EXISTENCE AND NONEXISTENCE OF NONTRIVIAL SOLUTIONS OF SEMILINEAR FOURTH AND SIXTH ORDER DIFFERENTIAL EQUATIONS

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Abstract. The existence of nontrivial periodic solutions of semilinear fourth and sixth order differential equations arising in phase transition models is discussed via variational methods.

Key words and phrases: Extended Fisher–Kolmogorov equation, Swift–Hohenberg equation, sixth-order equations, Clark's theorem, eigenvalue problems, variational methods.

1 Introduction

Bistable systems play an important role in the study of spatial patterns. A typical example, which appears in population dynamics, leads to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3,$$

referred to as the classical Fisher-Kolmogorov (FK) equation (cf. [FMcL], [KPP]). The term "bistable" refers here to the fact the stationary states $u_\pm = \pm 1$ are stable as solutions of the equation $\frac{du}{dt} = u - u^3$.

Recently, interest has turned to higher-order model equations involving bistable dynamics, such as the extended Fisher-Kolmogorov (EFK) equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad \gamma > 0, \tag{1.1}$$

proposed by Coullet, Elphick & Repaux [CER] and Dee & van Saarloos [DS]. The EFK equation has appeared in studies of phase transitions, for instance near a Lifshitz point (cf. Zimmerman [Z]). Another well known equation of this type is the Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = \alpha u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + u - u^3, \qquad \alpha > 0, \tag{1.2}$$

which was proposed by Swift & Hohenberg [SH] in studies of Rayleigh-Bénard convection (see also [CE] and [PM]).

With appropriate changes of variables, stationary solutions of these equations lead to the equation

$$u^{iv} - pu'' - u + u^3 = 0, \tag{1.3}$$

in which p is a constant. If $p > 0$ this equation corresponds through an appropriate tranformation, to the EFK equation and if $p < 0$ it corresponds to the SH equation. Solutions of equation (1.3) which are bounded on the real line have recently been extensively studied by a variety of methods such as topological shooting methods, phase-plane analysis and variational methods. (see [Be], [BPT], [KV], [KKBV], [MPT], [PT1,2] and [PTV]).

Whereas the EFK equation and the SH equation involve fourth order spatial derivatives, certain phase field models lead to even higher-order spatial gradients. We mention the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^2 u}{\partial x^2} + u - u^3, \tag{1.4}$$

studied by Gardner & Jones [GJ] and Caginalp & Fife [CF], in which A and B are constants.

In this paper we extend results obtained for equation (1.3) in two directions:

(i) We study a *nonautonomous* generalisation of equation (1.3):

$$\text{(I)} \qquad u^{iv} - pu'' - a(x)\,u + b(x)\,u^3 = 0, \tag{1.5}$$

in which the coefficients $a(x)$ and $b(x)$ are even continuous positive $2L$–periodic functions.

(ii) We study the sixth-order equation

$$\text{(II)} \qquad u^{vi} + Au^{iv} + Bu'' + Cu - b(x)\,u^3 = 0, \tag{1.6}$$

which results from equation (1.4) when we seek stationary solutions. Here A and B are arbitrary constants, while C is assumed to be a *positive* constant. As in equation (1.5) we assume that $b(x)$ is an even continuous positive $2L$–periodic function.

Henceforth we denote by (P_1) the boundary value problem for equation (I) with the boundary conditions

$$u\left(0\right)=u\left(L\right)=u''\left(0\right)=u''\left(L\right)=0,$$

and by (P_2) the boundary value problem for equation (II) with the boundary conditions

$$\begin{aligned} u\left(0\right) &= u''\left(0\right)=u^{iv}\left(0\right)=0, \\ u\left(L\right) &= u''\left(L\right)=u^{iv}\left(L\right)=0. \end{aligned}$$

We shall prove the existence of $2L$–periodic solutions of both both equations (I) and (II) which are antisymmetric with respect to $x = 0$ and $x = L$, by extending the solutions u of the problems (P_1) and (P_2) on $[0, L]$ as odd functions on $[-L, L]$ and then continuing the resulting function periodically over the real line.

Problem (P_1) was studied in recent papers of Tersian & Chaparova [TCh1] and [Ch] by means of variational methods, and Problem (P_2) was studied in [TCh2] under the assumptions that $A > 0$, $A^2 < 4B$ and $C = 1$.

In this paper we shall also use variational methods: Problems (P_1) and (P_2) both have a variational structure. The weak solutions of Problem (P_1) can be found as critical points of the functional

$$I\left(u;L\right):=\int_0^L\left\{\frac{1}{2}\left(u''^2+pu'^2-a\left(x\right)u^2\right)+\frac{1}{4}b\left(x\right)u^4\right\}dx \tag{1.7}$$

in the space $X\left(L\right):=H^2\left(0,L\right)\cap H_0^1\left(0,L\right)$; those of Problem (P_2) can be found as a critical point of the functional

$$J\left(u;L\right):=\int_0^L\left\{\frac{1}{2}\left(u'''^2-Au''^2+Bu'^2-u^2\right)+\frac{1}{4}b\left(x\right)u^4\right\}dx \tag{1.8}$$

in the space

$$Y\left(L\right):=\{u\in H^3\left(0,L\right):u\left(0\right)=u''\left(0\right)=u\left(L\right)=u''\left(L\right)=0\}. \tag{1.9}$$

In finding the critical points of the functionals I and J our main tool will be Clark's theorem (cf. Rabinowitz [R], Theorem 9.1, Clark [C]).

Theorem (Clark). *Let X be a real Banach space, and let $E \in C^1(X, \mathbf{R})$ be even and bounded from below, and let E satisfy the (PS) condition. Suppose that $E(0) = 0$ and that there exists a set $K \subset X$ which is homeomorphic to $\mathbf{S}^{m-1}$ by an odd map, and $\sup_K E < 0$. Then E possesses at least m distinct pairs of critical points.*

Before stating the main results of this paper we recall the results found in [TCh1], [Ch], [TCh2]. The are collected in the following theorem.

Theorem A. *Let $p \neq 0$, let a and b be positive continuous functions on $[0, L]$, and let A* and *B be constants such that $A^2 < 4B$. Then there exist numbers L_1 and L_2 $(0 < L_2 < L_1)$, such that the problems (P_1) and (P_2) have only the trivial solution if $0 < L \leq L_2$ and at least m distinct pairs of solutions if $L > mL_1$ for $m \in \mathbf{N}$.*

A natural question is what happens if $L \in (L_2, L_1)$. Our first main result is concerned with answering this question for Problem (P_1).

