TRIANGLES, SQUARES, ORANGES AND CUBOIDS

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Abstract. We discuss three problems in number theory that are of 'Pythagorean' nature. The first is a more or less classical one, and we present various approaches to its solution. The second one was only settled relatively recently using 20th century mathematics. The third problem is still widely open at the moment of writing, and probably requires 21th century mathematics for its solution.

Pythagoras

Most people know the name of Pythagoras of Samos from his theorem on the lengths of sides of right angled triangles. The modern statement of this theorem, which is '$a^2+b^2=c^2$', is so compact that it rivals Einstein's $E=mc^2$ in popularity. In fact, there are many people who know both formulas, but have no clue as to what the c stands for in either expression!

Pythagoras, who lived from around 570 BC to about 475 BC, is often regarded as the first pure mathematician, and certainly as the first number theorist. As none of his writings have survived, it is however hard to get a clear picture of his mathematical achievements. From ancient Greek biographies, we know that in his time, he was primarily viewed as a philosopher, a prophet who led a 'society' of scientific-religious nature in Southern Italy. In his younger years, Pythagoras had traveled to Egypt and Babylon, and his society was known for 'oriental' customs as vegetarianism and a refusal to eat beans or to wear cloths made from animal skins.

The society had a code of secrecy and practised communalism, so it is hard to distinguish between the discoveries of the master and of the inner circle of followers known as 'mathematikoi'. What certainly goes back to Pythagoras is the philosophical principle that harmony in nature consists of numerical ratios:

everything is ordered by number.

The inspiration for this belief appears to be based on the discovery that strings produce harmonious tones when the ratios of the lengths of the strings are whole numbers. When a string is shortened to half its length, the tone it produces is an octave higher. Similarly, the ratios 3: 2 and 4: 3 correspond to the intervals of the fifth and the fourth. The numbers 1, 2, 3, 4 themselves, from which the basic

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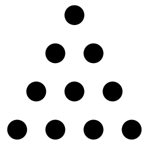
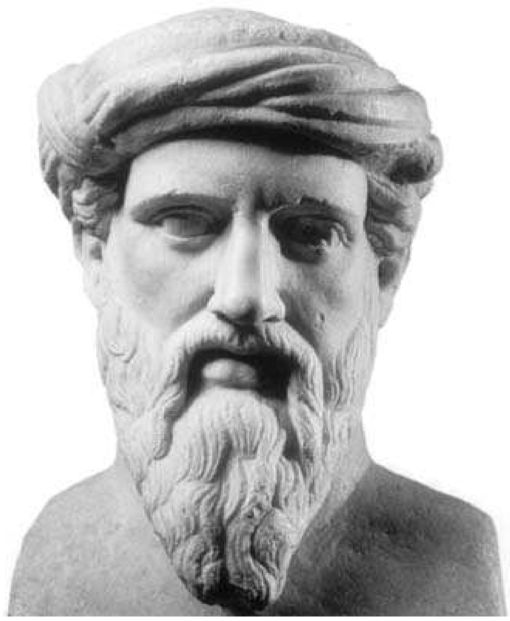
harmonic intervals are formed, constitute the *tetractys*, the 'scale of the sirens' that can be represented by the *perfect triangle*



or by the *triangular number*

$$10 = 1 + 2 + 3 + 4.$$

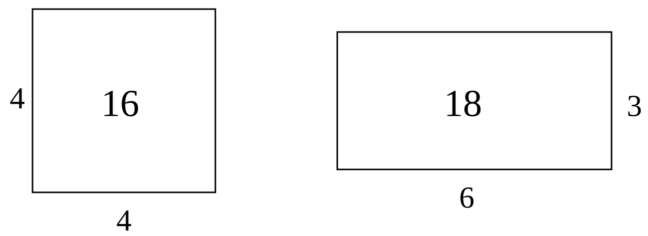
The Pythagoreans viewed 10 as a perfect number since it represented the tetractys—that it is also the number of fingers of the human species may be a coincidence. The left picture below, which is taken from a book of Boethius, shows Pythagoras analyzing the harmony of tones. It is rather different from the bust of Pythagoras on the right, which is remarkable for the 'oriental' head covering.



