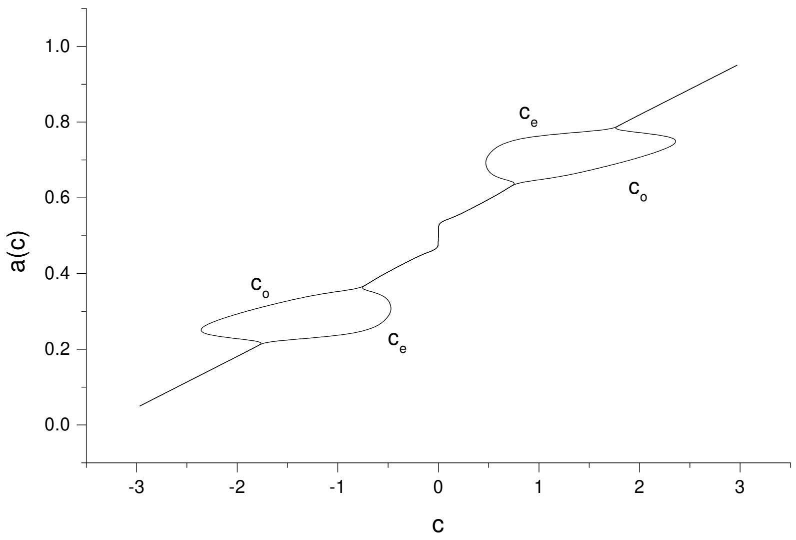
Analysis of Newton's Method to Compute Travelling Wave Solutions to Lattice Differential Equations

By

Hermen Jan Hupkes



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ANALYSIS OF NEWTON'S METHOD TO COMPUTE TRAVELLING WAVE SOLUTIONS TO LATTICE DIFFERENTIAL EQUATIONS

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Table of Contents

Abstract

We present a variant of Newton's Method for computing travelling wave solutions to bistable lattice differential equations. We prove that the method converges to a solution, obtain existence and uniqueness of solutions to such equations with a small second order term and study the limiting behaviour of such solutions as this second order term tends to zero. The robustness of the algorithm will be discussed using numerical examples. These results will also be used to illustrate phenomena like propagation failure, which are encountered when studying lattice differential equations. We finish by outlining some properties of higher dimensional systems, including a period two bifurcation.

Chapter 1

Introduction

The purpose of this thesis is to present and implement a numerical method to solve families of bistable differential difference equations of the form

$$-\gamma\phi''(\xi) - c\phi'(\xi) = F\big(\phi(\xi), \phi(\xi + r_1), \ldots, \phi(\xi + r_N), \rho\big). \tag{1.1}$$

Here $\gamma > 0$ is a fixed parameter, c is an unknown wavespeed, ρ can be thought of as a detuning parameter and the diagonal function $-F(x, \ldots, x, \rho)$ is an N-shaped function which depends C^1-smoothly on ρ. The numbers r_i are shifts which may have either sign. The condition $\gamma > 0$ is a technical restriction imposed by the numerical method, as we shall discuss later on.

The algorithm we present, was originally proposed by Elmer and Van Vleck in [13]. Our contribution here is to give a detailed analysis of the method. In particular, we shall show that it converges to a solution of (1.1) and use numerical examples to discuss some of the issues connected to solving (1.1). In addition, we shall obtain existence and uniqueness of solutions to (1.1) and prove that these solutions depend C^1-smoothly on the detuning parameter ρ. These results extend earlier results in [24], where the $\gamma = 0$ case was treated. To relate this case to our situation where $\gamma > 0$, we shall also prove that a sequence of solutions to (1.1) with γ tending to zero converges to a solution with $\gamma = 0$.

Equation (1.1) arises naturally when studying so-called lattice differential equations, which are infinite systems of ordinary differential equations indexed by points on a spatial lattice, such as the D-dimensional integer lattice $\mathbb{Z}^D$. Consider, for example, the infinite system

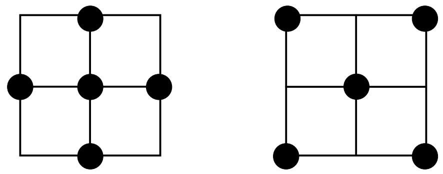
$$\dot{u}_{i,j} = \alpha(L_D u)_{i,j} - f(u_{i,j}, q), \quad (i,j) \in \mathbb{Z}^2, \tag{1.2}$$

on the lattice $\mathbb{Z}^2$. Here $f : \mathbb{R} \times (-1, 1) \to \mathbb{R}$ typically is a bistable nonlinearity of the form

$$f(u, q) = (u - q)(u^2 - 1) \tag{1.3}$$

for some parameter $-1 < q < 1$ and L_D is a discrete Laplacian, which could be given either by

$$\begin{aligned}(L_D u)_{i,j} &= (\Delta^+ u)_{i,j} \equiv u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}, \qquad \text{or}\\ (L_D u)_{i,j} &= (\Delta^\times u)_{i,j} \equiv u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j}.\end{aligned} \tag{1.4}$$



(a) (b)

The discrete Laplacian Δ^+ is depicted in Figure 1.1(a) and involves only the nearest neighbours of a point on the lattice, while $\Delta^\times$ involves the next nearest neighbours as illustrated in Figure 1.1(b).

Figure 1.1: *In (a) the discrete Laplacian Δ^+ is depicted, while (b) describes $\Delta^\times$.*

Equation (1.2) with $\alpha = h^{-2}$ arises, for example, when one discretizes the continuous reaction diffusion equation on $\mathbb{R}^2$,

$$u_t = \Delta u - f(u, q), \tag{1.5}$$

to a rectangular lattice with spacing h. Here the Laplace operator Δ is given by $\Delta u = u_{xx} + u_{yy}$. We shall see in the sequel that away from the continuous limit, i.e., for small values of α, the dynamical behaviour of (1.2) is quite different than that of its continuous counterpart (1.5).

Travelling wave solutions to the PDE (1.5) have played a crucial role in the analysis of partial differential equations and have been studied extensively. For example, the classic work of Fife and McLeod [15] is concerned with solutions to (1.5) of the form $u(x,t) = \phi(k \cdot x - ct)$. Here the unit vector k indicates the direction in which the wave propagates and c is the unknown wavespeed which has to be determined along with the waveprofile ϕ. Upon substitution of this ansatz in (1.5), one obtains the second order ordinary differential equation

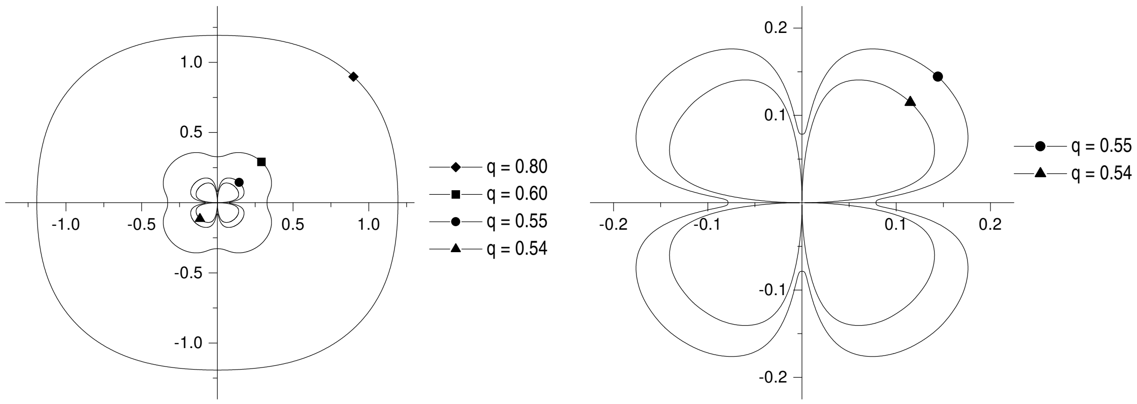
$$-c\phi'(\xi) = \phi''(\xi) - f(\phi(\xi), q), \quad \xi \in \mathbb{R}. \tag{1.6}$$

Following this approach, we can also study travelling wave solutions to equation (1.2). Substituting the travelling wave ansatz $u_{i,j}(t) = \phi(ik_1 + jk_2 - ct)$ into (1.2) with Δ^+, we arrive at the differential difference equation

$$-c\phi'(\xi) = \alpha\big(\phi(\xi + k_1) + \phi(\xi - k_1) + \phi(\xi + k_2) + \phi(\xi - k_2) - 4\phi(\xi)\big) - f(\phi(\xi), q), \tag{1.7}$$

which is a special case of (1.1).

The discretisation of partial differential equations away from the continuous limit can be extremely important when one studies interactions which are nonlocal, i.e., take place at a finite, but nonzero length scale. A crystal presents a perfect example to illustrate this fact. The ions interact

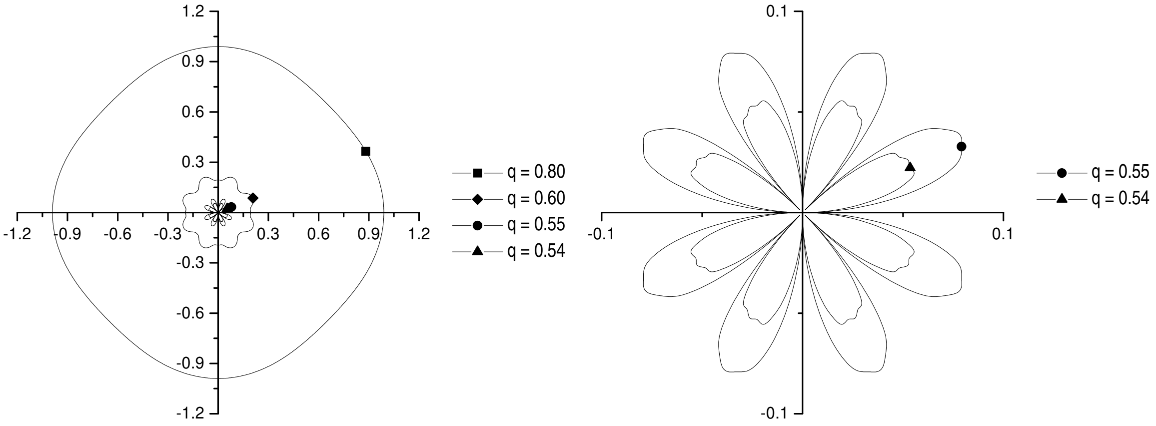


(a) (b)

Figure 1.2: *These figures are polar plots representing the* $c(\theta)$ *relation for solutions to (1.7), with* $\alpha = 1$ *and cubic nonlinearity* $f(x, q) = 10x(x-1)(x-q)$*, at different values of the detuning parameter* q*. Here* θ *indicates the angle of propagation through the lattice, i.e.,* $k = (\cos\theta, \sin\theta)$*. Figure (b) is just a magnification of (a) to illustrate the behaviour for small values of the wavespeed* c *in greater detail. The solutions were required to connect the equilibrium solutions* $\phi = 0$ *and* $\phi = 1$*, i.e., to satisfy the limits* $\lim_{\xi\to-\infty}\phi(\xi) = 0$ *and* $\lim_{\xi\to\infty}\phi(\xi) = 1$*. The results were obtained by using the numerical method discussed in this thesis. For computational purposes, a small term* $-10^{-5}\phi''(\xi)$ *was added to the left-hand side of (1.7). Notice that for* $q = 0.80$ *the wavespeed is almost independent of the propagation angle* θ*, while for values of* q *closer to* $q = 0.5$ *this is clearly not the case.*

with each other but are situated on a lattice in $\mathbb{R}^3$ at finite distances h from one another. To simplify the model (and the boundary conditions), one often takes the limit $h \to 0$ to arrive at a continuous PDE like (1.5). However, a lot of information is lost in this process. For instance, a lattice does not look the same from every angle, hence one expects that travelling wave solutions to (1.2) will depend on the direction of motion, while in the continuous limit this dependence is obviously lost. One can immediately see this by comparing equations (1.6) and (1.7) for the travelling wave. The former is independent of the direction of propagation k, while the latter clearly is not. Numerical results in this direction can be found in Figures 1.2 and 1.3, where travelling wave solutions to the discretized reaction diffusion equation (1.2) were computed for different directions of propagation, $k = (\cos\theta, \sin\theta)$. From the plots of the wavespeed c as a function of the propagation angle θ the lattice anisotropy is clearly visible. It is also interesting to see how the choice of the discrete Laplacian L_D affects the spatial structure, which is a further indication of the rich structure that lattice differential equations possess.

Another example of a property which distinguishes lattice differential equations from their continuous counterparts, is the phenomenon of propagation failure. In the discrete case (1.7), a nontrivial interval of the detuning parameter q can exist in which the wavespeed satisfies $c = 0$. This means the



(a) (b)

Figure 1.3: *These figures are polar plots representing the $c(\theta)$ relation for travelling solutions to (1.2) with $\alpha = \frac{1}{4}$ and discrete Laplacian $(L_D u)_{i,j} = \big((\Delta^+ u)_{i,j} + (\Delta^\times u)_{i,j}\big)$. The nonlinearity was again given by $f(x,q) = 10x(x-1)(x-q)$. As in Figure 1.2, (b) is a magnification of (a).*

waveform $\phi(\xi)$ does not propagate and thus the solution $u_{i,j}(t) = \phi(ik_1 + jk_2 - ct) = \phi(ik_1 + jk_2)$ to (1.2) remains constant in time. This behaviour does not occur for the continuous reaction diffusion equation (1.5). This phenomenon has been studied extensively in [5], where one replaces the cubic nonlinearity f by an idealized nonlinearity to obtain analytic solutions to (1.7). For each propagation angle θ, the quantity $q^*(\theta)$ is defined to be the supremum of values $q > 0$ for which the wavespeed satisfies $c(q,\theta) = 0$. It is proven that this critical value $q^*(\theta)$ typically satisfies $q^* > 0$, depends continuously on θ when $\tan\theta$ is irrational and is discontinuous when $\tan\theta$ is rational or infinite. Numerical investigations in [13] and the present work suggest that the phenomenon of propagation failure is not just an artifact of the idealized nonlinearity f, but also occurs in the case of a cubic nonlinearity. This has recently been confirmed by Mallet-Paret in [26].

It should now be clear that the discrete counterpart (1.2) of the reaction diffusion equation (1.5) has a much richer structure than the continuous variant. This is why lattice differential equations are so interesting. At present, models involving lattice differential equations can be found in many scientific disciplines, including chemical reaction theory [14, 21], image processing and pattern recognition [9], material science [4] and biology [2]. Early papers on the subject by Chi, Bell and Hassard [7] and by Keener [20] were followed by many other which developed the basic theory; see, for example, [5, 8, 16, 18, 19, 22, 24, 29, 31, 32]. The early work by Chi, Bell and Hassard [7] already contained computations of solutions to lattice differential equations and Elmer and Van Vleck have performed extensive calculations on equations of the form (1.1) in [10, 11, 12, 13]. In their early works [10, 11], the nonlinearity f was replaced by an idealized nonlinearity. However, in [13] they developed a method for arbitrary nonlinearities f, which we further investigate in this thesis. At present, see [1], they are developing a general purpose numerical solver for functional differential equations of mixed type, based on the method which we study in this thesis.

