Discrete scattering and simple non-simple face-homogeneous random walks

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Abstract

In this paper we will derive some results for characterising the almost closed sets of a face-homogeneous random walk. We will present a conjecture on the relation between discrete scattering of the fluid limit and the absence of non-atomic almost closed sets. We will illustrate the conjecture with random walks with both simple *and* non-simple decomposition into almost closed sets.

1 Introduction

Let be given a discrete time, irreducible Markov chain $\{\xi_t\}_{t=0,1,\dots}$, on a countable state space $\boldsymbol{S}$ with stationary transition probabilities

$$p_{xy} = \mathsf{P}\{\xi_{t+1} = y \,|\, \xi_t = x\}.$$

For unravelling the transient behaviour of this Markov chain, it seems of interest to study the *almost closed sets*, i.e. subsets $A \subset \boldsymbol{S}$ for which $\limsup\{\xi_n \in A\}$ and $\liminf\{\xi_n \in A\}$ are a.s. equal and have positive probability. If this probability is 0, the corresponding set is said to be *transient*.

It is known (cf. [2]) that the state space partitions into an at most denumerable collection of disjoint, almost closed sets that together absorb all probability mass in the long run. All, except at most one, are atomic, that is they do not contain two disjoint almost closed subsets; the non-atomic one, if present, does not contain any atomic subsets. The collection

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is essentially unique. It determines a decomposition of the process itself, by identifying the "directions" into which the process disappears in the long run. The Markov chain is called *simple* whenever the state space consisting of a single almost closed set.

Additionally, there is an one-one correspondence between the *bounded harmonic functions* and the almost closed sets from the decomposition. This immediately shows that the process is simple and atomic whenever the chain is recurrent.

Blackwell has used the almost closed set structure for determining the structure of the invariant σ-algebra. A consequence is a characterisation of the Poisson boundary of the Markov chain (cf. [9]), which (essentially) is the probability space restricted to the sub-σ-algebra of invariant sets with induced measure.

Here we would like to highlight the relation between almost closed sets and scattering properties of a certain class of Markov chains over its sets of fluid or Euler paths. This class consists of so-called face-homogeneous random walks. Extensions to more general walks are easily possible, but involve more technical descriptions that might have obscured the exposition.

A Markov chain ξ_t is said to be *face-homogeneous*, when the state space $\boldsymbol{S}$ can be partitioned into a finite collection of disjoint subsets $\boldsymbol{\Lambda}_i \subset \boldsymbol{S}$, $i = 1, \ldots, k$, where ξ_t behaves as a homogeneous random walk on each of these. More precisely, to each $\boldsymbol{\Lambda}$ corresponds a discrete probability distribution $p^{\boldsymbol{\Lambda}}$, such that

$$p_{xy} = p^{\boldsymbol{\Lambda}}_{y-x}, \quad \text{any } x \in \boldsymbol{\Lambda}_i, i = 1, \ldots, k.$$

The subsets $\boldsymbol{\Lambda}$ are called *faces*. Denote $m(x) = \mathsf{E}\{\xi_{t+1} - \xi_t \,|\, \xi_t = x\} = \sum_y (y-x) p_{xy}$ for the drift from state x. By face-homogeneity, the drift only depends on the face $\boldsymbol{\Lambda}$ to which x belongs. Henceforth, we will mostly write $m^{\boldsymbol{\Lambda}} = \sum_y y p^{\boldsymbol{\Lambda}}_y$ instead.

We would like to focus on the following aspect of transient facehomogeneous random walks. Consider the time-space scaled process $\xi_{[tN]}([xN])/N$, as $N \to \infty$, where $\xi_0 = [xN]$ denotes the initial position. Assume that it has a limit (in distribution), $\mathbf{u}(x;t)$ say. This limit may be stochastic. Any realisation $u(x;\cdot)$ is called a fluid path or Euler path starting at point x. Assuming $\boldsymbol{S} \subset \mathbf{Z}^p$, then $u(x;\cdot) \subset \mathbf{R}^p$. As a natural extension of a set $C \subset \mathbf{Z}^p$ to $\mathbf{R}^p$ we will take for instance the convex hull $\mathsf{conv}(C)$ of C in $\mathbf{R}^p$.

Let $\{C_i\}_{i=0}^{\infty}$ be the almost closed sets from the decomposition. Our conjecture has the following form (modulo technical conditions).

Conjecture 1.1. *Assume that there exist finitely many almost closed sets and no non-atomic ones. Then to each almost closed set C corresponds* *precisely one path $u(0;\cdot)$, such that $u(0;t) \subset \mathsf{conv}(C)$, for $t \geq T$, for some finite time T. Moreover,*

$$\mathsf{P}\Big\{\lim_{N\to\infty} \frac{\xi_{[tN]}(x)}{N} = u(0;t)\Big\} = \mathsf{P}\Big\{\liminf_{N\to\infty}\{\xi(x)_N \in C\}\Big\}, \qquad (1.1)$$

that is, the probability of selecting a given path is given by the absorption probability of the corresponding almost closed set. These probabilities are called the scattering probabilities.

We will call this 'discrete scattering'. In case of an uncountably infinite number of Euler paths starting at a given point, we have 'continuous scattering'. This occurs for instance in the case of the transient, face-homogeneous random walk on $\mathbf{Z}^2$ discussed in [11]. In this paper, there are uncountably many Euler paths starting at the origin 0, that all cycle off to infinity. Then the set of realisations $u(0;1)$ form a closed curve around 0. The usual scaling by time, does not produce any convergence results for a fixed initial point. If one identifies each Euler path with one point on this curve, we conjecture convergence of the unscaled process $\xi_{[tN]}(x)$ to a random variable on this curve. Time scaling and identification are equivalent in the case of denumerably many Euler paths. The intuition behind this is the following. We expect that one can only have at most a denumerable amount of Euler paths starting at a given point, when in the long run each path belongs to one face. Both approaches map such path to precisely one point.

The paper's aim is to bring into the limelight a connection between the invariant set structure (i.e. the bounded harmonic functions) and Euler paths. This relation is implicit in for instance the computation of the Poisson boundary by Kurkova [10] for face-homogeneous random walks in mainly dimension 2. It is therefore not surprising, that the proof techniques used in [10] seem often to be similar to the ones used here.

In order to present the main idea of discrete scattering, we discuss some simple examples with both simple and non-simple decomposition, using proof techniques that seem to apply to more general cases. These proof techniques are based on the existence of well-behaved Lyapunov functions that can be turned into contractive ones implying exponential convergence properties. Papers [11] and [13] discuss an example of continuous scattering. If neither courage nor stamina fail, we will later address the more general problem, which we fear, is bound to be quite technical. This is mostly because the construction of Lyapunov functions seems quite involved even in relatively simple models like face-homogeneous random walks.

The following section provides the basic definitions and results from [2] and [3] concerning almost closed sets. In section 3 we will derive a number of tools that seem basic to us for studying the almost closed set structure and Euler limit behaviour for face-homogeneous random walks in general. These use certain techniques from Feller [5] and Lyapunov function techniques from [4].

As an application, section 4 will study face-homogeneous random walks on the integer line.

Finally section 5 addresses the problem of face-homogeneous random walks on the quarter plane. In [7] large deviation bounds are derived for face-homogenous random walks on the quarter plane. As an application of [7], the local rate function is derived in [8] in explicit form for the path identically 0 in exponential queueing networks corresponding to a coupled processors system. Hidden in the analysis of these papers, but essential for the derivation of their result, is the assertion of Conjecture 1.1, applied to the appropriate random walk. Indeed, the local rate function for the path identically 0, is precisely the maximum of the local rates for the path identically 0, conditioned on the almost closed sets containing the origin of space. Section 5 aims to show validity of the conjecture for two versions of the models studied in [8]. The first is the "coupled processors system with switched off processors whenever a queue is empty". The second is the same model with additional input whenever a queue is empty. To be clear, for all models in [8] the conjecture can be shown to hold. However, the derivations are analogous to the one presented here and we prefer to focus on the two versions mentioned.

2 Almost closed sets and invariant σ-algebra

We recall definitions and results from Chung's exposition [3], §I.17. Assume the Markov chain ξ_t to be *aperiodic*. For μ_0 the inital distribution of ξ_t, i.e. $\mu_0(A) = \mathsf{P}\{\xi_0 \in A\}$, write P_{μ_0} for the corresponding probability measure on the space space

$$\Omega = \boldsymbol{S}^\infty = \{(x_0, x_1, \ldots) \,|\, x_n \in \boldsymbol{S}\},$$

endowed with the σ-algebra $\mathcal{F}$ generated by the one-point cylinder sets

$$\{(x_0, x_1, \ldots, x_n) \times \boldsymbol{S} \times \boldsymbol{S} \times \ldots\}, \quad x_i \in \boldsymbol{S},\, i \leq n,\, n = 0, 1, \ldots.$$

When $\mu_0(x) = 1$ for a given state $x \in \boldsymbol{S}$, we will simply write P_x and E_x for the corresponding probability and expectation operators. Let $\mathcal{A} \subset \mathcal{F}$ by any sub-σ-algebra of sets. A set $C \in \mathcal{A}$ is called *atomic* with respect to $\mathcal{A}$, if $\mathsf{P}_{\mu_0}(C) > 0$ and C does not contain two disjoint subsets in $\mathcal{A}$ with positive probability. It is called *completely non-atomic*, if $\mathsf{P}_{\mu_0}(C) > 0$ and it does not contain any atomic subsets (in $\mathcal{A}$).

The following lemma is well-known (cf. [1], [3]).

Lemma 2.1. *The path space Ω can be represented by means of a denumerable collection of disjoint sets belonging to $\mathcal{A}$:*

$$\Omega = \cup_{n=0} C_n,$$

where some of the C_n may be absent. If present, then C_0 is completely non-atomic, and C_i, $i \geq 1$, are atomic. This decomposition is unique modulo sets of zero measure. Hence $\sum_i \mathsf{P}_{\mu_0}\{C_i\} = 1$.

We will call $\mathcal{A}$ trivial if Ω is atomic w.r.t. $\mathcal{A}$, i.e. $C_1 = \Omega$. Note that $\mathsf{P}_{\mu_0}(C)$ being positive or zero does *not* depend on the initial measure μ_0, only its particular value.

Bearing in mind our interest in the long run behaviour of the Markov chain, we will consider the sub-σ-algebra of invariant sets. To this end, introduce the time shift T on Ω:

$$T(x_0, x_1, \ldots) = (x_1, x_2, \ldots).$$

A set $C \in \mathcal{F}$ is called *invariant* whenever $T^{-1}C = C$. The class of invariant sets is a sub-σ-algebra of $\mathcal{F}$ denoted by $\mathcal{G}$.

