Fronts on a lattice

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1 Introduction

In this paper we study the propagation of fronts in the Cauchy Problem for a sequence of functions $\{u_n(t)\}_{-\infty}^{\infty}$ defined on a one-dimensional infinitely extended lattice:

$$\begin{cases} u_n' = -u_n + u_{n-1}^2, & n \in \mathbb{Z},\ t > 0 \qquad (1.1a) \\ u_n(0) = a_n & n \in \mathbb{Z}, \qquad (1.1b) \end{cases}$$

where $a_n \in [0,1]$ for every $n \in \mathbb{Z}$.

Problem (1.1) arises in the context of a clock model for a dilute gas of N particles with short range interactions, in which every particle carries a clock with a discrete time $k \in \mathbb{Z}$ which is advanced at every collision. This happens according to the following rule: when two particles collide, they *both* reset their respective clock values, say k and ℓ, to either $k+1$ or $\ell+1$, whichever is the largest. Thus, if we denote the number of particles with clock value k by N_k, we obtain the following dynamical equation:

$$\frac{dN_k}{dt} = -\sum_{\substack{\ell=-\infty \\ \ell \neq k}}^{\infty} R_{k,\ell} - 2R_{k,k} + 2\sum_{\ell=-\infty}^{k-1} R_{k-1,\ell},$$

where $R_{k,\ell}$ denotes the rate by which collisions occur between particles with clock values k and ℓ. We assume this rate to be proportional to $N_k N_\ell / N^2$ when $k \neq \ell$ and to $N_k^2/(2N^2)$ when two particles with equal clock value k collide. Then, writing $f_k = N_k/N$ and scaling the time appropriately, we are led to the equation

$$\frac{df_k}{dt} = -f_k + f_{k-1}^2 + 2f_{k-1}C_{k-2} = -f_k + C_{k-1}^2 - C_{k-2}^2,$$

where we have set

$$C_k = \sum_{\ell=-\infty}^{k} f_\ell.$$

Adding the equations for f_ℓ for all values of $\ell \leqslant k$ then yields

$$\frac{dC_k}{dt} = -C_k + C_{k-1}^2. \tag{1.2}$$

For further details of this model we refer to [8].

We can write equation (1.1) as

$$u_n' = -u_n + u_{n-1} - f(u_{n-1}), \tag{1.3}$$

where

$$f(s) = s(1-s). \tag{1.4}$$

Viewing the first two terms on the right of (1.3) as a "diffusion" operator, we can draw the analogy between (1.3) and the classical *Fisher equation* (see e.g. [1] and [7])

$$w_t = w_{xx} - w(1-w). \tag{1.5}$$

It is well known that this equation has a monotone travelling front of the form

$$u(x,t) = \varphi(\xi), \qquad \xi = x - ct \qquad c \in \mathbb{R},$$

between the two constant solutions $w = 0$ and $w = 1$ such that

$$\varphi(-\infty) = 0 \qquad \text{and} \qquad \varphi(+\infty) = 1, \tag{1.6}$$

for every wave speed $c \geqslant 2$. This warrants the conjecture that equation (1.1) also has increasing fronts for a half-infinite interval of wave speeds c. Such fronts would be of the form

$$u_n(t) = \varphi(\xi), \qquad \xi = n - ct. \tag{1.7}$$

This is also observed numerically (see [4]).

In our first result we assume the existence of such a monotone front and establish a lower bound for the wave speed c. Specifically we prove that

$$c > \frac{1}{\log 2}. \tag{1.8}$$

We then study the existence of travelling waves for an analogous equation in which the function f in (1.4) has been replaced by a piece-wise linear function:

$$f(s) = \begin{cases} s & \text{for} \quad s \leqslant \frac{1}{2}, \\ 1-s & \text{for} \quad s \geqslant \frac{1}{2}. \end{cases} \tag{1.9}$$

For the original variable u_n this yields the equation

$$u_n' = -u_n + g(u_{n-1}), \tag{1.10}$$

where

$$g(s) = \begin{cases} 0 & \text{for} \quad s \leqslant \frac{1}{2}, \\ 2s - 1 & \text{for} \quad s \geqslant \frac{1}{2}. \end{cases} \tag{1.11}$$

Before stating the main existence theorem, we introduce a *critical wave speed* c_0: it is the unique positive root of the equation

$$c\, e^{1/c} = 2\, e \qquad (c_0 = 4.31107\ldots). \tag{1.12}$$

We establish the following result:

Theorem 1.1 *For each $c \geqslant c_0$ equation* (1.3), *with f given by* (1.9) *has a unique monotone travelling wave $u_n(t) = \varphi(\xi)$ of the form given in* (1.7), *such that $\varphi(-\infty) = 0$ and $\varphi(\infty) = 1$.*

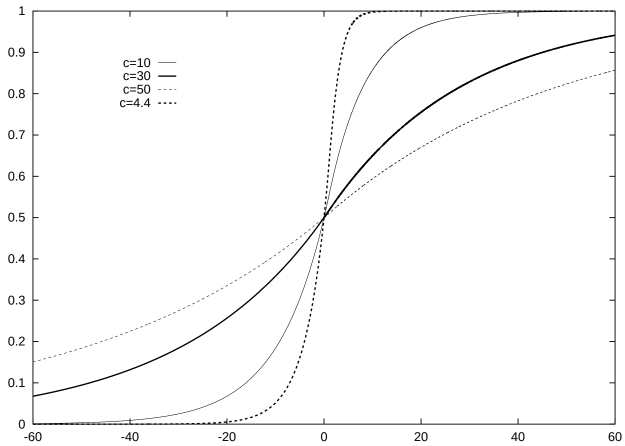


Figure 1: Monotone fronts for some values of $c > c_0$

Plainly,

$$u_n'(t) = -c\, \varphi'(\xi) \quad \text{and} \quad u_{n-1}(t) = \varphi(\xi - 1),$$

so that the function $\varphi(\xi)$ needs to satisfy the following delay-differential equation:

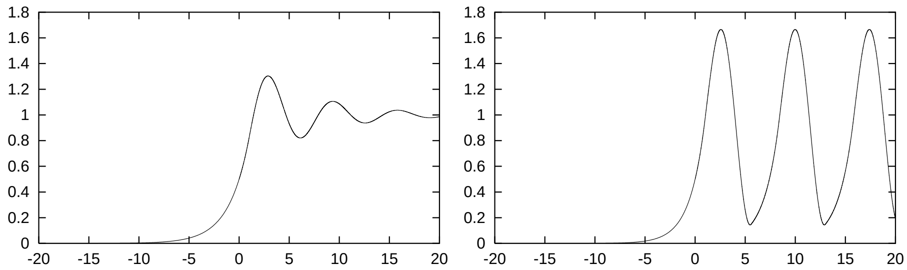
$$c\varphi'(\xi) = \varphi(\xi) - g\big(\varphi(\xi - 1)\big), \qquad \xi \in \mathbb{R}. \tag{1.13}$$

Let us briefly describe the proof of this theorem. Since the function f given by (1.9) is piecewise linear and the monotone travelling wave φ increases from 0 to 1, there is no loss of generality when we can fix $\varphi(0) = \frac{1}{2}$. Then, for $\xi < 1$,

φ verifies a linear ODE that we solve explicitly and for $\xi > 1$ it verifies a linear DDE that we will analyse in detail via Laplace Transform. Theorem 1.1 is a consequence of the analysis of $\varphi'(\xi)$.

The plan of the paper is the following: in Section 2 we first prove the lower bound (1.8), and in Section 3 we give a detailed outline of the existence proof. The roots of the characteristic equation of the linearised equation at $u = 1$ will play a central role in the analysis; this equation will be discussed in Section 4. In Section 5 we discuss the corresponding fundamental solution. In Section 6 we prove a monotonicity property when $c \geqslant c_0$, and in Section 7 we present an explicit expression for the front for $c \geqslant c_0$. In Figure 1 we show the graph of this front for a few values of $c > c_0$. In Section 8, via analytical arguments and numerical experiments, we discuss the behaviour of *all* the fronts with $0 < c < c_0$ that solve (1.3) such that $\varphi(-\infty) = 0$. There we show the existence of different types of orbits: nonmonotone fronts such that $\varphi(\infty) = 1$, as in figure 2(a), bounded orbits where $\varphi(\infty)$ is not defined, *eventually periodic orbits*, as in figure 2(b) and unbounded orbits.

Equations such as (1.10) also arises in the context of Cellular Neural Networks. For details we refer to [6] and the references cited there.



(a) Damped oscillation for $c = 2$

(b) Eventually periodic orbit for $c = 1.5$

Figure 2: Graphs of orbits for different values of c

2 A lower bound for the wave speed c

In this section we consider increasing travelling wave solutions of the equation

$$u_n' = -u_n + u_{n-1}^2,$$

and establish a lower bound for the wave speed. As explained in the Introduction we therefore need to study the problem

$$\begin{cases} cu'(x) = u(x) - u^2(x-1) \qquad \text{for} \quad x \in \mathbb{R}, & \text{(2.1a)} \\ u(-\infty) = 0, \quad u(0) = \dfrac{1}{2}, \quad u(+\infty) = 1, & \text{(2.1b)} \end{cases}$$

where we have eliminated the translation invariance by pinning the solution at the origin.