Notice that (1.7) contains no second derivative term, while (1.1) does. This second order term has been introduced to allow us to use a boundary value solver for ordinary differential equations like COLMOD [6]. This issue is discussed in further depth in Section 6.2. Since the behaviour of solutions in the singular perturbation limit $\gamma \to 0$ and $c \to 0$ is very interesting, one hopes that if one chooses the parameter γ to be small enough, one will see the generic behaviour associated to propagation failure. To this end we prove in Theorem 4.3.3 that solutions to (1.1) with increasingly small γ converge to a solution with $\gamma = 0$. There is also a physical reason to introduce a second order term in (1.1). Such a term arises naturally if we consider systems which have local as well as nonlocal interactions and it allows us to perform continuation from systems with a continuous Laplacian to systems with a discrete Laplacian.

This paper is organized as follows. In Chapter 2 we introduce the general Fredholm theory developed in [23] for linear functional equations of mixed type. In Chapter 3 basic properties of solutions to (1.1) are established, including comparison principles which we shall use frequently in later chapters. Also, a detailed Fredholm result, Theorem 3.5.1, is presented there for a class of linear differential difference equations.

In Chapter 4, we set out to establish existence and uniqueness of solutions to (1.1). We introduce the operator $\mathcal{G} : W_0^{2,\infty} \times \mathbb{R} \times V \to L^\infty$ associated to (1.1) and given by

$$\mathcal{G}(\phi, c, \rho)(\xi) = -\gamma\phi''(\xi) - c\phi'(\xi) - F\big(\phi(\xi), \phi(\xi + r_1), \ldots, \phi(\xi + r_N), \rho\big). \tag{1.8}$$

Solutions to (1.1) correspond to zeroes of $\mathcal{G}$. In the first part of Chapter 4, Theorem 3.5.1 is used to prove that the Frechet derivative $D_{1,2}\mathcal{G}$ of $\mathcal{G}$, evaluated at a solution (P, c) to (1.1) at some parameter ρ_0, is in fact an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ to L^∞ (Proposition 4.2.6). This allows us to make a smooth local continuation $(P(\rho), c(\rho))$ of solutions around ρ_0. In the second part of Chapter 4 we will establish the uniqueness of solutions and prove Theorem 4.3.3, which enables us to turn the local continuation from the first part into a global continuation. In order to obtain the existence of solutions we solve an explicit equation of the form (1.1) and use a homotopy of systems to extend this solution to an arbitrary family (1.1).

In Chapter 5 the algorithm we used is discussed and we prove its convergence to a solution of (1.1). The algorithm is a modified Newton iteration, which uses the inverse of a linear operator $D_{1,2}\mathcal{F}$, where $\mathcal{F}$ is closely associated to the operator $\mathcal{G}$, but with a relaxation on the shifted terms. Our analysis of the method relies heavily on the isomorphism result in Proposition 4.2.6, which can be extended to the operator $D_{1,2}\mathcal{F}$. In Chapter 6 we use our algorithm to calculate solutions to a specific family (1.1). The results will be used to illustrate some of the technical difficulties involved in the application of our method. Considerable attention will be devoted to the phenomenon of propagation failure and the issue of approaching the solutions in the singular perturbation limit $\gamma \to 0$ and $c \to 0$. Finally, in the last chapter, we will address some issues connected to the generalization of the algorithm and theory to higher dimensions.

Chapter 2

Linear Functional Differential Equations of Mixed Type

2.1 Preliminaries and Notation

In this chapter we present and extend the results obtained by Mallet-Paret in [23] concerning the linear functional differential equation of mixed type

$$x'(\xi) = \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j) + h(\xi). \tag{2.1.1}$$

Linear equations of the form (2.1.1) arise when one considers the linearization of (1.1) around a particular solution $\phi(\xi)$. In order to investigate the nonlinear equation (1.1) it will turn out to be crucial to understand the properties of the associated linear differential difference equation. Results in this direction will be given in the next section, while in this section we will introduce the terminology we shall need.

Throughout this chapter we will assume that the complex matrix coefficients $A_j : J \to \mathbb{C}^{d\times d}$ are measurable and uniformly bounded on some (usually infinite) interval J and that the inhomogeneity $h : J \to \mathbb{C}^d$ is locally integrable. The quantities r_j, the so-called shifts, can have either sign. As a technical restriction we shall assume $r_0 = 0$ and $r_i \neq r_j$ whenever $i \neq j$. For convenience we demand that $N \geq 1$. It should be noted that in this case this is not a restriction on (2.1.1), as we can always take any matrix coefficient of A_j to vanish identically on J. We define the quantities

$$\begin{aligned} r_{\min} &= \min \{r_j \mid j = 0 \dots N\}, \\ r_{\max} &= \max \{r_j \mid j = 0 \dots N\} \end{aligned} \tag{2.1.2}$$

and observe that $r_{\min} \leq 0 \leq r_{\max}$ and $r_{\min} < r_{\max}$.

It will be convenient to introduce the function $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^d)$ defined by $x_\xi(\theta) = x(\xi + \theta)$ for $\theta \in [r_{\min}, r_{\max}]$. This function is called the state of the solution at ξ. We can then rewrite equation (2.1.1) as

$$x'(\xi) = L(\xi)x_\xi + h(\xi), \tag{2.1.3}$$

where $L(\xi)$, for almost every $\xi \in J$, denotes the linear functional

$$L(\xi)\phi = \sum_{j=0}^{N} A_j(\xi)\phi(r_j), \quad \phi \in C([r_{\min}, r_{\max}], \mathbb{C}^d) \tag{2.1.4}$$

from $C([r_{\min}, r_{\max}], \mathbb{C}^d)$ into $\mathbb{C}^d$. When the function h is absent, we have the homogeneous system

$$x'(\xi) = L(\xi)x_\xi. \tag{2.1.5}$$

A special case of (2.1.4) occurs when all the matrix functions $A_j(\xi)$ are constants. We then have the constant coefficient operator

$$L_0(\phi) = \sum_{j=0}^{N} A_{j,0}\phi(r_j) \tag{2.1.6}$$

and the homogeneous constant coefficient system

$$x'(\xi) = L_0 x_\xi. \tag{2.1.7}$$

Definition 2.1.1. A solution to equation (2.1.3) on an interval J is a continuous function $x : J^\# \to \mathbb{C}^d$, defined on the larger interval

$$J^\# = \{\xi + \theta \mid \xi \in J \text{ and } \theta \in [r_{\min}, r_{\max}]\}, \tag{2.1.8}$$

such that x is absolutely continuous on J and satisfies (2.1.3) for almost every $\xi \in J$. □

From now on we shall assume $J = \mathbb{R}$, unless explicitly stated otherwise. Throughout this thesis we shall use the shorthand L^p for the space $L^p(\mathbb{R}, \mathbb{C}^d)$ of L^p vector-valued functions on the line and assume that $1 \leq p \leq \infty$. We also define the spaces

$$\begin{aligned} W^{1,p} &= \{f \in L^p \mid f \text{ is absolutely continuous and } f' \in L^p\}, \\ W^{2,p} &= \left\{f \in L^p \mid f \text{ is absolutely continuous and } f' \in W^{1,p}\right\}. \end{aligned} \tag{2.1.9}$$

If X is a Banach space, then we shall denote the norm of an element $x \in X$ by $\|x\|_X$, or more shortly $\|x\|$ when the underlying space is obvious. If also Y is a Banach space and $T : X \to Y$ is a bounded linear operator, then we denote the operator norm by $\|T\|$. We will refer to the kernel and range of T respectively by

$$\mathcal{K}(T) = \{x \in X \mid Tx = 0\}, \quad \mathcal{R}(T) = \{y \in Y \mid y = Tx \text{ for some } x \in X\}. \tag{2.1.10}$$

We recall that T is a Fredholm operator in case

- the kernel $\mathcal{K}(T) \subseteq X$ is finite dimensional;
- the range $\mathcal{R}(T) \subset Y$ is closed; and
- $\mathcal{R}(T)$ has finite codimension in Y.

For such an operator the Fredholm index is defined to be the integer

$$\operatorname{ind}(T) = \dim \mathcal{K}(T) - \operatorname{codim} \mathcal{R}(T). \tag{2.1.11}$$

Associated to the homogeneous equation (2.1.5) we have the linear operator $\Lambda_L : W^{1,p} \to L^p$ defined by

$$(\Lambda_L x)(\xi) = x'(\xi) - L(\xi)x_\xi. \tag{2.1.12}$$

Using the assumption that the coefficients $A_j(\xi)$ are uniformly bounded, one sees that $\Lambda_L x$ is indeed an element of L^p if $x \in W^{1,p}$ and that Λ_L is a bounded linear operator from $W^{1,p}$ into L^p.

The adjoint equation of (2.1.5) is the equation

$$y'(\xi) = L^*(\xi)y_\xi, \tag{2.1.13}$$

in which

$$L^*(\xi)\phi = -\sum_{j=0}^{N} A_j(\xi - r_j)^* \phi(-r_j), \quad \phi \in C([-r_{\max}, -r_{\min}], \mathbb{C}^d), \tag{2.1.14}$$

with $A_j(\xi - r_j)^*$ denoting the conjugate transpose of the matrix $A_j(\xi - r_j)$. We define the adjoint operator Λ_L^* of Λ_L to be

$$(\Lambda_L^*)(\xi) = -y'(\xi) + L^*(\xi)y_\xi, \tag{2.1.15}$$

that is, $\Lambda_L^* = -\Lambda_{L^*}$. Indeed, using partial integration it is easy to verify that

$$(x, \Lambda_L y) = (\Lambda_L^* x, y), \tag{2.1.16}$$

where $(,)$ denotes the standard inner product $(x, y) = \int_{-\infty}^{\infty} x^*(\xi)y(\xi)d\xi$.

Associated to the constant coefficient system (2.1.7) is the characteristic equation, given by

$$\det \Delta_{L_0}(s) = 0, \tag{2.1.17}$$

where Δ_{L_0}, called the characteristic function, is given by

$$\Delta_{L_0}(s) = sI - \sum_{j=0}^{N} A_{j,0} e^{sr_j}. \tag{2.1.18}$$

We recall that a number $\lambda \in \mathbb{C}$ is an eigenvalue of the constant coefficient system (2.1.7) if and only if it satisfies the characteristic equation, i.e., $\det \Delta_{L_0}(\lambda) = 0$. Elementary solutions $y(\xi)$ of the constant coefficient system (2.1.7) corresponding to the eigenvector λ can be written as $y(\xi) = e^{\lambda\xi}p(\xi)$, for some $\mathbb{R}^d$ valued polynomial p. We will also refer to these solutions as eigensolutions.

Definition 2.1.2. The constant coefficient system (2.1.7) (or more simply L_0) is hyperbolic in case

$$\det \Delta_{L_0}(i\eta) \neq 0 \tag{2.1.19}$$

for all $\eta \in \mathbb{R}$, i.e., there are no eigenvalues on the imaginary axis. $\square$

We shall often write the operator $L(\xi)$ in (2.1.4) as a sum

$$L(\xi) = L_0 + M(\xi) \tag{2.1.20}$$

of a constant coefficient operator L_0 and a perturbation operator $M(\xi) : C([r_{\min}, r_{\max}], \mathbb{C}^d) \to \mathbb{C}^d$, given by

$$M(\xi)\phi = \sum_{j=0}^{N} B_j(\xi)\phi(\xi + r_j). \tag{2.1.21}$$

We shall be specially interested in cases where $M(\xi)$ vanishes as $\xi \to \pm\infty$.

Definition 2.1.3. The system (2.1.5) (or more simply L) is asymptotically autonomous at $+\infty$ if there exist L_0 and M as in (2.1.20), for which

$$\lim_{\xi\to\infty} \|M(\xi)\| = 0. \tag{2.1.22}$$

In this case of course

$$\lim_{\xi\to\infty} A_j(\xi) = A_{j,0}, \quad 0 \leq j \leq N \tag{2.1.23}$$

and (2.1.7) is called the limiting equation at $+\infty$. If in addition this limiting equation is hyperbolic, then we say that (2.1.5) is asymptotically hyperbolic at $+\infty$. These concepts are defined analogously at $-\infty$. If (2.1.5) is asymptotically autonomous at both $\pm\infty$, then we say that (2.1.5) is asymptotically autonomous. Similarly, if (2.1.5) is asymptotically hyperbolic at both $\pm\infty$, then we say that (2.1.5) is asymptotically hyperbolic. $\square$

2.2 Fredholm results for Λ_L

In this section the three main theorems from [23] concerning the operator Λ_L are stated. Also two important results concerning the asymptotic behaviour of solutions to (2.1.3) are included.

The first result states that the linear operator Λ_L associated to asymptotically hyperbolic equations is a Fredholm operator and relates the kernel and range of Λ_L to those of the adjoint operator Λ_L^*.

Theorem 2.2.1 (The Fredholm Alternative [23, Theorem A]). *Assume the homogeneous equation (2.1.5) is asymptotically hyperbolic. Then for each p with* $1 \le p \le \infty$*, the operator* Λ_L *from* $W^{1,p}$ *to* L^p *is a Fredholm operator. The kernel* $\mathcal{K}_L^p \subseteq W^{1,p}$ *of* Λ_L *is independent of p, so we write* $\mathcal{K}_L^p = \mathcal{K}_L$*, and similarly* $\mathcal{K}_{L^*}^p = \mathcal{K}_{L^*}$ *for the kernel of the operator* Λ_{L^*} *associated to the adjoint* L^**. The range* $\mathcal{R}_L^p \subseteq L^p$ *of* Λ_L *is given by*

$$\mathcal{R}_L^p = \left\{ h \in L^p \mid \int_{-\infty}^{\infty} \overline{y(\xi)} h(\xi) d\xi = 0 \text{ for all } y \in \mathcal{K}_{L^*} \right\}. \tag{2.2.1}$$

In particular,

$$\dim \mathcal{K}_{L^*} = \operatorname{codim} \mathcal{R}_L^p, \quad \dim \mathcal{K}_L = \operatorname{codim} \mathcal{R}_{L^*}^p, \quad \operatorname{ind}(\Lambda_L) = -\operatorname{ind}(\Lambda_{L^*}), \tag{2.2.2}$$

where ind *denotes the Fredholm index. Finally, when* $L = L_0$ *is a hyperbolic constant coefficient operator (2.1.6), we have*

$$\operatorname{codim} \mathcal{R}_{L_0} = 0, \quad \dim \mathcal{K}_{L_0} = 0, \quad \operatorname{ind}(\Lambda_{L_0}) = 0 \tag{2.2.3}$$

and so Λ_{L_0} *is an isomorphism.*

Theorem 2.2.2 (The Cocycle Property [23, Theorem B]). *Assume the homogeneous equation (2.1.5) is asymptotically hyperbolic. Then the Fredholm index of* Λ_L *depends only on the limiting operators* $L_\pm$*, namely the limits of* $L(\xi)$ *as* $\xi \to \pm\infty$*. Denoting*

$$\operatorname{ind}(\Lambda_L) = \imath(L_-, L_+), \tag{2.2.4}$$

we have that

$$\imath(L_1, L_2) + \imath(L_2, L_3) = \imath(L_1, L_3) \tag{2.2.5}$$

for any triple L_1*,* L_2*,* L_3*, of hyperbolic constant coefficient operators.*

The next theorem provides an effective way of calculating the Fredholm index of the operator Λ_L in terms of the crossing number, which roughly is the difference between the number of eigenvalues crossing the imaginary axis from right to left and the number of eigenvalues crossing from left to right, along any generic homotopy which takes the limiting system at $-\infty$ to the one at ∞.