Theorem B. *Let $p \in \mathbf{R}$, and let a and b be positive continuous functions on $[0, L]$. Then there exists a number $L^* \in (L_2, L_1)$ such that Problem (P_1) has only the trivial solution if $0 < L \leq L^*$ and a nontrivial solution if $L > L^*$.*

To formulate the answer to this question for Problem (P_2) we need to introduce the symbol of the linear part of equation (1.6):

$$P(\xi) = \xi^6 - A\xi^4 + B\xi^2 - C. \tag{1.10}$$

We prove

Theorem C. *Let $A, B \in \mathbf{R}$ and $C > 0$, and let $P(\xi)$ have exactly one positive zero ξ_0. We write $L_0 = \pi/\xi_0$. Let Y_m be the subspace of $Y(L)$:*

$$Y_m = \left\{ \sin\left(\frac{\pi x}{L}\right), ..., \sin\left(\frac{m\pi x}{L}\right) \right\}^{\perp}.$$

Then Problem (P_2) has only the trivial solution if $0 < L \leq L_0$, and it has at least m pairs of nontrivial solutions if $mL_0 < L < (m+1)L_0$. If $L = (m+1)L_0$ there exist has at least m pairs of nontrivial solutions which are not in the subspace Y_{m+1}.

The paper is organized as follows. In Section 2 we prove two abstract theorems about nontrivial critical points of functionals in Hilbert spaces. In Section 3 we consider the solvability of Problem (P_1) , and in Section 4 we deal with the Problem (P_2) .

2 Abstract results

In this section we prove two abstract theorems for critical points of functionals on Hilbert spaces.

Let H be a Hilbert space, and let $E : H \to \mathbf{R}$ be a functional of the form

$$E(u) = a(u,u) - b(u,u) + \psi(u), \tag{2.1}$$

where a and b are bilinear forms which satisfy the assumptions

$$(A_1) : \begin{cases} a : H \times H \to \mathbf{R} \text{ is a scalar product in } H, \\ b : H \times H \to \mathbf{R} \text{ is a symmetric bilinear form, } b(u,u) > 0, u \neq 0, \\ b \text{ is weakly continuous, i.e. } u_k \rightharpoonup u \;\Rightarrow\; b(u_k,u_k) \to b(u,u), \end{cases}$$

and ψ satisfies

$$(A_2) : \begin{cases} \psi \in C^1(H, \mathbf{R}) \text{ is a convex functional,} \\ \psi(u) > 0 \text{ if } u \neq 0 \text{ and } \psi(0) = 0. \end{cases}$$

It is known that under assumption (A_1) (see [S, p. 29], [W, p. 91]) the eigenvalue problem

$$a(u,v) = \lambda\, b(u,v) \qquad \text{for} \qquad v \in H$$

has a countable set of positive eigenvalues and eigenvectors (λ_n, e_n). The set of eigenvectors $\{e_1, ..., e_m, ...\}$ is an orthogonal base in H. Let $H_m := \{e_1, ..., e_m\}$. Then the eigenvalues are characterized by:

$$\lambda_1 = \min_{u \in H \setminus \{0\}} \frac{a(u,u)}{b(u,u)}, \tag{2.2}$$

and

$$\lambda_m = \min_{u \in H_{m-1}^{\perp} \setminus \{0\}} \frac{a(u,u)}{b(u,u)}, \qquad m \geq 2. \tag{2.3}$$

Thus

$$0 < \lambda_1 \leq ... \leq \lambda_m \leq ....$$

We prove

Theorem 2.1. *Let the bilinear forms a and b on the Hilbert space H satisfy (A_1) and let $\psi : H \to \mathbf{R}$ satisfy assumption (A_2).*

(a) *Let $\lambda_1 > 1$. Then the functional*

$$E(u) = a(u,u) - b(u,u) + \psi(u)$$

has only the trivial critical point.

(b) *Let $0 < \lambda_1 < 1$ and let ψ satisfy in addition the condition*

$$\psi(tu) = |t|^k \psi(u), \qquad k > 2.$$

Then, if the functional E has a bounded minimizing sequence it has a non-trivial critical point .

Proof. Let $\lambda_1 > 1$. Then $(a(u,u) - b(u,u))^{1/2}$ is an equivalent norm in H and a strictly convex function because

$$c_1\|u\|^2 \geq a(u,u) - b(u,u) \geq \left(1 - \frac{1}{\lambda_1}\right) a(u,u) \geq c_2\|u\|^2.$$

It follows that E is a strictly convex positive functional, $E(0) = 0$ and its unique critical point is 0.

Next, let $0 < \lambda_1 < 1$ and let e_1 be an eigenvector corresponding to λ_1. Then

$$\begin{aligned} E(te_1) &= a(te_1, te_1) - b(te_1, te_1) + \psi(te_1) \\ &= t^2(a(e_1, e_1) - b(e_1, e_1)) + t^k \psi(e_1) \\ &= t^2 \left(1 - \frac{1}{\lambda_1}\right) a(e_1, e_1) + t^k \psi(e_1) < 0, \end{aligned}$$

for sufficiently small t. The functional E is weakly lower semi-continuous as a sum of sequentially weakly continuous and convex functionals (cf. [B, p. 301] and has a bounded minimizing sequence. Therefore, E has a minimum on H and there exists a function $u_0 \neq 0$ such that $E(u_0) = \min E < 0$ and $E'(u_0) = 0$.

THEOREM 2.2. *Let the bilinear forms a and b on the Hilbert space H satisfy (A_1) and let $0 < \lambda_m < 1$. Let $\psi : H \to \mathbf{R}$ satisfy assumption (A_2) and let $\psi(tu) = |t|^k \psi(u), k > 2$. Suppose that the functional $E(u) = a(u,u) - b(u,u) + \psi(u)$ is bounded from below and even, that $E(0) = 0$, and that E satisfies the (PS)* condition. *Then E has at least m distinct pairs of critical points. If $\lambda_m < 1 < \lambda_{m+1}$ there exist at least m distinct pairs of critical points of E which are not in the subspace H_m, and if $\lambda_m < 1 = \lambda_{m+1} < \lambda_{m+2}$, there are at least m distinct pairs of critical points of E which are not in the subspace H_{m+1},*

Proof. We apply Clark's theorem and show that there exists a set $K \subset H_m = \{e_1, ..., e_m\}$ which is homeomorphic to $\mathbf{S}^{m-1}$ by an odd map, and that $\sup_K I < 0$.