One can find the decomposition of Ω w.r.t $\mathcal{G}$ through a decomposition of the state space $\boldsymbol{S}$ into so-called almost closed sets. Let $A \subset \boldsymbol{S}$ and define two corresponding invariant sets by

$$\begin{aligned}
\underline{\mathcal{L}}(A) &= \liminf_{n\to\infty}\{\xi_n \in A\} = \cup_{m\geq 0} \cap_{t\geq m} \{\omega \mid \xi_t(\omega) = \omega_t \in A\} \\
\overline{\mathcal{L}}(A) &= \limsup_{n\to\infty}\{\xi_n \in A\} = \cap_{m\geq 0} \cup_{t\geq m} \{\omega \mid \xi_t(\omega) \in A\}.
\end{aligned}$$

We call a set A *transient* if $\mathsf{P}_{\mu_0}\{\overline{\mathcal{L}}(A)\} = 0$. The set A is called *almost closed* if it is not transient and

$$\mathsf{P}_{\mu_0}\{\underline{\mathcal{L}}(A)\} = \mathsf{P}_{\mu_0}\{\overline{\mathcal{L}}(A)\} \, (> 0). \tag{2.1}$$

By $\mathcal{A}$ denote the class of almost closed and transient sets. Then for any $A \in \mathcal{A}$ and any initial state x one has

$$\mathsf{P}_x\{\underline{\mathcal{L}}(A)\} = \mathsf{P}_x\{\overline{\mathcal{L}}(A)\} = \lim_{n\to\infty} \mathsf{P}_x\{\xi_n \in A\}, \tag{2.2}$$

and so the limit probability of the chain being in set A exists.

Clearly $\boldsymbol{S}$ itself is an almost closed set. The following properties will be used in the sequel. The proof is elementary.

Lemma 2.2. **i)** *Suppose that $C \subset A$, for sets $A, C \in \mathcal{A}$. Then $A \setminus C \in \mathcal{A}$. Moreoever, $\lim_{n\to\infty} \mathsf{P}_x\{\xi_n \in A\} = \lim_{n\to\infty} \mathsf{P}_x\{\xi_n \in C\}$ if and only if $A \setminus C$ is transient.*

ii) *Suppose that A, $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$. In particular, if A and B are both transient, then $A \cup B$ is transient.*

iii) *Any subset of a transient set is transient.*

The lemma implies that $\mathcal{A}$ is an algebra of sets and transient sets are an ideal of this algebra. The following theorem exhibits the relation between $\mathcal{G}$ and $\mathcal{A}$.

Theorem 2.3. *To each invariant set $C \in \mathcal{G}$ there corresponds a transient or almost closed set $A \in \mathcal{A}$, unique upto transient sets, such that*

$$C = \{\overline{\mathcal{L}}(A)\} = \{\underline{\mathcal{L}}(A)\},$$

according as C is null or not. The correspondence is an isomorphism of algebras. In particular, one can choose the set A by setting

$$A = \big\{x \in \boldsymbol{S} \mid \mathsf{P}_x\{C\} > a\big\},$$

for $0 < a < 1$ arbitrary.

The above correspondence with invariant sets, motivates calling an almost closed set $A \in \mathcal{A}$ *atomic*, if does not contain two disjoint almost closed sets, and *completely non-atomic*, if it does not contain any atomic subset. As a consequence of Lemma 2.1 and Theorem 2.3 the state space $\boldsymbol{S}$ can be partitioned into a denumerable set of disjoint almost closed sets

$$\boldsymbol{S} = \cup_{n=0}^{\infty} A_n,$$

where A_0 (if present) is completely non-atomic and A_i, $i \geq 1$, is atomic (if present). The decomposition is unique modulo transient sets. Moreover, one has

$$1 = \sum_{n=0}^{\infty} \mathsf{P}_x\{\underline{\mathcal{L}}(A_n)\} = \sum_{n=0}^{\infty} \mathsf{P}_x\{\overline{\mathcal{L}}(A_n)\} = \sum_{n=0}^{\infty} \lim_{t\to\infty} \mathsf{P}_x\{\xi_t \in A_n\}. \tag{2.3}$$

3 Tools

Sojourn sets and well-behaved Lyapunov functions
For characterising almost closed sets, and consequently invariant sets, the concept 'sojourn set' (cf. [5], [3]) seems the more manageable one.

The set S of states is called a *sojourn set* iff $\mathsf{P}_{\mu_0}\{\underline{\mathcal{L}}(S)\} > 0$. In this case, for every a, $0 < a < 1$, define the set

$$S^{(a)} = \{x \in \boldsymbol{S} \,|\, \mathsf{P}_x\big\{\underline{\mathcal{L}}(S)\} > a\big\}.$$

The following theorem (cf. [3], §I.17) holds.

Theorem 3.1. *If S is a sojourn set, then for every a, $0 < a < 1$, the set $S \cap S^{(a)}$ is almost closed, $S^{(a)} - S \cap S^{(a)}$ is transient and*

$$\mathsf{P}_x\{\underline{\mathcal{L}}(S)\} = \lim_{t\to\infty} \mathsf{P}_x\{\xi_t \in S \cap S^{(a)}\}. \tag{3.1}$$

The right-hand side of (3.1), as a function of the initial state x, has the nice property of being a *bounded harmonic function*. Remind that a function $f : \boldsymbol{S} \to \mathbf{R}$ is called *harmonic* w.r.t. a given transition matrix P if

$$f(x) = \sum_y p_{xy} f(y), \quad x \in \boldsymbol{S}. \tag{3.2}$$

In essence, Theorem 3.1 reduced the problem of the almost closed set structure by the more easier one of determining the sojourn set structure (or the collection of bounded harmonic functions..) By means of the following lemma (cf. [4], [14]) certain sets attracting probability mass can be identified as sojourn sets. For convenience we will give the proof here, since it does not occur in explicit form in these references. The proof itself is a standard construction of transforming sub-additive Lyapunov functions into contractive ones, as will be repeatedly used.

Lemma 3.2. *Suppose there exist a function $f : \boldsymbol{S} \to \mathbf{R}$, a set $B \subset \boldsymbol{S}$, a step function $k : \boldsymbol{S} \to \mathbf{Z}_+$ and constants d, $\epsilon > 0$, $C \geq 0$, such that*

i) *the f-jumps are bounded by d, i.e. $|f(\xi_{t+1}) - f(\xi_t)| \leq d$, a.s.;*

ii) *the step function is uniformly bounded, i.e. $\sup_x k(x) < \infty$;*

iii) *$B \subset \{x \,|\, f(x) \leq C\} \neq \boldsymbol{S}$;*

iv) *the f-drift outside B is strictly positive, i.e.*

$$\mathsf{E}\{f\big(\xi_{t+k(\xi_t)}\big) - f(\xi_t)\,|\,\xi_t = x\} \geq \epsilon, \quad x \in \boldsymbol{S} \setminus B. \tag{3.3}$$

Then the set $\{x\,|\,f(x) > C + c\}$ is almost closed for any $c \geq 0$.

Proof. Denote $B' = \{x\,|\,f(x) \leq C\}$, then $B' \supset B$. Denote the entrance time of B' by τ, i.e. $\tau = t$ iff $t = \inf\{n \geq 0\,|\,\xi_{n-1} \notin B', \xi_n \in B'\}$, and $\tau = \infty$, whenever $\xi_t \notin B'$ for all t.

First assume that $k(x) \equiv 1$ and note that $\exp\{y\} < 1 + y + 3y^2/2$, whenever $|y| < 1$. We can use a standard argument to deduce the desired result (see [4]). Using the drift condition (3.3), we have for any sufficiently small constant $h > 0$ with $h, hd < 1$, and $x \notin B'$ that

$$\begin{aligned}
&\mathsf{E}\big\{\exp\{-h\big(f(\xi_{t+1}) - f(\xi_t)\big)\}\,|\,\xi_t = x\big\} \\
&\leq \quad \mathsf{E}\Big\{1 - h\big(f(\xi_{t+1}) - f(\xi_t)\big) + \frac{3h^2\big(f(\xi_{t+1}) - f(\xi_t)\big)^2}{2}\,|\,\xi_t = x\Big\} \\
&\leq \quad 1 - h\epsilon + \frac{3(hd)^2}{2} \leq \exp\{-\gamma\},
\end{aligned}$$

for some positive constant $\gamma > 0$. Hence, denoting $g(x) = \exp\{-hf(x)\}$, and iterating, we have for $\xi_0 = x \notin B'$

$$\mathsf{E}_x\{g(\xi_t)\mathbf{1}_{\{\tau > t-1\}}\} \leq g(x)\exp\{-\gamma t\}, \quad t > 0.$$

Since $\{\tau > t - 1\} \supset \{\tau = t\}$ and $g \geq 0$, it follows that

$$\mathsf{E}_x\{g(\xi_t)\mathbf{1}_{\{\tau = t\}}\} \leq g(x)\exp\{-\gamma t\}, \quad t > 0.$$

For $y \in B'$ we have $g(y) \geq \exp\{-hC\}$, so that

$$\mathsf{P}_x\{\tau = t\}\exp\{-hC\} \leq g(x)\exp\{-\gamma t\}, \quad t > 0.$$

Multiplying both sides by $\exp\{hC\}$ and taking the summation over $t \geq 1$, we find that

$$\mathsf{P}_x\{\tau < \infty\} \leq \frac{\exp\{hC - hf(x)\}}{1 - \exp\{-\gamma\}}. \tag{3.4}$$

By the positive drift condition, necessarily the set $\{x \notin B'\}$ is infinite and $\sup_{x \notin B'} f(x) = \infty$. This implies $\{x \in \boldsymbol{S}\,|\,f(x) > c\} \neq \emptyset$, for any constant c.

Let $C' > C$, such that $\alpha = 1 - \exp\{h(C - C')\}/\big(1 - \exp\{-\gamma\}\big) > 0$. For x with $f(x) > C'$, we have for $A = \{x \in \boldsymbol{S} \,|\, f(x) > C\}$ that

$$\begin{aligned} \mathsf{P}_x\{\underline{\mathcal{L}}(A)\} &= \mathsf{P}_x\big\{\liminf_{t\to\infty}\{f(\xi_t) > C\}\big\} \\ &\geq \mathsf{P}_x\big\{f(\xi_t) > C, t > 0\} = \mathsf{P}_x\{\tau = \infty\} \geq \alpha. \end{aligned} \tag{3.5}$$

By irreducibility, it follows that A is a sojourn set. Next we would like to show almost closed-ness. This will follow from Theorem 3.1, if we can show that $A = A \cap A^{(a)}$ for some $a > 0$.

By condition (i) and (iv), for any $x \in A$, the probability of reaching the set $\{y \in \boldsymbol{S} \,|\, f(y) \geq f(x) + \epsilon/2\}$ after the next jump is at least $\epsilon/(2d - \epsilon)$. Hence, for any $x \in A$,

$$\mathsf{P}_x\{f(\xi_l) > C'\} \geq \big(\frac{\epsilon}{2d - \epsilon}\big)^l, \text{ with } l = \big\lfloor \frac{2(C' - C}{\epsilon} \big\rfloor + 1.$$

This yields for any $x \in A$ that

$$\mathsf{P}_x\{\underline{\mathcal{L}}(A)\} \geq \sum_{\{y \,|\, f(y) > C'\}} \mathsf{P}_x\{\xi_l = y\}\mathsf{P}_y\{\underline{\mathcal{L}}(A)\} \geq \big(\frac{\epsilon}{2d - \epsilon}\big)^l \alpha, \tag{3.6}$$

and, consequently the desired assertion holds for $a = (\epsilon/(2d - \epsilon))^l \cdot \alpha$. Clearly, by condition (iii) the exception set B is contained in $\boldsymbol{S} \setminus A$, and so one may take any constant $C + c$, $c \geq 0$, instead of C.