Theorem 2.2.3 (The Spectral Flow Property [23, Theorem C]). *Let* L_0^ρ*, for* $-1 \le \rho \le 1$*, be a continuously varying one-parameter family of constant coefficient operators (2.1.6) and suppose* *the operators $L_\pm = L_0^{\pm 1}$ are hyperbolic. Further suppose that there are only finitely many values*

$$\{\rho_1, \rho_2, \ldots, \rho_J\} \subseteq (-1, 1) \tag{2.2.6}$$

of ρ for which L_0^ρ is not hyperbolic. Then

$$\imath(L_-, L_+) = -\mathrm{cross}(L^\rho), \tag{2.2.7}$$

in which $\mathrm{cross}(L^\rho)$ *is the net number of eigenvalues of (2.1.7) which cross the imaginary axis from left to right as ρ increases from -1 to $+1$.*

To define $\mathrm{cross}(L^\rho)$ more precisely, fix any ρ_j as in (2.2.6) in the statement of Theorem 2.2.3 and let $\{\lambda_{j,k}\}_{k=1}^{K_j}$ denote the eigenvalues of the corresponding constant coefficient systems (2.1.7), with $L_0 = L_0^{\rho_j}$, on the imaginary axis, $\mathrm{Re}\,\lambda_{j,k} = 0$. We list these eigenvalues with repetitions, according to their multiplicity as roots of the characteristic equation. Let M_j denote the sum of their multiplicities. For ρ near ρ_j, with $\pm(\rho - \rho_j) > 0$, this equation has exactly M_j eigenvalues (counting multiplicity) near the imaginary axis, $M_j^{L\pm}$ with $\mathrm{Re}\,\lambda < 0$ and $M_j^{R\pm}$ with $\mathrm{Re}\,\lambda > 0$, where $M_j^{L\pm} + M_j^{R\pm} = M_j$. The net crossing number of eigenvalues at $\rho = \rho_j$ is thus $M_j^{R+} - M_j^{R-}$. We can now define

$$\mathrm{cross}(L^\rho) = \sum_{j=1}^{J} (M_j^{R+} - M_j^{R-}). \tag{2.2.8}$$

The next proposition will turn out to be extremely useful when obtaining asymptotic estimates on solutions to (1.1). It enables us to turn the detailed information about the eigenvalues of (2.1.5) which we shall obtain for our class of differential difference equations into very precise statements concerning the decay rate of our solutions.

Proposition 2.2.4. *Let $x : J^\# \to \mathbb{C}^d$ be a solution to equation (2.1.5) on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Assume that (2.1.5) is asymptotically autonomous at $+\infty$, with L written as in (2.1.20). Also assume for some real number a and some positive number $k > 0$, that*

$$x(\xi) = O(e^{-a\xi}), \quad \|M(\xi)\| = O(e^{-k\xi}), \quad \xi \to \infty. \tag{2.2.9}$$

Then either

- *there exist $b \geq a$ and $\epsilon > 0$ such that*

$$x(\xi) = y(\xi) + O(e^{-(b+\epsilon)\xi}), \quad \xi \to \infty, \tag{2.2.10}$$

where y is a nontrivial eigensolution of the limiting equation (2.1.7) corresponding to the nonempty set of eigenvalues with $\mathrm{Re}\,\lambda = -b$*; or else*

- *for each $b \in \mathbb{R}$ we have that*

$$\lim_{\xi \to \infty} e^{b\xi} x(\xi) = 0. \tag{2.2.11}$$

This final proposition shows that solutions to (2.1.5) decay exponentially. Note that it is not required here that the coefficients $A_j(\xi)$ approach their limits exponentially fast.

Proposition 2.2.5. *Assume that equation (2.1.3) is asymptotically hyperbolic at $+\infty$. Then there exist positive quantities K, K' and a such that for all pairs of functions $x \in W^{1,p}$ and $h \in L^p$ which satisfy $\Lambda_L x = h$, the estimate*

$$|x(\xi)| \leq K e^{-a\xi} \|x\|_{L^\infty} + K' \|h\|_{L^p} \tag{2.2.12}$$

holds for all $\xi \geq 0$.

2.3 One Dimensional Systems

Up to this point all of the theory has been presented for systems. We will now specialize the Fredholm results to the case at hand, which is the scalar equation

$$-\gamma x''(\xi) - cx'(\xi) = L(\xi)x_\xi + h(\xi), \tag{2.3.1}$$

where $\gamma \neq 0$ and $L(\xi)$ is again defined by

$$L(\xi)\phi = \sum_{j=0}^{N} A_j(\xi)\phi(r_j). \tag{2.3.2}$$

Here x, A_j and h represent real valued functions, i.e., we take $d = 1$ in the terminology of the previous section. Note that in this section we also allow negative values of γ.

In the homogeneous case (2.3.1) reduces to

$$-\gamma x''(\xi) - cx'(\xi) = L(\xi)x_\xi. \tag{2.3.3}$$

The linear operator $\Lambda_{c,\gamma,L} \colon W^{2,\infty} \to L^\infty$ associated to (2.3.1) is given by

$$\big(\Lambda_{c,\gamma,L}(x)\big)(\xi) = -\gamma x''(\xi) - cx'(\xi) - L(\xi)x_\xi. \tag{2.3.4}$$

We define the adjoint operator $\Lambda^*_{c,\gamma,L} \colon W^{2,\infty} \to L^\infty$ to be

$$\big(\Lambda^*_{c,\gamma,L}(y)\big)(\xi) = -\gamma y''(\xi) + cy'(\xi) + L^*(\xi)y_\xi. \tag{2.3.5}$$

We note here that the results in Section 2.2 do not apply directly to the operators $\Lambda_{c,\gamma,L}$ and $\Lambda^*_{c,\gamma,L}$. However, in Proprosition 2.3.1 we shall embed the second order equation (2.3.3) into a first order two-dimensional system which is covered by the theorems in Section 2.2. By comparing solutions of (2.3.3) to solutions of the corresponding two-dimensional system, we can establish that all the results in Section 2.2 also hold for the operators $\Lambda_{c,\gamma,L}$ and $\Lambda^*_{c,\gamma,L}$.

This embedding will also immediately allow us to read off how hyperbolicity should be defined in the current setting. We shall see that in the constant coefficient case, where (2.3.3) reduces to

$$-\gamma x''(\xi) - cx'(\xi) = L_0 x_\xi, \tag{2.3.6}$$

for some constant coefficient operator L_0 given by (2.1.6), the characteristic function corresponding to (2.3.6), which is now a scalar, is given by

$$\Delta_{c,\gamma,L_0}(s) = -\gamma s^2 - cs - \sum_{j=0}^{N} A_{j,0} e^{sr_j}. \tag{2.3.7}$$

Correspondingly, we say that the constant coefficient equation (2.3.3) with $L = L_0$ is hyperbolic if

$$\Delta_{c,\gamma,L_0}(i\eta) \neq 0 \tag{2.3.8}$$

for all $\eta \in \mathbb{R}$. With this definition it is clear what is meant by the notion of asymptotic hyperbolicity in the present setting.

Proposition 2.3.1. *Assume the homogeneous equation (2.3.3) with $\gamma \neq 0$ is asymptotically hyperbolic. Then all the results in the statement of Theorems 2.2.1, 2.2.2 and 2.2.3 hold for the operator $\Lambda_{c,\gamma,L}$, with the convention that the adjoint of this operator is given by (2.3.5) and the function $\imath(\,,\,)$ defined in Theorem 2.2.2 also depends on the parameters c and γ.*

Proof. Consider the first order homogeneous system

$$\begin{cases} x_1'(\xi) = x_2(\xi) \\ x_2'(\xi) = -\frac{1}{\gamma} L(\xi)(x_1)_\xi - \frac{c}{\gamma} x_2(\xi), \end{cases} \tag{2.3.9}$$

and let $\widetilde{\Lambda}_{c,\gamma,L}$ be the corresponding operator from $W^{1,p}(\mathbb{R}, \mathbb{R}^2)$ to $L^p(\mathbb{R}, \mathbb{R}^2)$. First note that the characteristic equations (2.1.17) for the limiting systems of (2.3.9) are equivalent to the characteristic equations $\Delta_{c,\gamma,L_\pm} = 0$, with Δ_{c,γ,L_0} given by (2.3.7). Thus, we see that the system (2.3.9) is asymptotically hyperbolic. The adjoint equation of (2.3.9) is given by

$$\begin{cases} y_1'(\xi) = -\frac{1}{\gamma} L^*(\xi)(y_2)_\xi \\ y_2'(\xi) = -y_1 + \frac{c}{\gamma} y_2. \end{cases} \tag{2.3.10}$$

Writing $\widetilde{\Lambda}^*_{c,\gamma,L}$ for the operator corresponding to (2.3.10), we see that also this system is asymptotically hyperbolic, allowing us to apply Theorem 2.2.1 to conclude that $\widetilde{\Lambda}_{c,\gamma,L}$ and $\widetilde{\Lambda}^*_{c,\gamma,L}$ are Fredholm operators. Now suppose that $x = (x_1, x_2) \in \mathcal{K}(\widetilde{\Lambda}_{c,\gamma,L})$. It is then immediately obvious from (2.3.9) that in fact $x_1 \in W^{2,p}(\mathbb{R}, \mathbb{R})$, as $x_1' = x_2$. Furthermore, we see that x_1 is a solution to (2.3.3). Conversely, supposing $x \in W^{2,p}(\mathbb{R}, \mathbb{R})$ is a solution to (2.3.3), we have that (x, x') is an element of the kernel $\mathcal{K}(\widetilde{\Lambda}_{c,\gamma,L})$. This gives us $\mathcal{K}(\Lambda_{c,\gamma,L}) \cong \mathcal{K}(\widetilde{\Lambda}_{c,\gamma,L})$, under the map $y \to (y, y')$ and thus proves that the kernel $\mathcal{K}(\Lambda_{c,\gamma,L})$ is finite dimensional. Now suppose that $y = (y_1, y_2) \in \mathcal{K}(\widetilde{\Lambda}^*_{c,\gamma,L})$. It follows immediately from (2.3.10) that y_2 satisfies the adjoint equation

$$-\gamma y_2''(\xi) + c y_2'(\xi) + L^*(\xi) y_2(\xi) = 0. \tag{2.3.11}$$

This means that we also have the isomorphism

$$\mathcal{K}(\Lambda^*_{c,\gamma,L}) \cong \mathcal{K}(\widetilde{\Lambda}^*_{c,\gamma,L}), \tag{2.3.12}$$

under the map $y \to (-y' + \frac{c}{\gamma} y, y)$.

Shifting our focus onto the range $\mathcal{R}(\widetilde{\Lambda}_{c,\gamma,L})$, suppose that $x = \widetilde{\Lambda}_{c,\gamma,L} v$ for some $v = (v_1, v_2)$, which satisfies $v_1' = v_2$. It is obvious that $x_1 = 0$ and in addition Theorem 2.2.1 implies the identity

$$\int_{-\infty}^{\infty} x(\xi) y(\xi) d\xi = 0 \tag{2.3.13}$$

for all $y \in \mathcal{K}(\widetilde{\Lambda}^*_{c,\gamma,L})$. In particular, using $x_1 = 0$ and the isomorphism (2.3.12), this means

$$\int_{-\infty}^{\infty} x_2(\xi) w(\xi) d\xi = 0 \tag{2.3.14}$$

for all $w \in \mathcal{K}(\Lambda^*_{c,\gamma,L})$. Conversely, suppose that a function $x \in L^p(\mathbb{R}, \mathbb{R})$ satisfies (2.3.14) for all $y \in \mathcal{K}(\Lambda^*_{c,\gamma,L})$. Then it follows that $(0, x) \in \mathcal{R}(\widetilde{\Lambda}_{c,\gamma,L})$ and thus also $x \in \mathcal{R}(\Lambda_{c,\gamma,L})$. From this, the claims about the range $\mathcal{R}(\Lambda_{c,\gamma,L})$ immediately follow. This establishes all the claims of Theorem 2.2.1 for the operator $\Lambda_{c,\gamma,L}$. In addition, as $\mathrm{ind}(\Lambda_{c,\gamma,L}) = \mathrm{ind}(\widetilde{\Lambda}_{c,\gamma,L})$ and the characteristic equations of the two operators are the same, we see that the claims in Theorems 2.2.2 and 2.2.3 also hold for the operator $\Lambda_{c,\gamma,L}$. $\square$

We remark here that it is now easy to see that Propositions 2.2.4 and 2.2.5 also hold for solutions to (2.3.3).

Chapter 3

Preliminary Results on Scalar Equations

The aim of this chapter is to provide some basic results on a special class of scalar differential difference equations, which will be used frequently throughout later chapters. We will mainly be concerned with linear equations (2.3.1), where the coefficients $A_j(\xi)$ for the shifted arguments, $j = 1 \ldots N$, satisfy some positivity or negativity condition. These conditions are satisfied by the systems which arise when linearizing (1.1) around solutions; thus the results developed here will turn out to be useful in that context. We shall also briefly be concerned with nonlinear equations in Section 3.3, where we use the results developed for linear systems to provide a number of comparison principles for nonlinear equations. In Section 3.1 we use a rescaling argument to rule out the existence of positive solutions to (2.3.1) which decay faster than exponentially. The ideas developed in this section will be further exploited in Sections 3.2 and 3.3, where comparison principles will be established for linear and nonlinear equations. Finally, in Sections 3.4 and 3.5 we shall establish asymptotic hyperbolicity for our class of linear equations (2.3.1) and prove the main theorem of this chapter. This theorem gives conditions under which the linear operator $\Lambda_{c,\gamma,L}$ associated to (2.3.1) is a Fredholm operator with a one dimensional kernel.