Step 1. We prove that

$$b\,(u,u) - a\,(u,u) \geq \left(1 - \frac{1}{\lambda_m}\right) \|u\|^2, \qquad u \in H_m.$$

In fact for $u = c_1 e_1 + ... + c_m e_m$ we have

$$\begin{aligned} b\,(u,u) - a\,(u,u) &= \sum_{j=1}^{m} c_j^2 \{b\,(e_j, e_j) - a\,(e_j, e_j)\} \\ &= \sum_{j=1}^{m} c_j^2 \left(1 - \frac{1}{\lambda_j}\right) a\,(e_j, e_j) \\ &\geq \left(1 - \frac{1}{\lambda_m}\right) \|u\|^2. \end{aligned}$$

Note that, $b\,(u,u) - a\,(u,u)$ is an equivalent norm in H_m because b is a continuous bilinear form in the finite dimensional space H_m, and that there exists a constant $c_3 > 0$ such that $b\,(u,u) \leq c_3 \|u\|^2$ for $u \in H_m$. Therefore

$$b\,(u,u) - a\,(u,u) \leq c_3 \|u\|^2 \quad \text{for} \quad u \in H_m.$$

Step 2. Under the assumptions on ψ *we have for* $c_1^2 + ... + c_m^2 = \rho^2$,

$$\psi\,(c_1 e_1 + ... + c_m e_m) \leq m^{k/2} \rho^k \{\psi(e_1) + ... + \psi(e_m)\}\,.$$

By the positivity and convexity of ψ, by Jensen's inequality and by the mean square inequality we have

$$\begin{aligned} &\psi\,(c_1 e_1 + ... + c_m e_m) \\ = \;& (|c_1| + ... + |c_m|)^k \, \psi \left(\frac{1}{|c_1| + ... + |c_m|} (c_1 e_1 + ... + c_m e_m) \right) \\ \leq \;& \left(\sqrt{m} \left(c_1^2 + ... + c_m^2 \right)^{1/2} \right)^k \sum_{j=1}^{m} \frac{|c_j| \psi\,(e_j)}{|c_1| + ... + |c_m|} \\ \leq \;& m^{k/2} \rho^k \left(\psi(e_1) + ... + \psi(e_m) \right). \end{aligned}$$

Step 3. The set

$$K_m = \{c_1 e_1 + ... + c_m e_m : c_1^2 + ... + c_m^2 = \rho^2\}$$

is homeomorphic to $\mathbf{S}^{m-1}$. *For sufficiently small values of* $\rho > 0$ *we have*

$$\sup_{K_m} E < 0.$$

By the previous steps we have

$$
\begin{aligned}
&E\left(c_1 e_1+\ldots+c_m e_m\right) \\
&\qquad \leq -\frac{1-\lambda_m}{\lambda_m}\rho^2+m^{k/2}\rho^k\left(\psi(e_1)+\ldots+\psi(e_m)\right)<0
\end{aligned}
$$

Step 4. If either $\lambda_m<1<\lambda_{m+1}$, or $\lambda_m<1=\lambda_{m+1}<\lambda_{m+2}$, there exist at least m distinct pairs of critical points of E which are not in the subspace H_m or in H_{m+1}.

If $\lambda_m<1<\lambda_{m+1}$, the form $a\left(u,u\right)-b\left(u,u\right)$ is a strictly convex functional in H_m because

$$
\begin{aligned}
c_1\|u\|^2 &\geq a\left(u,u\right)-b\left(u,u\right) \\
&\geq \left(1-\frac{1}{\lambda_{m+1}}\right)a\left(u,u\right)\geq c_3\|u\|^2, \qquad u\in H_m.
\end{aligned}
$$

Therefore, $E_m=E|_{H_m}$ is a strictly convex positive functional, $E_m\left(0\right)=0$ and its unique critical point is 0. It follows from Step 3 that E has at least m distinct pairs of critical points which do not belong to the subspace H_m.

If $\lambda_m<1=\lambda_{m+1}<\lambda_{m+2}$, then

$$
c_1\|u\|^2\geq a(u,u)-b(u,u)\geq\left(1-\frac{1}{\lambda_{m+2}}\right)a(u,u), \qquad u\in H_{m+1}.
$$

Then $E_{m+1}=E|_{H_{m+1}}$ is a strictly convex functional, $E_{m+1}(0)=0$ and it has only the trivial critical point. Respectively E has at least m pairs of critical points which are not in H_{m+1}.

3 Solvability of Problem (P_1)

We first study the solvability of Problem (P_1) assuming that the functions $a(x)$ and $b(x)$ satisfy

$$
0<a_1\leq a(x)\leq a_2, \qquad 0<b_1\leq b(x)\leq b_2, \tag{3.1}
$$

where a_j and b_j $(j=1,2)$ are constants.

Let $X(L)=H^2\left(0,L\right)\cap H_0^1\left(0,L\right)$ be the Hilbert space with inner product

$$
(u,v)=\int_0^L\left(u''v''+u'v'+uv\right)dx.
$$

A function $u \in X(L)$ is said to be a weak solution of Problem (P_1) if for every function $v \in X(L)$ we have

$$\int_0^L \left(u''v'' + pu'v' - a(x)\,uv + b(x)\,u^3 v\right) dx = 0.$$

Weak solutions of (P_1) are critical points of the functional $I : X \to \mathbf{R}$

$$I(u) = \frac{1}{2}\int_0^L \left(u''^2 + pu'^2 - a(x)\,u^2\right) dx + \frac{1}{4}\int_0^L b(x)\,u^4 dx.$$

From Assumption (3.1) it follows that $I \in C^1(X(L), \mathbf{R})$ and that

$$\langle I'(u), v\rangle = \int_0^L \left(u''v'' + pu'v' - a(x)\,uv + b(x)\,u^3 v\right) dx$$

for all $v \in X(L)$. Here $\langle \cdot, \cdot \rangle$ denotes the duality between $X(L)$ and its dual $X^*(L)$. We recall from [TCh1] that a weak solution $u \in X(L)$ of Problem (P_1) is also a classical solution of (P_1).