Let us now assume $k(\cdot)$ to be a general step function. We will give the proof again for the constant C. Consider the embedded Markov chain $\tilde{\xi}_t$ at successive instants $0, k(\xi_0), k\big(\xi_{k(\xi_0)}\big), \ldots$. The above derivation implies (3.5) for the embedded chain, for the constants $C + d \cdot \sup_x k(x)$, and C', such that $\alpha = 1 - \exp\{h(C + d \cdot \sup_x k(x) - C')\}/\big(1 - \exp\{-\gamma\}\big) > 0$.

Boundedness conditions (i) and (ii) imply that

$$\mathsf{P}_x\{\inf_{t>0} f(\xi_t) > C\} \geq \mathsf{P}_x\{\inf_{t>0} f(\tilde{\xi}_t) > C + d \cdot \sup_x k(x)\},$$

thus implying (3.5) for the chain ξ_t and constant C, for the constants C' and α that have been chosen for the embedded chain $\tilde{\xi}_t$. The remainder follows as before. $\square$

A sufficient condition for transience of an infinite set can be derived in a similar fashion.

Lemma 3.3. *Suppose there exist a function $f : \boldsymbol{S} \to \mathbf{R}$, a non-empty set $A \subset \boldsymbol{S}$ and a finite step function $k : \boldsymbol{S} \to \mathbf{Z}_+$, such that*

i) *the step function is uniformly bounded, i.e.* $\sup_x k(x) < \infty$*;*

ii) *for some* $0 < a < 1$ *we have* $\big\{x \in A \,|\, \mathsf{P}_x\{\underline{\mathcal{L}}(A)\} \le a\big\} = \emptyset$*;*

iii) *the* f*-drift outside* A *is strictly negative, i.e.*

$$\mathsf{E}\{f(\xi_{t+k(\xi_t)}) - f(\xi_t) \,|\, \xi_t = x\} \le -\epsilon, \quad x \in \boldsymbol{S} \setminus A; \tag{3.7}$$

iv) $f(x) \ge 0$ *on*

$$\big(\boldsymbol{S} \setminus A\big) \cup \{y \in A \,|\, p_{zy} > 0, \text{for some } z \in \boldsymbol{S} \setminus A\}.$$

Then the set $\boldsymbol{S} \setminus A$ *is transient.*

Proof. Note that if the set A would be finite, then (3.7) is simply a generalised version of Lyapunov-Foster criterion for positive recurrence of the Markov chain. Checking that proof, it follows that the time τ to hit A (from a state $x \notin A$) is finite a.s. and has finite expectation, whether A be finite or not, provided that (iv) holds.

The set A is almost closed by (ii) and Theorem 3.1. Denote $A^c = \boldsymbol{S} \setminus A$ and let $x \in A^c$ be given. Then taking into account the fact that τ is finite with probability 1, using iteration and the fact that $\mathsf{P}_{\cdot}\{\xi_t \in A \text{ i.o.}\}$ is a *harmonic function*, $\mathsf{P}_x\{\xi_t \in A \text{ i.o.}\} = \sum_y p_{xy} \mathsf{P}_y\{\xi_t \in A \text{ i.o.}\})$, we find for $x \in A^c$

$$\mathsf{P}_x\{\xi_t \in A \text{ i.o.}\} = \sum_{y \in A} \mathsf{P}_x\{\xi_\tau = y\} \mathsf{P}_y\{\xi_t \in A \text{ i.o.}\} \ge a. \tag{3.8}$$

Hence, $A^{(a)} = \big\{y \in \boldsymbol{S} \,|\, \mathsf{P}_y\{\underline{\mathcal{L}}(A)\} > a\big\} = \boldsymbol{S}$ and $A^{(a)} \cap A = A$. By Theorem 3.1, $A^c = \boldsymbol{S} \setminus A = A^{(a)} \setminus A = A^{(a)} \setminus A \cap A^{(a)}$ is transient. □

In applications as face-homogeneous random walks, one often can construct a suitable Lyapunov function on the whole state space, satisfying (3.3) outside at most a compact set. This one may be used for piecing together both Lemmas 3.2 and Lemma 3.3.

Lemma 3.4. *Suppose there exist a function* $f : \boldsymbol{S} \to \mathbf{R}$*, a finite set* $B \subset \boldsymbol{S}$*, a step function* $k : \boldsymbol{S} \to \mathbf{Z}^+$*, and constants* d*,* C*,* $\epsilon > 0$*, such that conditions (i) and (ii) of Lemma 3.2 are satisfied as well as (iv) for* $x \in \boldsymbol{S} \setminus B$*. For any constant* $C \ge \max_{x \in B} f(x)$*, let* $A = \{x \in \boldsymbol{S} \,|\, f(x) > C\}$*. If* $A \ne \emptyset$*, then it is almost closed and* $\boldsymbol{S} \setminus A$ *is transient.*

Proof. Almost closed-ness of the set A follows by a direct application of Lemma 3.2.

We will check transience of $\boldsymbol{S} \setminus A$. Note that the conditions of the lemma are in fact a well-known Lyapunov function criterion for transience of the Markov chain (cf. [4]). This implies that any *finite* set is a transient set (as in Blackwells definition).

Note that condition (i) of the previous Lemma 3.3 is satisfied. Since $B \subset \{x \mid f(x) \leq C\} = \boldsymbol{S} \setminus A$, by (3.6) it follows that condition (ii) of Lemma 3.3 applies with the set A. Condition (iv) of Lemma 3.3 holds for the function $g = -f + C + d$. Finally, (iii) holds for the function g with exception set B, in particular

$$\mathsf{E}\{g(\xi_{t+k(\xi_t)}) - g(\xi_t) \mid \xi_t = x\} \leq -\epsilon, \quad x \in \boldsymbol{S} \setminus B.$$

Define $\tau_A = 1$, for $\xi_1 \in A$; $\tau_A = t$, $t \geq 2$, iff $t = \inf\{n \geq 2 \mid \xi_{n-1} \notin A, \xi_n \in A\}$, and $\tau_A = \infty$ otherwise. Intuitively it is clear that τ_A is a.s. finite, for any initial state $x \in \boldsymbol{S} \setminus A$, since B is a finite, transient set. Once the validity of the statement has been settled, the further arguments proceed as in the proof of the previous lemma. We will now provide the arguments for τ_A being a.s. finite.

Condition (iii) and (iv) of Lemma 3.3 do hold for the set $A \cup B$ and the function g defined above. Analogously to τ_A, we define $\tau_{A \cup B} \geq 1$ to be the first hitting time of $A \cup B$.

Then, as in the proof of Lemma 3.3, $\tau = \tau_{A \cup B}$ is a.s. finite, and has finite expectation, for any initial state $x \notin A \cup B$. This extends directly to all initial states $x \in B$.

In turn, this implies non-defectiveness of the embedded, finite state Markov chain η_t on the set $B \cup \alpha$, where

$$\mathsf{P}\{\eta_{t+1} = y \mid \eta_t = x\} = \begin{cases} \mathsf{P}_x\{\xi_\tau = y\}, & x, y \in B \\ \mathsf{P}_x\{\xi_\tau \in A\} & x \in B, y = \alpha \\ 1, & x, y = \alpha. \end{cases}$$

So, this Markov chain has one absorbing state, which is reached with positive probability from any other. It follows that the state α is reached with probability 1 from any other state, and in finite expected time.

Now, attach a reward $\tau_{xy} = t\mathbf{1}_{\{\tau = t, \xi_\tau = y\}}$ to the transition $x \to y$ in this finite chain, $x, y \in B$. Then τ_A is the total reward till absorption in state α. Since τ is a.s. finite with finite expectation for each initial state $x \in B$, it follows that τ_{xy} is a.s. finite with finite expectation. It is now standard from finite reward chain theory that τ_A is a.s. finite with finite expectation for each initial state $x \in B$.

By a decomposition argument to set B, this extends to all initial states $x \in \boldsymbol{S} \setminus A$. $\square$

So far, we have not said anything on how to check atomicity of the almost closed sets from Lemma 3.2. We will turn to that problem now, and derive atomicity of these sets for a special subclass of face-homogeneous random walks. The main idea is that 'far away into' an almost closed set, the transition probabilities at the boundary of and outside the almost closed set are irrelevant for the structure of the almost closed set itself. Hence, one might change these in such way as to determine atomicity, or lack of it, of the almost closed set in a simple way. As a result, the almost closed set in the original must have the same structure as well.

To set this up, we would like to slightly dwell on how to compute the probabilities of sojourn sets, as has been extensively studied in Feller [5]. The proofs of the statements mentioned below before Remark 3.1, can be found in his paper.

For quoting the necessary details, we prefer to introduce notation allowing the analytic approach used by Feller. First write P for the transition matrix of the Markov chain to be considered. In what follows, it is allowed to be a *substochastic* matrix. The restricted probability matrix P_A to set A is defined by

$$P_{A,xy} = \begin{cases} p_{xy}, & x, y \in A \\ 0, & \text{otherwise} \end{cases}$$

The n-step (restricted) transition probabilities and matrix are denoted by superscript (n).

For any $A \subset \boldsymbol{S}$ the following limits exist as vectors on $\boldsymbol{S}$

$$\begin{aligned} \boldsymbol{\sigma}_A &= \lim_{t\to\infty} P_A^{(t)} \mathbf{1}_{\{A\}} \\ s_A &= \lim_{t\to\infty} P^{(t)} \boldsymbol{\sigma}_A. \end{aligned}$$

Then, upto a constant factor, σ_A is the maximum bounded harmonic function on A with respect to the restricted transition matrix P_A. The probabilistic interpretation is that $\boldsymbol{\sigma}_A(x) = \mathsf{P}_x\{\xi_t \in A, t \geq 0\}$. As a consequence, $s_A(x)$ denotes the probability that $\xi_t(x) \in A$ eventually, i.e.

$$s_A(x) = \mathsf{P}_x\{\underline{\mathcal{L}}(A)\}$$

(harmonicity of $\boldsymbol{\sigma}_A$ w.r.t. P_A should be used for showing this directly). Thus A is a sojourn set if and only if $s_A \neq 0$ and then we will refer to it as the *sojourn solution* corresponding to A. Additionally, $s_A \geq \sigma_A$ is harmonic on

$\boldsymbol{S}$ with respect to P. One has that $\sup_{x\in\boldsymbol{S}} s_A(x) = \sup_{x\in\boldsymbol{S}} \boldsymbol{\sigma}_A(x) = 1$, for sojourn set A.

Note that this easily implies a sojourn set to be an infinite set. Indeed, by irreducibility $\lim_{t\to\infty} P_B^{(t)} = 0$, for any finite set B. Hence also (by finiteness) $\sigma_B = 0$. Consequently $s_B = 0$ and so B cannot be a sojourn set.

The sojourn set $B \subset \boldsymbol{S}$ is said to be *representative* whenever there is $0 < \eta < 1$, such that $s_B(x) > \eta$ for all $x \in B$. A representative set B is almost closed by Theorem 3.1. Hence, it enjoys the property that the probability of eventually ending up in B equals the limiting probability of B (cf. (2.2)), i.e. $s_B(x) = \lim_{t\to\infty} \mathsf{P}_x\{\xi_t \in B\}$! This does not hold for an arbitrary sojourn set, since sojourn sets need not be almost closed. For any sojourn set A and any $0 < \eta < 1$ put

$$\begin{aligned} A^\eta &= A^{(\eta)} \cap A = \{x \in A \,|\, s_A(x) > \eta\} \\ A_\eta &= \{x \,|\, \boldsymbol{\sigma}_A(x) > \eta\}. \end{aligned}$$

Then $s_A = s_{A^\eta} = s_{A_\eta}$, so that A^η is representative by definition. By Theorem 3.1, it is almost closed.