3.1 Superexponential Decay

In this section, we will show that the linear equation (2.3.3) admits no positive solutions which decay faster than exponentially, under appropriate conditions on the coefficients. This is especially useful in combination with Proposition 2.2.4, as in the absence of superexponentially decaying solutions this Proposition allows us to obtain asymptotic descriptions of the solutions to (2.3.3). We first define exactly what we mean by solutions that decay faster than exponentially.

Definition 3.1.1. Let $x : J \to \mathbb{R}$ be a continuous function on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Then we say x decays superexponentially or has superexponential decay at $+\infty$ if

$$\lim_{\xi \to \infty} e^{b\xi} x(\xi) = 0 \tag{3.1.1}$$

for every $b \in \mathbb{R}$. We define superexponential decay at $-\infty$ analogously. We will drop the distinction "at $\pm\infty$" if this is clear from the context. Functions decaying superexponentially are often also called small solutions. □

We shall consider equation (2.3.3) on an interval $J = [\tau, \infty)$ and assume that the coefficients are bounded according to

$$\alpha_j \leq A_j(\xi) \leq \beta_j, \quad 0 \leq j \leq N, \quad \xi \in J. \tag{3.1.2}$$

We now introduce two conditions on the linear equation (2.3.3), which concern the bounds in (3.1.2).

Assumption 3.1.1. *The lower bounds α_j in (3.1.2) satisfy*

$$\alpha_j \geq 0, \quad 1 \leq j \leq N. \tag{3.1.3}$$

In addition, if $\alpha_0 \leq 0$ we have the strict inequality

$$c^2 > 4\gamma\alpha_0. \tag{3.1.4}$$

Assumption 3.1.2. *The upper bounds β_j in (3.1.2) satisfy*

$$\beta_j \leq 0, \quad 1 \leq j \leq N. \tag{3.1.5}$$

In addition, if $\beta_0 \geq 0$ we have the strict inequality

$$c^2 > 4\gamma\beta_0. \tag{3.1.6}$$

Note that $A_0(\xi)$ may have either sign.

We remark here that we shall use the result established in the next lemma only in the case where $\gamma > 0$ and Assumption 3.1.1 holds. However, an analysis of the operator $\Lambda_{c,\gamma,L}$ which is deeper than the one we give here is required for a more thorough analysis of the algorithm. The absence of superexponentially decaying solutions will then turn out to be crucial, which is why the result is established for the general case. See Remark 5.3.1 for further details.

Lemma 3.1.1. *Let $x : J^{\#} \to \mathbb{R}$ be a solution to (2.3.3) on $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$, with $\gamma \neq 0$. Assume further that $x(\xi) \geq 0$ for all $\xi \in J^{\#}$, that the coefficients are bounded according to (3.1.2) and that either Assumption 3.1.1 or 3.1.2 holds. Then*

$$\lim_{\xi \to \infty} e^{b\xi} x(\xi) = 0 \tag{3.1.7}$$

cannot hold for every $b \in \mathbb{R}$ unless $x(\xi) = 0$ identically for all $\xi \geq \tau + R$, for some $R > 0$. The analogous result for $J = (-\infty, \tau]$ also holds.

Proof. Throughout this proof we shall assume $\gamma > 0$, as the case $\gamma < 0$ can be handled by multiplying by -1 and switching the assumption on the coefficients. Without loss we shall also assume $J = [\tau, \infty)$, as the case of $J = (-\infty, \tau]$ can be treated by a change of variables $\xi \to -\xi$. For convenience, we introduce the quantity $\widetilde{\alpha}_0 = \min(\alpha_0, \frac{c^2}{4\gamma} - \epsilon)$, where $\epsilon > 0$ is an arbitrary number.

We start out by noting that we can rescale equation (2.3.3) by defining $y(\xi) = e^{\lambda\xi} x(\xi)$, where λ can be chosen appropriately. It is easy to see that $y(\xi)$ satisfies the following differential difference equation

$$y''(\xi) = (2\lambda - \frac{c}{\gamma})y'(\xi) - \lambda(\lambda - \frac{c}{\gamma})y(\xi) - \frac{1}{\gamma}\sum_{j=0}^{N} A_j(\xi)e^{-\lambda r_j} y(\xi + r_j). \tag{3.1.8}$$

First assume that Assumption 3.1.1 holds. Then, because also $y(\xi) \geq 0$ for all $\xi \in J^{\#}$, we have the inequality

$$y''(\xi) \leq (2\lambda - \frac{c}{\gamma})y'(\xi) + \Big(-\lambda(\lambda - \frac{c}{\gamma}) - \frac{\widetilde{\alpha}_0}{\gamma}\Big)y(\xi). \tag{3.1.9}$$

Now choosing $\lambda = \frac{c}{2\gamma}$ we obtain

$$y''(\xi) \leq By(\xi), \quad \xi \in J, \tag{3.1.10}$$

in which $B = \frac{c^2}{4\gamma^2} - \frac{\widetilde{\alpha}_0}{\gamma} > 0$. Due to Lemma A.1, we have that for arbitrary $\xi_0 \in J$,

$$y(\xi) \leq C_1 e^{\sqrt{B}(\xi - \xi_0)} + C_2 e^{-\sqrt{B}(\xi - \xi_0)} \tag{3.1.11}$$

holds for all $\xi \geq \xi_0$. The coefficients C_1 and C_2 in this expression are given by

$$\begin{aligned} C_1 &= \tfrac{1}{2\sqrt{B}}\Big(y'(\xi_0) + \sqrt{B}y(\xi_0)\Big), \\ C_2 &= \tfrac{1}{2\sqrt{B}}\Big(-y'(\xi_0) + \sqrt{B}y(\xi_0)\Big). \end{aligned} \tag{3.1.12}$$

From the nonnegativity of $y(\xi)$ we see that we must have $C_1 \geq 0$, as otherwise (3.1.11) would imply that $y(\xi) < 0$ for sufficiently large ξ. From this we conclude

$$y'(\xi_0) \geq -\sqrt{B}y(\xi_0), \quad \xi_0 \in J, \tag{3.1.13}$$

which immediately implies that $y(\xi)$ and hence $x(\xi)$ cannot have superexponential decay.

We now consider the case in which Assumption 3.1.2 holds. Assuming that $x(\xi)$ decays superexponentially, we set out to find a contradiction. As before, we define $y(\xi) = e^{\lambda\xi}x(\xi)$. Again, because $y(\xi) \geq 0$ for all $\xi \in J^{\#}$, we obtain the inequality

$$y''(\xi) \geq (2\lambda - \frac{c}{\gamma})y'(\xi) + \Big(- \lambda(\lambda - \frac{c}{\gamma}) - \frac{\beta_0}{\gamma}\Big)y(\xi),\ \xi \in J. \tag{3.1.14}$$

Choosing $\lambda = \frac{c}{2\gamma}$, this inequality becomes

$$y''(\xi) \geq By(\xi), \quad \xi \in J, \tag{3.1.15}$$

in which

$$B = \frac{c^2}{4\gamma^2} - \frac{\beta_0}{\gamma}. \tag{3.1.16}$$

Due to the assumption that $c^2 > 4\gamma\beta_0$ whenever $\beta_0 \geq 0$, we see that $B > 0$ holds. As above, we may now conclude that for all $\xi \geq \xi_0$, where $\xi_0 \in J$ is arbitrary,

$$y(\xi) \geq C_1 e^{\sqrt{B}(\xi - \xi_0)} + C_2 e^{-\sqrt{B}(\xi-\xi_0)}, \tag{3.1.17}$$

in which C_1 and C_2 are again given by (3.1.12). By the definition of superexponential decay we see that $\lim_{\xi\to\infty} y(\xi) = 0$, which means that we must have $C_1 \leq 0$. This in turn implies

$$y'(\xi_0) \leq -\sqrt{B}y(\xi_0), \quad \xi_0 \in J. \tag{3.1.18}$$

A direct consequence of this fact is that $y'(\xi) \leq 0$ for all $\xi \in J$.

First consider the case that $r_j > 0$ for $j = 1 \ldots N$. Due to the fact that $y(\xi)$ is non-increasing, we can deduce

$$y''(\xi) \leq \left(\frac{c^2}{4\gamma^2} - \frac{1}{\gamma}\widetilde{\alpha}_0 - \frac{1}{\gamma} \sum_{\substack{j=1 \\ \alpha_j < 0}}^{N} \alpha_j e^{-\frac{c}{2\gamma}r_j} \right) y(\xi). \tag{3.1.19}$$

Since the term between brackets in the above equation is strictly positive, the inequality (3.1.13) follows as before, contradicting the superexponential decay of $y(\xi)$ and hence that of $x(\xi)$.

We now treat the case that $r_j < 0$ for some $j \in \{1, \ldots, N\}$. From (3.1.18) we see that

$$y(\xi) \leq -\frac{1}{\sqrt{B}}y'(\xi), \quad \xi \in J. \tag{3.1.20}$$

Now define

$$z(\xi) = e^{\mu\xi}x(\xi) = e^{(\mu-\lambda)\xi}y(\xi), \tag{3.1.21}$$

where μ will be chosen appropriately. Observe that we have

$$
\begin{aligned}
&z'(\xi) = e^{(\mu-\lambda)\xi} y'(\xi) + (\mu-\lambda) e^{(\mu-\lambda)\xi} y(\xi)\\
&\le e^{(\mu-\lambda)\xi}\big(-\sqrt{B}y(\xi)\big) + (\mu-\lambda) e^{(\mu-\lambda)\xi} y(\xi)\\
&= (\mu-\lambda-\sqrt{B}) z(\xi),
\end{aligned} \tag{3.1.22}
$$

from which it immediately follows that when $\mu < \lambda$,

$$
z(\xi) \le (\mu-\lambda-\sqrt{B})^{-1} z'(\xi), \quad \xi \in J. \tag{3.1.23}
$$

Now define $\epsilon = \max\{|r_j| \mid r_j < 0\}$ and fix $\xi_1 \ge \tau + 2\epsilon$. When $\mu < \lambda$, we see from (3.1.22) that $z(\xi)$ is nonincreasing, which allows us to conclude that $z(\xi + r_j) \le z(\xi_1 - 2\epsilon)$ when $\xi_1 - \epsilon \le \xi \le \xi_1$. Referring back to the scaled equation (3.1.8), we see that when $\xi_1 - \epsilon \le \xi \le \xi_1$, we have

$$
\begin{aligned}
z''(\xi) \quad \le \quad & (2\mu - \tfrac{c}{\gamma}) z'(\xi) + \Big(-\mu(\mu - \tfrac{c}{\gamma}) - \tfrac{1}{\gamma}\alpha_0 - \tfrac{1}{\gamma} \sum_{\substack{r_j<0\\ \alpha_j<0}} \alpha_j e^{-\mu r_j}\Big) z(\xi_1 - 2\epsilon)\\
& - \Big(\tfrac{1}{\gamma} \sum_{\substack{r_j>0\\ \alpha_j<0}} \alpha_j e^{-\mu r_j}\Big) z(\xi)
\end{aligned} \tag{3.1.24}
$$

and thus using (3.1.23),

$$
\begin{aligned}
&z''(\xi) \le D z'(\xi) + C z(\xi_1 - 2\epsilon),\\
&C = -\mu(\mu - \tfrac{c}{\gamma}) - \tfrac{1}{\gamma}\alpha_0 - \tfrac{1}{\gamma} \sum_{\substack{r_j<0\\ \alpha_j<0}} \alpha_j e^{-\mu r_j},\\
&D = 2\mu - \tfrac{c}{\gamma} - \tfrac{1}{\gamma}(\mu - \lambda - \sqrt{B})^{-1} \sum_{\substack{r_j>0\\ \alpha_j<0}} \alpha_j e^{-\mu r_j}.
\end{aligned} \tag{3.1.25}
$$

We can rewrite (3.1.25) as

$$
\Big(e^{-D\xi} z'(\xi)\Big)' \le e^{-D\xi} C z(\xi_1 - 2\epsilon). \tag{3.1.26}
$$

Integrating this from $\xi_1 - \epsilon$ to ξ we obtain

$$
e^{-D\xi} z'(\xi) - e^{-D(\xi_1-\epsilon)} z'(\xi_1 - \epsilon) \le C z(\xi_1 - 2\epsilon)(-\frac{1}{D})(e^{-D\xi} - e^{-D(\xi_1-\epsilon)}). \tag{3.1.27}
$$

Discarding the second term on the left, which is positive by (3.1.18), and integrating from $\xi_1 - \epsilon$ to ξ_1, we get

$$
z(\xi_1) - z(\xi_1 - \epsilon) \le -\frac{1}{D} C\Big(\epsilon - \frac{1}{D}(e^{D\epsilon} - 1)\Big) z(\xi_1 - 2\epsilon). \tag{3.1.28}
$$

Dropping the first term on the left, we arrive at

$$
z(\xi_1 - \epsilon) \ge \frac{1}{D} C\Big(\epsilon - \frac{1}{D}(e^{D\epsilon} - 1)\Big) z(\xi_1 - 2\epsilon). \tag{3.1.29}
$$

From (3.1.25) and the fact that for $\epsilon > 0$

$$\lim_{D\to-\infty} \frac{1}{D}(e^{D\epsilon} - 1) = 0, \tag{3.1.30}$$

we easily see that there exists $\mu^* < \min(0, \lambda)$ such that $C_\mu < 0$, $D_\mu < 0$ and $\epsilon - \frac{1}{D}(e^{D\epsilon} - 1) > 0$ when $\mu \leq \mu^*$. Here we have made the μ dependence $C = C_\mu$ and $D = D_\mu$ explicit. Thus fixing $\mu = \mu^*$, we obtain from (3.1.29)

$$z(\xi_1 - 2\epsilon) \leq C'' z(\xi_1 - \epsilon), \quad C'' > 0, \tag{3.1.31}$$

where $\xi_1 \geq \tau + 2\epsilon$ is arbitrary. We now see that when $r_j < 0$,

$$z(\xi + r_j) \leq z(\xi - \epsilon) \leq C'' z(\xi) \tag{3.1.32}$$

holds for all $\xi \geq \tau + \epsilon$. This gives us the inequality

$$y(\xi + r_j) \leq C''' y(\xi) \tag{3.1.33}$$

for $r_j < 0$, where $C''' = C'' e^{(\lambda-\mu)r_j} > 0$. Finally, we have

$$y''(\xi) \leq B y(\xi) \tag{3.1.34}$$

for all $\xi \geq \tau + \epsilon$, where

$$B = \frac{c^2}{4\gamma^2} - \frac{1}{\gamma}\widetilde{\alpha}_0 - \frac{1}{\gamma}\left(\sum_{\substack{r_j<0 \\ \alpha_j<0}} \alpha_j e^{\frac{-c}{2\gamma} r_j} C''' + \sum_{\substack{r_j>0 \\ \alpha_j<0}} \alpha_j e^{\frac{-c}{2\gamma} r_j} \right), \tag{3.1.35}$$

from which we obtain a contradiction to the superexponential decay of $y(\xi)$. □

3.2 Comparison Principles

The following two lemma's will turn out to be crucial in establishing comparison principles for solutions to the homogeneous and inhomogeneous differential difference equations (2.3.3) and (2.3.1) respectively. The rescaling argument introduced in the proof of Lemma 3.1.1 will again be used to establish the claims.