In our analysis the critical lengths:

$$L_j := \frac{\pi}{\sqrt{2a_j}}\sqrt{p + \sqrt{p^2 + 4a_j}}, \qquad j = 1, 2, \tag{3.2}$$

will play an important role. An elementary calculation shows that

$$L_2 < L_1 \qquad \text{if} \qquad a_1 < a_2.$$

The proof of Theorem A for Problem (P_1) is divided into a sequence of lemmata about the lengths L_1 and L_2. They are proved in [TCh1] and [Ch]. We have:

Lemma 3.1. *The functional I is bounded from below on $X(L)$, it has a bounded minimizing sequence, and there exists a minimizer u of I.*

Lemma 3.2. *If $L \leq L_2$ then Problem (P_1) has the trivial solution only.*

Lemma 3.3. *The functional I satisfies the Palais-Smale condition (PS).*

Lemma 3.4. *Let $L > mL_1$ for some $m \in \mathbf{N}$. Then there exists a set $K \subset X(L)$ which is homeomorphic to $\mathbf{S}^{m-1}$ by an odd map, and* $\sup_K I < 0$.

We will now discuss the solvability of Problem (P_1) when $L_2 < L < L_1$. In considering Problem (P_1) we treat the two cases $p \geq 0$ and $p < 0$ separately.

Suppose that $p \geq 0$. We make a change of variables, and put $t = \frac{x}{L}$ and $u(t) = u\left(\frac{x}{L}\right)$. Problem (P_1) is then seen to be equivalent to

$$(P_1') : \begin{cases} u^{iv} - L^2 p u'' - L^4 \overline{a} a + L^4 \overline{b} u^3 = 0, & 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $\overline{a}(t) = a(tL)$ and $\overline{b}(t) = b(tL)$, and weak solutions are critical points of the functional

$$I_1(u) = \frac{1}{2} \int_0^1 \left(u''^2 + L^2 p u'^2 - L^4 \overline{a} u^2 \right) dt + \frac{1}{4} \int_0^1 \overline{b} u^4 dt$$

on the space $X_1 = X(1)$. The functional I_1 can be written in the form

$$I_1(u) = a_1(u,u) - b_1(u,u) + \psi_1(u), \tag{3.3}$$

where

$$a_1(u,v) = \frac{1}{2} \int_0^1 \left(u''v'' + L^2 p u'v' \right) dt$$

and

$$b_1(u,v) = \frac{1}{2} L^4 \int_0^1 \overline{a} uv \, dt, \quad \psi_1(u,L) = \frac{1}{4} L^4 \int_0^1 \overline{b} u^4 \, dt.$$

The forms a_1 and b_1 are symmetric positive bilinear forms, which satisfy (A_1) in X_1, $\psi_1(u)$ is a positive convex functional, and

$$\lambda_1(L) := \min_{u \in X_1 \setminus \{0\}} \frac{\int_0^1 \left(u''^2 + L^2 p u'^2 \right) dx}{L^4 \int_0^1 a u^2 dx} > 0. \tag{3.4}$$

We have

Lemma 3.5. *There exists a constant $L_1^* > 0$ such that $\lambda_1(L_1^*) = 1$ and*

$$(\lambda_1(L) - 1)(L - L_1^*) < 0 \qquad \textit{for} \quad L \neq L_1^*. \tag{3.5}$$

Proof. To keep the notation simple, we shall drop the bars on a and b. We first prove that $\lambda_1(L)$ is strictly deceasing:

$$\lambda_1(L_a) > \lambda_1(L_b) \qquad \text{if} \qquad L_a < L_b.$$

Let u_a and u_b be functions in $X_1\backslash\{0\}$ at which the minimum in (3.4) is achieved. Then, since $L_b < L_a$, we obtain

$$\begin{aligned}
\lambda_1(L_a) &= \min_{u\in X_1\backslash\{0\}} \frac{\int_0^1 \left(u''^2 + L_a^2 p u'^2\right) dx}{L_a^4 \int_0^1 a u^2\, dx} = \frac{\int_0^1 \left(u_a''^2 + L_a^2 p u_1'^2\right) dx}{L_a^4 \int_0^1 a u_a^2\, dx} \\
&> \frac{\int_0^1 \left(u_a''^2 + L_b^2 p u_a'^2\right)\, dx}{L_b^4 \int_0^1 a u_a^2\, dx} \geq \frac{\int_0^1 \left(u_b''^2 + L_b^2 p u_b'^2\right) dx}{L_b^4 \int_0^1 a u_b^2\, dx} \\
&= \min_{u\in X_1\backslash\{0\}} \frac{\int_0^1 \left(u''^2 + L_b^2 p u'^2\right)\, dx}{L_b^4 \int_0^1 a u^2\, dx} = \lambda_1(L_b)\,.
\end{aligned}$$

Suppose that $L \in (0,1)$. Then, since λ_1 is decreasing, $\lambda_1(L) > \lambda_1(1)$. Note that

$$\begin{aligned}
\frac{a_1(u,u)}{b_1(u,u)} &= \frac{1}{L^2} \frac{\int_0^1 \left(u''^2/L^2 + p u'^2\right) dx}{\int_0^1 a u^2 dx} \\
&> \frac{1}{L^2} \frac{\int_0^1 \left(u''^2 + p u'^2\right) dx}{\int_0^1 a u^2 dx} \geq \frac{1}{L^2}\lambda_1(1)\,,
\end{aligned}$$

which implies that $\lambda_1(L) > \frac{1}{L^2}\lambda_1(1)$. Thus, $\lambda_1(L) \to +\infty$ as $L \to 0^+$.

Next, let $L > 1$. Let $u_0 \neq 0$ be such that $\lambda_1(1) = \frac{a_1(u_0,u_0)}{b_1(u_0,u_0)}$. Then

$$\lambda_1(L) \leq \frac{\int_0^1 \left(u_0''^2 + L^2 p u_0'^2\right) dx}{L^4 \int_0^1 a u_0^2 dx} < \frac{1}{L^2} \frac{\int_0^1 \left(u_0''^2 + p u_0'^2\right)\, dx}{\int_0^1 a u_0^2 dx} = \frac{1}{L^2}\lambda_1(1)\,,$$

and $\lambda_1(L) \to 0$ as $L \to +\infty$.

Therefore, there exists unique L_1^* such that $\lambda_1(L_1^*) = 1$ and (3.5) holds.

In the next theorem we show that the length L_1^* found in Lemma 3.5 is the critical length below which there are no nontrivial solutions, and above which there are.