Furthermore, $A^\eta \supset A_\eta$, as $s_A \geq \boldsymbol{\sigma}_A$ and $A_\eta \subset A$. As a consequence, $\overline{\mathcal{L}}(A^\eta) \supset \overline{\mathcal{L}}(A_\eta) \supset \underline{\mathcal{L}}(A_\eta)$. Given $\xi_0 = x$, the left and right events have the same probability $s_A(x)$. Thus $\mathsf{P}_x\{\overline{\mathcal{L}}(A_\eta)\} = \mathsf{P}_x\{\underline{\mathcal{L}}(A_\eta)\} = s_A(x)$, by which A_η is almost closed.

Similarly, for any set B, $A \supset B \supset A_\eta$ we have $s_B = s_A$! We give an example of representative sets with a nice structure.

Example 3.1. *The conditions of Lemma 3.2 or those of Lemma 3.4 imply that the set $A = \{f(x) > C + c\}$ is representative, i.e. $A = A^\eta$ for some $0 < \eta < 1$. This follows from the proof of Lemma 3.2, equation (3.6).*

The next statement is evident, but has not been explicitly proved in [3], [2] nor [5]. We will need it in the sequel.

Lemma 3.5. *Let two almost closed sets A, $B \subset \boldsymbol{S}$ be given. Then the symmetric difference of A and B is a transient set if and only if $s_A = s_B$.*

Proof. Suppose that A and B differ by at most a transient set. By virtue of Lemma 2.2 (ii) $A\cup B$ is almost closed. Note that $A\cap B = A\cup B \setminus \{(A\setminus B)\cup (B\setminus A)\}$. The union of two transient sets is transient by Lemma 2.2 (ii) and so by (i) we have that $A\cap B$ is almost closed and $s_{A\cup B}(x) = s_{A\cap B}(x)$. By mononpremonicity

$$s_{A\cap B}(x) \leq s_A(x), s_B(x) \leq s_{A\cup B}(x),$$

and so it follows that $s_A(x) = s_B(x)$.

We will not explicitly refer to Lemma 2.2 anymore.

Next, suppose that $s_A = s_B$. Then $A^{(\eta)} = B^{(\eta)}$. By Theorem 3.1 with $S = A$ and $S = B$, the sets $A^{(\eta)} \setminus A^\eta$ and $A^{(\eta)} \setminus B^\eta = B^{(\eta)} \setminus B^\eta$ are transient. Since $A^\eta \setminus B^\eta \subset A^{(\eta)} \setminus B^\eta$, this implies that $A^\eta \setminus B^\eta$ is transient.

Now, note that A^η and B^η are both almost closed. By Lemma 2.2, $A^\eta \cap B^\eta$ is either transient or almost closed. If it is transient, then A^η must be transient as the union of two transient sets $A^\eta \setminus B^\eta$ and $A^\eta \cap B^\eta$. This contradicts the fact that A^η is almost closed. Hence, $A^\eta \cap B^\eta$ is almost closed. Additionally, since both $A^\eta \setminus B^\eta$ and $A \setminus A^\eta$ are transient, their union $A \setminus (A^\eta \cap\ B^\eta)$ is transient. Similarly, we find that $B \setminus (A^\eta \cap\ B^\eta)$ is transient. And consequently the union $A \cup B \setminus (A^\eta \cap\ B^\eta)$ is transient as well. The symmetric difference of A and B is contained in this set and is therefore transient. $\square$

Our method for checking atomicity relies on the lemma following here.

Lemma 3.6. *Let $A \subset \boldsymbol{S}$. Let $\{\tilde{\xi}_t\}_{t\geq 0}$ be an irreducible and aperiodic Markov chain on a denumerable space $\widetilde{\boldsymbol{S}} \supset \boldsymbol{S}$, such that*

$$p_{xy} = \tilde{p}_{xy}, \quad x \in A, \quad y \in \boldsymbol{S},$$

where $\tilde{p}_{xy} = \mathsf{P}\{\tilde{\xi}_{t+1} = y \mid \tilde{\xi}_t = x\}$, $x, y \in \widetilde{\boldsymbol{S}}$, denote the transition probabilities of $\tilde{\xi}_t$. Then for any $B \subset A$, we have that B is a sojourn set for ξ_t if and only if it is a sojourn set for $\tilde{\xi}_t$. In particular, B being a sojourn set for one chain, implies that B_η is almost closed for both chains, $0 < \eta < 1$.

Proof. Denote all quantifiers for the chain $\tilde{\xi}_t$ by $\tilde{\cdot}$. For $B \subset A$, denote $\tilde{B}^\eta = \{x \in B \mid \tilde{s}_B(x) > \eta\}$ and similarly $\tilde{B}_\eta = \{x \mid \tilde{\boldsymbol{\sigma}}_B(x) > \eta\}$, for $0 < \eta < 1$.

Suppose, that $B \subset S$ is sojourn for ξ_t. By assumption $\boldsymbol{\sigma}_B = \tilde{\boldsymbol{\sigma}}_B$. Since $\tilde{s}_B \geq \tilde{\boldsymbol{\sigma}}_B \neq 0$, B is sojourn for the chain $\tilde{\xi}_t$. The analogous argument applies when assuming that B is sojourn for $\tilde{\xi}_t$. This proves the first statement. The second statement follows from the above that B_η is almost closed if B is sojourn. $\square$

For proving atomicity by means of the previous lemma, we will restrict to the class of so-called face-homogeneous random walks.

Let $\boldsymbol{S} = \prod_{i=1}^p \boldsymbol{S}_i$, with $\boldsymbol{S}_i \in \{\mathbf{Z}, \mathbf{Z}^+\}$. Define the following vector: $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_p\}$ with values $\boldsymbol{\lambda}_i \in \{0, +\}$ when $S_i = \mathbf{Z}^+$ and $\boldsymbol{\lambda}_i \in \{0, +, -\}$, whenever $\boldsymbol{S}_i = \mathbf{Z}$. Associated with $\boldsymbol{\lambda}$ is the set $\boldsymbol{\Lambda}(\boldsymbol{\lambda}) \subset \boldsymbol{S}$ given by

$$\boldsymbol{\Lambda}(\boldsymbol{\lambda}) = \{x \in \boldsymbol{S} \mid \operatorname{sgn}(x_i) = \boldsymbol{\lambda}_i, i = 1, \ldots, p\}.$$

We call $\boldsymbol{\Lambda}(\boldsymbol{\lambda})$ a *face* of the space. Here we consider only faces $\boldsymbol{\Lambda}$, for which there exists a defining vector $\boldsymbol{\lambda}$. So there is a one-one correspondence between faces and defining vectors $\boldsymbol{\lambda}$. Denote $\boldsymbol{\lambda}(\boldsymbol{\Lambda})$ by the vector defining the face $\boldsymbol{\Lambda}$.

Define the projection operator $\mathsf{proj}^{\boldsymbol{\Lambda}} : \boldsymbol{S} \to \boldsymbol{\Lambda}$ by

$$\left(\mathsf{proj}^{\boldsymbol{\Lambda}}(x)\right)_i = \begin{cases} x_i, & \boldsymbol{\lambda}(\boldsymbol{\Lambda})_i \neq 0 \\ 0 & \boldsymbol{\lambda}(\boldsymbol{\Lambda})_i = 0. \end{cases}$$

For any face $\boldsymbol{\Lambda}$, we define the induced chain $\xi_t^{\boldsymbol{\Lambda}}$ on the state space $\boldsymbol{S}^{\boldsymbol{\Lambda}}$ as follows. If $\boldsymbol{\lambda}(\boldsymbol{\Lambda})_i = 0$, then $\boldsymbol{S}_i^{\boldsymbol{\Lambda}} := \boldsymbol{S}_i$. If $\boldsymbol{\lambda}(\boldsymbol{\Lambda})_i \neq 0$, then $\boldsymbol{S}_i^{\boldsymbol{\Lambda}} := \{0\}$. Fix $x_0 \in \boldsymbol{\Lambda}$ and define $\boldsymbol{S}^{\boldsymbol{\Lambda}} = x_0 + \prod \boldsymbol{S}_i^{\boldsymbol{\Lambda}}$. This space is *orthogonal* to x_0. States from this space are denoted by superscript $\boldsymbol{\Lambda}$, for instance $x^{\boldsymbol{\Lambda}}$ is a state of $\boldsymbol{S}^{\boldsymbol{\Lambda}}$.

The transition probabilities are now obtained from the transition probabilities of ξ_t by orthogonal projection onto $\boldsymbol{S}^{\boldsymbol{\Lambda}}$: for $x_0 + x^{\boldsymbol{\Lambda}} \in \boldsymbol{S}^{\boldsymbol{\Lambda}}$

$$\mathsf{P}\{\xi_{t+1}^{\boldsymbol{\Lambda}} = x_0 + y^{\boldsymbol{\Lambda}} \,|\, \xi_t^{\boldsymbol{\Lambda}} = x_0 + x^{\boldsymbol{\Lambda}}\} = \sum_{\substack{y \in \boldsymbol{S}: \\ y-(x_0+y^{\boldsymbol{\Lambda}}) \in \boldsymbol{\Lambda}}} p_{x_0+x^{\boldsymbol{\Lambda}},y}.$$

It is also convenient to be an ordering of faces: $\boldsymbol{\Lambda}' \succeq \boldsymbol{\Lambda}$, if $\boldsymbol{\lambda}(\boldsymbol{\Lambda}')_i = \boldsymbol{\lambda}(\boldsymbol{\Lambda})_i$, whenever $\boldsymbol{\lambda}(\boldsymbol{\Lambda})_i \neq 0$.

At this point, the evolution of $\xi_t^{\boldsymbol{\Lambda}} - x_0$ may depend on the choice of basis point $x_0 \in \boldsymbol{\Lambda}$. However, we will restrict our analysis to random walks that are *face-homogeneous* w.r.t the above defined faces. That is, we assume that

$$p_{xy} = p_{z\,z-x+y}, \quad \text{for all } z, x \in \boldsymbol{\Lambda}, y \in \boldsymbol{S}.$$

Hence, we can represent the transition probabilities on each $\boldsymbol{\Lambda}$ by a probability distribution $p^{\boldsymbol{\Lambda}}$ on $\mathbf{Z}^p$. That is, $p_{x\,x+y} = p_y^{\boldsymbol{\Lambda}}$ for all $x \in \boldsymbol{\Lambda}$ and $y \in \mathbf{Z}^p$, such that $x+y \in \boldsymbol{S}$. As a consequence, the behaviour of the process $\xi_t^{\boldsymbol{\Lambda}} - x_0$ does not depend on the choice of x_0 anymore.

Given a face-homogeneous random walk, we will construct a transformed Markov chain on an extended state space, for which the state space consists of one almost closed class, and such that Lemma 3.6 may be applied. This transformation will leave the jump behaviour on pre-determined face intact.

Let face $\boldsymbol{\Lambda}$ be given. We may assume that $\boldsymbol{\lambda}(\boldsymbol{\Lambda})_i \neq 0$ for $i = 1, \ldots, s$ and $\boldsymbol{\lambda}(\boldsymbol{\Lambda})_i = 0$, $i = s+1, \ldots, p$ (the case $s = p$ is allowed). The $\boldsymbol{\Lambda}$-transformation $\tilde{\xi}_t$ of ξ_t is a Markov chain on the state space $\widetilde{\boldsymbol{S}} = \mathbf{Z}^s \times \prod_{i=s+1}^p \boldsymbol{S}_i$. All quantifiers of this transformed Markov chain will denoted by $\tilde{\cdot}$.

