18$. Indeed we can find the values of ε^* and x^* as the solution of the following system in q,ε and x:

$$H(q,\varepsilon) = 0, \qquad u'(x,\varepsilon) = 0, \qquad u''(x,\varepsilon) = 0. \tag{6.9}$$

Explicit computations show that $\varepsilon^* \approx 0.17$ and $x^* \approx 3.33$.

Thus, for $0 < \varepsilon < \varepsilon^*$, u_+ is decreasing and at $\varepsilon = \varepsilon^*$ a *point of inflection* x^* appears. We expect that for $\varepsilon > \varepsilon^*$ two local extrema appear in the graph of u, namely a local minimum and a local maximum. Indeed this is the case. Let us show that this happens.

Fix $x = x^*$ and consider $u'_+(x^*, \varepsilon)$ for $\varepsilon > \varepsilon^*$, a function depending on ε only. Computations show that this function is increasing for $\varepsilon > \varepsilon^*$ and hence is positive for $\varepsilon > \varepsilon^*$. As $u'_+(x^*, \varepsilon)$ is negative for $x = \frac{\pi}{a}$ and $x = \zeta$, we conclude that it has at least two zeros, one corresponding to a local minimum and one to a maximum of $u'_+(x, \varepsilon)$. We denote the minimum by $x_m(\varepsilon)$. We know that $x_m(\varepsilon) < x^*$.

We are interested in studying the behaviour of $u(x_m(\varepsilon), \varepsilon)$ for $\varepsilon > \varepsilon^*$. We can check, again via Maple, that $u(x^*, \varepsilon)$ crosses the line $u = \frac{1}{2}$ for $\varepsilon \in (0, \frac{1}{2})$. This implies that there exists some $\varepsilon \in (0, \frac{1}{2})$ for which $u(x_m(\varepsilon), \varepsilon) = \frac{1}{2}$. Thus, we define

$$\varepsilon^+ = \sup\{\varepsilon \in \left(\varepsilon^*, \frac{1}{2}\right) : u(x_m(\varepsilon), \varepsilon) > \frac{1}{2}\}. \tag{6.10}$$

This value of ε is the value where the bifurcation diagram breaks. It can be found as the solution of the system

$$H(q, \varepsilon) = 0, \qquad u'(x, \varepsilon) = 0, \qquad u(x, \varepsilon) = \frac{1}{2}. \tag{6.11}$$

We find numerically that $\varepsilon^+ \approx 1.9826$.

Appendix

A Proof of Lemma 2.3

Lemma *Let $u(x)$ be a nontrivial homoclinic orbit to the origin that solves the differential equation* (1.3) *where the nonlinearity f is given by the cubic polynomial in* (1.6)*. Then*

$$q \geqslant \frac{2\sqrt{7}}{3}. \tag{A.1}$$

Proof. We will make use of the interpolation inequality

$$\int_{-\infty}^{\infty} u'(x)^2 dx \leqslant \lambda \int_{-\infty}^{\infty} u(x)^2 dx + \frac{1}{4\lambda} \int_{-\infty}^{\infty} u''(x)^2 dx \tag{A.2}$$

which holds for every $\lambda > 0$.

Multiply (1.3) by $u(x)$ and integrate over $x \in (-\infty, \infty)$. Then we obtain the equality

$$6 \int_{-\infty}^{\infty} u''(x)^2 dx - 2q \int_{-\infty}^{\infty} u'(x)^2 dx - \int_{-\infty}^{\infty} u(x)^4 dx + 2 \int_{-\infty}^{\infty} u(x)^2 dx = 0. \tag{A.3}$$

On the other hand, multiplying (1.3) by $u'(x)$ and integration over $x \in (-\infty, \infty)$ gives us

$$\int_{-\infty}^{\infty} u''(x)^2 dx - q \int_{-\infty}^{\infty} u'(x)^2 dx + \int_{-\infty}^{\infty} u(x)^4 dx - \int_{-\infty}^{\infty} u(x)^2 dx = 0. \quad \text{(A.4)}$$

We can eliminate the term $\int_{-\infty}^{\infty} u(x)^4 dx$ by adding both expressions (A.3) and (A.4)

$$\int_{-\infty}^{\infty} u'(x)^2 dx = \frac{1}{3q} \int_{-\infty}^{\infty} u(x)^2 dx + \frac{7}{3q} \int_{-\infty}^{\infty} u''(x)^2 dx. \quad \text{(A.5)}$$

A necessary condition about q is that it must be a positive parameter.

Comparison between (A.2) and (A.5) motivates that we take $\lambda = \frac{3q}{28}$. Thus, we obtain

$$\frac{1}{3q} \int_{-\infty}^{\infty} u(x)^2 dx + \frac{7}{3q} \int_{-\infty}^{\infty} u''(x)^2 dx \leqslant \frac{3q}{28} \int_{-\infty}^{\infty} u(x)^2 dx + \frac{7}{3q} \int_{-\infty}^{\infty} u''(x)^2 dx. \quad \text{(A.6)}$$

or equivalently

$$\left(\frac{3q}{28} - \frac{1}{3q}\right) \int_{-\infty}^{\infty} u(x)^2 dx \geqslant 0. \quad \text{(A.7)}$$

As u is a nontrivial solution, the bound (A.1) follows.