THEOREM 3.1. *There exists a number $L_1^* \in [L_2, L_1]$, where L_1 and L_2 are given by* (3.2) *such that Problem* (P_1) *with $p \geq 0$ has only the trivial solution if $0 < L \leq L_1^*$ and a nontrivial solution if $L > L_1^*$.*

Proof. Problem (P_1) is equivalent to Problem (P_1'). Its solutions are critical points of the functional I_1, which satisfies the assumptions of Theorem 2.2.

If $0 < L < L_1^*$ it follows from Lemma 3.5 that $\lambda_1(L) > 1$, and from the second part of Theorem 2.1 and Lemma 3.1, we conclude that I_1 has a nontrivial critical point.

If $L > L_1^*$ then by Lemma 3.5, $\lambda_1(L) < 1$ and from the first part of Theorem 2.1 it follows that I_1 has only the trivial critical point. Note that if $L = L_1^*$ the functional I_1 is still strictly convex because $a_1(u,u) - b_1(u,u)$ is nonnegative and convex and $\psi_1(u)$ is a strictly convex functional. Therefore, 0 is also the unique critical point of I_1.

That $L_1^* \in [L_2, L_1]$ follows at once from Theorem A.

Next, we assume that $p < 0$ and for convenience we write $p = -q$, so that $q > 0$. We make the same change of variables: $t = \frac{x}{L}$ and $u(t) = u\left(\frac{x}{L}\right)$. Problem (P_1) is now equivalent to the problem

$$(P_1''): \begin{cases} u^{iv} + L^2 q u'' - L^4 a u + L^4 b u^3 = 0, & 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

and its weak solutions are critical points of the functional

$$I_2(u) = \frac{1}{2}\int_0^1 \left(u''^2 - L^2 q u'^2 - L^4 a u^2\right) dt + \frac{1}{4}\int_0^1 b u^4 dt,$$

which we now write in the form

$$I_2(u) = a_2(u,u) - b_2(u,u) + \psi_2(u),$$

where

$$a_2(u,v) = \frac{1}{2}\int_0^1 u''v'' dt$$

and

$$b_2(u,v) = \frac{1}{2}\int_0^1 \left(L^2 q u'v' + L^4 a u v\right) dt, \quad \psi_2(u) = \frac{1}{4}L^4 \int_0^1 b u^4 \, dt.$$

The forms a_2 and b_2 are symmetric positive bilinear forms, which satisfy (A_1) in X_1, and $\psi_2(u)$ is a positive convex functional. Therefore, there exists

$$\overline{\lambda}_1(L) = \min_{u \in X_1 \setminus \{0\}} \frac{\int_0^1 u''^2 \, dt}{\int_0^1 \left(L^2 q u'^2 + L^4 a u^2\right) dt} > 0.$$

Note that $a_2(u,v)$ generates an equivalent norm in X_1 as can be seen by applying the Poincaré inequality, and the inequality

$$\int_0^1 u'^2 dt = -\int_0^1 u u'' dt \le \frac{1}{2}\left(\int_0^1 u^2 dt + \int_0^1 u''^2 dt\right).$$

As before we can prove

Lemma 3.6. *There exists a constant* $L_2^* > 0$ *such that* $\overline{\lambda}_1(L_2^*) = 1$ *and*

$$\left(\overline{\lambda}_1(L_2^*) - 1\right)(L - L_2^*) < 0, \qquad \textit{for} \quad L \neq L_2^*.$$

and

Theorem 3.2. *There exists a number* $L_2^* \in [L_2, L_1]$, *where* L_1 *and* L_2 *are given by (3.2), such that Problem* (P_1) *with* $p < 0$ *has only the trivial solution if* $0 < L \leq L_2^*$ *and a nontrivial solution if* $L > L_2^*$.

If the coefficient a is a constant, we can compute the eigenvalues and eigenfunctions of the corresponding eigenvalue problems explicitly, and Theorem 2.2 enables us to improve Theorem A for the problem (P_1).

Let $p \geq 0$. The eigenvalue problem

$$(Q_1') : \begin{cases} u^{iv} - L^2 p u'' = \lambda L^4 a u, & 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

has positive eigenvalues

$$\lambda_m^{(1)}(L) = \frac{1}{a}\left\{\left(\frac{m\pi}{L}\right)^4 + p\left(\frac{m\pi}{L}\right)^2\right\}$$

and corresponding eigenfunctions

$$e_m(t) = \sin(m\pi t).$$

The function $\lambda_m^{(1)}(L)$ is strictly decreasing and the unique positive solution of the equation $\lambda_m^{(1)}(L) = 1$ is given by

$$L_m^{(1)} = m\pi\left\{\frac{1}{2a}\left(p^2 + 4a\right)^{1/2} + p\right\}^{1/2}.$$

Next, let $p = -q < 0$. The eigenvalue problem then becomes

$$(Q_2') : \begin{cases} u^{iv} = \lambda\left(-L^2 q u'' + L^4 a u\right), & 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

and it has positive eigenvalues

$$\lambda_m^{(2)}(L) = \frac{(m\pi)^4}{aL^4 + q(Lm\pi)^2}$$

and the corresponding eigenfunctions are

$$e_m(t) = \sin(m\pi t).$$

The function $\lambda_m^{(2)}(L)$ is strictly decreasing and the unique positive solution of the equation $\lambda_m^{(2)}(L) = 1$ is given by

$$\begin{aligned} L_m^{(2)} &= m\pi \left\{ \frac{1}{2a} \left(q^2 + 4a \right)^{1/2} - q \right\}^{1/2} \\ &= m\pi \left\{ \frac{1}{2a} \left(p^2 + 4a \right)^{1/2} + p \right\}^{1/2}. \end{aligned}$$

Let

$$L_0 = \pi \left\{ \frac{1}{2a} \left(p^2 + 4a \right)^{1/2} + p \right\}^{1/2}. \tag{3.6}$$

Note that in both cases $L_m^{(j)} = mL_0, \quad j = 1, 2$. Now, in view of Lemma 3.1, Lemma 3.3 and Theorem 2.2, we have

THEOREM 3.3. *Let a be a positive constant, and let b be a positive continuous function on $[0, L]$. Let L_0 be defined by (3.6), and*

$$X_m = \left\{ \sin\left(\frac{\pi x}{L}\right), ..., \sin\left(\frac{m\pi x}{L}\right) \right\}^{\perp} \subset X(L).$$

If $mL_0 < L < (m+1)L_0$ then Problem (P_1) has at least m pairs of nontrivial solutions which are not in the subspace X_m. If $L = (m+1)L_0$ then Problem (P_1) has at least m pairs of nontrivial solutions which are not in the subspace X_{m+1}.