Number mysticism

Pythagoras approach of numbers was rather different from the approach that is common in modern number theory courses. Not only did the Pythagoreans regard the number 10 above as a perfect number, most other small numbers had an emotional value as well. Even numbers were regarded as feminine, and odd numbers as masculine. As the 'unit' 1 did not count as a number for the ancient Greeks, the sum $5 = 2 + 3$ of the first feminine and the first masculine number was an obvious symbol of marriage.

In a similar way, 17 was a horrible number, being right between the particularly beautiful numbers 16 and 18.



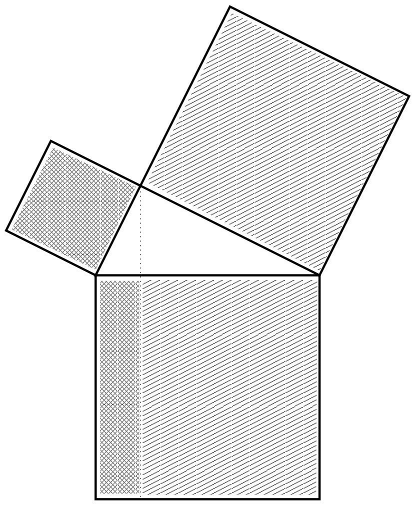
In case you did not realize this right away: the rectangles above having area 16 and 18 also have perimeter 16 and 18, respectively.

Exercise. Show that no other rectangles with integral sides have this property.

Already in Pythagoras's days some people made fun of the mystical beliefs of the Pythagoreans, and nowadays number mysticism is no longer part of mathematics. Superstition however remains: multi-story apartment buildings in the United States often have their 14th floor on top of the 12th!

Theorem of Pythagoras

Already the Babylonians knew around 1700 BC about the theorem that is now ascribed to Pythagoras. In fact, we do not know if Pythagoras actually discovered the theorem for himself, and if so, how he did this. The classical way to present the theorem (and its proof!) is the picture below,



in which the area of the lower square is the sum of the areas of the other two squares. In our efficient modern notation, which mostly goes back to the 18th century mathematician Leonhard Euler (1707–1783), one simply writes

$$a^2 + b^2 = c^2$$

for the relation between the sides. The idea of using letters ('variables') to denote various quantities is in fact a rather recent one in the history of mathematics! It has enabled us to solve geometric questions 'by algebraic means'. For this reason, geometry and algebra are inextricably linked in modern mathematics.

Long before Pythagoras people already knew that a triangle having sides of length 3, 4 and 5 was right angled since $(a, b, c) = (3, 4, 5)$ is, as we would say nowadays, a 'solution' to the 'equation' $a^2 + b^2 = c^2$. With some trial and error, one easily finds other pleasing solutions as $(5, 12, 13)$ or $(23, 264, 265)$. Many of these *Pythagorean triples* were known long ago: there is a Babylonian clay tablet in the Plimpton Collection at Columbia University that lists a couple of them, and its authors probably used a systematic method for finding them.

One can make as many solutions as one likes, as the triple

$$(2mn, m^2 - n^2, m^2 + n^2)$$

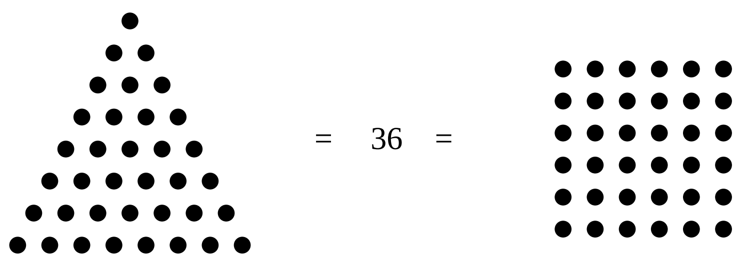
is a Pythagorean triple for all choices of integers $m > n > 0$. In fact every Pythagorean triple is a multiple of a triple of this form, as one can easily deduce by viewing the triple (a, b, c) as a point $(\frac{a}{c}, \frac{b}{c})$ on the unit circle in the plane with equation $x^2 + y^2 = 1$ having positive coordinates.