We first prove a few bounds and estimates for increasing solutions of Problem (2.1).

Theorem 2.1 *Let u be an increasing solution of Problem* (2.1). *Then*

1.
$$u(x) \leqslant \frac{e^{x/c}}{1+e^{x/c}} \qquad \textit{for} \quad x < 0,$$
and hence $u(x) = O(e^{x/c})$ as $x \to -\infty$.

2.
$$1 - u(x) \leqslant \frac{1}{1+e^{x/c}} \qquad \textit{for} \quad x > 0,$$
and hence $1 - u(x) = O(e^{-x/c})$ as $x \to +\infty$.

3.
$$\lim_{x\to-\infty} u(x)\, e^{-x/c} \stackrel{\text{def}}{=} \ell \in (\tfrac{1}{2}, 1).$$

Proof. The main ingredients of the proof are the monotonicity and the bounds $0 < u < 1$. From the differential equation we obtain

$$cu'(x) = u(x) - u^2(x-1) \geqslant u(x) - u^2(x).$$

Let $y > x$. Then, integrating this inequality over the interval (x, y) yields

$$\frac{u(y)}{1-u(y)}\, e^{-y/c} \geqslant \frac{u(x)}{1-u(x)}\, e^{-x/c}. \tag{2.2}$$

For $x < 0$ and $y = 0$ we obtain Part 1 and for $x = 0$ and $y > 0$ we obtain Part 2. To prove Part 3, we transform to the variable $v(x) = u(x)e^{-x/c}$. Plainly,

$$v'(x) = \{u'(x) - \frac{1}{c}u(x)\}\, e^{-x/c} = -\frac{1}{c}\, u^2(x-1)\, e^{-x/c} < 0.$$

Hence, v is a decreasing function. By the first inequality v is bounded above by 1. Therefore

$$\lim_{x\to-\infty} v(x) = \ell_1 > v(0) = \frac{1}{2}.$$

Next, we define the function

$$w(x) = \frac{u(x)}{1-u(x)}\, e^{-x/c}.$$

We have shown in (2.2) that w is a nondecreasing function. Therefore

$$\lim_{x\to-\infty} w(x) = \ell_2 < w(0) = 1,$$

and

$$\lim_{x\to-\infty} v(x) = \lim_{x\to-\infty} w(x) < 1,$$

as asserted in Part 3.

We are now ready to prove the lower bound for the wave speed.

Theorem 2.2 *Let u be an increasing solution of Problem* (2.1)*. Then*

$$c > \frac{1}{\log 2}.$$

Proof. We return to the function $v(x) = u(x)e^{-x/c}$. Because $u < 1$, we have the upper bound

$$v(x) < e^{-x/c} \qquad \text{for} \qquad x \in \mathbb{R}. \tag{2.3}$$

An elementary computation shows that v is a solution of the integral equation

$$v(x) = \frac{1}{2} - \int_0^x \frac{1}{c}\, e^{(t-2)/c}\, v^2(t-1)dt, \tag{2.4}$$

and that it is bounded above by

$$v_+(x) = \begin{cases} 1 & x \leqslant -c\log 2 \\ \frac{1}{2}\, e^{-x/c} & -c\log 2 < x < 0 \\ \frac{1}{2} & 0 \leqslant x \leqslant c\log 2 \\ e^{-x/c} & x > c\log 2. \end{cases}$$

When we substitute this upper bound for v into the integral equation (2.4), we obtain the following lower bound on the half line $x > 1 + c\log 2$:

$$v(x) > e^{-x/c} + \frac{1}{2} - \frac{3}{2}e^{-1/c} + e^{-2/c}. \tag{2.5}$$

From (2.3) and (2.5) we conclude that

$$\frac{1}{2} - \frac{3}{2}e^{-1/c} + e^{-2/c} < 0,$$

which implies that $c > \frac{1}{\log 2}$. This completes the proof of the theorem.

3 Outline

In order to prove Theorem 1.1 we need to find a *strictly increasing* solution of the two-point boundary value problem

$$\begin{cases} u'(x) = \frac{1}{c}\{u(x) - g\big(u(x-1)\big)\}, \qquad x \in \mathbb{R}, & (3.1a) \\ u(-\infty) = 0, & (3.1b) \\ u(+\infty) = 1. & (3.1c) \end{cases}$$

Since u is strictly increasing from 0 to 1, there exists a unique value of x, where its graph crosses the line $u = \frac{1}{2}$. Without loss of generality we may choose this value to be the origin. Then $u(x-1) < \frac{1}{2}$ for $x < 1$, and we find that $u(x)$ must satisfy

$$\begin{cases} u'(x) = \frac{1}{c}u(x) \qquad \text{for} \quad -\infty < x < 1, & (3.2a) \\ u(-\infty) = 0 \quad \text{and} \quad u(0) = \dfrac{1}{2}. & (3.2b) \end{cases}$$

Therefore

$$u(x) = \frac{1}{2}\,e^{x/c} \qquad \text{for} \quad -\infty < x < 1. \tag{3.3}$$

Since it is required that $u(x) \to 0$ as $x \to -\infty$, it is evident that c needs to be *positive.*

For $x > 1$ we have $u(x-1) > \frac{1}{2}$ and we must solve the Problem

$$\begin{cases} u'(x) = \frac{1}{c}\{u(x) + 1 - 2u(x-1)\} \qquad \text{for} \quad 1 < x < \infty, & (3.4a) \\ u(x) = \frac{1}{2}\,e^{x/c}, \qquad \text{for} \quad 0 \leqslant x \leqslant 1, & (3.4b) \\ u(\infty) = 1. & (3.4c) \end{cases}$$

It will be convenient to transform this problem, and introduce the variables

$$t = x - 1 \qquad \text{and} \qquad y(t) = 1 - u(x). \tag{3.5}$$

Problem (3.4) then becomes

$$\begin{cases} y'(t) = \frac{1}{c}\{y(t) - 2y(t-1)\} \qquad \text{for} \quad 0 < t < \infty, & (3.6a) \\ y(t) = \eta(t) \qquad \text{for} \quad -1 \leqslant t \leqslant 0, & (3.6b) \\ y(\infty) = 0, & (3.6c) \end{cases}$$

where

$$\eta(t) \overset{\text{def}}{=} 1 - u(1+t) = 1 - \frac{1}{2}e^{(1+t)/c}. \tag{3.7}$$

We now need to identify values of $c > 0$ for which Problem (3.6) has a solution with the property

$$y(t) < \frac{1}{2} \qquad \text{for} \quad 0 < t < \infty. \tag{3.8}$$

Then $u(x-1) > \frac{1}{2}$ for $x > 1$, as assumed at the outset.

Problem (3.6) is a linear problem, which can be solved by means of the Laplace Transform. Thus, we write formally

$$\hat{y}(s) = \mathcal{L}(y)(s) \stackrel{\text{def}}{=} \int_0^\infty y(t)\, e^{-st}\, dt.$$

Transformation of equation (3.6a) then yields

$$\begin{aligned}
\{s\hat{y}(s) - y(0)\} &= \frac{1}{c}\hat{y}(s) - \frac{2}{c}\int_0^\infty y(t-1)\, e^{-st}\, dt. \\
&= \frac{1}{c}\hat{y}(s) - \frac{2}{c}\int_0^1 y(t-1)\, e^{-st}\, dt - \frac{2}{c}\int_1^\infty y(t-1)\, e^{-st}\, dt \\
&= \frac{1}{c}\hat{y}(s) - \frac{2}{c}\int_0^\infty \psi(t)\, e^{-st}\, dt - \frac{2}{c}e^{-s}\hat{y}(s),
\end{aligned}$$

where

$$\psi(t) = \begin{cases} \eta(t-1) = 1 - e^{t/c}, & 0 < t < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Thus,

$$\left(\frac{2}{c}e^{-s} + s - \frac{1}{c}\right)\hat{y}(s) = y(0) - \frac{2}{c}\hat{\psi}(s). \tag{3.10}$$

Since

$$y(0) = \eta(0) = 1 - \frac{1}{2}e^{1/c}, \tag{3.11}$$

this yields the expression

$$\hat{y}(s) = \left(1 - \frac{1}{2}e^{1/c}\right)\hat{\Gamma}(s) - \frac{2}{c}\hat{\psi}(s)\hat{\Gamma}(s), \tag{3.12}$$

where

$$\hat{\Gamma}(s) = \frac{1}{\frac{2}{c}e^{-s} + s - \frac{1}{c}}. \tag{3.13}$$

Assuming for the moment that we can carry out the inverse Laplace Transform, we formally obtain for $y(t)$:

$$y(t) = \left(1 - \frac{1}{2}e^{1/c}\right)\Gamma(t) - \frac{2}{c}(\Gamma * \psi)(t), \tag{3.14}$$

where $*$ denotes the convolution product

$$(\Gamma * \psi)(t) = \int_0^t \psi(s)\ \Gamma(t-s)\, ds.$$

If it can be shown that $y(t)$ is a decreasing function, then substitution into (3.5) yields an increasing solution $u(x)$ of Problem (3.1).