4 Solvability of Problem (P_2)

In this section we turn to Problem (P_2) which also has a variational structure; its weak solutions are critical points of the functional

$$J(u; L) = \int_0^L \left\{ \frac{1}{2} \left(u'''^2 - Au''^2 + Bu'^2 - Cu^2 \right) + \frac{1}{4} b(x) u^4 \right\} dx \tag{4.1}$$

in the Hilbert space

$$Y(L) = \{H^3(0, L) : u(0) = u''(0) = u(L) = u''(L) = 0\},$$

with scalar product

$$(u,v)=\int_0^L\left(u'''v'''+u''v''+u'v'+uv\right)dx,$$

and norm

$$||u||=(u,u)^{1/2}\,.$$

A function $u\in Y\,(L)$ is said to be a *weak solution* of Problem (P_2) if

$$\int_0^L\left(u'''v'''-Au''v''+Bu'v'-Cuv+bu^3v\right)dx=0,\quad\text{for all}\quad v\in Y.$$

The functional J is differentiable,

$$\langle J'\,(u;L)\,,v\rangle=\int_0^L\left(u'''v'''-Au''v''+Bu'v'-Cuv+bu^3v\right)dx,$$

and its critical points are weak solutions of Problem (P_2).

We make three sets of hypotheses about the constants A, B and C.

(**H1**) $\quad A\le 0,\quad B>0,\quad C>0,$

(**H2**) $\quad A>0,\quad A^2\le 3B,\quad B>0,\quad C>0,$

(**H3**) $\quad A>0,\quad B>0,\quad A^2>3B,\quad C\in(0,C_-)\cup(C_+,\infty),$

where

$$C_\pm=C_\pm(A,B):=\frac{1}{27}\left\{9AB-2A^3\pm 2\left(A^2-3B\right)^{3/2}\right\}.$$

(**H4**) $\quad A\in\mathbf{R},\quad B\le 0,\quad C>0.$

We recall the following lemma, proved in [TCh2] :

LEMMA 4.1. *We have for every $u\in Y\,(L)$:*

$$\int_0^L u^2dx\le\frac{L^{2k}}{\pi^{2k}}\int_0^L u^{(k)}dx,\qquad k=1,2,3 \tag{4.2}$$

where $u^{(k)}$ denotes the k^{th} derivative of u.

With this lemma we prove

LEMMA 4.2. (a) *The scalar product*

$$\langle u,v\rangle:=\int_0^L u'''v'''dx$$

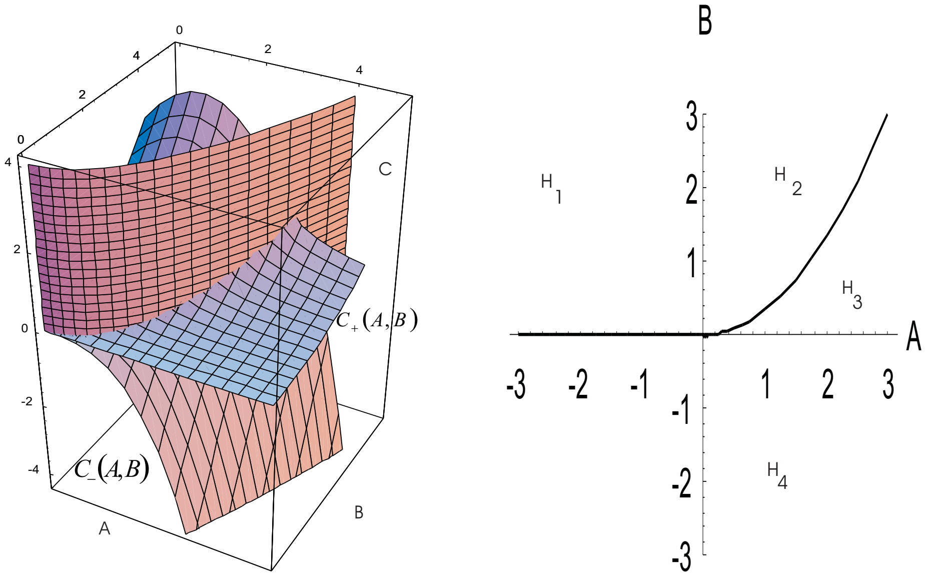

Figure 1: Graphs of the functions $C_+(A,B)$ and $C_-(A,B)$ and the cylinder $A^2 = 3B$ (left) and the projections of the domains H_1, H_2, H_3 and H_4 on the (A,B)-plane (right).

induces an equivalent norm $[u] = \langle u, u\rangle^{1/2}$ *in the space* $Y(L)$.
(b) *The set of functions* $\{\sin\left(\frac{n\pi x}{L}\right) : n \in \mathbf{N}\}$ is a complete orthogonal basis in $Y(L)$.

Proof. (a) We have

$$\int_0^L u'^2 dx = -\int_0^L uu'' dx \le \frac{1}{2}\int_0^L u^2 dx + \frac{1}{2}\int_0^L u''^2 dx \tag{4.3}$$

and

$$\int_0^L u''^2 dx = -\int_0^L u'u''' dx \le \frac{1}{2}\int_0^L u'^2 dx + \frac{1}{2}\int_0^L u'''^2 dx. \tag{4.4}$$

Hence, for every $u \in Y(L)$

$$\int_0^L u'^2 dx + \int_0^L u''^2\, dx \le \frac{1}{2}\|u\|^2.$$

By (4.3), (4.4),

$$\int_0^L u'^2 dx + \int_0^L u''^2 dx \le \int_0^L u^2 dx + \int_0^L u'''^2 dx,$$

which means, in view of Lemma 4.1, that

$$[u] = \langle u, u\rangle^{1/2} = \left(\int_0^L u'''^2\, dx\right)^{1/2}$$

is an equivalent norm in $Y(L)$.