Exercise. Show that Pythagorean triples correspond to rational points P with positive coordinates on the unit circle as indicated above, provided that we identify a triple (a, b, c) with its scalar multiples (ka, kb, kc). Show that such P are obtained as intersection points of the lines $y = \frac{m}{n}x - 1$ through $(0, -1)$ with rational slope $\frac{m}{n} > 1$ with the unit circle. Deduce that we have $P = (\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2})$.

Finding integral solutions to various equations still lies very much at the heart of modern number theory. Whenever we have solved a problem, a new one naturally suggests itself.

Figural harmony

Our first Pythagorean problem concerns the following surprising property of the number 36.



It shows that in arithmetic, triangles can be squares!

From a Pythagorean point of view, a number that is both triangular and a square certainly deserves to be called *harmonic*, and anyone playing with a pile of coins will find that it is rather exceptional for a number to be harmonic. Our first Pythagorean problem will now be obvious.

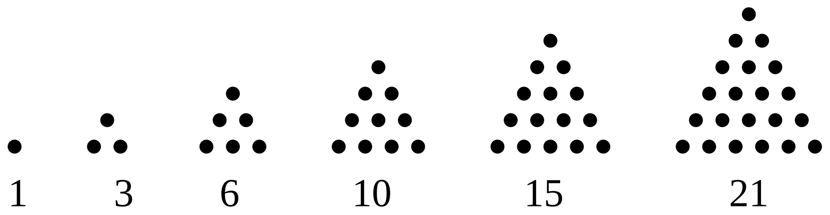
Question. *Which integers are harmonic?*

More specifically, we would like to know whether there are harmonic integers at all besides 36, and come up with a method to produce a complete list of them. For all we know, there can be finitely many or infinitely many harmonic numbers, and in the second case the best we can hope for is an efficient *algorithm* to produce all harmonic numbers in ascending order or, even better, a simple *formula* for the n-th harmonic number.

As for the existence of harmonic integers besides 36, the mathematically inclined reader may already have noticed that the number 1 is actually both a triangular number and a square! One might be inclined to call this a *trivial* example of a harmonic number, and actually a true Pythagorean would contend that 1 is not to be considered a number at all—it is the unit by which the true numbers are measured. The modern reader, who knows about relatively recent inventions as the number zero and negative numbers as well, may feel this Pythagorean distinction of 1 to be only awkward.

Using your computer

When confronted with the question of existence of small harmonic numbers, the modern mathematician has a tool that often makes life easy: the computer. It is not too difficult to ask a computer to look for harmonic numbers, provided that we can tell it how to decide whether a number is triangular.



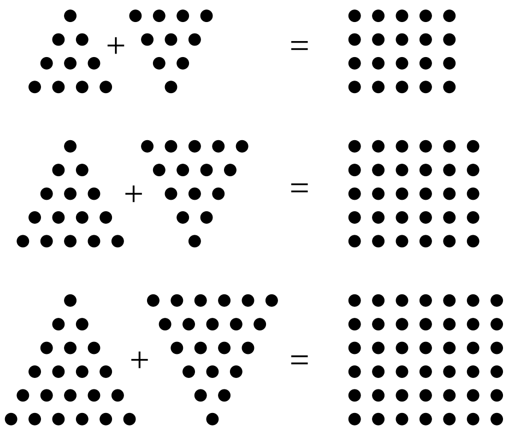
Looking at the first few triangular numbers, one can try to come up with an easy formula for the m-th number in the sequence