Now, for any $\tilde{x} \in \widetilde{\boldsymbol{S}}$, there exists a unique face $\boldsymbol{\Lambda}' \subset \boldsymbol{S}$ with $\boldsymbol{\Lambda}' \succeq \boldsymbol{\Lambda}$, such that $\operatorname{sgn}(\tilde{x}_i) = \operatorname{sgn}(\boldsymbol{\lambda}(\boldsymbol{\Lambda}')_i)$ for $i > s$. Put

$$\tilde{p}_{\tilde{x},\tilde{x}+y} = p_y^{\boldsymbol{\Lambda}'}, \text{ for all } y.$$

By construction, the transition probabilities of ξ_t and $\tilde{\xi}_t$ coincide on the subset $\bigcup_{\boldsymbol{\Lambda}' \succeq \boldsymbol{\Lambda}} \boldsymbol{\Lambda}'$. Moreover, the jump probabilities from any two points $\tilde{x}$ and $\tilde{y}$ with $\operatorname{sgn}(\tilde{x}_i) = \operatorname{sgn}(\tilde{y}_i)$ for $i > s$, are equal. Hence, the homogeneity faces $\widetilde{\boldsymbol{\Lambda}}$ are characterised by $(p-s)$-dimensional vectors $\tilde{\boldsymbol{\lambda}}$, with $\tilde{\boldsymbol{\lambda}}_i \in \{0, +\}$ or $\in \{0, +, -\}$ depending on whether $\boldsymbol{S}_{i+s} = \mathbf{Z}_+$ or $\mathbf{Z}$. That is, given $\tilde{\boldsymbol{\lambda}}$, the associated homogeneity face is defined by

$$\tilde{\boldsymbol{\Lambda}}(\tilde{\boldsymbol{\lambda}}) = \{\tilde{x} \in \widetilde{S} \mid \operatorname{sgn}(\tilde{x}_{i+s}) = \tilde{\boldsymbol{\lambda}}_i, \quad i = 1, \ldots, p-s\}.$$

One may view the constructed chain as a maximally homogeneous extension of the Markov chain ξ_t restricted to $\boldsymbol{\Lambda}$.

Lemma 3.7. *Suppose that the induced chain $\xi_t^{\boldsymbol{\Lambda}}$ is recurrent. Assume that the $\boldsymbol{\Lambda}$-transformation $\tilde{\xi}_t$ is an irreducible and aperiodic Markov chain on $\widetilde{\boldsymbol{S}}$. Then, upto transient sets, the state space $\widetilde{\boldsymbol{S}}$ forms a single almost closed class for $\tilde{\xi}_t$, which is therefore atomic.*

Proof. Since sojourn solutions are bounded harmonic functions, for our purpose it is sufficient to show the existence of only one bounded harmonic function (up to a multiplicative factor). This follows from Lemma 3.5.

We wish to reduce the proof to the similar statement on homogeneous random walks on $\mathbf{Z}^s$. To this end, define the embedded Markov chain on $\widetilde{\boldsymbol{S}}^{\mathsf{e}} = \mathbf{Z}^s \times \{0\} \times \cdots \times \{0\}$:

$$\tilde{p}_{xy}^e = \mathsf{P}\{\tilde{\xi}_\tau = y \mid \tilde{\xi}_0 = x\}, \quad x, y \in \widetilde{\boldsymbol{S}}^{\mathsf{e}},$$

with $\tau = \inf\{t \geq 1 \mid \tilde{\xi}_n \notin \widetilde{\boldsymbol{S}}^{\mathsf{e}}, 1 \leq n < t, \tilde{\xi}_t \in \widetilde{\boldsymbol{S}}^{\mathsf{e}}\}$ the first entrance time of $\widetilde{\boldsymbol{S}}^{\mathsf{e}}$. Recurrence of the induced chain $\xi_t^{\boldsymbol{\Lambda}}$ implies that $\tau < \infty$ with probability 1, that is, the embedded chain is a non-defective Markov chain. The embedded chain is a homogeneous random walk on $\widetilde{\boldsymbol{S}}^{\mathsf{e}}$. Since $\tilde{\xi}_t$ is aperiodic and irreducible, the embedded chain must be aperiodic and irreducible as well. Hence the conditions of Theorem 24.1 of Spitzer's book [15] are satisfied. It follows that there is only one bounded harmonic function, f^e say, for the embedded chain, and it is constant, say $f^e \equiv 1$.

Going back to the chain $\tilde{\xi}_t$ and using that τ is finite with probability 1, we get by iteration that any bounded harmonic function, f say, for $\tilde{\xi}_t$ satisfies

$$f(x) = \sum_y \tilde{p}_{xy}^{(\tau)} f(y), \tag{3.9}$$

and so f is harmonic for the embedded chain. A more formal argument leading to the above statement is by noting that $f(\tilde{\xi}_t)$ is a uniformly bounded martingale. Equation (3.9) follows as a result of the martingale optional stopping theorem.

The function f being harmonic for the embedded chain, implies it must be equal to 1 on $\widetilde{\boldsymbol{S}}^{\mathsf{e}}$ (up to a factor). Equation (3.9) implies that $f \equiv 1$ on $\tilde{\boldsymbol{S}}$. As a consequence, the chain $\tilde{\xi}_t$ has a simple and atomic state space. □

Clearly sojourn sets need not be equal to homogeneity faces. So we need some condition as to make the above derivation work for a more general sojourn set.

Lemma 3.8. *Suppose that ξ_t is a face-homogeneous random walk on $\boldsymbol{S} = \prod_{i=1}^p \boldsymbol{S}_i$, with the above specified state space and homogeneity faces. Suppose that $\xi_t^{\boldsymbol{\Lambda}}$ is recurrent, for some face $\boldsymbol{\Lambda}$. Assume the existence of a sojourn set $A \subset \boldsymbol{S}$, with $\mathsf{proj}^{\boldsymbol{\Lambda}}(A) \subset \boldsymbol{\Lambda}$. Assume that the $\boldsymbol{\Lambda}$-transformation $\tilde{\xi}_t$ of ξ_t is an aperiodic and irreducible Markov chain.*

Then, up to transient sets, the representative set $A^\eta \subset A$ is the unique almost closed set contained in A, and so it is atomic.

Proof. By Theorem 3.1, $A^\eta = \{x \in A \,|\, s_A(x) > \eta\}$ is almost closed for $0 < \eta < 1$. We have to show that, up to transient sets, A does not contain any other almost closed set. But this follows immediately by taking the $\boldsymbol{\Lambda}$-transformation of ξ_t, and by applying Lemmas 3.5, 3.6 and 3.7. □

So far, we have considered the problem of suitable methods for determining the almost closed set structure. Now we would like to turn to the problem of determining Euler limit paths and convergence to these of the time-scaled process.

Martingales

The other main tool to be used is a.s. convergence to 0 of a time scaled sequence of random vectors, the components of which form a martingale sequence. This will be applied as follows. Let $\xi_0 = x$ be given, then

$$U(x;t) = x + \sum_{n=1}^{t} \mathsf{E}_x\{\xi_n - \xi_{n-1} \,|\, \xi_{n-1}\}, \quad t = 0, 1, \ldots,$$

which is a random discrete time path in $\mathbf{R}^p$ when ξ_t is a vector of dimension p. The sequence $\xi_t(x) - U(x;t)$ is a sequence with martingale components. A.s. convergence of the time scaled process to 0, together with a.s. convergence of $U(x;[tN])/N$, $N \to \infty$, implies that $\xi_{[tN]}(x)/N$ a.s. converges to the same limit, where argument x denotes the initial position. Scaling by time is not always the right scaling, so we quote Theorem 2.18 from [6] in general form.

Theorem 3.9. *Let $(\Omega', \mathcal{F}', \mathsf{P}')$ be a probability space and let $\{\mathcal{F}'_n\}_{n=1,\ldots}$ be an increasing sequence of sub-σ-fields of $\mathcal{F}'$. Let $\{M_n, \mathcal{F}'_n, n \geq 1\}$ be a martingale and $\{T_n\}_{n=1,\ldots}$ a non-decreasing sequence of positive random variables such that T_n is $\mathcal{F}'_{n-1}-$measurable for each n. Then $M_n/T_n \to 0$, $n \to \infty$, a.s. on the set where $\big\{ \lim_{n\to\infty} T_n = \infty, \sum_{n=1}^{\infty} \mathsf{E}\{(M_n - M_{n-1})^2 \,|\, \mathcal{F}'_{n-1}\}/T_n^2 < \infty \big\}$.*

An immediate consequence for face homogeneous random walks is summarised below.

Corollary 3.10. *Let ξ_t be a face-homogeneous random walk with bounded jumps, i.e. there is a constant d such that $\|\xi_{t+1} - \xi_t\| \leq d$, a.s. Then $\lim_{N\to\infty} \big(\xi_{[tN]}(x) - U(x;[tN])\big)/N = 0$, a.s.*

Here $\|\cdot\|$ is the ℓ^2-norm. Note, that the corollary is a statement on vector processes, whereas Theorem 3.9 is a statement on one-dimensional processes. Since the space where the vector processes live, has finite dimension, generalising Theorem 3.9 is a straightforward business.

The final step is now to study the limits (provided they exist) of the random variables $U(x;[tN])/N$ along paths of the process $\xi_t(x)$. We again will restrict to face-homogeneous random walks as described earlier.

We assume that the induced chain $\xi_t^{\mathbf{\Lambda}}$ is ergodic. Say it has the stationary measure $\pi^{\mathbf{\Lambda}}$. Let $x_0 \in \mathbf{\Lambda}$. Write $m^{\mathbf{\Lambda}} = \sum_y y p_y^{\mathbf{\Lambda}}$ for the expected jumps (drift) from points in $\mathbf{\Lambda}$ and

$$v^{\mathbf{\Lambda}} = \sum_{\mathbf{\Lambda}' \succeq \mathbf{\Lambda}} \sum_{x_0 + y^{\mathbf{\Lambda}} \in \mathbf{\Lambda}'} \pi^{\mathbf{\Lambda}}(x_0 + y^{\mathbf{\Lambda}}) m^{\mathbf{\Lambda}'}.$$

This is called the *second vector field on* $\mathbf{\Lambda}$ ([4]). By plugging in the definition of the drifts $m^{\mathbf{\Lambda}'}$, it follows that $v_i^{\mathbf{\Lambda}} = 0$, when $\boldsymbol{\lambda}_i = 0$. For convenience, denote $\Omega_x = \{\omega \in \Omega \,|\, \omega_0 = x\}$.