(b) It is clear that the system $\{\sin\left(\frac{n\pi x}{L}\right)\}$ is orthogonal in $Y(L)$ with respect to the scalar product $\langle\cdot,\cdot\rangle$. We show that it is a complete orthogonal basis in $Y(L)$. Let $v \in Y(L)$ be such that

$$\left\langle v, \sin\left(\frac{n\pi x}{L}\right)\right\rangle = 0 \qquad \text{for every} \quad n \in \mathbf{N}.$$

Then

$$0 = \int_0^L v'''(x)\left(\sin\left(\frac{n\pi x}{L}\right)\right)''' dx = -\int_0^L v(x)\left(\sin\left(\frac{n\pi x}{L}\right)\right)^{vi} dx,$$

and

$$\int_0^L v(x)\sin\left(\frac{n\pi x}{L}\right) dx = 0 \qquad \text{for every} \quad n \in \mathbf{N}.$$

Since $Y\,(L) \subset L^2\,(0,L)$ and $\{\sin \frac{n\pi x}{L}\}$ is an orthogonal basis in $L^2\,(0,L)\,,$ it follows that $v = 0$, which means that $\{\sin \frac{n\pi x}{L}\}$ is a complete orthogonal basis in $Y\,(L)\,.$

LEMMA 4.3. *The polynomial*

$$P(\xi) = \xi^6 - A\xi^4 + B\xi^2 - C$$

has exactly one positive zero ξ_0 if either $(H1)$ or $(H2)$ or $(H3)$ or $(H4)$ holds.

Proof. Put $\xi^2 = \lambda$ and consider the polynomial

$$r\,(\lambda) = \lambda^3 - A\lambda^2 + B\lambda - C \qquad \text{for} \quad \lambda \geq 0.$$

Because $r(0) = -C < 0$ and $r\,(+\infty) = +\infty$ it is clear that $r(\lambda)$ has a zero. Below we show that the hypotheses (H1-4) guarantee that r has only one zero.

If $A \leq 0$ and $B > 0$ – case (H1) – then $r(\lambda)$ is seen to be strictly increasing for $\lambda > 0$. This means that $r(\lambda)$ has exactly one zero.

Thus, suppose that $A > 0$. Then we need to investigate

$$r'\,(\lambda) = 3\lambda^2 - 2A\lambda + B.$$

If $D := A^2 - 3B \leq 0$ – case (H2) – then $r'(\lambda)$ is nonnegative for all $\lambda > 0$ and as before, $r(\lambda)$ has only one zero.
If $D = A^2 - 3B > 0$ – case (H3) – then $r'(\lambda)$ does change sign at $\lambda = \lambda_\pm$, where

$$\lambda_\pm = \frac{1}{3}\left(A \pm \sqrt{A^2 - 3B}\right).$$

Since $r'(\lambda) > 0$ for λ sufficiently large, $r(\lambda)$ has a local minimum at λ_+ and a local maximum at λ_-. Thus, $r(\lambda)$ has precisely one zero if either

$$r(\lambda_+) > 0 \qquad \Longleftrightarrow \qquad C < r_0(\lambda_+) := C_-(A,B),$$

or

$$r(\lambda_-) < 0 \qquad \Longleftrightarrow \qquad C > r_0(\lambda_-) := C_+(A,B),$$

where

$$r_0(\lambda) = \lambda^3 - A\lambda^2 + B\lambda.$$

Suppose that $B \leq 0$, as in (H4). If $B < 0$, then $D > 0$, and $r(\lambda)$ has two critical points λ_+ and λ_-. If $A \geq 0$, then $\lambda_- < 0 < \lambda_+$ and $r(\lambda_+) < -C < 0$.

If $A < 0$, then $\lambda_- < \lambda_+ < 0$ and $r(\lambda)$ is increasing for $\lambda \geq 0$. In both cases, $r(\lambda)$ has exactly one zero on $\mathbf{R}^+$.

Finally, suppose that $B = 0$. If $A > 0$, then the critical points of $r(\lambda)$ are $\lambda_- = 0$ and $\lambda_+ = \frac{2}{3}A > 0$. As before $r(\lambda_+) < 0$, and so $r(\lambda)$ has only one positive zero. If $A \leq 0$, then $\lambda_- = \frac{2}{3}A \leq 0$ and $\lambda_+ = 0$, so that $r(\lambda)$ is strictly incerasing for $\lambda > 0$, so that again, r has only one positive zero.

LEMMA 4.4. *Supose that $A, B \in \mathbf{R}$ and C is positive a constant. Then the functional $J(u, L)$ is bounded from below, it has a bounded minimizing sequence and it satisfies the (PS) condition.*

Proof. Using Lemma 4.2 and Fourier series arguments we obtain for $u \in Y\,(L)$

$$u = \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi x}{L}\right),$$

$$J(u, L) = \frac{1}{2}\sum_{k=1}^{\infty} c_k^2 P\left(\frac{k\pi x}{L}\right) + \frac{1}{4}\int_0^L bu^4\,dx.$$

The polynomial $P\,(\xi)$ is bounded from below for $\xi \geq 0$ by a negative constant $-\alpha$. Thus we have

$$\begin{aligned} J(u, L) &\geq -\frac{\alpha}{2}\left(\sum_{k=1}^{\infty} c_k^2 + \frac{b_1}{4}\int_0^L u^4 dx\right) \\ &\geq -\frac{\alpha}{2}\|u\|_{L^2}^2 + \frac{b_1}{4L^2}\|u\|_{L^2}^4 \\ &\geq -\frac{\alpha^2 L^2}{4b_1}, \end{aligned}$$

so that the functional J is bounded from below.

Let $(u_n)_n$ be a minimizing sequence and let $C_1 > J\,(u_n)$. In what follows C_j will denote various positive constants. We have

$$J(u) = \frac{1}{4}\int_0^L u'''^2 dx + \frac{1}{2}\widetilde{J}(u),$$

where

$$\widetilde{J}(u) = \int_0^L \left\{\frac{1}{2}\left(u'''^2 - 2Au''^2 + 2Bu'^2 - 2Cu^2\right) + \frac{1}{2}b\,(x)\,u^4\right\} dx.$$

As before, the corresponding polynomial $\widetilde{P}$ for $\widetilde{J}$ is bounded from below for $\xi \geq 0$. By the above considerations and Lemma 3.2 we have

$$C_1 > J\,(u_n) \geq \frac{1}{4}\int_0^L u'''^2 dx - C_2,$$

which implies that the sequence $(u_n)_n$ is bounded in $Y(L)$.

Let $(u_n)_n$ be a (PS) sequence. Arguing as before we show that $(u_n)_n$ is a bounded sequence. Following [TCH2], Theorem 3, Step 1 we concude that $(u_n)_n$ has a convergent subsequence.