4 The characteristic equation

The Laplace Transform $\hat{\Gamma}(s)$ of the fundamental solution $\Gamma(t)$ has poles at the zeros of the *characteristic equation*:

$$\frac{2}{c}\, e^{-\lambda} + \lambda - \frac{1}{c} = 0. \tag{4.1}$$

In this section we investigate these zeros. Note that if λ is a root of equation (4.1) then so is its complex conjugate $\overline{\lambda}$. Hence, we need only consider zeros with nonnegative imaginary part.

The complex equation (4.1) for $\lambda = a + ib$ is equivalent to the following system

$$\begin{cases} \frac{2}{c}\, e^{-a} \cos b = \frac{1}{c} - a, & \text{(4.2a)} \\ \frac{2}{c}\, e^{-a} \sin b = b. & \text{(4.2b)} \end{cases}$$

We note that (4.2b) is trivially satisfied for $b = 0$. For a we then obtain from (4.2a) the equation

$$\frac{2}{c} e^{-a} + a - \frac{1}{c} = 0. \tag{4.3}$$

An elementary analysis leads to the following theorem about the real roots of equation (4.1):

Theorem 4.1 *There exists a unique constant $c_0 > 0$ verifying*

$$c_0\, e^{1/c_0} = 2e \qquad (c_0 = 4.31107...)$$

such that

(a) For $c < c_0$, equation (4.3) *has no real solution.*

(b) For $c = c_0$, equation (4.3) *has exactly one negative solution of multiplicity two:*

$$a_0 = -1 + \frac{1}{c_0} = -\log\Big(\frac{c_0}{2}\Big) \qquad (a_0 = -0.768....). \tag{4.4}$$

(c) For $c > c_0$, equation (4.3) *has exactly two negative solutions $a_1(c)$ and $a_2(c)$ which verify $a_2(c) < a_0 < a_1(c) < 0$, and*

$$a_1(c) = -\frac{1}{c-2} + O\Big(\frac{1}{c^3}\Big) \qquad \textit{as} \quad c \to \infty.$$

In the next theorem we discuss the location of the complex roots of the characteristic equation. We make two preliminary observations:

(a) From (4.2b) we deduce the inequality

$$e^a = \frac{2}{c}\,\frac{\sin b}{b} < \frac{2}{c}.$$

Hence, for $c > 2$ all the complex solutions have negative real part. Actually, this lower bound can be improved.

Definition. The critical wave speed c_{bif} will be defined as the biggest value for which the characteristic equation has purely imaginary roots.

We see from (4.2) that

$$c_{\text{bif}} = \frac{3\sqrt{3}}{\pi} \approx 1.654. \tag{4.5}$$

In this section, as well as in Sections 5, 6 and 7, we will always consider $c > c_{\text{bif}}$. In Section 8 we shall explore the range $c \in (0, c_{\text{bif}})$.

(b) Since by assumption, $c > 0$ and $a < 0$, it follows from (4.2a) that $\cos(b) > 0$ and from (4.2b) that $\sin(b) > 0$. Hence

$$b \in I_k = \Big(2(k-2)\pi, 2(k-2)\pi + \frac{\pi}{2}\Big) \qquad \text{for some} \quad k \in \{2, 3, 4...\}.$$

Elementary analysis shows that for $k = 2$, there exists a nonreal root iff $c < c_0$.

Theorem 4.2 *For each $k \geqslant 3$ there exists a unique nonreal complex root $\lambda_k = (a_k, b_k)$ of equation (4.1) with positive imaginary part such that if $c \geqslant c_0$, then $a_k(c) < a_2(c) < 0$, where $a_2(c)$ is the smallest real solution of (4.1) found in Theorem 4.1, and*

$$-\log\Big(\frac{c}{2}\Big) - \log\Big(2(k-2)\pi + \frac{\pi}{2}\Big) < \; a_k \; < -\log\Big(\frac{c}{2}\Big) - \log\big(2(k-2)\pi\big), \tag{4.6}$$

$$2(k-2)\pi < \; b_k \; < 2(k-2)\pi + \frac{\pi}{2} \tag{4.7}$$

for $k = 3, 4, 5, \ldots$ In particular $a_k = O(-\log k)$ as $k \to \infty$.

5 The expansion of the fundamental solution

In the last section we have investigated the poles of $\hat{\Gamma}(s)$ that we will need, in order to invert the Laplace operator. The inversion formula is given by

$$\Gamma(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-iR}^{iR} e^{st}\, \hat{\Gamma}(s) ds \tag{5.1}$$

if $e^{st}\hat{\Gamma}(s)$ is integrable. In Lemma 5.1 it is shown that we may apply the residues formula, and express the integral in (5.1) as a series of residues computed at the poles of $\hat{\Gamma}(s)$, i.e. at the zeros $(a_k, \pm b_k)$ of the characteristic function

$$h(s) = \frac{2}{c}e^{-s} + s - \frac{1}{c}.$$

For convenience, let us introduce the following notation: for each $c > 0$, we will denote the set of *all* the roots of the characteristic function $h(s)$ by $\{z_k = z_k(c) : k = 1, 2, \ldots\}$. In last section we have found that they correspond to

$$\begin{aligned} \{a_1, a_2, a_k \pm i b_k : k = 3, 4, \ldots\} \qquad & \text{for} \quad c > c_0, \\ \{a_k \pm i b_k : k = 2, 3, \ldots\} \qquad & \text{for} \quad c < c_0. \end{aligned}$$

Lemma 5.1 *Let $\{z_k\}$ be the sequence of all the roots of the function h. We have*

$$\Gamma(t) = \sum_{k=1}^{\infty} \operatorname{Res}(h^{-1}(s), s = z_k)\, e^{z_k t}. \tag{5.2}$$

Proof. We consider the sequence of semicircles S_n of radius R_n:

$$S_n = \Big\{ R_n e^{i\theta} : \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2} \Big\}.$$

Here $R_n \to \infty$ as $n \to \infty$. In order to prove (5.2) we need to check that the integral taken along S_n tends to zero when $n \to \infty$.

From (4.7) we know that $b_k \in \big(2(k-3)\pi, 2(k-3)\pi + \frac{\pi}{2}\big)$, $k = 3, 4, 5 \ldots.$ By choosing the radii R_k appropriately, we can ensure that on the semicircles S_k the denominator of the integrand $h(s)$ is bounded away from zero. Specifically, we choose $R_k = (2k+1)\pi$. Then

$$|R_k - \lambda_k| \geqslant \big| R_k - |\lambda_k| \big| = \big| (2k+1)\pi - |b_k| \sqrt{1 + \theta_k}\, \big|,$$

where

$$\theta_k = \frac{a_k^2}{b_k^2} \sim \Big(\frac{\log k}{2\pi k} \Big)^2 \qquad \text{as} \quad k \to \infty.$$

Therefore

$$\liminf_{k \to \infty} \; |R_k - \lambda_k| \geqslant \frac{\pi}{2}.$$

With this choice of radii R_n, and with n large enough, each successive semidisk contains one more zero and on S_n the denominator is bounded away from zero as $n \to \infty$.

The next step is to estimate the integrand on S_n and to show that it tends to zero as $n \to \infty$ sufficiently fast. To that end we divide S_n into two parts:

$$\begin{aligned} S_n^1 &= \{ s \in S_n : \operatorname{Re}(s) \leqslant -\mu \log |s| \}, \\ S_n^2 &= \{ s \in S_n : \operatorname{Re}(s) > -\mu \log |s| \}, \end{aligned}$$

where μ is a positive constant.

We take $\mu > 1$. Then on S_n^1 we have

$$|se^s| \leqslant e^{(1-\mu)\log|s|} \to 0 \quad \text{as} \quad |s| \to \infty,$$

so that

$$|e^{-s}h^{-1}(s)| = \frac{c}{|2 + cse^s - e^s|} \to \frac{c}{2} \quad \text{as} \quad |s| \to \infty.$$

Therefore,

$$\sup_{s\in S_n^1} |h^{-1}(s)| \sim \frac{c}{2} \sup_{s\in S_n^1} |e^s| = \frac{c}{2}\, e^{-\mu \log R_n} \quad \text{as} \quad R_n \to \infty.$$

Since the length of S_n^1 is less than πR_n, and $|e^{st}| \leqslant 1$ on S_n^1, it follows that

$$\Big|\int_{S_n^1} h^{-1}(s)e^{st}ds\Big| \leqslant O\big(e^{(1-\mu)\log R_n}\big) \quad \text{as} \quad R_n \to \infty.$$

Because $\mu > 1$ the integral over S_n^1 converges to zero uniformly for $t \geqslant 0$.