Lemma 3.11. *Assume the conditions of Lemma 3.8, that is suppose that ξ_t is a face-homogeneous random walk on $\boldsymbol{S} = \prod_{i=1}^p \boldsymbol{S}_i$, with the state space and homogeneity faces specified in the previous paragraph. Suppose that $\xi_t^{\mathbf{\Lambda}}$*

is ergodic, for some face $\mathbf{\Lambda}$*. Assume the existence of a sojourn set* $A \subset \boldsymbol{S}$*, with* $\mathsf{proj}^{\mathbf{\Lambda}}(A) \subset \mathbf{\Lambda}$*. Then*

$$\lim_{N\to\infty} \frac{U(x;[tN])(\omega)}{N} = v^{\mathbf{\Lambda}} \cdot t, \qquad (3.10)$$
$$\textit{for } \mathsf{P}_x\text{-almost all } \omega \in \underline{\mathcal{L}}(B) \cap \Omega_x, \quad \textit{for } t \in \mathbf{R}_+,\ x \in \boldsymbol{S},$$

with $B \subset A$ *almost closed.*

Proof. We will use the $\mathbf{\Lambda}$-transformation $\tilde{\xi}_t$ of ξ_t, defined in the above.

Let reward $r(\tilde{x}) = \mathsf{E}\{\tilde{\xi}_n - \tilde{\xi}_{n-1} \,|\, \xi_{n-1} = \tilde{x}\}$ be paid, whenever the process is in state $\tilde{x} \in \tilde{\boldsymbol{S}}$. Then the random vectors denoting the accumulated reward earned between the n-th and $(n+1)$-th visits to $\widetilde{\boldsymbol{S}}^{\mathsf{e}}$, $n = 0, \dots$ are i.i.d. vectors. Similarly, the random variables denoting the time elapsed time between n-th and $(n+1)$-th visit to $\widetilde{\boldsymbol{S}}^{\mathsf{e}}$, $n = 0, \dots$, are i.i.d. random variables.

Also, to $x \in \mathbf{\Lambda}'$ we assign a reward $r(x) = m^{\mathbf{\Lambda}'}$, for all faces $\mathbf{\Lambda}'$ of $\tilde{\boldsymbol{S}}$. Now the $\mathbf{\Lambda}$-transformation has been defined in such a way that the total accumulated reward between two successive visits to $\widetilde{\boldsymbol{S}}^{\mathsf{e}}$ equals the total accumulated reward in the induced Markov chain $\xi_t^{\mathbf{\Lambda}}$ between two successive visits to the fixed reference point $x_0 \in \mathbf{\Lambda}$, orthogonally to which the induced chain lives. The expectation of the latter, and hence of the former, equals $v^{\mathbf{\Lambda}}/\pi^{\mathbf{\Lambda}}_{x_0}$. Similarly, the expected return time to x_0 equals $1/\pi^{\mathbf{\Lambda}}_{x_0}$, which is hence equal to the expected time elapsed between two successive visits of $\widetilde{\boldsymbol{S}}^{\mathsf{e}}$.

By a delayed version of the so-called Renewal Reward theorem (cf. [12]), one has for the chain $\tilde{\xi}_t$ that

$$\frac{\tilde{U}(x;[tN])}{N} = \frac{x + \sum_{n=1}^{[tN]} \mathsf{E}_x\{\tilde{\xi}_n - \tilde{\xi}_{n-1} \,|\, \tilde{\xi}_{n-1}\}}{N} \to v^{\mathbf{\Lambda}} \cdot t, \quad N \to \infty,\ \tilde{\mathsf{P}}_x\text{-a.s.}$$

Note that, the Renewal Reward theorem has been formulated in [12] for one-dimensional processes. Extension to finite-dimensional processes is straightforward. For any subset $\widetilde{\Omega}'_x$ of the path space $\widetilde{\Omega}_x$ of the transformed chain, it follows that the symmetric difference of the sets

$$\Big\{\tilde{\omega} \,\Big|\, \lim_{N\to\infty} \frac{\tilde{U}(x;[tN])(\tilde{\omega})}{N} = v^{\mathbf{\Lambda}} \cdot t\Big\} \cap \widetilde{\Omega}'_x \quad \text{and} \quad \widetilde{\Omega}'_x$$

is a $\tilde{\mathsf{P}}_x$-null-set for the chain $\tilde{\xi}_t$. Let $x \in A$ and put $\widetilde{\Omega}'_x = A^\infty \cap \Omega_x$. The paths of ξ_t and $\tilde{\xi}_t$ restricted to $\widetilde{\Omega}'_x$ have equal probabilities. Hence the symmetric difference of the sets

$$\Omega_x(A) = \Big\{\omega \,\Big|\, \lim_{N\to\infty} \frac{U(x;[tN])(\omega)}{N} = v^{\mathbf{\Lambda}}\cdot t\Big\} \cap A^\infty \cap \Omega_x \quad \text{and} \quad A^\infty \cap \Omega_x$$

is a P_x-null-set for the chain ξ_t. Therefore,

$$\mathsf{P}_x\{(A^\infty \cap \Omega_x) \setminus \Omega_x(A)\} = 0. \tag{3.11}$$

Let $x \in \boldsymbol{S}$ be arbitrary and let $B \subset A$ be almost closed. By Lemma 3.8 the sets B and A^η differ a transient set, and so by Lemma 3.5, we have $s_B = s_{A^\eta}$. Hence, $s_B = s_{A^\eta} = s_A$. As a consequence, the symmetric difference of sets $\underline{\mathcal{L}}(A)\cap\Omega_x$ and $\underline{\mathcal{L}}(B)\cap\Omega_x$ is a P_x-null-set, since their probabilities are equal, and the second set is contained in the first. So, it suffices to prove the assertion of the lemma for the set $\underline{\mathcal{L}}(A)\cap\Omega_x$. For $\omega \in \underline{\mathcal{L}}(A)$ there exists a finite time n_ω such that $\omega_n \in A$, for $n \geq n_\omega$ and either $\omega_{n_\omega - 1} \notin A$ or $n_\omega = 1$. Write $\omega_A = (\omega_{n_\omega}, \omega_{n_\omega+1}, \ldots\}$. We will show that the set

$$\Omega_{A,x} = \Big\{\omega \in \underline{\mathcal{L}}(A)\cap\Omega_x \mid \frac{U(\omega_{n_\omega};[tN])(\omega_A)}{N} \not\to v^{\mathbf{\Lambda}}\cdot t, \quad N\to\infty\Big\}$$

is a P_x-null-set. Indeed,

$$\begin{aligned}
&\mathsf{P}_x\{\Omega_{A,x}\} \\
&\quad\leq \sum_{n=1}^{\infty}\sum_{y\in A} \mathsf{P}_x\Big\{\omega\in\Omega_x \,:\, n_\omega = n, \omega_n = y, \\
&\qquad\qquad \frac{U(y;[tN])(\omega_k,\omega_{k+1},\ldots)}{N} \not\to v^{\mathbf{\Lambda}}\cdot t,\, N\to\infty\Big\} \\
&\quad\leq \sum_{n=1}^{\infty}\sum_{y\in A} \mathsf{P}_x\{\xi_n = y\}\mathsf{P}_y\{(A^\infty\cap\Omega_y)\setminus\Omega_y(A)\}.
\end{aligned}$$

The last probability equals 0 by the (3.11), and so $\Omega_{A,x}$ is a P_x-null-set. For almost all $\omega \in \underline{\mathcal{L}}(A)\cap\Omega_x$ and N large enough so that $[tN] > n_\omega$

$$\begin{aligned}
\frac{U(x;[tN])(\omega)}{N} &= \frac{x + \sum_{n=0}^{n_\omega-1}\mathsf{E}\{\xi_{n+1}-\xi_n \mid \xi_n = \omega_n\}}{N} - \frac{\omega_{n_\omega}}{N} \\
&\quad + \frac{U(\omega_{n_\omega};[tN]-n_\omega)(\omega_A) - U(\omega_{n_\omega};[tN])(\omega_A)}{N} + \frac{U(\omega_{n_\omega};[tN])(\omega_A)}{N}.
\end{aligned}$$

Now letting N tend to infinity, for P_x-almost all $\omega \in \underline{\mathcal{L}}(A)\cap\Omega_x$ the first three terms converge to 0 and the latter converges to $v^{\mathbf{\Lambda}}\cdot t$. $\square$

We finally piece together the results.

Corollary 3.12. *Under the conditions of Lemma 3.8, we have for any initial state x that*

$$\lim_{N\to\infty} \frac{\xi_{[tN]}(\omega)}{N} = v^{\mathbf{\Lambda}} \cdot t, \quad \text{for } \mathsf{P}_x\text{-}\textit{almost all } \omega \in \underline{\mathcal{L}}(B) \cap \Omega_x,$$

for $B \subset A$ almost closed. In other words,

$$\mathsf{P}_x\{\lim_{N\to\infty} \frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}} \cdot t \,|\, \underline{\mathcal{L}}(A) \cap \Omega_x\} = 1.$$

4 Random walk on the integers

Consider an irreducible, face-homogeneous random walk ξ_t on the integers $\mathbf{Z}$ with three homogeneity faces:

$$\mathbf{\Lambda}^+ = \{1, 2, \ldots\}, \quad \mathbf{\Lambda}^0 = \{0\}, \quad \mathbf{\Lambda}^- = \{-1, -2, \ldots\}.$$

This means that the transition probabilities take three different forms:

$$p_{xy} = \begin{cases} p^+_{y-x}, & x > 0 \\ p^0_y, & x = 0 \\ p^-_{y-x}, & x < 0. \end{cases}$$

The corresponding means jumps will be denoted by m^+, m^0 and m^- respectively. We assume that none equals 0. We also assume that the jumps are bounded.

The induced chain on the face $\mathbf{\Lambda}^+$ has a one-point state space, and consequently is always ergodic. An immediate consequence is that $v^{\mathbf{\Lambda}^+} = m^+$. On the other hand, for $x > 0$ sufficiently large and t comparatively small, the random walk inside $\mathbf{\Lambda}^+$ behaves like a homogeneous one. Corollary 3.12 can be used to show for the space-time scaled process that $\xi_{[tN]}([xN])/N \to x + tm^+$, $N \to \infty$, almost surely.

The same observations apply to the face $\mathbf{\Lambda}^-$. Therefore we associate with it the following dynamical system on $\mathbf{R}$. Let $u : \mathbf{R} \setminus \{0\} \times \mathbf{R} \to \mathbf{R}$ be a continuous mapping with

$$\frac{d}{dt} u(x;t) = \begin{cases} m^+, & u(x;t) > 0 \\ m^-, & u(x;t) < 0 \end{cases}, \quad t \neq 0,$$

for initial condition $u(x;0) = x$. For $u(x;t) = 0$, it is defined by continuity, and so the mean drift at point 0 does not play a role.

Clearly this defines $u(x;t)$ uniquely, whenever m^+ and m^- are both positive or both negative. Suppose that m^+, $m^- > 0$. Then, similar to the derivation in [11], one can prove by virtue of a generalised Kolmogorov inequality or Azuma Hoeffding inequality (see [16]), convergence in probability to

$$\begin{aligned} u(x;t) &= \lim_{N\to\infty} \frac{\xi_{[tN]}([xN])}{N} \\ &= \begin{cases} x + tm^+, & x \geq 0 \\ x + tm^-, & x < 0, t \leq t_0 = |x/m^-| \\ x + t_0 m^- + (t - t_0)m^+, & x < 0, t > t_0, \end{cases} \end{aligned}$$

which we will call the Euler limit. As has been pointed out in the above, by means of Corollary 3.12, almost sure convergence in the first and second cases seems immediate. Showing almost sure convergence in the third one seems to require some extra work.