B Weak convergence

Consider the equations

$$\frac{1}{q^2} v^{iv}(y) + v''(y) + f(v(y)) = 0, \quad \text{(B.1)}$$

and

$$v''(y) + f(v(y)) = 0, \quad \text{(B.2)}$$

where f is the piecewise linear function defined in (1.7). It is known that for $q = q_k$ there exists a positive symmetric homoclinic orbit to the origin that crosses once the line $v = \frac{1}{2}$ at $y = \zeta = \frac{3\pi}{4} + o(1)$. The form of the solution for $|y| < z$ is

$$v_k(y) = 1 + A_k \cos(a_k y) + B_k \cos(b_k y), \quad \text{(B.3)}$$

where

$$A_k \sim \frac{1}{k^3}, \qquad a_k \sim k, \qquad B_k \sim \frac{1}{\sqrt{2}}, \qquad b_k \sim 1, \qquad \text{as } k \to \infty.$$

Consider V the solution of equation (B.2) with boundary conditions

$$v'(0) = 0, \qquad v\left(\frac{3\pi}{4}\right) = \frac{1}{2}, \quad \text{(B.4)}$$

in other words,

$$V(y) = \begin{cases} 1 + \dfrac{1}{\sqrt{2}}\cos y & |y| \leqslant \dfrac{3\pi}{4}, \\ \dfrac{1}{2}e^{\frac{3\pi}{4}-y} & y > \dfrac{3\pi}{4}. \end{cases} \tag{B.5}$$

It was shown in Lemma 4.3 that $v_k \to V$ uniformly on $C^2(\mathbb{R})$. However, $v_k'''(y)$ does not converge even pointwise.

Call $L = \frac{3\pi}{4}$ and $I = (-L, L)$. The main result is the following:

Theorem B.1 *v_k''' converges to V''' in the following weak sense: for every bounded continuous function h defined on I we have*

$$\lim_{k\to\infty} \int_{-L}^{L} v_k'''(y)h(y)dy = \int_{-L}^{L} V'''(y)h(y)dy.$$

Proof. To prove this theorem, we note that v_k''' and V''' are given by

$$v_k'''(y) = A_k a_k^3 \sin(a_k y) + B_k b_k^3 \sin(b_k y), \tag{B.6}$$

$$V'''(y) = \frac{1}{\sqrt{2}}\sin y. \tag{B.7}$$

Thus, by an application of Lebesgue's dominated convergence theorem, we have

$$\lim_{k\to\infty} \int_{-L}^{L} B_k b_k^3 \sin(b_k y)h(y)dy = \int_{-L}^{L} \frac{1}{\sqrt{2}}\sin(y)h(y)dy.$$

It remains to show that

$$\lim_{k\to\infty} \int_{-L}^{L} A_k a_k^3 \sin(a_k y)h(y)dy = 0, \tag{B.8}$$

for every bounded continuous function h on I.

Let $\alpha > 0$ and consider

$$f_\alpha(x) = \sin(\alpha x), \qquad f_{+,\alpha}(x) = \max(f_\alpha(x), 0), \qquad f_{-,\alpha}(x) = \max(-f_\alpha(x), 0),$$

and call

$$k = \max\{n \geqslant 0 : n \leqslant \frac{3\alpha}{8}\}.$$

Thus, by the algorithm of division, we have

$$\alpha L = 2\pi k + r \qquad \text{with } 0 \leqslant r < 2\pi.$$

In particular,

$$\frac{3}{8} \leqslant \frac{k}{\alpha} < \frac{3}{8} + \frac{1}{\alpha}.$$

We compute the integral of $f_{+,\alpha}$ over I

$$A_\alpha = \int_{-L}^{L} f_{+,\alpha}(x)dx = \frac{1}{\alpha}\int_{-\alpha L}^{\alpha L} f_{+,1}(y)dy =$$
$$\frac{1}{\alpha}\sum_{j=-k}^{k-1}\left(\int_{2j\pi}^{(2j+1)\pi}\sin(y)dy + \int_{(2j+1)\pi}^{(2j+2)\pi} 0\right) +$$
$$\frac{1}{\alpha}\int_{-L}^{-2k\pi} f_{+,1}(y)dy + \frac{1}{\alpha}\int_{2k\pi}^{L} f_{+,1}(y)dy =$$
$$\frac{1}{\alpha}\sum_{j=-k}^{k-1} 2 + O(\frac{1}{\alpha}) = \frac{4k}{\alpha} + O(\frac{1}{\alpha}) = \frac{3}{2} + O(\frac{1}{\alpha}).$$

Call $g_{+,\alpha}(x) = \frac{f_{+,\alpha}(x)}{A_\alpha}$. Then $g_{+,\alpha}(x)$ is a probability density on I for every $\alpha > 0$. We are interested in the behaviour for $\alpha \to \infty$.