Next, we show that the integral over S_n^2 converges to zero as well. We write S_n^2 in complex notation:

$$S_n^2 = \Big\{R_n e^{\pm i\theta} : \frac{\pi}{2} < \theta < \arccos\Big(-\mu\frac{\log(R_n)}{R_n}\Big)\Big\}.$$

The Taylor expansion

$$\arccos(x) = \frac{\pi}{2} - x + O(x^3) \quad \text{as} \quad x \to 0$$

implies that, as $R_n \to \infty$, the length of S_n^2 is given asymptotically by

$$|S_n^2| = 2\mu\log(R_n) + O\Big(\frac{\log^3(R_n)}{R_n^2}\Big).$$

Since the exponential factor is bounded in modulus by one, and $h^{-1}(s) = O(R_n^{-1})$ on S_n^2, we conclude that

$$\Big|\int_{S_n^2} h^{-1}(s)e^{st}ds\Big| \leqslant O\Big(\frac{\log R_n}{R_n}\Big) \quad \text{as} \quad R_n \to \infty.$$

Thus, we have shown that

$$\lim_{R_n\to\infty} \int_{S_n} e^{st}\,\hat{\Gamma}(s)ds = 0$$

and we can conclude by the residues formula that

$$\Gamma(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-iR}^{iR} e^{st}\,\hat{\Gamma}(s)ds = \sum_{k=1}^{\infty} \mathrm{Res}(h^{-1}(s), s = z_k)e^{z_k t}.$$

Since we have shown the validity of the residues formula, next we will compute the residues. We note that if $c > c_0$, then all the roots of $h(s)$ are simple and with negative real part and if $c = c_0$, then $h(s)$ has one double root. We will treat both cases separately.

Lemma 5.2 *Suppose that $c > c_0$. Then*

$$\Gamma(t) = \sum_{k=1}^{2} \frac{1}{h'(a_k)}\, e^{a_k t} + \sum_{k=3}^{\infty} \Big(\frac{1}{h'(\lambda_k)}\, e^{\lambda_k t} + cc \Big). \tag{5.3}$$

Since

$$h'(\lambda_k) = \lambda_k + 1 - \frac{1}{c},$$

the expansion (5.3) *can be written as*

$$\Gamma(t) = \sum_{k=1}^{2} \frac{e^{a_k t}}{a_k + 1 - \frac{1}{c}} + 2 \sum_{k=3}^{\infty} \frac{(a_k + 1 - \frac{1}{c})\cos(b_k t) + b_k \sin(b_k t)}{(a_k + 1 - \frac{1}{c})^2 + b_k^2}\, e^{a_k t}. \tag{5.4}$$

Proof. The lemma is an immediate consequence of the residues formula (5.2). When $c > c_0$ the residues of $\hat{\Gamma}(s) = \frac{1}{h(s)}$ are given by $\frac{1}{h'(\lambda_k)}$.

Lemma 5.3 *Suppose that $c = c_0$. Then*

$$\Gamma(t) = 2\Big(t + \frac{1}{3}\Big)\, e^{a_0 t} + \sum_{k=3}^{\infty} \Big(\frac{1}{h'(\lambda_k)}\, e^{\lambda_k t} + \mathrm{cc} \Big), \tag{5.5}$$

and the expansion (5.5) *can be written as*

$$\Gamma(t) = 2\Big(t + \frac{1}{3}\Big)\, e^{a_0 t} + 2 \sum_{k=3}^{\infty} \frac{(a_k + 1 - \frac{1}{c_0})\cos(b_k t) + b_k \sin(b_k t)}{(a_k + 1 - \frac{1}{c_0})^2 + b_k^2}\, e^{a_k t}. \tag{5.6}$$

Proof. When $c = c_0$, the computation made for the zeros λ_k which are not real valued is still valid since these zeros are all simple. However, in this case the zero a_0 (see Theorem 4.1) has multiplicity two and the computation is different.

Since h is analytic we have the Taylor expansion

$$h(s) = \alpha_0 (s - a_0)^2 + \alpha_1 (s - a_0)^3 + \cdots \tag{5.7}$$

which is valid in the disk with circumference $S_r = \{|s - a_0| = r\}$ for some sufficiently small r, and $\alpha_0 \neq 0$. Therefore

$$\frac{1}{h(s)} = \frac{1}{\alpha_0} \frac{1}{(s - a_0)^2} - \frac{\alpha_1}{\alpha_0^2} \frac{1}{s - a_0} + \text{ analytic function}$$

and, writing $w = s - a_0$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{S_r} \frac{e^{st} ds}{h(s)} &= \frac{e^{a_0 t}}{2\pi i} \int_{S_r} \frac{e^{wt} ds}{h(s)} = \frac{e^{a_0 t}}{2\pi i} \frac{1}{\alpha_0} \int_{S_r} e^{wt} \Big(\frac{1}{w^2} - \frac{\alpha_1}{\alpha_0} \frac{1}{w} + \cdots \Big)\, ds \\ &= \frac{e^{a_0 t}}{2\pi i} \frac{1}{\alpha_0} \int_{S_r} \Big(\frac{1}{w^2} + \frac{t}{w} - \frac{\alpha_1}{\alpha_0} \frac{1}{w} + \cdots \Big)\, ds \\ &= \frac{1}{\alpha_0} \Big(t - \frac{\alpha_1}{\alpha_0} \Big) e^{a_0 t}. \end{aligned} \tag{5.8}$$

It remains to compute α_0 and α_1. We find:

$$\alpha_0 = \lim_{s\to a_0} \frac{h(s)}{(s-a_0)^2} = \frac{1}{2}\, h''(a_0) = \frac{1}{2},$$

and

$$\alpha_1 = \lim_{s\to a_0} \frac{h(s) - a_0(s-a_0)^2}{(s-a_0)^3} = \frac{1}{6}\, h'''(a_0) = -\frac{1}{6}.$$

If we use these expressions in (5.8) we obtain the desired expansions (5.5) and (5.6).

6 Monotonicity

The question that remains is to check whether the solution given in (3.14), with $\Gamma(t)$ given by the expansion formula in (5.3) or (5.5), is monotone. This step is necessary because in our construction of the travelling wave solution of the nonlinear problem (3.1) we have assumed that it was monotone and hence obeys (3.4). The answer to this question is affirmative.

As in Section 3, we tranform to the vatiables t and y introduced in (3.5):

$$t = x - 1 \qquad \text{and} \qquad y(t) = 1 - u(x).$$

We shall prove the following monotonicity result.

Theorem 6.1 *For every $c \geqslant c_0$ the solution $y(t)$ of the Problem*

$$\begin{cases} y'(t) = \frac{1}{c}\{y(t) - 2y(t-1)\} & \textit{for} \quad 0 < t < \infty, & (6.1\text{a}) \\ y(t) = \eta(t) = 1 - \frac{1}{2}e^{(1+t)/c} & \textit{for} \quad t \leqslant 0. & (6.1\text{b}) \end{cases}$$

is strictly decreasing.

Proof. The proof is carried out again by means of the Laplace Transform, as in Section 3. We begin with a preliminary result.

Lemma 6.1 *Let $y(t)$ be the solution of Problem* (6.1)*. Then*

$$y'(t) = -\frac{1}{2c}\,\Gamma(t+1) \qquad \textit{for} \quad t > 0. \tag{6.2}$$

Proof. We multiply equation (6.1a) by e^{-st}, integrate over $t \in (0,\infty)$, and express all the integrals in terms of $\int_{-1}^{\infty} e^{-st} y'(t)dt$. After an elementary computation this yields

$$h(s) \int_{-1}^{\infty} e^{-st} y'(t)dt = -\frac{1}{2c}\, e^{s}.$$

Since

$$\int_{-1}^{\infty} e^{-st} y'(t)dt = \int_{-1}^{0} e^{-st}\eta'(t)dt + \mathcal{L}(y')(s),$$

we obtain for the Laplace Transform of y'

$$\mathcal{L}(y')(s) = -\frac{1}{2c}\,\frac{e^s}{h(s)} - \int_{-1}^{0} e^{-st}\eta'(t)dt = -\frac{1}{2c}I_1(s) - I_2(s). \tag{6.3}$$

An elementary computation yields for $I_2(s)$:

$$I_2(s) = \int_{-1}^{0} e^{-st}\eta'(t)dt = -\frac{1}{2c}\,\frac{e^s - e^{(1/c)}}{s-(1/c)}. \tag{6.4}$$

To find an expression for $I_1(s)$, we compute the Laplace Transform of $\Gamma(t+1)$

$$\mathcal{L}(\Gamma(t+1))(s) = \int_0^{\infty} \Gamma(t+1)e^{-st}\,dt = e^s\Big(\int_0^{\infty} - \int_0^1\Big)\Gamma(\tau)e^{-s\tau}\,d\tau.$$

From [5], pages 19 and 20, we know that $\Gamma(t)$ is the solution of equation (6.1a) with initial data on $[-1,0]$ given by

$$y(t) = 0 \quad \text{for} \quad -1 \leqslant t < 0, \quad \text{and} \quad y(0) = 1.$$

Therefore, $\Gamma(t) = e^{t/c}$ for $0 < t < 1$, and hence

$$\mathcal{L}(\Gamma(1+t)(s) = \frac{e^s}{h(s)} - \frac{e^s - e^{1/c}}{s-(1/c)} = \frac{e^s}{h(s)} + 2cI_2(s). \tag{6.5}$$

Thus, putting (6.4) and (6.5) into (6.3) we can conclude

$$\mathcal{L}\big(y'(\cdot)\big)(s) = \mathcal{L}\big(\Gamma(1+\cdot)\big)(s) \tag{6.6}$$

and (6.2) follows.

Thus, it suffices to show that $\Gamma(t)$ is positive for $t > 1$. We will again treat the cases $c > c_0$ and $c = c_0$ separately.