Lemma 3.2.1. *Let $x : J^\# \to \mathbb{R}$ be a solution to (2.3.3) on $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$, with $\gamma > 0$. Assume further that $x(\xi) \geq 0$ for all $\xi \in J^\#$, that the coefficients are bounded according to (3.1.2) and that the positivity assumption 3.1.1 holds on the coefficients. Then if $x(\xi_0) = 0$ for some $\xi_0 \in J$, we have $x(\xi) = 0$ for all $\xi \geq \xi_0$.*

Proof. From the proof of Lemma 3.1.1, we see that $y(\xi) = e^{\frac{c}{2\gamma}\xi}x(\xi) \geq 0$ satisfies

$$y(\xi) \leq C_1 e^{\sqrt{B}(\xi-\xi_0)} + C_2 e^{-\sqrt{B}(\xi-\xi_0)} \tag{3.2.1}$$

for some $B > 0$ and all $\xi \geq \xi_0$. The coefficients C_1 and C_2 in this expression are given by

$$\begin{aligned} C_1 &= \tfrac{1}{2\sqrt{B}}\Big(y'(\xi_0) + \sqrt{B}y(\xi_0)\Big), \\ C_2 &= \tfrac{1}{2\sqrt{B}}\Big(-y'(\xi_0) + \sqrt{B}y(\xi_0)\Big). \end{aligned} \tag{3.2.2}$$

Noticing that $y(\xi_0) = 0$ and $y'(\xi_0) = 0$ (as we cannot have $y'(\xi_0) \neq 0$ because $y(\xi) \geq 0$ on J), we see that

$$y(\xi) \leq 0, \quad \xi \geq \xi_0. \tag{3.2.3}$$

This immediately gives us $y(\xi) = 0$ for all $\xi \geq \xi_0$, as desired. □

Lemma 3.2.2. *Let $x : J^{\#} \to \mathbb{R}$ be a solution to (2.3.1) on $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$, with $\gamma > 0$ and $h : J \to \mathbb{R}$ a continuous function satisfying $h(\xi) \geq 0$ for all $\xi \in J$. Assume further that $x(\xi) \geq 0$ for all $\xi \in J^{\#}$, that the coefficients are bounded according to (3.1.2) and that the positivity assumption 3.1.1 holds on the coefficients. Then if $x(\xi_0) = 0$ for some $\xi_0 \in J$, we have $x(\xi) = 0$ for all $\xi \geq \xi_0$.*

Proof. In the inhomogeneous case we can also rescale equation (2.3.1) by defining $y(\xi) = e^{\lambda\xi}x(\xi)$. It is easy to see that $y(\xi)$ satisfies the differential difference equation

$$y''(\xi) = (2\lambda - \frac{c}{\gamma})y'(\xi) - \lambda(\lambda - \frac{c}{\gamma})y(\xi) - \frac{1}{\gamma}\sum_{j=0}^{N} A_j(\xi)e^{-\lambda r_j}y(\xi + r_j) - \frac{1}{\gamma}h(\xi)e^{\lambda\xi}. \tag{3.2.4}$$

Choosing $\lambda = \frac{c}{2\gamma}$ and using the fact that $h(\xi) \geq 0$, we can again write

$$y''(\xi) \leq By(\xi), \quad \xi \in J \tag{3.2.5}$$

for some $B > 0$. We can now proceed as in the proof of Lemma 3.2.1 to prove the claim. □

3.3 Basic Properties of Nonlinear Equations

In this section, our focus shifts to nonlinear differential difference equations of the form

$$-\gamma x''(\xi) - cx'(\xi) = G\big(\xi, x(\xi), x(\xi + r_1), \ldots, x(\xi + r_N)\big). \tag{3.3.1}$$

In the automatous case we write

$$-\gamma x''(\xi) - cx'(\xi) = F\big(x(\xi), x(\xi + r_1), \ldots, x(\xi + r_N)\big). \tag{3.3.2}$$

We require $\gamma > 0$, $r_i \neq r_j$ when $i \neq j$ and $r_i \neq 0$. We also require the presence of at least one shifted argument, i.e., we take $N \geq 1$. However, some results in this section can be easily seen to hold for $N = 0$ as well, in which case (3.3.1) is just an ordinary differential equation. For convenience we will set $r_0 = 0$ and define the quantities

$$\begin{aligned} r_{\min} &= \min\{r_j \mid j = 0 \ldots N\}, \\ r_{\max} &= \max\{r_j \mid j = 0 \ldots N\}, \end{aligned} \tag{3.3.3}$$

which will be used in the proofs below.

Two crucial assumptions on G and F will be required to hold at several places. The assumptions are stated for the function G, but can of course also be applied to the function F in the automatous case, as this is just a special case of (3.3.1).

Assumption 3.3.1. *The function $G : \mathbb{R}\times\mathbb{R}^{N+1} \to \mathbb{R}$, written as $G(\xi, u)$, where $u = (u_0, u_1, \ldots, u_N)$, is continuous and locally Lipschitz in u.*

Assumption 3.3.2. *For every $\xi \in \mathbb{R}$ we have that*

$$\frac{\partial G(\xi, u)}{\partial u_j} > 0, \quad u \in \mathbb{R}^{N+1}, \quad 1 \leq j \leq N. \tag{3.3.4}$$

The following lemma roughly states that solutions to (3.3.1) are uniquely specified by their initial conditions.

Lemma 3.3.1. *Assume that Assumptions 3.3.1 and 3.3.2 both hold. Let $x_j : J \to \mathbb{R}$ for $j = 1, 2$ be two solutions of equation (3.3.1) with the same parameters c and γ on some interval J. Assume that*

$$x_1(\xi) = x_2(\xi), \quad \tau + r_{\min} \leq \xi \leq \tau + r_{\max}, \tag{3.3.5}$$

for some $\tau \in J$ for which $[\tau + r_{\min}, \tau + r_{\max}] \subseteq J$. Then

$$x_1(\xi) = x_2(\xi), \quad \xi \in J^{\#}. \tag{3.3.6}$$

Proof. We shall only prove that

$$x_1(\xi) = x_2(\xi), \quad \xi \in (-\infty, \tau + r_{\max}] \cap J^{\#}, \tag{3.3.7}$$

as the statement can be proved analogously for the case $\xi \in [\tau + r_{\min}, \infty) \cap J^{\#}$.

We start out by noting that the set $J^{\#}$ is an interval, as J contains a closed interval of length $r_{\max} - r_{\min}$. With $x_1(\xi)$ and $x_2(\xi)$ as in the statement of the Lemma, assume that (3.3.7) fails. Then there exists $\tau_0 \in (-\infty, \tau + r_{\min}] \cap J^{\#}$ such that $x_1(\xi) = x_2(\xi)$ for all $\xi \in [\tau_0, \tau + r_{\max}]$, but that for every $\epsilon > 0$ we have $x_1(\xi) \neq x_2(\xi)$ for some $\xi \in (\tau_0 - \epsilon, \tau) \cap J^{\#}$. Note that $\tau_0 \in J^{\#}$ is not the left-hand endpoint of $J^{\#}$, hence $\tau_0 - r_{\min} \in J$ is not the left-hand endpoint of J.

Consider the case $r_{\min} < 0$ and suppose that $r_{\min} = r_{j_0} < 0$ for some $1 \leq j_0 \leq N$. Consider equation (3.3.1) for $\xi \in (\tau_0 - r_{j_0} - \epsilon, \tau_0 - r_{j_0}) \equiv I$, where $\epsilon > 0$ is chosen so small that $\xi + r_j > \tau_0$ when $\xi \in I$ and $j \neq j_0$ and also that $I \subseteq J$. Then we have $x_1(\xi + r_j) = x_2(\xi + r_j)$ for $\xi \in I$ whenever $j \neq j_0$. Because both $x_1(\xi)$ and $x_2(\xi)$ are twice differentiable and I has a non-empty interior, we also have $x_1'(\xi) = x_2'(\xi)$ and $x_1''(\xi) = x_2''(\xi)$ for $\xi \in I$. From (3.3.1) we now see that for $\xi \in I$ we have

$$\begin{aligned} &G\big(\xi, x_1(\xi), x_1(\xi + r_1), \ldots, x_1(\xi + r_{j_0}), \ldots, x_1(\xi + r_N)\big) \\ &= G\big(\xi, x_1(\xi), x_1(\xi + r_1), \ldots, x_2(\xi + r_{j_0}), \ldots, x_1(\xi + r_N)\big). \end{aligned} \tag{3.3.8}$$

The strict inequality of the derivatives, guaranteed by Assumption 3.3.2, implies that we must also have $x_1(\xi + r_{j_0}) = x_2(\xi + r_{j_0})$, which contradicts the definition of τ_0.

Now suppose that $r_{\min} > 0$. Then equation (3.3.1) is an advanced functional differential equation, which can be converted into a delay differential equation by changing variables $\xi \to -\xi$. Solutions of such equations are unique in the backward direction, see for example [17], and thus, by regarding (3.3.5) as an initial value for such a problem, we immediately obtain (3.3.7). □

Remark 3.3.1. *In the case of an ordinary differential equation, where $N = 0$, the above lemma continues to hold if one adds the condition $x_1'(\tau) = x_2'(\tau)$ in the statement of the lemma. Indeed, one sees immediately that, due to the Lipschitz Assumption 3.3.1, equation (3.3.1) has a unique solution in the backward and forward direction if an initial condition $x(\tau)$ and $x'(\tau)$ is specified.*

Suppose that x_1 and x_2 are both bounded solutions of the nonlinear automatous differential difference equation (3.3.2) with the same parameters c and γ, where $\gamma > 0$. Then the difference $y(\xi) = x_1(\xi) - x_2(\xi)$ satisfies the linear equation (2.3.3) with coefficients

$$A_j(\xi) = \int_0^1 \left. \frac{\partial F(u)}{\partial u_j} \right|_{u = t\pi(x_1,\xi) + (1-t)\pi(x_2,\xi)} dt. \tag{3.3.9}$$

Here π is the state projection

$$\pi(\phi, \xi) = (\phi(\xi + r_0), \ldots, \phi(\xi + r_N)) \in \mathbb{R}^{N+1}. \tag{3.3.10}$$

This can be seen by using the formula

$$F(v,\rho) - F(w,\rho) = \int_0^1 \frac{dF(tv+(1-t)w,\rho)}{dt}dt = \sum_{j=0}^{N} \left(\int_0^1 \frac{\partial F(tv+(1-t)w,\rho)}{\partial u_j} dt. \right) (v_j - w_j). \tag{3.3.11}$$

If Assumption 3.3.2 holds for the equation (3.3.2), we can conclude that $A_j(\xi) > 0$ for all $\xi \in \mathbb{R}$ and for all $1 \leq j \leq N$. If in addition Assumption 3.3.2 holds, we can use the fact that $x_1(\xi)$ and $x_2(\xi)$ are uniformly bounded, together with the continuity of the derivatives $\frac{\partial F(u)}{\partial u_j}$, to establish that the coefficients $A_j(\xi)$ are uniformly bounded for $0 \leq j \leq N$. This means that our linear equation (2.3.3) with coefficients (3.3.9) satisfies all the assumptions of Lemma 3.2.1. In particular, the assumption that $c^2 > 4\gamma\alpha_0$ if $\alpha_0 \leq 0$ follows easily from the fact that $\gamma > 0$.

With these observations we can use the results developed in Section 3.2 for linear equations to establish comparison principles for solutions to (3.3.1).

Lemma 3.3.2. *Assume that Assumptions 3.3.1 and 3.3.2 both hold. Let $x_j : J^{\#} \to \mathbb{R}$ for $j = 1, 2$ be two bounded solutions of the nonlinear automatous differential difference equation (3.3.2) with the same parameters c and γ on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Assume also that $\gamma > 0$ and that*

$$x_1(\xi) \geq x_2(\xi), \quad \xi \in J^{\#}. \tag{3.3.12}$$

Then if $x_1(\xi_0) = x_2(\xi_0)$ for some ξ_0, we have $x_1(\xi) = x_2(\xi)$ for all $\xi \geq \xi_0$.

Proof. The result follows immediately by applying Lemma 3.2.1 to the difference $y(\xi) = x_1(\xi) - x_2(\xi)$. □

Combining the previous lemma's gives us the useful result we shall use widely in subsequent sections.

Corollary 3.3.3. *Consider the linear differential difference equation (2.3.3) with $\gamma > 0$ and assume that the coefficients $A_j(\xi)$ are bounded according to (3.1.2), where the lower bounds α_j for the shifted arguments are strictly positive, i.e., $\alpha_j > 0$ for $j = 1, \ldots N$. Let $x_j : J \to \mathbb{R}$ for $j = 1, 2$ be two solutions of equation (2.3.3) with the same parameters c and γ on the interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. If for all $\xi \in J^{\#}$ we have*

$$x_1(\xi) \geq x_2(\xi), \tag{3.3.13}$$

with equality $x_1(\xi_0) = x_2(\xi_0)$ for some $\xi_0 \in J$, then we have

$$x_1(\xi) = x_2(\xi), \quad \xi \in J. \tag{3.3.14}$$

Proof. The statement follows directly from Lemma's 3.3.2 and 3.3.1, using the assumptions on the coefficients $A_j(\xi)$. $\square$

Remark 3.3.2. *When $N = 0$, we have $x_1'(\xi_0) = x_2'(\xi_0)$, since the difference $y(\xi) = x_1(\xi) - x_2(\xi)$ satisfies $y(\xi) \geq 0$ for all $\xi \in J$ and $y(\xi_0) = 0$, which implies $y'(\xi_0) = 0$. This allows us to use the remark after Lemma 3.3.1 to conclude the equality $x_1(\xi) = x_2(\xi)$ for all $\xi \in J$, under the conditions of Corollary 3.3.3.*

In order to establish uniqueness of solutions to (1.1), we shall need a comparison principle for solutions to (3.3.2) which have different wavespeeds.