Let us compute the Fourier transform of $g_{+,\alpha}(x)$.

$$\begin{aligned}
\mathcal{F}(g_{+,\alpha})(t) &= \int_{-L}^{L} e^{itx} g_{+,\alpha}(x)dx = \\
&= \frac{1}{A_\alpha}\frac{1}{\alpha}\sum_{j=-k}^{k-1}\left(\int_{2j\pi}^{(2j+1)\pi} e^{ity/\alpha}\sin(y)dy + \int_{(2j+1)\pi}^{(2j+2)\pi} 0\right) + \\
&+ \frac{1}{A_\alpha}\frac{1}{\alpha}\int_{-L}^{-2k\pi} f_{+,1}(y)dy + \frac{1}{A_\alpha}\frac{1}{\alpha}\int_{2k\pi}^{L} f_{+,1}(y)dy = \\
&= \frac{1}{A_\alpha}\frac{1}{\alpha}\frac{\alpha^2}{\alpha^2 - t^2}(1 + e^{it\pi/\alpha})\sum_{j=-k}^{k-1} e^{it2j\pi/\alpha} + O(\frac{1}{\alpha}) = \\
&= \frac{1}{A_\alpha}\frac{1}{\alpha}\frac{\alpha^2}{\alpha^2 - t^2}(1 + e^{it\pi/\alpha})\frac{e^{it2k\pi/\alpha} - e^{-it2k\pi/\alpha}}{e^{it2\pi/\alpha} - 1} + O(\frac{1}{\alpha}) = \\
&= \frac{1}{A_\alpha}\frac{\alpha^2}{\alpha^2 - t^2}\frac{1 + e^{it\pi/\alpha}}{e^{it\pi/\alpha}}\frac{\sin(2k\pi t/\alpha)}{\alpha\sin(\pi t/\alpha)} + O(\frac{1}{\alpha}) = \\
&= \frac{\sin(Lt)}{Lt} + o(1) = \mathcal{F}(g_\infty)(t) + o(1),
\end{aligned}$$

where

$$g_\infty(x) = \begin{cases} \dfrac{1}{2L} & |x| \leqslant L \\ 0 & \textit{otherwise.} \end{cases}$$

Arguing similarly as before, we proceed to define

$$B_\alpha = \int_{-L}^{L} f_{-,\alpha}(x)dx = \frac{3}{2} + O(\frac{1}{\alpha}).$$

Call $g_{-,\alpha}(x) = \frac{f_{-,\alpha}(x)}{B_\alpha}$. Again $g_{-,\alpha}(x)$ is a probability density on I for every $\alpha > 0$. We compute its Fourier transform

$$\mathcal{F}(g_{-,\alpha})(t) = \mathcal{F}(g_\infty)(t) + o(1)$$

Thus, as $\alpha \to \infty$, both the Fourier transform of $g_{+,\alpha}(x)$ and $g_{-,\alpha}(x)$ tend to the Fourier transform of g_∞.

By the use of Levy's Theorem (see, for instance, [2]), we know that for every bounded continuous function h on I

$$\lim_{\alpha\to\infty}\int_{-L}^{L} g_{+,\alpha}(x)h(x)dx = \lim_{\alpha\to\infty}\int_{-L}^{L} g_{-,\alpha}(x)h(x)dx = \int_{-L}^{L} g_\infty(x)h(x)dx.$$

Since

$$f_\alpha = f_{+,\alpha} - f_{-,\alpha} = A_\alpha g_{+,\alpha} - B_\alpha g_{-,\alpha},$$

we conclude that for every bounded continuous function h on I

$$\lim_{\alpha\to\infty}\int_{-L}^{L} f_\alpha(x)h(x)dx = 0.$$

From this last equality, (B.8) follows.

C Values of q_k, ζ_k and of u_k and u_k'' at the origin

k	q_k	$\zeta(q_k)$	$u(0)$	$u''(0)$
1	2.676238584	3.404608234	1.907870236	-.5694105416
2	3.782513691	4.339444241	1.719342998	-.1321149080
3	5.000217193	5.113599136	1.733251153	-.1940547773
4	6.264233336	5.787981648	1.712938707	-.0910032657
5	7.551688580	6.392938791	1.716282398	-.1142824189
6	8.852533501	6.946103324	1.710412301	-.0684516828
7	10.16170325	7.458758840	1.711669783	-.0804604621
8	11.47638465	7.938627940	1.709215602	-.0546513892
9	12.79489648	8.391264245	1.709815095	-.0619473193
10	14.11617480	8.820817639	1.708563923	-.0454182263

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