$$1, 3, 6, 10, 15, 21, 28, 36, \ldots .$$

You may have heard the famous story in this context about the German mathematician Carl Friedrich Gauss (1777–1855). When he was still a small boy in elementary school, the teacher tried to keep his class busy for a while by ordering them to compute the sum $1 + 2 + 3 + \ldots + 99 + 100$ of all positive integers up to 100. In other words, the children had to compute the 100-th triangular number, which takes a while by hand if one actually performs the 99 required additions. Gauss thought for a few seconds before writing down the correct answer. In geometric terms, he had realized that doubling the m-th triangular number yields the 'rectangular number' $m(m + 1)$. The resulting simple formula

$$\Delta_m = \frac{m(m+1)}{2}$$

for the m-th triangular number Δ_m is obvious if one draws the right picture.



Using Gauss's formula, one may write a little computer program that, depending on the language one uses, will more or less resemble the following program in pseudo-code.

```
for m = 1 to 1000:
  do Δm = m(m+1)/2;
  if issquare Δm print Δm;
endif; enddo; endfor;
```

You will find:

$$\frac{49 \cdot 50}{2} = 1225 = 35^2$$

and

$$\frac{288 \cdot 289}{2} = 41616 = 204^2.$$

So there actually are non-trivial harmonic numbers besides 36!

Reading computer output

The complete list of harmonic numbers $\Delta_m = n^2$ with $m < 1\,000\,000$ that are squares is easily produced by the computer. It reads as follows.

$\Delta_m = n^2$	m	n
1	1	1
36	8	6
1225	49	35
41616	288	204
1413721	1681	1189
48024900	9800	6930
1631432881	57121	40391
55420693056	332928	235416

The question is: what can we deduce from such a table? It seems that there are many harmonic numbers, but that they grow rapidly when written down in ascending order. Can we predict from the table above where the first harmonic number Δ_m with $m > 1\,000\,000$ is to be expected?

It appears that the growth in the table is quite regular, as n gets about 6 times as large if we move from a harmonic n-value in the table to the next one. More precisely, a pocket calculator suffices to tell us that subsequent n's in our table have the following quotients:

$$\begin{aligned}&6.000000000;\\&5.833333333;\\&5.828571428;\\&5.828431372;\\&5.828427249;\\&5.828427128;\\&5.828427124.\end{aligned}$$

These quotients seem to converge, and we therefore expect the next value of n to have the approximate value

$$\begin{aligned}n &\approx 5.828427124 \cdot 235416\\ &= 1372104.9998\ldots.\end{aligned}$$

And indeed, for $n = 1372105$ we find

$$\begin{aligned}n^2 &= 1372105^2 = 1882672131025\\ &= \frac{1940449 \cdot 1940450}{2} = \Delta_{1940449}.\end{aligned}$$

The corresponding quotient $\frac{1372105}{235416} = 5.8284271247$ yields, correct up to 5 decimals:

$$\frac{1372105}{235416} \cdot 1372105 \approx 7997214$$

as the next harmonic value of n. Is it a surprise that our expectations are met again?

$$\begin{aligned}7997214^2 &= 63955431761796\\ &= \frac{11309768 \cdot 11309769}{2} = \Delta_{11309768}.\end{aligned}$$

By what one might call 'empirical math', we have arrived at a method to predict the next harmonic number from the previous ones in the list. Those with some faith in nature will feel the method is bound to work in the infinitely many cases further down the list, so that we have found a method to generate all harmonic numbers in ascending order. But the cynical mathematician will observe that there is no guarantee that our method will always work, and that we can only hope that no harmonic numbers will be missed by our method.

Looking for structure

From a mathematical point of view, we have only scratched the surface, as our *understanding* of why our 'method' works is still non-existent. Let us try to gain a deeper understanding of what is going on by looking in a more *structural* way at the integers occurring in our table. What are their striking properties from an arithmetic point of view?