(a) **Positivity of $\mathbf{\Gamma(t)}$ for the case $\mathbf{c > c_0}$**

We recall from (5.3) that $\Gamma(t)$ is given by

$$\Gamma(t) = \frac{e^{a_1 t}}{a_1 + 1 - 1/c} + \frac{e^{a_2 t}}{a_2 + 1 - 1/c} + 2\sum_{k=3}^{\infty} A_k(t)\frac{1}{|h'(\lambda_k)|}\,e^{a_k t},$$

where a_1 and a_2 are such that $a_1 + 1 - 1/c > 0$ and $a_2 + 1 - 1/c < 0$, and $|A_k(t)| \leqslant 1$ for every $k \geqslant 3$. To show that $\Gamma(t)$ is positive we need a bound on the third term $R(t)$. By Theorem 4.2, the exponent $a_k t$ can be estimated by

$$a_k t = -t\log\Big(\frac{c}{2}\Big) + t\log\frac{\sin b_k}{b_k} \leqslant -t\log\Big(\frac{c}{2}\Big) - t\log\big(2(k-2)\pi\big).$$

Hence

$$
\begin{aligned}
|R(t)| &\leqslant 2\sum_{k=3}^{\infty} \frac{1}{2(k-2)\pi}\, e^{a_k t} \leqslant \sum_{k=3}^{\infty} \frac{1}{(k-2)\pi}\, e^{-t\log(c/2)-t\log(2(k-2)\pi)} \\
&= \frac{1}{\pi}\, e^{-t\log(\pi c)} \sum_{k=3}^{\infty}\Big(\frac{1}{k-2}\Big)^{t+1}.
\end{aligned}
$$

We are interested in the range $t \geqslant 1$. Then

$$
|R(t)| \leqslant \frac{1}{\pi}\, e^{-t\log(\pi c)} \sum_{k=3}^{\infty}\Big(\frac{1}{k-2}\Big)^{2} = \frac{\pi}{6}\, e^{-t\log(\pi c)},
$$

so that

$$
\Gamma(t) \geqslant \frac{e^{a_1 t}}{a_1+1-1/c} + \frac{e^{a_2 t}}{a_2+1-1/c} - \frac{\pi}{6}\, e^{-t\log(\pi c)} \stackrel{\text{def}}{=} Q(t,c) \qquad \text{for} \quad t \geqslant 1.
$$

Proposition 6.1 *If $c > c_0$, then $Q(t,c) > 0$ for $t \geqslant 1$.*

Proof. We consider the equivalent problem of showing the positivity of the function $Q(t,c)e^{-a_1 t}$, i.e. of

$$
Q(t,c)e^{-a_1 t} = \frac{1}{a_1+1-1/c} + \frac{e^{(a_2-a_1)t}}{a_2+1-1/c} - \frac{\pi}{6}\, e^{-t(a_1+\log(\pi c))}
$$

for $c > c_0$ and $t \geqslant 1$. This function is seen to be increasing with respect to t. Therefore, its minimum is reached at $t = 1$. Hence, it remains to show that the function

$$
Q(1,c) = \frac{1}{a_1+1-1/c} + \frac{e^{a_2-a_1}}{a_2+1-1/c} - \frac{\pi}{6}\, e^{-(a_1+\log(\pi c))}
$$

is positive for every $c > c_0$. This can easily be checked by the use of a computer program.

(b) **Positivity of $\mathbf{\Gamma(t)}$ for the case $\mathbf{c = c_0}$**

We first consider the range $t \geqslant 1$. There

$$
\Gamma(t) \geqslant 2\Big(t+\frac{1}{3}\Big)e^{a_0 t} - \frac{\pi}{6}\, e^{-t\log(\pi c_0)} \stackrel{\text{def}}{=} Q_0(t).
$$

As in the previous case, the function $e^{-a_0 t}Q_0(t)$ is seen to be increasing for $1 \leqslant t < \infty$, and hence $e^{-a_0 t}Q_0(t) \geqslant e^{-a_0}Q_0(1) > 0$. Thus $\Gamma(t) > 0$, and we conclude that $y(t)$ is a decreasing function for $t \geqslant 0$.

7 The final solution

The solution of Problem (3.1) can be expressed in the form of an expansion.

Theorem 7.1 *If $c > c_0$, the solution $u(x)$ of Problem* (3.1) *can be written as*

$$u(x) = \begin{cases} \dfrac{1}{2}\, e^{x/c} & \text{for} \quad x \leqslant 1, \\ 1 - \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty} \dfrac{1}{(-z_k)(c + cz_k - 1)}\, e^{z_k x} & \text{for} \quad x > 1. \end{cases} \tag{7.1}$$

where $\{z_k : k = 1, 2, 3, \ldots\}$ is the set of roots of the characteristic equation (4.1).

If $c = c_0$ we have

$$u(x) = \begin{cases} \dfrac{1}{2}\, e^{x/c_0} & \text{for} \quad x \leqslant 1, \\ 1 - \dfrac{1}{2}\big(x - b\big)e^{a_0 x} - \dfrac{1}{2}\displaystyle\sum_{k=2}^{\infty} \dfrac{1}{(-z_k)(c_0 + c_0 z_k - 1)}\, e^{z_k x} & \text{for} \quad x > 1, \end{cases} \tag{7.2}$$

where $\{z_k\}$ denotes all the nonreal complex roots of the characteristic function $h(s)$ and

$$b = \frac{3c_0^2 + c_0 - 1}{3(c_0 - 1)^2} \approx 1.796$$

Proof. Recall the change of variables used in (3.5). One way is to express the solution is via the equality (3.14). Another possibility is to use the equality (6.2) obtained in Lemma 6.1. We will use the latter.

Thus, in (6.2) we have transformed the original delay-differential equation (3.6) into an ordinary differential equation

$$\begin{cases} y'(t) = -\dfrac{1}{2c}\,\Gamma(t+1), & \text{for} \quad 0 < t < \infty, \\ y(0) = \eta(0) = 1 - \dfrac{1}{2}\, e^{1/c}. \end{cases} \tag{7.3}$$

The solution of (7.3) is given formally by

$$y(t) = \eta(0) - \frac{1}{2c}\int_0^t \Gamma(s+1)\, ds \qquad \text{for} \quad t \geqslant 0. \tag{7.4}$$

We make use of the expansion for Γ obtained in (5.3) and (5.5) for the cases $c > c_0$ and $c = c_0$ respectively.

The expansion for the case $\mathbf{c > c_0}$.

We recall from (5.3) that $\Gamma(t)$ was given by

$$\Gamma(t) = \sum_{k=1}^{\infty} \frac{1}{h'(z_k)}\, e^{z_k t} = \sum_{k=1}^{\infty} \frac{1}{\frac{1}{c} - 1 - z_k}\, e^{z_k t}.$$

We substitute this expression into (7.4). An elementary computation shows that

$$y(t) = \eta(0) + \sum_{k=1}^{\infty} \alpha_k (e^{z_k t} - 1), \tag{7.5}$$

where the constants α_k are given by

$$\alpha_k = \frac{1}{z_k(1 - cz_k)(1 - c - cz_k)}. \tag{7.6}$$

We can simplify expression (7.5). We know that there exist a limit of (7.5) as t tends to infinity, which can only be a steady state solution of (3.6a), this is, zero. This means

$$\sum_{k=1}^{\infty} \alpha_k = \eta(0) = 1 - \frac{1}{2}\, e^{1/c} \tag{7.7}$$

so that (7.5) becomes

$$y(t) = \sum_{k=1}^{\infty} \alpha_k e^{z_k t}. \tag{7.8}$$

Returning to the original variables x and $u(x)$ we obtain (7.1) as required.

The expansion for the case $\mathbf{c = c_0}$.

We recall from (5.5) that $\Gamma(t)$ is given by

$$\Gamma(t) = 2\Big(t + \frac{1}{3}\Big) e^{a_0 t} + \sum_{k=2}^{\infty} \Big(\frac{1}{h'(z_k)}\, e^{z_k t}\Big).$$

When we put this expression into (7.5), with $c = c_0$, we obtain the solution y of Problem (7.3):

$$y(t) = \eta(0) + \frac{1}{c_0}\Big(t - \frac{7c_0 - 4}{3(c_0 - 1)^2}\Big)\, e^{a_0 t} + \frac{7c_0 - 4}{3c_0(c_0 - 1)^2} + \sum_{k=2}^{\infty} \big\{\alpha_k (e^{z_k t} - 1)\big\}. \tag{7.9}$$

We can simplify this expression (7.9) for $y(t)$. We take the limit as t tends to ∞, which will be zero. This yields the relation

$$\sum_{k=2}^{\infty} \alpha_k = \eta(0) + \frac{7c_0 - 4}{3c_0(c_0 - 1)^2} = 1 - \frac{e}{c_0} + \frac{7c_0 - 4}{3c_0(c_0 - 1)^2}, \tag{7.10}$$

and (7.9) becomes

$$y(t) = \frac{1}{c_0}\Big(t - \frac{7c_0 - 4}{3(c_0 - 1)^2}\Big) e^{a_0 t} + \sum_{k=2}^{\infty} \alpha_k e^{z_k t}. \tag{7.11}$$

Return to the original variables x and $u(x)$ yields (7.1).