Lemma 3.3.4. *Assume that Assumptions 3.3.1 and 3.3.2 both hold. Let $x_j : J^{\#} \to \mathbb{R}$ for $j = 1, 2$ be two bounded solutions of the nonlinear automatous differential difference equation (3.3.2) with parameters $\gamma = \gamma_j$ and $c = c_j$ on some interval $J = [\tau, \infty)$ for some $\tau \in \mathbb{R}$. Assume that $\gamma_1 = \gamma_2 > 0$, but that $c_1 > c_2$. Also assume that*

$$x_1(\xi) \geq x_2(\xi), \quad \xi \in J^{\#} \tag{3.3.15}$$

and that $x_2(\xi)$ is monotonically increasing. Then if $x_1(\xi_0) = x_2(\xi_0)$ for some ξ_0, we have that $x_1(\xi) = x_2(\xi)$ is constant for all $\xi \geq \xi_0$.

Proof. We start out by noticing that the difference $y(\xi) = x_1(\xi) - x_2(\xi)$ satisfies the linear equation

$$y''(\xi) = -\frac{c_1}{\gamma} x_1'(\xi) + \frac{c_2}{\gamma} x_2'(\xi) - \frac{1}{\gamma} \sum_{j=0}^{N} A_j(\xi) y(\xi + r_j), \tag{3.3.16}$$

where the coefficients A_j are again given by (3.3.9).

We have already seen that the coefficients $A_j(\xi)$ are uniformly bounded for $0 \leq j \leq N$ and that $A_j(\xi) > 0$ for all $\xi \in \mathbb{R}$ and for $1 \leq j \leq N$. We can thus write $A_0(\xi) \geq \alpha_0$, for some $\alpha_0 < 0$. Now using the fact that $x_2'(\xi) \geq 0$, we have

$$\frac{c_2}{\gamma} x_2'(\xi) \leq \frac{c_1}{\gamma} x_2'(\xi), \tag{3.3.17}$$

which allows us to conclude

$$y''(\xi) \leq -\frac{c_1}{\gamma} y'(\xi) - \frac{\alpha_0}{\gamma} y(\xi). \tag{3.3.18}$$

Upon defining $z(\xi) = e^{\frac{c_1}{2\gamma}\xi} y(\xi)$, we obtain

$$z''(\xi) \leq (\frac{c_1^2}{4\gamma^2} - \frac{\alpha_0}{\gamma}) z(\xi) = B z(\xi), \tag{3.3.19}$$

where $B > 0$. We now proceed as in the proof of Lemma 3.2.1 to conclude that $z(\xi) = 0$ for all $\xi \geq \xi_0$, which implies $x_1(\xi) = x_2(\xi)$ for all $\xi \geq \xi_0$. Referring back to (3.3.2), we see that for $\xi \geq \xi_0 + r_{\min}$ we must have $c_1 x_1'(\xi) = c_2 x_2'(\xi)$. However, as also $x_1'(\xi) = x_2'(\xi)$, we must have $x_1'(\xi) = x_2'(\xi) = 0$. This establishes the claim. □

In light of Proposition 2.2.4, the following lemma will be useful when studying the asymptotic behaviour of solutions to the linear homogeneous equation (2.3.3).

Lemma 3.3.5. *Consider a real-valued function $x : [\tau, \infty) \to \mathbb{R}$ of the form*

$$x(\xi) = y(\xi) + O(e^{-(b+\epsilon)\xi}), \quad \xi \to \infty, \tag{3.3.20}$$

for some $b \in \mathbb{R}$ and $\epsilon > 0$, where y is a nontrivial solution of the constant coefficient system (2.3.6) with $\gamma \neq 0$, given by a finite sum of eigensolutions corresponding to a set Λ of eigenvalues λ, all of which satisfy $\operatorname{Re}\lambda = -b$. If $\operatorname{Im}\lambda \neq 0$ for all $\lambda \in \Lambda$, then there exist arbitrarily large ξ for which $x(\xi) > 0$ and arbitrarily large ξ for which $x(\xi) < 0$. On the other hand, if $\Lambda = \{-b\}$, then $x(\xi) \neq 0$ for all large ξ. The analogous result for $\xi \to -\infty$ also holds.

Proof. First suppose that $\operatorname{Im}\lambda \neq 0$ for all $\lambda \in \Lambda$. Then we may write y as a linear combination of functions $\xi \to e^{-b\xi}\sin(\eta\xi + \kappa)p(\xi)$, for various $\eta > 0$, $\kappa \in \mathbb{R}$ and real polynomials p. Thus, for some integer $J \geq 0$, we have that $y(\xi) = \xi^J e^{-b\xi}\big(q(\xi) + O(\xi^{-1})\big)$ as $\xi \to \infty$, where q is a nontrivial quasiperiodic function of mean value zero. In particular,

$$\liminf_{\xi\to\infty} q(\xi) < 0 < \limsup_{\xi\to\infty} q(\xi), \tag{3.3.21}$$

and it follows that $\xi^{-J}e^{b\xi}x(\xi) = q(\xi) + O(\xi^{-1})$ must assume both positive and negative values for arbitrarily large values of ξ, as claimed.

Now suppose that $\Lambda = \{-b\}$. Then $y(\xi) = e^{-b\xi}p(\xi)$ for some polynomial p, hence the limit

$$\lim_{\xi\to\infty} \xi^{-J}e^{b\xi}x(\xi) = \alpha \neq 0 \tag{3.3.22}$$

exists and is nonzero for some integer $J \geq 0$. □

3.4 Hyperbolicity

In this section we will be interested in the constant coefficient equation

$$-\gamma x''(\xi) - cx'(\xi) - L_0(x_\xi) = 0, \tag{3.4.1}$$

where the constant coefficient operator L_0, given by

$$L_0(\phi) = \sum_{j=0}^{N} A_{j,0}\phi(r_j), \tag{3.4.2}$$

satisfies one of the following conditions.

Assumption 3.4.1.

$$A_{j,0} \geq 0, \quad 1 \leq j \leq N. \tag{3.4.3}$$

Assumption 3.4.2.

$$A_{j,0} \leq 0, \quad 1 \leq j \leq N. \tag{3.4.4}$$

For convenience we shall again demand $N \geq 1$, although in this setting this is not an actual restriction on the constant coefficient operator L_0.

Our goal will be to obtain detailed information about the eigenvalues of the constant coefficient equation (3.4.1). In particular, in Lemma 3.4.1 we relate the existence of complex eigenvalues of (3.4.1) to the sign of the characteristic function $\Delta_{c,\gamma,L_0}(s)$ for real values of s. This will allow us to rule out the existence of eigenvalues in a large part of the complex plane. Lemma 3.4.2 provides a useful sufficient condition for the system (3.4.1) to be hyperbolic. To this end, we introduce the quantity

$$A_\Sigma = -\Delta_{c,\gamma,L_0}(0) = \sum_{j=0}^{N} A_{j,0}, \tag{3.4.5}$$

associated to the constant coefficient operator L_0. Lemma 3.4.2 connects the sign of the quantity A_Σ to the hyperbolicity of (3.4.1) and, together with Lemma 3.4.3, gives us information about the real eigenvalues of (3.4.1). In Section 3.5 the results from this section will be combined with Proposition 2.2.4 to obtain very specific asymptotic descriptions of solutions to the nonautomatous linear equation (2.3.3).

Lemma 3.4.1. *Suppose that*

$$c^2 \geq 4\gamma A_{0,0}. \tag{3.4.6}$$

Assume that the nonnegativity assumption 3.4.1 holds for the constant coefficient equation (3.4.1) and suppose that $a \in \mathbb{R}$ is such that $\Delta_{c,\gamma,L_0}(a) > 0$. Then there do not exist any eigenvalues $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda = a$. Now suppose that $\Delta_{c,\gamma,L_0}(a) = 0$ for some $a \in \mathbb{R}$. Then there do not exist any eigenvalues $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda = a$, except for $\lambda = a$ itself.

Assuming that the nonpositivity assumption 3.4.2 holds for (3.4.1), the above results also hold if we replace the condition $\Delta_{c,\gamma,L_0}(a) > 0$ by $\Delta_{c,\gamma,L_0}(a) < 0$.

Proof. We start by writing $\lambda = a + i\eta$ with $a, \eta \in \mathbb{R}$ and computing

$$\begin{aligned}
\left|c\lambda + \gamma\lambda^2 + A_{0,0}\right|^2 &= \left|ca + \gamma a^2 - \gamma\eta^2 + A_{0,0} + i(2a\gamma\eta + c\eta)\right|^2 \\
&= \left|ca + \gamma a^2 + A_{0,0}\right|^2 + \eta^2(\eta^2\gamma^2 + 2a\gamma c + 2a^2\gamma^2 + c^2 - 2\gamma A_{0,0}) \\
&= \left|ca + \gamma a^2 + A_{0,0}\right|^2 + \eta^2 p(a),
\end{aligned}$$

where p is a second degree polynomial. It is elementary to see that if (3.4.6) holds, then

$$p(a) \geq \frac{1}{2}c^2 + \eta^2\gamma^2 - 2\gamma A_0 \geq \eta^2\gamma^2 \geq 0.$$

We thus have

$$\left|c\lambda + \gamma\lambda^2 + A_{0,0}\right| \geq \left|ca + \gamma a^2 + A_{0,0}\right|, \tag{3.4.7}$$

with equality if and only if $\lambda = a$.

Assuming the nonnegativity condition 3.4.1, suppose that $\lambda = a + i\eta$ satisfies $\Delta_{c,\gamma,L_0}(\lambda) = 0$ for some real η and that $\Delta_{c,\gamma,L_0}(a) \geq 0$. Then using (3.4.7), we arrive at

$$\left|ca + \gamma a^2 + A_{0,0}\right| \leq \left|c\lambda + \gamma\lambda^2 + A_{0,0}\right| = \left|\sum_{j=1}^{N} A_{j,0}e^{\lambda r_j}\right| \leq \sum_{j=1}^{N} A_{j,0}e^{ar_j} \leq -(ca + \gamma a^2 + A_{0,0}). \tag{3.4.8}$$

By examining the first and last terms in (3.4.8), we see that the three inequalities have to be equalities. In particular, the third inequality (when read as an equality) implies that $\Delta_{c,\gamma,L_0}(a) = 0$, proving the first claim. Using the fact that (3.4.7) is an equality only if $\lambda = a$, the second claim immediately follows.

Similarly, we can consider the case in which the nonpositivity condition 3.4.2 holds. Assuming $\Delta_{c,\gamma,L_0}(\lambda) = 0$ and $\Delta_{c,\gamma,L_0}(a) \leq 0$, we now obtain

$$\begin{aligned}
\left|ca + \gamma a^2 + A_{0,0}\right| &\leq \left|c\lambda + \gamma\lambda^2 + A_{0,0}\right| = \left|\textstyle\sum_{j=1}^{N} A_{j,0}e^{\lambda r_j}\right| \leq -\textstyle\sum_{j=1}^{N} A_{j,0}e^{ar_j} \\
&\leq (ca + \gamma a^2 + A_{0,0}).
\end{aligned} \tag{3.4.9}$$

By considering the first and last terms in (3.4.9), we again see that all three inequalities have to be equalities, from which the claims in the lemma follow as above. $\square$

Lemma 3.4.2. *Assume that $A_\Sigma < 0$ holds for the equation (3.4.1) and that in addition the nonnegativity assumption 3.4.1 holds. Assume also that $\gamma > 0$. Then equation (3.4.1) is hyperbolic. Furthermore, there exists precisely one real positive eigenvalue $\lambda^+ \in (0, \infty)$ and precisely one real* *negative eigenvalue* $\lambda^- \in (-\infty, 0)$ *and each of these eigenvalues is simple. The eigenvalues* λ^- *and* λ^+ *depend* C^1 *smoothly on* c *and the coefficients* $A_{j,0}$*. In addition, we have that*

$$\frac{\partial \lambda^-}{\partial c} < 0 \quad \text{and} \quad \frac{\partial \lambda^+}{\partial c} < 0. \tag{3.4.10}$$

All the remaining eigenvalues satisfy

$$\operatorname{Re} \lambda \in (-\infty, \lambda^-) \cup (\lambda^+, \infty), \quad \operatorname{Im} \lambda \neq 0. \tag{3.4.11}$$

Proof. Using Lemma 3.4.1 and the fact that $\Delta_{c,\gamma,L_0}(0) = -A_\Sigma > 0$, we immediately see that there are no eigenvalues on the imaginary axis and thus equation (3.4.1) is hyperbolic. Due to the nonnegativity of the coefficients $A_{j,0}$ for $j \geq 1$, we see that $\lim_{s \to \pm\infty} \Delta(s) = -\infty$ and so there exist at least one negative and one positive eigenvalue. Using the fact that

$$\Delta''(s) = -2\gamma - \sum_{j=1}^{N} r_j^2 A_{j,0} e^{s r_j} < 0, \quad s \in \mathbb{R}, \tag{3.4.12}$$

we see that there can be at most two real eigenvalues, which we will denote as λ^+ and λ^-, as in the statement of the lemma. From the fact that $\Delta(0) > 0$ we see that we must have

$$\Delta'(\lambda^+) < 0, \quad \Delta'(\lambda^-) > 0. \tag{3.4.13}$$

Thus $\lambda^\pm$ are both simple eigenvalues and depend C^1-smoothly on c and on the coefficients $A_{j,0}$. The inequalities in (3.4.10) follow directly by implicit differentiation. Finally, using

$$\Delta(s) > 0, \quad s \in (\lambda^-, \lambda^+) \tag{3.4.14}$$

and Lemma 3.4.1, one sees that the statement about the remaining eigenvalues follows. □

Lemma 3.4.3. *Assume that* $A_\Sigma > 0$ *holds for the equation (3.4.1) and that in addition the nonnegativity assumption 3.4.1 holds. Assume also that* $\gamma > 0$*. Then either all real eigenvalues of (3.4.1) lie in* $(0, \infty)$*, or else they all lie in* $(-\infty, 0)$*.*

Proof. The statement follows directly from the observations that $\Delta''(s) < 0$ and $\Delta(0) = -A_\Sigma < 0$. □

3.5 A One Dimensional Kernel

We are now ready to state and prove the main theorem of this chapter.

Theorem 3.5.1. *Consider the linear operator given in (2.3.4) and assume that there exist quantities*

$$\alpha_j,\ \beta_j \in \mathbb{R}, \quad j = 0 \dots N, \tag{3.5.1}$$

$$\alpha_j > 0, \quad j = 1 \dots N, \tag{3.5.2}$$

such that the coefficients satisfy the bounds

$$\alpha_j \le A_j(\xi) \le \beta_j, \quad \xi \in \mathbb{R}, \quad j = 0 \dots N. \tag{3.5.3}$$

We demand that there is at least one shifted argument, i.e. we assume $N \ge 1$.