If one looks at the parity of m and n, one sees that m and n have the same parity, which alternates between consecutive values. The odd values of m are easily recognized as squares: 1, $49 = 7^2$ and $1681 = 41^2$. A calculator gives us $57121 = 239^2$. The even values are not squares, but one less than a square: $8 = 3^2 - 1$, $288 = 17^2 - 1$, $9800 = 99^2 - 1$. The calculator confirms this: $332928 = 577^2 - 1$. One may also recognize the even values of m as being twice a square: $8 = 2 \cdot 2^2$, $288 = 2 \cdot 12^2$, $9800 = 2 \cdot 70^2$ and finally $332928 = 2 \cdot 408^2$.

Having found that the harmonic values of m are alternatingly of the form x^2 and $x^2 - 1$, the next observation is that x is a *divisor* of the corresponding n-value. This leads us to rewrite the table in the following form.

$\Delta_m = n^2$	m	n
1	1^2	$1 \cdot 1$
36	$3^2 - 1$	$3 \cdot 2$
1225	7^2	$7 \cdot 5$
41616	$17^2 - 1$	$17 \cdot 12$
1413721	41^2	$41 \cdot 29$
48024900	$99^2 - 1$	$99 \cdot 70$
1631432881	239^2	$239 \cdot 169$
55420693056	$577^2 - 1$	$577 \cdot 408$

Those who chose to write the even values of m as $m = 2y^2$ would also note that this y divides n, leading to $n = x \cdot y$ with x as before. Either way, one finds that every harmonic number is of the form

$$x^2 \cdot y^2$$

with x and y taking on the following values:

$$\begin{array}{ll} x: & 1, 3, 7, 17, 41, 99, 239, \dots \\ y: & 1, 2, 5, 12, 29, 70, 169, \dots \end{array}$$

We are now down to the level to much smaller integers, and in order to predict the next values of x and y from the previous ones one needs a final discovery: the sequences for x and y are *recurrent sequences*. More precisely, for $k \geq 2$ their $k+1$-th term is given by

$$\begin{aligned} x_{k+1} &= 2x_k + x_{k-1} \\ y_{k+1} &= 2y_k + y_{k-1}. \end{aligned}$$

This is satisfactory as we have now gotten rid of non-integers such as the mysterious limit $5.8284271247\ldots$. But do we really understand what is going on?

Solving equations

We get a better understanding if we realize that what we are doing is finding the integer solutions of the equation

$$\frac{m(m+1)}{2} = n^2.$$

No general method for solving equations in integers exists—but in many cases we can do it! In this case, it helps to read the left hand side as

$$m \cdot \frac{(m+1)}{2} \qquad \text{for } m \text{ odd;}$$

$$\frac{m}{2} \cdot (m+1) \qquad \text{for } m \text{ even.}$$

The integers m and $m+1$ have no common factors as they differ by 1. It follows that the two integers on the left hand side of our equation are *coprime*. Now anyone familiar with the factorization of integers into primes will have no problem to prove the following fundamental fact.

Fact. *If a product ab of two coprime positive integers a and b is a square, then both a and b are squares.*

It follows immediately that if Δ_m is a square, then we have $\Delta_m = x^2y^2$ with

$$m = x^2 \quad \text{and} \quad m+1 = 2y^2$$

if m is odd, and

$$m+1 = x^2 \quad \text{and} \quad m = 2y^2$$

if m is even. As x^2 and $2y^2$ differ in both cases by 1, we see that x and y are solutions to the equation

$$x^2 - 2y^2 = \pm 1.$$

There is an amazing trick due to Euler to produce infinitely many integer solutions to this equation out of the 'trivial' solution $x = y = 1$. To do this, one boldly leaves the realm of the integers and writes the equation in terms of 'algebraic integers' of the form $a + b\sqrt{2}$ with a and b integers as

$$(x + y\sqrt{2})(x - y\sqrt{2}) = \pm 1.$$

The trivial solution $x = y = 1$ yields the identity

$$(1 + \sqrt{2})(1 - \sqrt{2}) = -1,$$

which we can raise to the power k to obtain

$$(1+\sqrt{2})^k(1-\sqrt{2})^k = (-1)^k.$$

Now every power of an 'integer' of the form $a+b\sqrt{2}$ is again of this form. Writing out the powers of $1+\sqrt{2}$, we encounter the values for x and y that occur in our final table!