Theorem 4.1. *Assume that m^+, $m^- > 0$. Then the set $\mathbf{\Lambda}^+$ is almost closed, and the set $\mathbf{\Lambda}^-$ is transient. Conjecture 1.1 holds and the process is simple and atomic.*

Proof. We wish to apply Lemmas 3.4 and 3.8, as well as Corollary 3.12. The proof is then reduced to constructing a suitable Lyapunov function and sojourn sets. Let $f(y) = t$, when $u(0;t) = y$, where t may also be negative. More explicitly, $f(y) = y/m^+$, for $y > 0$ and $f(y) = y/m^-$ for $y \leq 0$.

Let $B = \{0\}$, $C = 0$, and $k \equiv 1$. Then the set A from Lemma 3.4 is precisely Λ^+. The conditions of this lemma hold, if (3.3) holds for some $\epsilon > 0$.

For $x > 0$ we have

$$\mathsf{E}\{f(\xi_{t+1}) - f(\xi_t) \,|\, \xi_t = x\} = \frac{1}{m^+}\mathsf{E}\{\xi_{t+1} - \xi_t \,|\, \xi_t = x\} = \frac{m^+}{m^+} = 1.$$

For $x < 0$, this expectation equals 1 as well. We can take $\epsilon = 1$, and so the conditions of Lemma 3.4 hold. Application of this lemma, yields almost closed-ness of Λ^+ and transience of $\Lambda^- \cup \{0\}$, hence of Λ^-.

By Example 3.1, $A = \Lambda^+$ is representative. Furthermore, $\mathsf{proj}^{\mathbf{\Lambda}^+}(A) \subset \mathbf{\Lambda}^+$. Ergodicity of the induced chain is trivial. Hence, the conditions of Lemma 3.8 hold for set $A = \mathbf{\Lambda}^+$ and face $\mathbf{\Lambda}^+$. As a consequence, $\mathbf{\Lambda}^+$ is atomic. This implies that the process is simple and atomic, and so $s_{\mathbf{\Lambda}^+} \equiv 1$. Moreover, by virtue of Corollary 3.12, for any initial state x

$$\mathsf{P}_x\Big\{\lim_{N\to\infty} \frac{\xi_{[tN]}}{N} \to v^{\mathbf{\Lambda}^+} t = u(0;t) \,|\, \underline{\mathcal{L}}(\mathbf{\Lambda}^+) \cap \Omega_x\Big\} = 1.$$

This implies that

$$
\begin{aligned}
& \mathsf{P}_x\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = u(0;t)\} \\
= \quad & \mathsf{P}_x\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = u(0;t) \,|\, \underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cap\Omega_x\}\mathsf{P}_x\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cap\Omega_x\} + \\
& \quad + \mathsf{P}_x\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = u(0;t) \,|\, \Omega_x \setminus \underline{\mathcal{L}}(\mathbf{\Lambda}^+)\}\mathsf{P}_x\{\Omega_x \setminus \underline{\mathcal{L}}(\mathbf{\Lambda}^+)\} \\
= \quad & \mathsf{P}_x\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cap\Omega_x\} = s_{\mathbf{\Lambda}^+}(x) = 1,
\end{aligned}
$$

since $\Omega_x \setminus \underline{\mathcal{L}}(\mathbf{\Lambda}^+)$ is a null-set. □

The more interesting case is, when $m^- < 0 < m^+$. Then we have the simpler form

$$
u(x;t) = \lim_{N\to\infty}\frac{\xi_{[tN]}([xN]}{N} = \begin{cases} x + tm^+, & x > 0 \\ x + tm^-, & x < 0, \end{cases} \quad , \quad \text{almost surely.}
$$

From point 0, two Euler paths start, one into either direction. This suggests the occurrence of two atomic almost closed sets: $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$. This is indeed the case.

Theorem 4.2. *Assume that $m^- < 0 < m^+$. The almost closed set decomposition is given by $\mathbf{Z} = \mathbf{\Lambda}^+ \cup \mathbf{\Lambda}^-$, with $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$ both atomic. Moreover, Conjecture 1.1 holds.*

Proof. Almost closed-ness of $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$ follows in the same way as almost closed-ness of $\mathbf{\Lambda}^+$ in the proof of the previous lemma, by application of Lemma 3.2. Use Example 3.1, to apply Lemma 3.8 in order to show that these are the only (modulo transient sets) almost closed sets, as well as atomicity, in the same manner as the proof of the previous theorem. Again note that the induced chains for $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$ are trivially ergodic.

In this case, $u(0;t)$ has two realisations: $v^{\mathbf{\Lambda}^+}t$ and $v^{\mathbf{\Lambda}^-}t$. Since Corollary 3.12 applies, (1.1) follows as in the proof of the previous theorem by a conditioning argument. For completeness we give it here for the path

$u(0;t) = v^{\mathbf{\Lambda}^+}t$:

$$\begin{aligned}
&\mathsf{P}_x\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}^+}t\} \\
&= \mathsf{P}_x\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}^+}t \,|\, \underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cap\Omega_x\}\mathsf{P}_x\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cap\Omega_x\} + \\
&\quad +\mathsf{P}_x\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}^+}t \,|\, \underline{\mathcal{L}}(\mathbf{\Lambda}^-)\cap\Omega_x\}\mathsf{P}_x\{\underline{\mathcal{L}}(\mathbf{\Lambda}^-)\cap\Omega_x\} + \\
&\quad +\mathsf{P}_x\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}^+}t \,|\, \Omega_x \setminus \big(\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cup\underline{\mathcal{L}}(\mathbf{\Lambda}^-)\big)\} \times \\
&\qquad \times\mathsf{P}_x\{\Omega_x\setminus\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cup\underline{\mathcal{L}}(\mathbf{\Lambda}^-)\} \\
&= \mathsf{P}_x\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cap\Omega_x\} = s_{\mathbf{\Lambda}^+}(x) = \mathsf{P}_x\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\},
\end{aligned}$$

since $\mathsf{P}_x\{\lim_{N\to\infty}\xi_{[tN]}/N = v^{\mathbf{\Lambda}^+}t \,|\, \underline{\mathcal{L}}(\mathbf{\Lambda}^-)\cap\Omega_x\} = 0$ and $\mathsf{P}_x\{\Omega_x\setminus\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\cup\underline{\mathcal{L}}(\mathbf{\Lambda}^-)\} = 0$. This completes the proof of Conjecture 1.1. $\square$

5 Coupled Processors system: a random walk on the quarter plane

We will illustrate the validity of Conjecture 1.1 for two special face-homogeneous random walks on the quarter plane. A characterisation of the almost closed set structure for face-homogeneous random walks on $\mathbf{Z}_+^2$ can be found in [10].

As in the previous section, the first version has only one atomic closed class and the second has two.

Coupled processors system with switched off processors whenever a queue is empty

Consider a system of two processors indexed by 1 and 2. Let $\lambda_1 > 0$ and $\lambda_2 > 0$ denote their input rates. Whenever both queues are non-empty, processor i works at speed μ_i, $i = 1,2$. The moment queue 1 empties, processor 2 is switched off, and vice versa. We consider the time-discretised version obtained by uniformisation, so that we may assume

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \le 1.$$

This model is a face-homogeneous random walk on $\mathbf{Z}_+^2$ with four homogeneity faces

$$\begin{aligned}
\mathbf{\Lambda}_0 &= \{0\}, & \mathbf{\Lambda}_3 &= \{x\in\mathbf{Z}^2 \,|\, x_1, x_2 > 0\}, \\
\mathbf{\Lambda}_1 &= \{x\in\mathbf{Z}^2 \,|\, x_1 > 0, x_2 = 0\}, & \mathbf{\Lambda}_2 &= \{x\in\mathbf{Z}^2 \,|\, x_1 = 0, x_2 > 0\}.
\end{aligned}$$

The jump probabilities from points in $\mathbf{\Lambda}_l$, $l = 0, 1, 2$ are given by

$$p_x^{\mathbf{\Lambda}_l} = \begin{cases} \lambda_1, & x_1 = 1, x_2 = 0, \\ \lambda_2, & x_1 = 0, x_2 = 1, \\ 1 - \lambda_1 - \lambda_2, & x_1 = x_2 = 0. \end{cases}$$

On face $\mathbf{\Lambda}_3$ they are given by

$$p_x^{\mathbf{\Lambda}_3} = \begin{cases} \lambda_1, & x_1 = 1, x_2 = 0, \\ \lambda_2, & x_1 = 0, x_2 = 1, \\ \mu_1, & x_1 = -1, x_2 = 0, \\ \mu_2, & x_1 = 0, x_2 = -1, \\ 1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2, & x_1 = x_2 = 0. \end{cases}$$

In the two-dimensional model, the drift vector $m^{\mathbf{\Lambda}}$ has two components $m_1^{\mathbf{\Lambda}}$ and $m_2^{\mathbf{\Lambda}}$. The same applies to the field $v^{\mathbf{\Lambda}}$ for an ergodic induced chain $\xi_t^{\mathbf{\Lambda}}$. So far, we have not made any assumptions on the parameters. Let us assume first that $\lambda_i < \mu_i$, $i = 1, 2$. Then the induced chains $\xi_t^{\mathbf{\Lambda}_1}$ and $\xi_t^{\mathbf{\Lambda}_2}$ are easily checked to be ergodic. They have a one dimensional state space. For instance, for $\mathbf{\Lambda}_1$ one can take $(1, 0) + (0, \mathbf{Z}_+)$ and so we can identify it with $\mathbf{Z}^+$. It has jump probabilities

$$\mathsf{P}\{\xi_{t+1}^{\mathbf{\Lambda}_1} = j \,|\, \xi_t^{\mathbf{\Lambda}_1} = i\} = \begin{cases} \lambda_2, & j = i + 1 \\ \mu_2 \mathbf{1}_{\{i>0\}}, & j = i - 1 \\ 1 - \lambda_2 - \mu_2 \mathbf{1}_{\{i>0\}}, & j = i, \end{cases}$$

and so ergodicity follows, since $\lambda_2 < \mu_2$. Moreover, $\pi_0^{\mathbf{\Lambda}_1} = 1 - \lambda_2/\mu_2$. Hence,

$$\begin{aligned} v^{\mathbf{\Lambda}_1} &= \big(1 - \frac{\lambda_2}{\mu_2}\big) m^{\mathbf{\Lambda}_1} + \frac{\lambda_2}{\mu_2} m^{\mathbf{\Lambda}_3} \\ &= \big(1 - \frac{\lambda_2}{\mu_2}\big) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \frac{\lambda_2}{\mu_2} \begin{pmatrix} \lambda_1 - \mu_1 \\ \lambda_2 - \mu_2 \end{pmatrix} \\ &= \frac{1}{\mu_2} \begin{pmatrix} \lambda_1 \mu_2 - \mu_1 \lambda_2 \\ 0 \end{pmatrix}. \end{aligned}$$

Similarly $\xi_t^{\mathbf{\Lambda}_2}$ is an ergodic induced chain with

$$v^{\mathbf{\Lambda}_2} = \frac{1}{\mu_1} \begin{pmatrix} 0 \\ \lambda_2 \mu_1 - \mu_2 \lambda_1 \end{pmatrix} = \frac{\mu_2}{\mu_1} \begin{pmatrix} 0 \\ -v_1^{\mathbf{\Lambda}_1} \end{pmatrix}.$$

Suppose that $v_1^{\mathbf{\Lambda}_1} > 0$, in other words, $\lambda_1/\mu_1 > \lambda_2/\mu_2$. Then $v_2^{\mathbf{\Lambda}_2} < 0$. Clearly, $\xi^{\mathbf{\Lambda}_3}$ is ergodic since it is a chain living on a one-point set. Hence,

$v^{\mathbf{\Lambda}_3} = m^{\mathbf{\Lambda}_3}$. The assertion in Corollary 3.12 suggests defining a continuous dynamical system $u(x;\cdot)$ on $\mathbf{R}^2$ satisfying

$$\frac{d^+}{dt}u(x;t) = v^{\mathbf{\Lambda}_l}, \quad u(x;t) \in \mathbf{\Lambda}_l, l \neq 0, \tag{5.1}$$

for initial condition $u(x;0) = x$, where d^+/dt denotes the right derivative. It is uniquely defined under the above conditions. Define $A_1 = \{x \in \mathbf{Z}_+^2 \,|\, x_2 < (\mu_2/\mu_1)x_1\}$. Note that it contains $\mathbf{\Lambda}_1$!