8 The range $c < c_0$

Having established the existence of mononotone fronts for $c \geqslant c_0$, we now turn to the regime $c < c_0$. We recall that c_0, given in Theorem 4.1, is the smallest value of c for which the characteristic equation has real roots: two for $c > c_0$ and a double root for $c = c_0$. For $c_{\mathrm{bif}} < c < c_0$, where c_{bif} was defined in (4.5), the roots still have negative real part and the uniform state $u = 1$ is asymptotically stable. Indeed, the value c_{bif} is precisely the value of c for which the set of roots of the characteristic equation reaches the imaginary axis. Thus, for $c_{\mathrm{bif}} < c < c_0$ we may still expect fronts to exist, but with oscillatory tails as $x \to \infty$. An important question for the analysis of these solutions is whether or not $u(x) \geqslant \frac{1}{2}$ for every $x > 1$. To answer this question, we did a series of numerical simulations, using the package xppaut ([3]). In this section we discuss the solutions we find, and their qualitative properties.

In addition to the range $c_{\mathrm{bif}} < c < c_0$, we investigate the properties of solutions for still smaller values of c, and we identify two further critical wave speeds c_{unb} and c_{per}:

$$0 < c_{\mathrm{unb}} < c_{\mathrm{per}} < c_{\mathrm{bif}} < c_0.$$

Let us give an overview of the results that we will describe in this section. The goal is to obtain a qualitative description of the solution $u(x, c)$ of the problem

$$\begin{cases} u'(x) = \frac{1}{c}\, u(x), & \text{if } x > 1 \quad \text{and} \quad u(x-1) \leqslant \frac{1}{2}, & (8.1\text{a}) \\ u'(x) = \frac{1}{c}\, \{u(x) + 1 - 2u(x-1)\} & \text{if } x > 1 \quad \text{and} \quad u(x-1) \geqslant \frac{1}{2}, & (8.1\text{b}) \\ u(x) = \frac{1}{2}\, e^{\frac{x}{c}} & \text{for} \quad x \leqslant 1, & (8.1\text{c}) \end{cases}$$

when c takes on different positive values, and we find the following:

1. If $c \geqslant c_0$, then $u(x, c)$ is strictly increasing and $u(\infty, c) = 1$.
2. If $c_{\mathrm{bif}} < c < c_0$, then $u(x, c)$ is nonmonotone and $u(\infty, c) = 1$.
3. If $c_{\mathrm{unb}} < c \leqslant c_{\mathrm{bif}}$, then $u(\cdot, c) > 0$ on $\mathbb{R}$ and bounded above. Moreover, if $c_{\mathrm{unb}} < c \leqslant c_{\mathrm{per}}$, then there exists $x_{\mathrm{per}}(c) \leqslant 0$ and $T(c) > 0$ such that
$$u(x, c) = u(x + T(c), c) \qquad \text{for every} \qquad x \geqslant x_{\mathrm{per}}(c).$$
4. If $c = c_{\mathrm{unb}}$, then there exists a constant $\tilde{x} > -1$ such that
$$u(\cdot, c_{\mathrm{unb}}) > 0 \quad \text{on} \quad (-\infty, \tilde{x}) \quad \text{and} \quad u(\cdot, c_{\mathrm{unb}}) = 0 \quad \text{on} \quad [\tilde{x}, \infty).$$
5. If $0 < c < c_{\mathrm{unb}}$, then
$$\lim_{x \to \infty} u(x, c) = -\infty.$$

Part 1 has been proved in the previous sections, Part 2 is the content of Conjecture 8.1 and Parts 3, 4 and 5 are shown in Proposition 8.2.

The range $c_{\rm bif} < c < c_0$.

For $c < c_0$, but close to c_0, small damped oscillations around the uniform state $u = 1$ appear. However the solution remains above the value $u = \frac{1}{2}$ for all $x > 1$. As we reduce c further, we find that the amplitude of the oscillations increases and there exists a value $c_1 \in (c_{\rm bif}, c_0)$ for which the graph of u touches the line $u = \frac{1}{2}$ ($c_1 \approx 1.664$). Specifically,

1. For $c_1 < c < c_0$ there exists a nonmonotone solution of the delay-differential equation (8.1) converging to 1 such that $u(x, c) > \frac{1}{2}$ for every $x > 1$.
2. For $c = c_1$ there exists a nonmonotone solution of the delay-differential equation (8.1) converging to 1 and there exists a value $x_1 > 1$ ($x_1 \approx 5.667$) such that $u(x_1, c_1) = \frac{1}{2}$ and $u(x, c_1) > \frac{1}{2}$ for every $x > 1$, $x \neq x_1$.
3. For $c < c_1$ the condition $u(\cdot, c) \geqslant \frac{1}{2}$ does not hold for x on $[1, \infty)$.

Thus, if $c \geqslant c_1$, and hence $u(x) \geqslant \frac{1}{2}$ for $x > 1$, an expansion similar to the one found in Section 7 holds:

$$u(x, c) = 1 - \sum_{k=1}^{\infty} \alpha_k \, e^{z_k(x-1)} \qquad \text{for} \quad x > 1, \tag{8.2}$$

where $\{z_k : k = 1, 2, \dots\}$ are *all* the complex roots of the characteristic equation found in Theorem 4.2 and α_k are given in (7.6).

A simple continuity argument shows that front solutions exist in a small-left neighbourhood of c_1.

Lemma 8.1 *There exists a constant $\varepsilon > 0$ such that for $c_1 - \varepsilon < c < c_1$ the solution $u(x, c)$ of Problem* (8.1) *has the property*

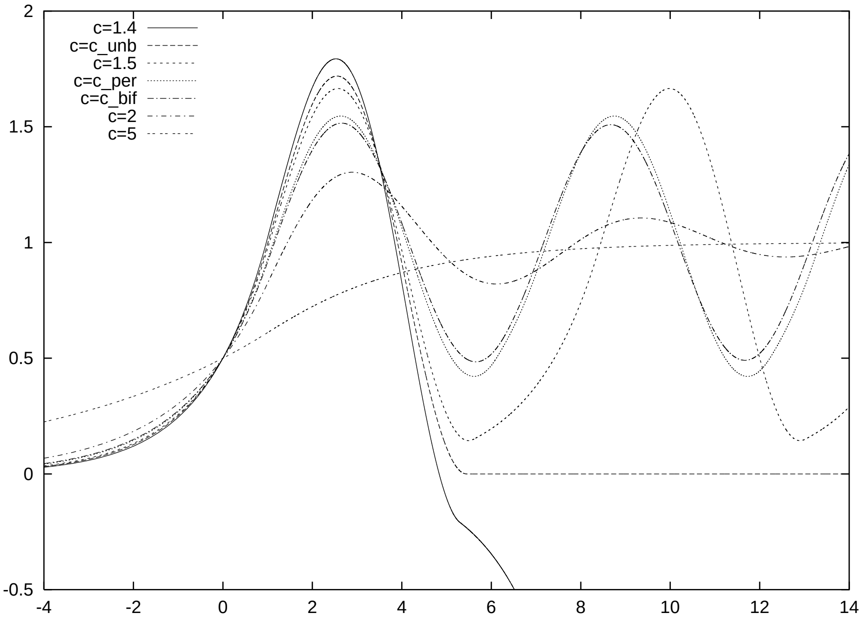
$$\lim_{x \to \infty} u(x, c) = 1.$$

A natural question is: how much smaller can we choose c for there to exist a front type solution. Notice that the difference between both c_1 and $c_{\rm bif}$ is small ($c_1 - c_{\rm bif} \approx 0.01$); thus, the regime where the question remains open is rather narrow. For values c *close* to $c_{\rm bif}$, the amplitude of the oscillations around $u = 1$ increases and the number of times that $u(x, c)$ crosses $u = \frac{1}{2}$ also increases. Motivated by numerical experiments, we formulate the following conjecture:

Conjecture 8.1

1. *There exists a sequence of values $c_1 > c_2 > \cdots > c_n > \cdots \geqslant c_{\rm bif}$ such that for $c_n > c > c_{n+1}$, the graph of the solution u crosses the line $u = \frac{1}{2}$ transversally, exactly $2n$ times. For $c = c_n$, u crosses the line $u = \frac{1}{2}$ transversally, exactly $2(n-1)$ times and once it is tangent to the line.*

Figure 3: Fronts for different values of c



2. *The sequence of values c_n verifies that*

$$c_\infty \stackrel{\text{def}}{=} \lim_{n\to\infty} c_n = c_{\text{bif}}.$$

3. *For $c_{\text{unb}} < c \leqslant c_\infty$, the number of points where the orbit $u(x,c)$ crosses the line $u = \frac{1}{2}$ is infinity. Each time the crossing is transversal.*

Remark. Conjecture 8.1 implies that for every $c > c_{\text{bif}}$, Problem (8.1) has a front type solution, because there exists a point $x_c > 1$ such that $u(\cdot, c)$ obeys the linear DDE (8.1b) on (x_c, ∞), and hence

$$\lim_{x\to\infty} u(x,c) = 1. \tag{8.3}$$

Part 3 of the Conjecture means that the set of values of c for which the boundary condition (8.3) holds is precisely $c > c_\infty$ and by Part 2, $c_\infty = c_{\text{bif}}$.