$$\begin{aligned}
(1+\sqrt{2})^2 &= 3+2\sqrt{2}\\
(1+\sqrt{2})^3 &= 7+5\sqrt{2}\\
(1+\sqrt{2})^4 &= 17+12\sqrt{2}\\
(1+\sqrt{2})^5 &= 41+29\sqrt{2}\\
(1+\sqrt{2})^6 &= 99+70\sqrt{2}\\
(1+\sqrt{2})^7 &= 239+169\sqrt{2}
\end{aligned}$$

Why does this work? Take for instance

$$(1+\sqrt{2})^{10} = 3363+2378\sqrt{2}.$$

Then we have

$$(1-\sqrt{2})^{10} = 3363-2378\sqrt{2}.$$

The product of these numbers is

$$\begin{aligned}
&(3363+2378\sqrt{2})(3363-2378\sqrt{2})\\
&\quad = 3363^2 - 2\cdot 2378^2.
\end{aligned}$$

It also equals

$$(1+\sqrt{2})^{10}(1-\sqrt{2})^{10} = (-1)^{10} = 1.$$

Thus $x=3363$ and $y=2378$ satisfy

$$x^2-2y^2=1,$$

and the square $3363^2\cdot 2378^2$ is harmonic: it is the (3363^2-1)-th triangular number.

More generally, we see that the integer pairs (x_k, y_k) defined by

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})^k$$

satisfy $x_k^2-2y_k^2=(-1)^k$ for all $k\geq 1$, and that the number $x_k^2\cdot y_k^2$ is harmonic, being the x_k^2-th or (x_k^2-1)-th triangular number.

With some extra effort, one can show that we actually find *all* harmonic numbers in this way, such that $x_k^2y_k^2$ is indeed the k-th harmonic number in the ordered list of all harmonic numbers.

From $(1+\sqrt{2})^2 = 2(1+\sqrt{2})+1$ we obtain $(1+\sqrt{2})^{k+1}(1+\sqrt{2})^k + (1+\sqrt{2})^{k-1}$ for all $k \geq 1$, $k \geq 1$, and therefore

$$x_{k+1} + y_{k+1}\sqrt{2} = (2x_k + x_{k-1}) + (2y_k + y_{k-1})\sqrt{2}.$$

This explains the recurrences we found for the sequences of the numbers x_k and y_k.

Exercise. Show that k-th harmonic number is obtained by squaring the integral part n of the real number

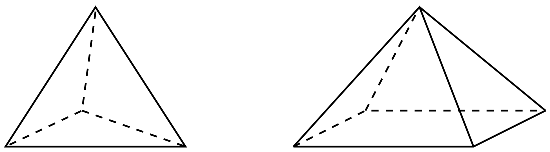
$$\frac{(3+2\sqrt{2})^k}{4\sqrt{2}}.$$

Deduce that the quotient of subsequent values of n tends to $3+2\sqrt{2} = 5.828427124746\ldots$.

The orange problem

Whenever we successfully solve a problem, there are always similar problems one may hope to be able to deal with in an analogous way. In the case of harmonic numbers, where triangular numbers are squares, one can move on to similar questions for pentagonal and hexagonal numbers, or think of an analogue in three-dimensional space. In the later case we run into the following greengrocer's problem.

Orange problem. *Which numbers of oranges can be piled up both as a 'triangular' and as a 'square' pyramid?*



As before, there is the trivial solution 1 corresponding to a single orange. We are of course interested in the larger solutions.