Assume that the equation (2.3.3) is asymptotically autonomous and that in addition the limiting equations are approached at an exponential rate, so

$$|A_j(\xi) - A_{j\pm}| = O(e^{-k|\xi|}), \quad \xi \to \pm\infty, \quad j = 0 \dots N \tag{3.5.4}$$

for some $k > 0$. *Also assume that each of the sums* $A_{\Sigma\pm}$ *given below, of the limiting coefficients at* $\pm\infty$, *is negative, namely*

$$A_{\Sigma\pm} = \sum_{j=0}^{N} A_{j\pm} < 0. \tag{3.5.5}$$

Finally, assume that there exists a nontrivial solution $x = p(\xi)$ *to the linear equation (2.3.3) which is nonnegative and bounded on* $\mathbb{R}$,

$$p \in L^\infty, \quad p(\xi) \ge 0, \quad \xi \in \mathbb{R}. \tag{3.5.6}$$

Then equation (2.3.3) is asymptotically hyperbolic and $\Lambda_{c,\gamma,L} : W^{2,\infty} \to L^\infty$ *is a Fredholm operator. Writing* $\mathcal{K}_{c,\gamma,L} = \mathcal{K}(\Lambda_{c,\gamma,L})$ *and* $\mathcal{R}_{c,\gamma,L} = \mathcal{R}(\Lambda_{c,\gamma,L})$, *we have*

$$\dim \mathcal{K}_{c,\gamma,L} = \dim \mathcal{K}_{c,\gamma,L^*} = \operatorname{codim} \mathcal{R}_{c,\gamma,L} = 1, \quad \operatorname{ind}(\Lambda_{c,\gamma,L}) = 0. \tag{3.5.7}$$

The element $p \in \mathcal{K}_{c,\gamma,L}$ *is strictly positive,*

$$p(\xi) > 0, \quad \xi \in \mathbb{R} \tag{3.5.8}$$

and there exists an element $p^* \in \mathcal{K}_{c,\gamma,L^*}$ *which is strictly positive,*

$$p^*(\xi) > 0, \quad \xi \in \mathbb{R}. \tag{3.5.9}$$

We will prove the theorem in four steps. In Lemma 3.5.2, the asymptotic hyperbolicity is established for the limiting equations at $\pm\infty$ and the asymptotic behaviour of solutions to (2.3.3) on the halfline will be determined. In Corollary 3.5.3 we establish the strict positivity of the solution $p(\xi)$ and in Lemma 3.5.4 the claims about the kernel $\mathcal{K}_{c,\gamma,L}$ will be proved. After these preliminaries the proof will be completed.

Lemma 3.5.2. *Assume that equation (2.3.3) satisfies all the conditions in the statement of Theorem 3.5.1, except possibly for the existence of the solution $x = p(\xi)$. Then equation (2.3.3) is asymptotically hyperbolic. There exist four quantities $\lambda^u_\pm, \lambda^s_\pm$ with*

$$-\infty < \lambda^s_\pm < 0 < \lambda^u_\pm < \infty, \tag{3.5.10}$$

such that $\lambda^u_\pm$ $\lambda^s_\pm$ are the only real eigenvalues of the limiting equations at $\pm\infty$. They are simple eigenvalues. If we only assume the asymptotic conditions of Theorem 3.5.1 at $+\infty$ and if $x(\xi)$ satisfies equation (2.3.3) on some interval $[\tau, \infty)$ and is bounded as $\xi \to \infty$, then

$$x(\xi) = C_+ e^{\lambda^s_+ \xi} + O(e^{(\lambda^s_+ - \epsilon)\xi}), \quad \xi \to \infty, \tag{3.5.11}$$

for some constant $C_+ \in \mathbb{R}$ and some $\epsilon > 0$. Also the asymptotic formulae for $x'(\xi)$ and $x''(\xi)$ obtained from (3.5.11) by formal differentiation (including the remainder term) hold. If moreover $x(\xi) \geq 0$ for all large ξ and $x(\xi)$ does not vanish identically for large ξ, then $C_+ > 0$.

The analogous results, in particular

$$x(\xi) = C_- e^{\lambda^u_- \xi} + O(e^{(\lambda^u_- + \epsilon)\xi}), \quad \xi \to -\infty, \tag{3.5.12}$$

hold for solutions which are bounded at $-\infty$.

Proof. The hyperbolicity of the limiting equations follows directly from Lemma 3.4.2 and equation (3.5.5), as do the claims about the eigenvalues. Now suppose that $x(\xi)$ is a solution to equation (2.3.3) on an interval $[\tau, \infty)$ as in the statement of this lemma. Then Proposition 2.2.4 implies that either

$$x(\xi) = y(\xi) + O(e^{-(b+\epsilon)\xi}), \quad \xi \to \infty, \tag{3.5.13}$$

where y is a nontrivial eigensolution corresponding to a set of eigenvalues with $\operatorname{Re}\lambda = -b \leq 0$, or $x(\xi)$ has superexponential decay. In the former case, we see from the statement about the eigenvalues in Lemma 3.4.2 that we either have $-b = \lambda^s_+$, in which case $y(\xi) = C_+ e^{\lambda^s_+ \xi}$ for some $C_+ \neq 0$, or else $-b < \lambda_+^s$, in which case we have (3.5.11) with $C_+ = 0$. In the case of superexponential decay we also obtain (3.5.11) with $C_+ = 0$.

The asymptotic expressions for $x'(\xi)$ and $x''(\xi)$ now follow by noticing that Proposition 2.2.4 applied to the system (2.3.9) also gives us an asymptotic expansion for $x'(\xi)$,

$$x'(\xi) = D_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), \quad \xi \to \infty, \tag{3.5.14}$$

for some constant $D_+ \in \mathbb{R}$. We have the relation $D_+ = \lambda_+^s C_+$. Substituting these asymptotic expressions into the right-hand side of the differential equation (2.3.3) yields the desired asymptotic expression for $x''(\xi)$, using the fact that $|A_j(\xi) - A_{j+}| = O(e^{-k\xi})$ and noting that the leading term $C_+ e^{\lambda_+^s \xi}$ is a solution to the limiting differential equation.

Now suppose that $x(\xi) \geq 0$, but not identically zero for all large ξ. Then we must have $C_+ \geq 0$ in (3.5.11). We wish to show that $C_+ > 0$, so assume to the contrary that $C_+ = 0$. We know from Lemma 3.1.1 that superexponential decay is impossible, so equation (3.5.13) must hold with $-b < \lambda_+^s$. But as all eigenvalues λ with $\operatorname{Re}\lambda = -b < \lambda_+^s$ must have nonzero imaginary part, we conclude from Lemma 3.3.5 that there exist arbitrary large ξ for which $x(\xi) > 0$ and arbitrary large ξ for which $x(\xi) < 0$. This contradicts the positivity of $x(\xi)$ and so we must have $C_+ > 0$, as desired. The proofs of the corresponding claims at $-\infty$ follow similar lines. □

Corollary 3.5.3. *Assume all the conditions of Theorem 3.5.1. Then the solution $p(\xi)$ in the statement of the theorem is strictly positive, i.e.,*

$$p(\xi) > 0, \quad \xi \in \mathbb{R}. \tag{3.5.15}$$

We also have the asymptotic expressions

$$p(\xi) = \begin{cases} C_- e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \\ C_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \end{cases} \tag{3.5.16}$$

for some ϵ, where both $C_\pm > 0$, with finite exponents

$$-\infty < \lambda_+^s < 0 < \lambda_-^u < \infty. \tag{3.5.17}$$

In addition, the asymptotic expressions for $p'(\xi)$ and $p''(\xi)$, obtained by formal differentiation of (3.5.16), including the remainder terms, hold.

Proof. We know that $p(\xi) \geq 0$. Now suppose that $p(\tau) = 0$ at some $\tau \in \mathbb{R}$. Then an application of Corollary 3.3.3 with $x_1 = p$ and $x_2 = 0$ immediately implies that $p(\xi) = 0$ for all $\xi \in \mathbb{R}$, which is a contradiction. We thus conclude conclude the strict positivity of p and the asymptotic expressions (3.5.16) now follow immediately from Lemma 3.5.2. $\square$

Lemma 3.5.4. *Assume all the conditions of Theorem 3.5.1. Then the operator $\Lambda_{c,\gamma,L}$ in the statement of the theorem is a Fredholm operator and the claims (3.5.7) about the kernel, range and Fredholm index of this operator and its adjoint hold.*

Proof. Lemma 3.5.2 gives the asymptotic hyperbolicity of equation (2.3.3) and Proposition 2.3.1 ensures that $\Lambda_{c,\gamma,L}$ is a Fredholm operator. Let us consider the kernel $\mathcal{K}_{c,\gamma,L}$ of $\Lambda_{c,\gamma,L}$. Suppose that $\dim \mathcal{K}_{c,\gamma,L} > 1$ and take any $x \in \mathcal{K}_{c,\gamma,L}$ which is linearly independent of $p \in \mathcal{K}_{c,\gamma,L}$. By Lemma 3.5.2 again, the solution x enjoys asymptotic estimates as in (3.5.16), but with generally different constants $C_\pm$ which need not necessarily be positive. However, by adding an appropriate multiple of p to x, we can ensure that the coefficient of $e^{\lambda_+^s \xi}$ in the asymptotic expressions for $x(\xi)$ as $\xi \to \infty$ vanishes, i.e.,

$$x(\xi) = \begin{cases} C_0 e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \\ O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \end{cases} \tag{3.5.18}$$

for some $C_0 \in \mathbb{R}$. By replacing x by $-x$ if necessary, we may assume that $C_0 \leq 0$. Because x is not identically zero, it follows from Lemma 3.3.1 that there exist arbitrarily large ξ for which $x(\xi) \neq 0$. If $x(\xi) \leq 0$ for all large ξ, then it follows from Lemma 3.5.2 applied to $-x$ that equation (3.5.11) holds for x with strictly negative C_+, contradicting equation (3.5.18). This means there even are arbitrarily large ξ for which $x(\xi) > 0$. From this it immediately follows that there exists $\mu_0 > 0$ such that

$$p(\xi) - \mu_0 x(\xi) < 0, \tag{3.5.19}$$

for some $\xi \in \mathbb{R}$. We now consider the family $p - \mu x \in \mathcal{K}_{c,\gamma,L}$ for $0 \leq \mu \leq \mu_0$. The asymptotic expressions for p and x ensure that there exist $\tau,\ K,\ \lambda \in \mathbb{R}$ such that

$$p(\xi) - \mu x(\xi) \geq K e^{-\lambda|\xi|} > 0, \quad |\xi| > \tau, \quad 0 \leq \mu \leq \mu_0. \tag{3.5.20}$$

Now define

$$\mu_* = \sup \left\{ \mu \in [0, \mu_0] \mid p(\xi) - \mu x(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R} \right\}. \tag{3.5.21}$$

By definition it follows from (3.5.19) that $\mu_* < \mu_0$. Obviously, we have the inequality $\mu_* x(\xi) \leq p(\xi)$ for all $\xi \in \mathbb{R}$, but actually we also have $\mu_* x(\xi_0) = p(\xi_0)$ for some $\xi_0 \in [-\tau, \tau]$. Indeed, assuming to the contrary that $\mu_* x(\xi) > p(\xi)$ for all $\xi \in [-\tau, \tau]$, we can define

$$\Gamma = \min \{p(\xi) - \mu_*(\xi) \mid \xi \in [-\tau, \tau]\} > 0, \tag{3.5.22}$$

because continuous functions attain their minima on compact intervals. If we now choose $\epsilon > 0$ such that $\mu_* + \epsilon < \mu_0$ and $\epsilon \|x\|_\infty < \Gamma$ then also $(\mu_* + \epsilon)x(\xi) \leq p(\xi)$ for all $\xi \in \mathbb{R}$ contradicting the definition of μ_*. From Corollary 3.3.3 it now immediately follows that $\mu_* x(\xi) = p(\xi)$, but this contradicts the linear independence of x and p. This contradiction completes the proof that $\dim \mathcal{K}_{c,\gamma,L} = 1$.

We next show that $\operatorname{ind} \Lambda_{c,\gamma,L} = 0$, from which we see that $\operatorname{codim} \mathcal{R}_{c,\gamma,L} = 1$ and from the Fredholm Alternative (Theorem 2.2.1) $\dim \mathcal{K}_{c,\gamma,L^*} = 1$, which completes the proof of the Lemma. By the Cocycle Property (Theorem 2.2.2) we know that the quantity $\operatorname{ind}(\Lambda_{c,\gamma,L})$ depends only on c, γ and the limiting operators $L_\pm$. Moreover, by the Spectral Flow Property (Theorem 2.2.3), we have that $\operatorname{ind}(\Lambda_{c,\gamma,L}) = -\operatorname{cross}(L^\rho)$, in which L^ρ is any generic homotopy of constant coefficient operators joining L_- at $\rho = -1$ to L_+ at $\rho = 1$, and $\operatorname{cross}(L^\rho)$ is the net number of roots $s = \lambda$ of the characteristic equation $\Delta_{c,\gamma,L^\rho}(s) = 0$ which cross the imaginary axis along this homotopy, keeping c and γ fixed. We choose the homotopy $L^\rho = ((1-\rho)L_- + (1+\rho)L_+)/2$. Using Lemma 3.4.2 and equation (3.5.5), we see that the corresponding equation (3.4.1) is hyperbolic for $-1 \leq \rho \leq 1$. This means that no eigenvalues cross the imaginary axis and thus $\operatorname{ind}(\Lambda_{c,\gamma,L}) = -\operatorname{cross}(L^\rho) = 0$. $\square$

We are now ready to complete the proof of Theorem 3.5.1.