The range $0 < c \leqslant c_{\text{bif}}$. Having discussed the orbits that verify

$$\lim_{x\to\infty} u(x,c) = 1,$$

and found that they correspond to speeds $c > c_{\mathrm{bif}}$, our next step is to investigate the behaviour of the orbits for $0 < c \leqslant c_{\mathrm{bif}}$. Numerical experiments show that both bounded and unbounded orbits exist, depending on the value of c. In fact we can define a critical speed $c_{\mathrm{unb}} < c_{\mathrm{bif}}$ such that for $c_{\mathrm{unb}} < c < c_{\mathrm{bif}}$ orbits are still bounded, but for $0 < c < c_{\mathrm{unb}}$ they become unbounded. For speeds in the range of bounded solutions, we find another critical speed c_{per} such that for $c_{\mathrm{unb}} < c \leqslant c_{\mathrm{per}}$, bounded orbits are eventually periodic, i.e., there exists a point $x_{\mathrm{per}}(c)$ such that $u(\cdot, c)$ is periodic for $x > x_{\mathrm{per}}(c)$.

Let

$$\xi(c) \stackrel{\mathrm{def}}{=} \sup\{x > 1 : u(\cdot, c) > 0 \text{ on } (-\infty, x)\}. \tag{8.4}$$

We shall show that there are three possibilities:

(a) $u(\cdot, c) > 0$ on $\mathbb{R}$;

(b) $u(\cdot, c) > 0$ on $\big(-\infty, \xi(c)\big)$ and $u(\cdot, c) = 0$ on $\big[\xi(c), \infty\big)$;

(c) $u(\cdot, c) > 0$ on $\big(-\infty, \xi(c)\big)$ and $u(\cdot, c) < 0$ on $\big(1 + \xi(c), \infty\big)$.

In case (c) we have $u(\cdot, c) \to -\infty$ as $x \to \infty$.

We begin with a general result which discusses the possible qualitative behaviours of the solution $u(x)$ in the two cases: $\xi = \infty$ and $\xi < \infty$.

Proposition 8.1 *We have*

1. *Suppose that $\xi(c) = \infty$. Then*

$$u(x, c) > 0 \qquad \textit{for every} \qquad x \in \mathbb{R}.$$

2. *Suppose that $\xi(c) < \infty$. Then, by definition,*

$$u(\xi(c), c) = 0 \qquad \textit{and} \qquad u(x, c) > 0 \quad \textit{for} \quad -\infty < x < \xi(c).$$

Here we distinguish two possibilities:

(a) *We have*

$$u(x, c) \leqslant \tfrac{1}{2} \qquad \textit{for} \qquad x \in [\xi(c) - 1, \xi(c)].$$

In that case:

$$u(x, c) = 0 \qquad \textit{for every} \qquad x \geqslant \xi(c).$$

(b) *We have*

$$u(x, c) > \tfrac{1}{2} \qquad \textit{for some} \qquad x \in [\xi(c) - 1, \xi(c)].$$

Then $u(x, c)$ is nonincreasing for every $x \geqslant \xi(c)$.

Moreover, if $u(\xi(c)-1,c)<\frac{1}{2}$*, then* $u(x,c)$ *is strictly decreasing for* $x \geqslant \xi$*.*

Define

$$A(c) \stackrel{\mathrm{def}}{=} -u\big(1+\xi(c),c\big)$$

Then $A(c)>0$ *and for* $x \geqslant 1+\xi(c)$ *we have*

$$u(x,c) = -A(c)\,\exp\Big(\frac{x-1-\xi(c)}{c}\Big). \tag{8.5}$$

In particular, $\lim_{x\to\infty} u(x,c) = -\infty$*.*

Proof. We write the proof in terms of the variables $t = x-1$ and $y = 1-u$, and we put $\tau = \xi - 1$.

Part 1 of Proposition 8.1 is obvious. Thus, suppose that $\tau < \infty$, then $y<1$ on $(-\infty,\tau)$ and $y(\tau)=1$. We have

$$cy'(t) = \quad y(t)-1 \qquad \text{if} \qquad y(t-1) \geqslant \frac{1}{2}, \tag{8.6}$$

$$cy'(t) = \quad y(t)-2y(t-1) \geqslant y(t)-1 \qquad \text{if} \qquad y(t-1) \leqslant \frac{1}{2}, \tag{8.7}$$

so that

$$cy'(t) \geqslant y(t)-1 \qquad \text{for} \qquad t \geqslant t_{\mathrm{unb}}. \tag{8.8}$$

Remembering that $y(\tau)=1$, we conclude that $y(t)\geqslant 1$ for all $t \geqslant \tau$.

In Case (2a) we have $y \geqslant \frac{1}{2}$ on $(\tau-1,\tau)$, so that we have equality in (8.8), and

$$y(t) = 1 \qquad \text{for all} \qquad t \geqslant \tau. \tag{8.9}$$

On the other hand, in Case (2b) we have $y<\frac{1}{2}$ on some subinterval of $(\tau-1,\tau)$. Let

$$t^* = \sup\Big\{t \geqslant \tau-1 : y \geqslant \frac{1}{2} \quad \text{on} \quad [\tau-1,t)\Big\}.$$

Then

$$y(t) > 1 \qquad \text{for} \qquad t \geqslant t^*+1 \geqslant \tau. \tag{8.10}$$

Thus, for $t > t^*+1$ we know that $y(t,c)>1$ and strictly increasing. For $t \geqslant 1+\tau(c)$, we know that $y(t,c)$ obeys the ODE (8.6). Solving it with the initial condition $y(1+\tau(c),c) = 1+A(c)$ gives us the expression (8.5).

In the following Proposition we summarise the results of a numerical investigation, identifying the values of the speed c with the three cases (a), (b) and (c) enumerated above.

Proposition 8.2 *There exists a critical speed* $c_{\mathrm{unb}} \in (0,c_{\mathrm{bif}})$ $(c_{\mathrm{unb}} \approx 1.455)$ *such that:*

1. *For* $c > c_{\mathrm{unb}} : u(x,c) > 0$ *for every* $x > 1$*.*

2. *For $c = c_{\text{unb}}$ there exists a point $\tilde{x} > 1$ ($\tilde{x} \approx 5.449$) such that $u(x, c_{\text{unb}}) > 0$ for $x < \tilde{x}$ and $u(x, c_{\text{unb}}) = 0$ for $x \geqslant \tilde{x}$.*

3. *For $0 < c < c_{\text{unb}}$ there exists a point $x_c < \tilde{x}$ such that $u(x_c, c) = \frac{1}{2}$ and $u(1 + x_c, c) < 0$. By Proposition 8.1, this implies that there exists a constant $B(c) > 0$ such that*

$$u(x, c) = -B(c) \exp\Big(\frac{x - (x_c + 1)}{c}\Big) \qquad \textit{for} \quad x \geqslant 1 + x_c.$$

In particular, $\lim_{x\to\infty} u(x, c) = -\infty$.

Remark. The value $(\tilde{x}, c_{\text{unb}})$ is found numerically by solving the following system:

$$\begin{cases} u(\tilde{x}, c_{\text{unb}}) = \dfrac{1}{2}, \\ u(\tilde{x} + 1, c_{\text{unb}}) = 0. \end{cases} \tag{8.11}$$

Summarising our results so far, we have found that:

(a) If $c > c_{\text{bif}}$, then $u(x, c) \to 1$ as $x \to \infty$ by Conjecture 8.1

(b) If $0 < c < c_{\text{unb}}$, then $u(x, c) \to -\infty$ as $x \to \infty$ by Proposition 8.2, and

(c) If $c = c_{\text{unb}}$, then the orbit reaches $u = 0$ at some $x = \tilde{x}$ and $u(x, c_{\text{unb}}) = 0$ for $x > \tilde{x}$, also by Proposition 8.2.

We now turn to values of c in the interval $(c_{\text{unb}}, c_{\text{bif}}]$. So far we can only say that $u(x, c)$ remains positive for all $x \in \mathbb{R}$. In the following Proposition, we show that for some values of c in this interval, there exists a point $x_{\text{per}}(c)$ such that the orbit $u(x, c)$ is periodic for $x \geqslant x_{\text{per}}(c)$.