Theorem 5.1. *Assume $\lambda_i < \mu_i$, $i = 1, 2$, and $\lambda_1\mu_2 > \lambda_2\mu_1$. Then the set A_1 is almost closed and atomic, and $\mathbf{Z}_+^2 \setminus A_1$ is transient. The process is simple and atomic. Conjecture 1.1 holds.*

Proof. Almost closed-ness of A_1 and transience of $\mathbf{Z}_+^2 \setminus A_1$ will follow from Lemma 3.4, by checking the conditions of this lemma.

Let $B = \{0\}$, $C = 0$, $k \equiv 1$ and $f(x) = \mu_2x_1 - \mu_1x_2$. Then $S_1 = \{x \in \mathbf{Z}_+^2 \,|\, f(x) > C\}$. We only need to check that (3.3) holds for all $x \neq 0$, and some $\epsilon > 0$. For any $x \neq 0$

$$\mathsf{E}\{f(\xi_{t+1}) - f(\xi_t) \,|\, \xi_t = x\} = \lambda_1\mu_2 - \lambda_2\mu_1 > 0,$$

so that (3.3) holds for $x \neq 0$ and $\epsilon = \lambda_1\mu_2 - \lambda_2\mu_1$.

Next, by Example 3.1, A_1 is representative. Also $\mathsf{proj}^{\mathbf{\Lambda}_1}(A_1) = \mathbf{\Lambda}_1$. The conditions of Lemma 3.8 are satisfied for set $A = A_1$ and face $\mathbf{\Lambda}_1$. It follows that A_1 is atomic, and so the process is simple and atomic.

The validity of Conjecture 1.1 finally follows from Corollary 3.12, by a similar decomposition as in the proof of Theorem 4.1. □

Coupled processors system with switched off processors whenever a queue is empty, and with additional input

The previous model has a nice 'simple' structure. In a system with arrival control, it seems not unnatural to allowing more customers or jobs to enter a queue, whenever it is empty. Indeed, this might reduce idle server time. Upon allowing this, 'non-simple' structures may appear.

Keeping the rates for 2 non-empty queues equal to the previous model, we allow arrival rates λ_i^l, $i = 1, 2$, on faces $\mathbf{\Lambda}_l$, $l = 0, 1, 2$. In this case, the jump probabilities from points in $\mathbf{\Lambda}_l$, $l = 0, 1, 2$ are given by

$$p_x^{\mathbf{\Lambda}_l} = \begin{cases} \lambda_1^l, & x_1 = 1, x_2 = 0, \\ \lambda_2^l, & x_1 = 0, x_2 = 1, \\ 1 - \lambda_1^l - \lambda_2^l, & x_1 = x_2 = 0. \end{cases}$$

Again assuming $\lambda_i < \mu_i$, the induced chains $\xi_t^{\mathbf{\Lambda}_l}$, $l = 1, 2$, are ergodic. Identifying their state space with $\mathbf{Z}_+$, the induced chain $\xi_t^{\mathbf{\Lambda}_1}$, has jump probabilities

$$\mathsf{P}\{\xi_{t+1}^{\mathbf{\Lambda}_1} = j \,|\, \xi_t^{\mathbf{\Lambda}_1} = i\} = \begin{cases} \lambda_2^1 \mathbf{1}_{\{i=0\}} + \lambda_2 \mathbf{1}_{\{i>0\}}, & j = i+1 \\ \mu_2 \mathbf{1}_{\{i>0\}}, & j = i-1 \\ 1 - \lambda_2^1 \mathbf{1}_{\{i=0\}} + \lambda_2 \mathbf{1}_{\{i>0\}} - \mu_2 \mathbf{1}_{\{i>0\}}, & j = i. \end{cases}$$

Now we have, $\pi_0^{\mathbf{\Lambda}_1} = (\mu_2 - \lambda_2)/(\mu_2 - \lambda_2 + \lambda_2^1)$. This yields

$$\begin{aligned} v^{\mathbf{\Lambda}_1} &= \frac{\mu_2 - \lambda_2}{\mu_2 - \lambda_2 + \lambda_2^1} m^{\mathbf{\Lambda}_1} + \frac{\lambda_2^1}{\mu_2 - \lambda_2 + \lambda_2^1} m^{\mathbf{\Lambda}_3} \\ &= \frac{\mu_2 - \lambda_2}{\mu_2 - \lambda_2 + \lambda_2^1} \begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \end{pmatrix} + \frac{\lambda_2^1}{\mu_2 - \lambda_2 + \lambda_2^1} \begin{pmatrix} \lambda_1 - \mu_1 \\ \lambda_2 - \mu_2 \end{pmatrix} \\ &= \frac{1}{\mu_2 - \lambda_2 + \lambda_2^1} \begin{pmatrix} \lambda_1^1(\mu_2 - \lambda_2) - \lambda_2^1(\mu_1 - \lambda_1) \\ 0 \end{pmatrix}. \end{aligned}$$

For $\mathbf{\Lambda}_2$ we get similarly,

$$v^{\mathbf{\Lambda}_2} = \frac{1}{\mu_1 - \lambda_1 + \lambda_1^2} \begin{pmatrix} 0 \\ -\lambda_1^2(\mu_2 - \lambda_2) + \lambda_2^2(\mu_1 - \lambda_1) \end{pmatrix}.$$

In order that both $v_1^{\mathbf{\Lambda}_1}$ and $v_2^{\mathbf{\Lambda}_2}$ are positive, it is sufficient to require

$$\frac{\lambda_1^2}{\lambda_2^2} < \frac{\mu_1 - \lambda_1}{\mu_2 - \lambda_2} < \frac{\lambda_1^1}{\lambda_2^1}. \tag{5.2}$$

Using (5.1) we can define a continuous dynamical system $u(x;t)$ for initial condition $u(x;0) = x$. However, it is not uniquely defined at the moment $u(x;t) = (0,0)$, for some $t \geq 0$. We have two possible realisations, both occurring with positive probability. Choose a_1 and a_2 satisfying

$$\frac{\lambda_1^2}{\lambda_2^2} < a_2 < \frac{\mu_1 - \lambda_1}{\mu_2 - \lambda_2} < \frac{1}{a_1} < \frac{\lambda_1^1}{\lambda_2^1}.$$

Let $A_1 = \{x \in \mathbf{Z}_+^2 \,|\, a_1 x_1 > x_2\}$ and $A_2 = \{x \in \mathbf{Z}_+^2 \,|\, x_1 < a_2 x_2\}$. Then $A_1 \cap A_2 \neq \emptyset$.

Theorem 5.2. *Assume $\lambda_i < \mu_i$, $i = 1, 2$, and (5.2). Then each set A_i is atomic and almost closed. The state space has the almost closed set decomposition $\mathbf{Z}_+^2 = A_1 \cup A_2$. Conjecture 1.1 holds.*

Proof. The first statement on almost closed-ness of S_1 and S_2 follows from a by now straightforward application of Lemma 3.2 with the functions $f_1(x) = a_1x_1 - x_2$ and $f_2(x) = -x_1 + a_2x_2$ respectively, step-function $k \equiv 1$, and constant $C = 0$.

Atomicity follows from Example 3.1 by applying Lemma 3.8 for sets A_1 and A_2 and faces $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ respectively. Note again that $\mathsf{proj}^{\mathbf{\Lambda}_i}(S_i) = \mathbf{\Lambda}_i$.

Since $A_1 \cup A_2 = \mathbf{Z}_+^2$, they are the unique almost closed sets, modulo transient ones. This proves that the almost closed set decomposition consists of two atomic sets.

The final proof of (1.1) is again by a conditioning argument along the same lines as in the proof of Theorem 4.2. $\square$

Conclusion

This paper has shown Conjecture 1.1 for interesting face-homogeneous random walks on $\mathbf{Z}$ and $\mathbf{Z}_+^2$. In particular, we have provided tools for characterising the almost closed set structure and fluid limits, in case of an almost closed set decomposition of the state space into *atomic* sets. The construction of suitable Lyapunov functions is the yet un-tackled Achilles' heel of completing the characterisation of such face-homogeneous random walks. Partial results exist, using in fact the continuous dynamical systems defined by the second vector field for sub-chains (cf. for instance the basic reference [4]).

References

[1] D. Blackwell (1942), On idempotent Markoff chains. *Ann. Mathematics* **43**, 560–567.

[2] D. Blackwell (1955), On transient Markov processes with a countable number of states and stationary transition probabilities. *Ann. Math. Stat.* **26**, 654–658.

[3] K.L. Chung (1960), *Markov Chains with Stationary Transition Probabilities.* Springer-Verlag, Berlin.

[4] G. Fayolle, V.A. Malyshev, and M.V. Menshikov (1995), *Constructive Theory of countable Markov Chains.* Cambridge University Press, Cambridge.

[5] W. Feller (1956), Boundaries induced by non-negative matrices. *Trans. Americ. Math. Soc.* **83**, 19–54.

[6] P. Hall and C.C. Heyde (1980), *Martingale limit Theory and its Application.* Academic Press, San Diego.

[7] A. Hordijk and N.V. Popov (2001), Large deviation bounds for face-homogeneous random walks in the quarter plane. Technical report MI 2001-32, University of Leiden.

[8] A. Hordijk and N.V. Popov (2002), Large deviation analysis for a coupled processors system. Technical report MI 2002-10, University of Leiden.

[9] V.A. Kaimanovich (1992), Measure-theoretic boundaries of Markov chains, 0–2 laws and entropy. In: *Harmonic Analysis and Discrete Potential Theory, Frascati, 1991*, 145–180. Plenum, New York.

[10] I.A. Kurkova (1999), The Poisson boundary for homogeneous random walks. *Russian Math. Surveys* **54**, 441–442.

[11] N. Popov and F.M. Spieksma (2002), Non-existence of a stochastic fluid limit for a cycling random walk. Technical report, Mathematical Institute, University of Leiden.

[12] S.M. Ross (1970), *Applied Probability Models with Optimization Applications.* Holden Day, San Francisco.

[13] F.M. Spieksma, Continuous scattering and non-atomicity of a face-homogeneous random walk. *In Preparation.*

[14] F.M. Spieksma, Lyapunov functions for Markov chains with applications to face-homogeneous random walks. *Internal communication.*

[15] F.L. Spitzer (1976), *Principles of Random Walk.* Springer Verlag, Berlin, 2d edition.

[16] D. Williams (1991), *Probability with Martingales.* Cambridge University Press, Cambridge.