Suppose now that A, B and C satisfy hypothesis (H1). Then we consider the eigenvalue problem

$$\begin{aligned} & u^{vi} + Au^{iv} + Bu'' + \lambda u = 0, \quad 0 < x < L, \\ & u\,(0) = u''\,(0) = u^{iv}\,(0) = 0 \\ & u\,(L) = u''\,(L) = u^{iv}\,(L) = 0, \end{aligned} \tag{4.5}$$

which is equivalent to the equation

$$\int_0^L \left(u'''v''' - Au''v'' + Bu'v'\right) dx = \lambda \int_0^L Cuv\, dx \quad \text{for} \quad v \in Y\,(L)\,. \tag{4.6}$$

A direct calculation shows that the functions $e_n\,(x) = \sin\left(\frac{n\pi x}{L}\right)$, $n \in \mathbf{N}$ are eigenfunctions of Problem (4.5) (respectively (4.6)) and the corresponding eigenvalues are

$$\mu_n^{(1)}\,(L) = \frac{1}{C}\left\{\left(\frac{n\pi}{L}\right)^6 - A\left(\frac{n\pi}{L}\right)^4 + B\left(\frac{n\pi}{L}\right)^2\right\}. \tag{4.7}$$

In view of Lemma 4.2, Problem (4.5) has no other eigenvalues except the numbers $\mu_n^{(1)}\,(L)$, $n \in \mathbf{N}$.

Problem (P_2) satisfies Assumption (A_1), with bilinear forms

$$\begin{aligned} a_3\,(u, v) &:= \int_0^L \left(u'''v''' - Au''v'' + Bu'v'\right)\, dx, \\ b_3\,(u, v) &:= \int_0^L Cuvdx, \end{aligned}$$

in the space $Y\,(L)$.

Suppose that either (H2) or (H3) hold and consider the eigenvalue problem

$$\begin{aligned} & u^{vi} + \lambda Au^{iv} + Bu'' + \lambda Cu = 0, \quad 0 < x < L, \\ & u\,(0) = u''\,(0) = u^{iv}\,(0) = 0 \\ & u\,(L) = u''\,(L) = u^{iv}\,(L) = 0, \end{aligned} \tag{4.8}$$

which is equivalent to the equation

$$\int_0^L \left(u'''v''' + Bu'v'\right) dx = \lambda \int_0^L \left(Au''v'' + Cuv\right) dx \quad \text{for} \quad v \in Y\,(L)\,. \tag{4.9}$$

As before the eigenfunctions are $e_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, $n \in \mathbf{N}$ and corresponding eigenvalues are

$$\mu_n^{(2)}(L) = \frac{\left(\frac{n\pi}{L}\right)^6 + B\left(\frac{n\pi}{L}\right)^2}{A\left(\frac{n\pi}{L}\right)^4 + C}. \tag{4.10}$$

In view of Lemma 4.2, Problem (4.8) has no other eigenvalues except the numbers $\mu_n^{(2)}(L)$, $n \in \mathbf{N}$.

Problem (P_2) satisfies Assumption (A_1), with bilinear forms

$$a_4(u,v) := \int_0^L \left(u'''v''' + Bu'v'\right) dx,$$
$$b_4(u,v) := \int_0^L \left(Au''v'' + Cuv\right) dx,$$

in the space $Y(L)$.

Suppose that (H4) holds, and consider the eigenvalue problem

$$\begin{array}{l} u^{vi} + \lambda(Au^{iv} + Bu'' + Cu) = 0, \quad 0 < x < L, \\ u(0) = u''(0) = u^{iv}(0) = 0 \\ u(L) = u''(L) = u^{iv}(L) = 0, \end{array} \tag{4.11}$$

if $A \geq 0$, and

$$\begin{array}{l} u^{vi} + Au^{iv} + \lambda(Bu'' + Cu) = 0, \quad 0 < x < L, \\ u(0) = u''(0) = u^{iv}(0) = 0 \\ u(L) = u''(L) = u^{iv}(L) = 0, \end{array} \tag{4.12}$$

if $A < 0$. The corresponding eigenvalues are

$$\mu_n^{(3)}(L) = \frac{\left(\frac{n\pi}{L}\right)^6}{A\left(\frac{n\pi}{L}\right)^4 + B\left(\frac{n\pi}{L}\right)^2 + C}, \tag{4.13}$$

and

$$\mu_n^{(4)}(L) = \frac{\left(\frac{n\pi}{L}\right)^6 - A\left(\frac{n\pi}{L}\right)^4}{-B\left(\frac{n\pi}{L}\right)^2 + C}, \tag{4.14}$$

and the corresponding bilinear forms are

$$a_5(u,v) := \int_0^L u'''v'''\, dx,$$
$$b_5(u,v) := \int_0^L (Au''v'' - Bu'v' + Cuv)\, dx,$$

and

$$a_6(u,v) := \int_0^L (u'''v''' - Au''v'')\,dx,$$

$$b_6(u,v) := \int_0^L (-Bu'v' + Cuv)\,dx.$$

They all satisfy Assumption (A1).

It follows directly from the above considerations that:

LEMMA 4.5. *Suppose that either* (H1) *or* (H2) *or* (H3) *or* (H4) *holds. Then the equation* $\mu_m^{(j)}(L) = 1$ *in which* $j = 1, 2, 3$ *or* 4, *has a unique positive solution* mL_0 *where* $L_0 = \pi/\xi_0$ *and* ξ_0 *is the unique positive solution of the equation* $P(\xi) = 0$.

In view of Theorem 2.1 and Theorem 2.2 the statement of Theorem C follows.

We conclude with a few remarks:

Remark 1. If $A^2 > 3B$ and

$$C_-(A,B) < C < C_+(A,B),$$

then the polynomial $P(\xi)$ is negative on a set of the form $(0,\xi_1) \cup (\xi_2, \xi_2)$, where $0 < \xi_1 < \xi_2 < \xi_3 < \infty$, and the set of values of L in which there exist nontrivial periodic solutions may be more complex. For the Swift-Hohenberg equation this situation has been investigated in [PR1,2]. In a subsequent paper we will return to this question.

Remark 2. In this paper we have focussed on the case when $C > 0$. If $C < 0$, then the equation $r(\lambda) = 0$ has either zero, one or two positive roots. If it has no roots, which is the case when

$$|C| > C_-(A,B),$$

then $P(\xi) > 0$ for all $\xi \in \mathbf{R}$ and the methods of this paper show that there are no nontrivial periodic solutions. If there are one or two roots then we are in a situation comparable to the one in the previous Remark.

Remark 3. The results established in this paper can readily be extended to equations in which the nonlinearity u^3 has been replaced by a more general odd nonlinearity $g(u)$, such as $g(u) = |u|^{p-1}u$ with $p > 2$.

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