We introduce the points

$$x_1(c) \overset{\text{def}}{=} \sup\big\{x > 1 : u(\cdot, c) > \frac{1}{2} \text{ on } (1, x)\big\}, \tag{8.12}$$

$$x_2(c) \overset{\text{def}}{=} \sup\big\{x > x_1(c) : u(\cdot, c) < \frac{1}{2} \text{ on } (x_1(c), x)\big\}, \tag{8.13}$$

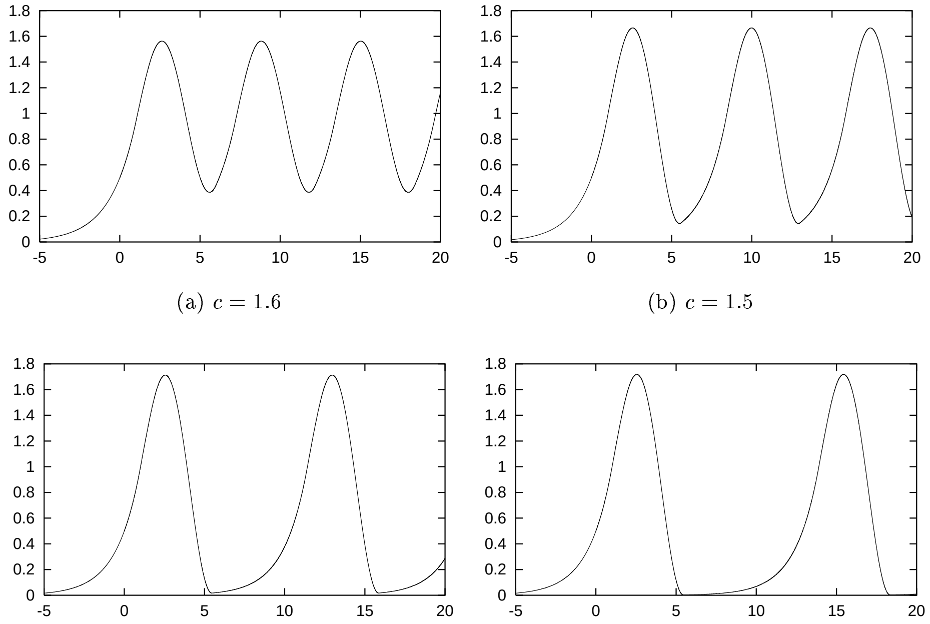
and the constant

$$L(c) \overset{\text{def}}{=} u\big(1 + x_1(c), c\big).$$

The points $x_1(c)$ and $x_2(c)$ are the first two positive values of x where $u(x, c)$ crosses the line $u = \frac{1}{2}$. As $c \leqslant c_{\text{bif}}$, we know that $x_1(c)$ is a well defined positive number. Since $c > c_{\text{unb}}$ then $L(c) > 0$ and by the *transversality* hypothesis formulated in Part 3 of Conjecture 8.1, $x_2(c)$ is well defined and $x_2(c) > x_1(c)$.

In the following Proposition, we prove the existence of the announced *eventually periodic solutions* under the condition that

$$x_2(c) - x_1(c) \geqslant 1. \tag{8.14}$$



(a) $c = 1.6$ (b) $c = 1.5$

(c) $c = 1.46$ (d) $c = 1.456$

Figure 4: Eventually periodic orbits for different values of c.

Note that this condition implies that $L(c) \leqslant \frac{1}{2}$. We characterize numerically the values of $c \in (c_{\rm unb}, \leqslant c_{\rm bif})$ for which this condition is satisfied, and in Figure 4 we show some eventually periodic orbits for different values of c. Note that the amplitude and the period both increase as c decreases. Some of those properties are described in Proposition 8.4.

Proposition 8.3 *Let $c_{\rm unb} < c \leqslant c_{\rm bif}$. Suppose that the condition* (8.14) *holds. Then there exists a point $x_{\rm per}(c) \leqslant 0$ such that*

$$u(x,c) = \frac{1}{2}\, e^{x/c} \qquad \textit{for} \qquad x \leqslant x_{\rm per}$$

and $u(x,c)$ is periodic for $x \geqslant x_{\rm per}(c)$. The period $T(c)$ is given by

$$T(c) = 1 + x_1(c) + c\log\Big(\frac{1}{2L(c)}\Big). \tag{8.15}$$

Proof. We write $t_i = x_i - 1$ $(i = 1, 2)$. Then, by assumption, $t_2 - t_1 \geqslant 1$, and $y(1+t_1, c) = 1 - L$. Since $y \geqslant \frac{1}{2}$ on (t_1, t_2), we can find $y(t)$ on (t_1+1, t_2+1) solving the problem

$$\begin{cases} cy'(t) = y(t) - 1 & \text{for} \quad t_1 + 1 < t < t_2 + 1, \\ y(1+t_1) = 1 - L, & y(t_2) = \frac{1}{2}. \end{cases}$$

This yields the solution we can find a relation between t_1, t_2 and L. We obtain

$$y(t) = 1 - \frac{1}{2}\exp\Big(\frac{t-t_2}{c}\Big) \qquad \text{for} \quad t_1 + 1 \leqslant t \leqslant t_2 + 1,$$

and the following relation between t_1, t_2 and L:

$$t_2 = 1 + t_1 + c\log\Big(\frac{1}{2L}\Big). \tag{8.16}$$

Now, consider the following auxiliary function:

$$z(t) = y(t - t_2 - 1) \qquad \text{for} \qquad t \geqslant t_2 + 1. \tag{8.17}$$

Then $z(t)$ verifies the same delay-differential equation (8.7) as $y(t)$, and

$$z(t) = y(t) \qquad \text{for} \qquad -1 \leqslant t \leqslant 0.$$

Hence, by uniqueness this implies that

$$z(t) = y(t - t_2 - 1) = y(t).$$

In other words, $y(t)$ is periodic and the period is given by $T = t_2 + 1$. From (8.16) we obtain that the period is given as in (8.15).

To complete the proof, we need to find $t_{\rm per}(c)$. This is found to be the value that satisfies $y(t_{\rm per}) = 1 - L$. A simple computation shows that $t_{\rm per}$ is given by

$$t_{\rm per}(c) = -1 - c\log\Big(\frac{1}{2L(c)}\Big). \tag{8.18}$$

Remark. Note that the condition (8.14) is essential. If it is not satisfied, we can still conclude is that $z(t)$ and $y(t)$ verify the same DDE (8.7). However, the initial condition is different. Indeed, in this case what we know is that there exists an interval of length $t_2 - t_1 < 1$ on which $y(t)$ and $z(t)$ coincide. Specifically, $z = y$ on $[0, t_2 - t_1]$, which is not enough for uniqueness.

To finish this section, it remains to characterize (topologically and numerically) which values of c verify the condition (8.14) to find the (eventually) periodic orbits found in Proposition 8.3.

Proposition 8.4 *There exists $c_{\rm unb} < c_{\rm per} < c_{\rm bif}$ ($c_{\rm per} \approx 1.618$) such that the condition (8.14) is satisfied if and only if $c_{\rm unb} < c \leqslant c_{\rm per}$. By Proposition 8.3, this implies that there exists a point $x_{\rm per}(c) \leqslant 0$ given by (8.18) and a constant $T(c) > 0$ given by (8.15) such that*

$$u(x, c) = u(x + T(c), c) \qquad \textit{for every} \quad x \geqslant x_{\rm per}(c).$$

Moreover,

$$\lim_{c \downarrow c_{\rm unb}} T(c) = \infty, \qquad \lim_{c \uparrow c_{\rm per}} T(c) = 1 + x_1(c_{\rm per}), \tag{8.19}$$

$$\lim_{c \downarrow c_{\rm unb}} x_{\rm per}(c) = -\infty, \qquad \lim_{c \uparrow c_{\rm per}} x_{\rm per}(c) = 0. \tag{8.20}$$

Proof. Consider the interval $I = (c_{\text{unb}}, c_{\text{bif}})$. Note that in this interval, the value $x_1(c)$ given by (8.12) is well defined and finite. This can be shown in two ways: by using Part 3 of Conjecture 8.1 or by recalling that the spectrum of the operator associated to the linear DDE (8.1b) contains eigenvalues with positive real part. Hence, the amplitude of the oscillations around the uniform steady state $u = 1$ increases and eventually the orbit crosses $u = \frac{1}{2}$.

On I we have that $x_2(c)$ given by (8.13) is also well defined and finite. The argument works as follows: the orbit crosses the line $u = \frac{1}{2}$ at $x = x_1$ and also $u(\cdot) > 0$ on $\mathbb{R}$ because $c > c_{\text{unb}}$, by Proposition 8.2. Suppose that the orbit verifies that $0 < u(\cdot, c) < \frac{1}{2}$ on $(x_1(c), \infty)$. Then, eventually it must satisfy the ODE (8.1a). The ODE forces the orbit to cross the line $u = \frac{1}{2}$, which contradicts our assumption. Hence, $x_2(c)$ is well defined and finite.

Thus, we have that $x_1(c)$ and $x_2(c)$ are well defined on I. By using continuity of solutions with respect to parameter c, we know that

$$\lim_{c \downarrow c_{\text{unb}}} x_2(c) - x_1(c) = \infty$$

because $x_2(c)$ tends to infinity. In particular, we can find a small-right neighbourhood of c_{unb} where condition (8.14) is satisfied. Specifically, there exists some $\varepsilon > 0$ such that condition (8.14) holds on $(c_{\text{unb}}, c_{\text{unb}} + \varepsilon]$. We define

$$c_{\text{per}} \stackrel{\text{def}}{=} \sup\{c \in I : \text{condition (8.14) holds on } (c_{\text{unb}}, c)\} \tag{8.21}$$

This definition gives us the criterium to find c_{per}: it must satisfy the equation

$$u\big(1 + x_1(c), c\big) = \frac{1}{2}. \tag{8.22}$$

The limits in (8.19) are obtained from (8.15) and (8.16) by using that

$$\lim_{c \downarrow c_{\text{unb}}} L(c) = 0 \qquad \text{and} \qquad \lim_{c \uparrow c_{\text{per}}} L(c) = \frac{1}{2}.$$

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