It is an interesting and not too difficult exercise to show that a triangular pyramid with m layers has

$$\frac{m(m+1)(m+2)}{6}$$

oranges. In a similar way, one finds that a square one with n layers has

$$\frac{n(n+1)(2n+1)}{6}.$$

Thus, we need to solve in positive integers x and y the equation

$$m(m+1)(m+2) = n(n+1)(2n+1).$$

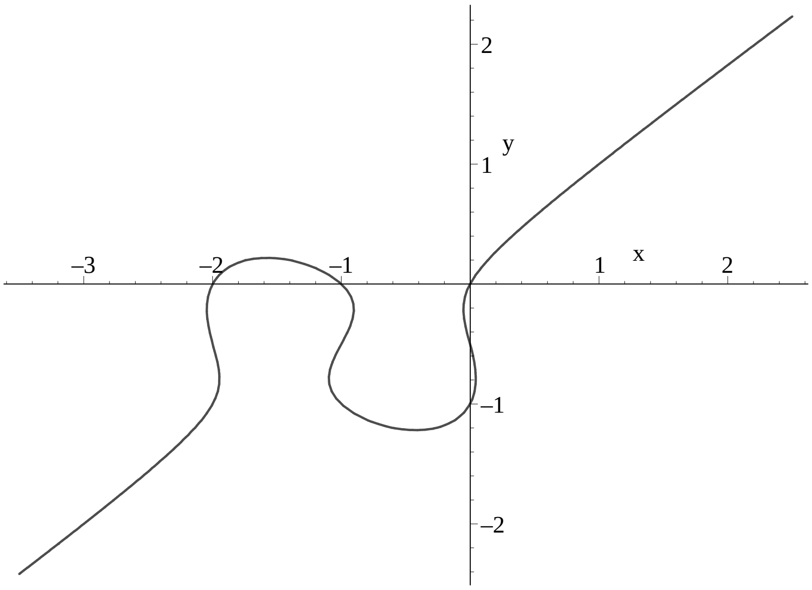
As in the case of harmonic numbers, one can write a slightly more complicated computer program to find small solutions, and the reader is invited to do so. However, no solutions $(m,n) \neq (1,1)$ are found if we check for all values for m up to

10^6 or even 10^9. Does this mean that there are no such solutions, or have we run into one of these equations for which the smallest non-trivial solution is too large to be found by a computer?

This time, there is no easy Euler-type trick such as the one we employed in the case of harmonic numbers. We suggestively rewrite the equation as

$$x(x+1)(x+2) = y(y+1)(2y+1),$$

such that the modern reader will think of this equation as describing a curve in the xy-plane. The given cubic equation describes what is called an *elliptic curve*. Its plot looks as follows.



We want to determine the points on this curve having positive integral coordinates.

The arithmetic theory of elliptic curves is very much a 20th century topic. It is an attractive blend of number theory, geometry and analysis. The theory is not yet complete, and some of the striking properties of elliptic curves, such as the famous Birch-Swinnerton-Dyer conjecture, are still unproved. Recently there have been several major breakthroughs. The *modularity* of elliptic curves, from which Fermat's Last Theorem was known to follow, was proved in 1995 by Wiles.

Effectively finding the integral points on an elliptic curve is something that can now be done—at least in principle. In 1988, the Dutch mathematicians Beukers and Top used the existing theory to prove the following theorem.

Theorem. *The only integral points on the curve*

$$x(x+1)(x+2) = y(y+1)(2y+1)$$

are the eight points

$$(-3,-2), \quad (-2,-1), \quad (-2,0), \quad (-1,0)$$
$$(-1,-1), \quad (0,-1), \quad (0,0), \quad (1,1).$$

In order to find a solution to the orange problem, we need integral points (x, y) with *positive* x and y. Thus the theorem tells us that it will never work out with more than a single orange. It may come as a surprise that the proof of this simple fact requires a fair amount of modern mathematics.

Pythagorean cuboids

In the case of Pythagorean triples solving the equation

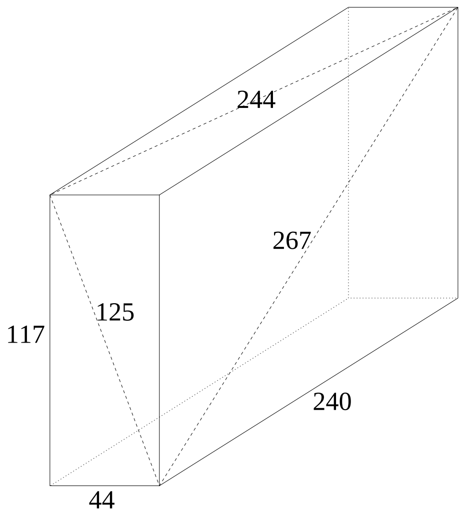
$$a^2 + b^2 = c^2,$$

we mentioned the complete classification of the integral solutions. They essentially correspond to the set of *rational* points on a speciic curve, the unit circle in the plane. When we try to use this complete knowledge to deal with 3-dimensional analogues, we are soon stuck with widely open problems.

A 3-dimensional generalization of the right-angled triangle having three sides a, b, c of integral length is the rectangular box ('cuboid') with sides a, b, c of integral length for which the three face diagonals x, y, z also have integral length. Here we have 'interrelated' Pythagorean triples, as each pair of sides of the box forms a Pythagorean triple with the corresponding face diagonal. More precisely, we need to solve the *set* of equations

$$\begin{aligned} a^2 + b^2 &= x^2 \\ b^2 + c^2 &= y^2 \\ a^2 + c^2 &= z^2. \end{aligned}$$

in positive integers a, b, c, x, y, z. By a clever manipulation of these formulas, Euler found solutions such as the cuboid below.



One can even show that there are infinitely many 'essentially different' solutions to this set of 3 equations in 6 unknowns.

Suppose now that we require that the full diagonal in such a cuboid has integral length as well. We then obtain a set of 4 equations in 7 unknowns:

$$\begin{aligned} a^2 + b^2 &= x^2 \\ b^2 + c^2 &= y^2 \\ a^2 + c^2 &= z^2 \\ a^2 + b^2 + c^2 &= w^2. \end{aligned}$$

In this more complicated case of *Pythagorean cuboids*, no solutions in positive integers are known to exist. Extensive computer calculations have shown that no 'small' solutions exist. But it would just be a wild guess to conjecture that they do not exist at all. In geometric terms, we are here in a situation where we need to find rational points on a *surface*. Finding the rational points on a curve is already a hard and intensively studied problem for which a complete solution is still lacking. For surfaces, the arithmetic theory barely exists, and we may only expect it to be developed in the 21-st century. The question as to whether Pythagorean cuboids exist at all is one of the yardsticks by which we can measure the progress of the theory.

References

For information on the life and work of Pythagoras (and many other mathematicians), one can consult the corresponding webpage at the history site of the University of St Andrews in Scotland:

http://www-history.mcs.st-andrews.ac.uk/history.

A very readable book on ancient mathematics that also devotes a dozen of pages to Pythagoras and his school is

B. L. van der Waerden, *Science Awakening*, Noordhoff, Groningen, 1954.

Euler's solution of the *Pell equation* $x^2 - 2y^2 = \pm 1$ that we gave appeared in

L. Euler, *De solutione problematum Diophanteorum per numeros integros*, Comm. Acad. Sci. Petropolitanae **6**, 175–188 (1732/33)

and can be found in Euler's collected works. If you have trouble reading Latin, try one of the more modern texts explaining algebraic number theory.

Finding rational points on elliptic curves is the topic of

J. Silverman, J. Tate, *Rational points on elliptic curves*, Springer, New York, 1992.

For the arithmetic theory of surfaces, we will still have to wait a while before the first textbooks become available.

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