Proof of Theorem 3.5.1. All that remains is to establish the positivity (3.5.9) of some $p^* \in \mathcal{K}_{c,\gamma,L^*}$. Again, by Corollary 3.3.3 and the definition of the adjoint equation (2.3.5), it is enough to show that $p^*(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Let us therefore suppose there exists $p^* \in \mathcal{K}_{c,\gamma,L^*}$ satisfying $p^*(\xi_1) > 0$ and $p^*(\xi_2) < 0$ for some ξ_1, $\xi_2 \in \mathbb{R}$ and seek a contradiction. First note that $p^*(\xi)$ does not vanish identically on any interval of length $r_{\max} - r_{\min}$, as otherwise Lemma 3.3.1 would imply that $p^*(\xi)$ is identically zero. Thus without loss we may assume $|\xi_1 - \xi_2| \leq r_{\max} - r_{\min}$. It now follows that there exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} p^*(\xi) h(\xi) d\xi = 0, \tag{3.5.23}$$

where h satisfies $h(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ and has compact support, in fact $\mathrm{supp}(h) \in [\tau_1, \tau_2]$ for some $\tau_1, \tau_2 \in \mathbb{R}$ satisfying $\tau_2 - \tau_1 < r_{\max} - r_{\min}$. By the Fredholm Alternative (Theorem 2.2.1) and the fact that $\dim \mathcal{K}_{c,\gamma,L^*} = 1$, we see that $h \in \mathcal{R}_{c,\gamma,L}$, which means there exists an $x \in W^{2,\infty}$ such that $\Lambda_{c,\gamma,L} x = h$. We know that adding a multiple of $p \in \mathcal{K}_{c,\gamma,L}$ to x yields any other such solution $x + \mu p$. As h has compact support, $x(\xi)$ satisfies the homogeneous equation (2.3.3) for large $|\xi|$ and so by Lemma 3.5.2 enjoys asymptotic estimates of the form (3.5.11) and (3.5.12), although with generally different constants $C_\pm$ than those in equation (3.5.16). As the constants $C_\pm$ in the asymptotic expressions (3.5.16) are both positive, we see that if μ is sufficiently large then $x(\xi) + \mu p(\xi) > 0$ for all $\xi \in \mathbb{R}$. We can thus define

$$\mu_* = \inf \{\mu \in \mathbb{R} \mid x(\xi) + \mu p(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}\}, \tag{3.5.24}$$

which is a finite quantity. We also introduce the function

$$y(\xi) = x(\xi) + \mu_* p(\xi). \tag{3.5.25}$$

By construction, we have $y(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. By Lemma 3.5.2 and the fact that $y(\xi) \geq 0$, we have either that $y(\xi) = D_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi})$ as $\xi \to \infty$, for some strictly positive $D_+ > 0$ and $\epsilon > 0$, in which case $y(\xi) > 0$ for large ξ, or else that $y(\xi) = 0$ identically for all large ξ. The corresponding statements for $\xi \to -\infty$ also hold. However, it is not the case that $y(\xi) = 0$ both for $\xi \to \infty$ and for $\xi \to -\infty$. Indeed, suppose to the contrary that $y(\xi) = 0$ for all large $|\xi|$. Noting that y satisfies the homogeneous equation (2.3.3) on the interval $J = [\tau_2, \infty)$, because $h(\xi) = 0$ there, we see from Lemma 3.3.1 that $y(\xi) = 0$ identically on $J^\# = [\tau_2 + r_{\min}, \infty)$. Using the same argument, $y(\xi) = 0$ identically on $(-\infty, \tau_1 + r_{\max}]$. However, since $\tau_2 - \tau_1 < r_{\max} - r_{\min}$, we see that

$$(-\infty, \tau_1 + r_{\max}] \cup [\tau_2 + r_{\min}, \infty) = \mathbb{R}, \tag{3.5.26}$$

hence y is identically zero on $\mathbb{R}$, contradicting the fact that $h = \Lambda_{c,\gamma,L} y$ is not the zero function.

Possibly after making a substitution $\xi \to -\xi$, which does not change the sign of γ, we can assume that $y(\xi) > 0$ for all large ξ. It is also true that $y(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Indeed, if $y(\xi) > 0$ for all large negative ξ than we can apply the same reasoning as in the proof of Lemma 3.5.4 to prove the claim. Now, using Lemma 3.2.2, we immediately obtain $y(\xi) = 0$ for all $\xi \geq \xi_0$, giving us the desired contradiction. This completes the proof of the theorem. $\square$

Chapter 4

Global Structure

4.1 Terminology

In this chapter we study the family of autonomous differential difference equations introduced in the introduction,

$$-cx'(\xi) - \gamma x''(\xi) = F\big(x(\xi + r_0), x(\xi + r_1), x(\xi + r_2), \dots, x(\xi + r_N), \rho\big), \tag{4.1.1}$$

in which $\gamma > 0$. As in the previous chapter, we demand that $r_0 = 0$, $r_i \neq r_j$ if $i \neq j$ and $r_i \neq 0$ for $i = 1 \dots N$, where $N \geq 1$. Here we take $\rho \in \overline{V}$ to be a parameter, where V is an open subset of $\mathbb{R}$. We shall prove existence and uniqueness of solutions to (4.1.1) under certain conditions and establish the C^1-dependence of the solutions on the parameter ρ.

We start out by making precise the requirements given in the introduction and give a list of conditions on the function F which we will assume to hold throughout this chapter.

(b1) The nonlinearity $F : \mathbb{R}^{N+1} \times \overline{V} \to \mathbb{R}$ is C^1-smooth in $\mathbb{R}^{N+1}$ and $\overline{V}$.

(b2) The derivative $D_1F : \mathbb{R}^{N+1} \times \overline{V} \to \mathbb{R}$ with respect to the first argument $v \in \mathbb{R}^{N+1}$ is locally Lipschitz in v.

(b3) For each $\rho \in \overline{V}$ and for $j = 1, \dots, N$, we have, writing $v = (v_0, v_1, \dots, v_N) \in \mathbb{R}^{N+1}$, that either

$$\tfrac{\partial F(v,\rho)}{\partial v_j} \equiv 0, \quad \text{or} \quad \tfrac{\partial F(v,\rho)}{\partial v_j} > 0, \tag{4.1.2}$$

that is, either F is totally independent of v_j or is strictly increasing in v_j. Furthermore, for each $\rho \in \overline{V}$ there is at least one j, satisfying $1 \leq j \leq N$, for which the nonlinearity F is not totally independent of v_j.

(b4) Let $\Phi : \mathbb{R} \times \overline{V} \to \mathbb{R}$ be defined as

$$\Phi(\phi, \rho) = F(\phi, \phi, \dots, \phi, \rho). \tag{4.1.3}$$

Then for some quantity $q = q(\rho) \in [-1, 1]$ we have that

$$\begin{aligned} &\Phi(-1, \rho) = \Phi(q(\rho), \rho) = \Phi(1, \rho) = 0 \\ &\Phi(\phi, \rho) > 0, \quad \phi \in (-\infty, -1) \cup (q, 1) \\ &\Phi(\phi, \rho) < 0, \quad \phi \in (-1, q) \cup (1, \infty) \end{aligned} \tag{4.1.4}$$

In case $\rho \in V$ we demand $q(\rho) \in (-1, 1)$.

(b5) We have for $q = q(\rho)$ that

$$\begin{aligned} &D_1\Phi(-1, \rho) < 0 \text{ if } q \neq -1, \\ &D_1\Phi(q, \rho) > 0 \text{ if } q \in (-1, 1), \\ &D_1\Phi(1, \rho) < 0 \text{ if } q \neq 1, \end{aligned} \tag{4.1.5}$$

with D_1 denoting the derivative with respect to the first argument $x \in \mathbb{R}$.

Condition (b3) allows us to consider families in which the shifts r_j may vary with ρ, by adding extra shifts r_j which do not affect the value of F for certain values of ρ.

In (4.1.1) the wavespeed c is an unknown parameter. From the above conditions we see that equation (4.1.1) has exactly three constant equilibrium solutions, namely $x = \pm 1$ and $x = q(\rho)$. We will be interested in solutions to (4.1.1) joining the two equilibrium points ± 1. As (4.1.1) is autonomous, we see that all translates of a solution $x(\xi)$ to (4.1.1) are also solutions. We can use this freedom to demand that $x(0) = 0$. It will turn out that after this normalization the solution to (4.1.1) is unique. We thus seek our solutions in the space

$$W_0^{2,\infty} = \left\{ x \in W^{2,\infty} \mid x(0) = 0 \right\}. \tag{4.1.6}$$

It will be useful to introduce the operator $\mathcal{G} : W_0^{2,\infty} \times \mathbb{R} \times V \to L^\infty$ defined by

$$\mathcal{G}(\phi, c, \rho)(\xi) = -\gamma\phi''(\xi) - c\phi'(\xi) - F\big(\phi(\xi + r_0), \phi(\xi + r_1), \dots, \phi(\xi + r_N), \rho\big). \tag{4.1.7}$$

We are now ready to define what we mean by a connecting solution to (4.1.1).

Definition 4.1.1. Given $\rho \in V$, a connecting solution to the nonlinear autonomous differential difference equation (4.1.1) is a pair $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ which satisfies (4.1.1) and joins the two equilibrium solutions ± 1, i.e., for which

$$\lim_{\xi \to \pm\infty} \phi(\xi) = \pm 1 \tag{4.1.8}$$

holds. □

Please note that we will continue to use the term "solution" to indicate a function in $x \in W^{2,\infty}$ satisfying the equation (4.1.1), but not necessarily joining the two equilibria ± 1 and not necessarily having $x(0) = 0$.

We are now in a position to state the main theorem of this chapter.

Theorem 4.1.1. *Consider the family of autonomous differential difference equations (4.1.1). There exist C^1-smooth functions $c : V \to \mathbb{R}$ and $P : V \to W_0^{2,\infty}$ such that for all $\rho_0 \in V$, the pair $\big(P(\rho_0), c(\rho_0)\big)$ is a connecting solution to equation (4.1.1). Moreover, these are the only connecting solutions to (4.1.1).*

The proof of Theorem 4.1.1 will be given in two parts. In Section 4.2 we shall concentrate on the existence of functions $P(\rho)$ and $c(\rho)$ as in the statement of Theorem 4.1.1 in a small neighbourhood of the detuning parameter ρ_0, given a connecting solution (P_0, c_0) for $\rho = \rho_0$. In Section 4.3 we show that this local continuation can be extended to all $\rho \in V$ and prove the existence and uniqueness claims in the statement of Theorem 4.1.1.

4.2 Local Continuation

We first recall that if $x : \mathbb{R} \to \mathbb{R}$ is any solution to (4.1.1) for some $\rho \in \overline{V}$, then $x'(\xi)$ is a solution of the linearization around x, that is the linear equation (2.3.3) with coefficients

$$A_j(\xi) = \frac{\partial F(u, \rho)}{\partial u_j}\bigg|_{u=\pi(x,\xi)}, \tag{4.2.1}$$

where π is the state projection given by (3.3.10). The linearization around the three equilibrium solutions $x = \pm 1$ and $x = q(\rho)$ are constant coefficient equations given by (3.4.1). We shall write L_+, L_- and $L_\diamond$ for the associated linear operators (2.1.6) and shall refer to the corresponding constant coefficients as

$$\begin{aligned} A_{j\pm}(\rho) &= \tfrac{\partial F(u,\rho)}{\partial u_j}\Big|_{u=\kappa(\pm 1)}, \\ A_{j\diamond}(\rho) &= \tfrac{\partial F(u,\rho)}{\partial u_j}\Big|_{u=\kappa(q(\rho))}, \end{aligned} \tag{4.2.2}$$

where κ is the diagonal map $\kappa(x) = (x, \ldots, x) \in \mathbb{R}^{N+1}$. Writing $A_{\Sigma\pm} = \sum_{j=0}^{N} A_{j\pm}$, we have the identity

$$A_{\Sigma\pm} = D_1\Phi(\pm 1, \rho). \tag{4.2.3}$$

Our aim in this section is to show that connecting solutions to (4.1.1) can be locally continued, i.e., if the system (4.1.1) has a connecting solution at some value $\rho_0 \in V$ of the detuning parameter, we shall prove that it also has connecting solutions for nearby values ρ. This is made precise in our main proposition.

Proposition 4.2.1. *Let* $(P_0, c_0) \in W_0^{2,\infty} \times \mathbb{R}$ *be a connecting solution to (4.1.1) for some* $\rho_0 \in V$ *and for some* $c_0 \in \mathbb{R}$. *Then for each* ρ *near* ρ_0 *there exists an unique* $(P, c) = \big(P(\rho), c(\rho)\big) \in W_0^{2,\infty} \times \mathbb{R}$, *that depends* C^1*-smoothly on* ρ, *for which* $\mathcal{G}\big(P(\rho), c(\rho), \rho\big) = 0$, *with* $c(\rho_0) = c_0$ *and* $P(\rho_0) = P_0$. *This function* $P(\rho)$ *satisfies the boundary conditions* $\lim_{\xi\to\pm\infty} x(\xi) = \pm 1$ *and thus* $\big(P(\rho), c(\rho)\big)$ *is a connecting solution to (4.1.1).*

Our approach to proving the main result will be to invoke the implicit function theorem on the operator $\mathcal{G}$ defined by (4.1.7). Consequently, in Proposition 4.2.6 we study the Frechet derivative of $\mathcal{G}$, which is given by

$$D_{1,2}\mathcal{G}(P_0, c_0, \rho_0)(\psi, b)(\xi) = -bP_0'(\xi) + (\Lambda_{c_0,\gamma,L}\psi)(\xi), \tag{4.2.4}$$

where $\Lambda_{c_0,\gamma,L}$ is the linear operator associated to the linearization of (4.1.1) around the solution P_0. We shall establish that Theorem 3.5.1 applies to the operator $\Lambda_{c_0,\gamma,L}$ and that the derivative P_0' is strictly positive (Lemma 4.2.5). In particular, this means that $P_0' \notin \mathcal{R}(\Lambda_{c_0,\gamma,L})$ and $\mathcal{K}(\Lambda_{c_0,\gamma,L}) \cap W_0^{2,\infty} = \emptyset$. From this it is easy to see that $D_{1,2}\mathcal{G}$ is an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ onto L^∞, which legitimizes the use of the implicit function theorem.

We start with a technical lemma.

Lemma 4.2.2. *Let* $x : \mathbb{R} \to \mathbb{R}$ *be a solution to (4.1.1) for some* $\rho \in \overline{V}$ *and* $c \in \mathbb{R}$. *Define*

$$\mu_- = \inf_{\xi\in\mathbb{R}} x(\xi), \quad \mu_+ = \sup_{\xi\in\mathbb{R}} x(\xi), \tag{4.2.5}$$

and assume that both $\mu_\pm$ *are finite. Then*

$$\mu_- \in [-1, q(\rho)] \cup 1, \quad \mu_+ \in -1 \cup [q(\rho) \cup [a(\rho), 1]. \tag{4.2.6}$$

The same conclusion (4.2.6) holds for

$$\mu_- = \liminf_{\xi\to\infty} x(\xi), \quad \mu_+ = \limsup_{\xi\to\infty} x(\xi) \tag{4.2.7}$$

and similarly for the $\liminf$ *and* $\limsup$ *at* $-\infty$.

Proof. With $\mu_\pm$ first as in (4.2.5), assuming that both $\mu_\pm$ are finite, we shall prove that

$$\Phi(\mu_-, \rho) \leq 0 \leq \Phi(\mu_+, \rho), \tag{4.2.8}$$

which by condition (b4) is equivalent to

$$\mu_- \in [-1, q(\rho)] \cup [1, \infty), \quad \mu_+ \in (\infty, -1] \cup [q(\rho), 1] \tag{4.2.9}$$

and thus yields (4.2.6) using the fact that $\mu_- \leq \mu_+$. We shall only prove the second inequality in (4.2.8), as the proof of the first is similar. Let $\xi_n \in \mathbb{R}$ be a sequence such that $x(\xi_n) \to \mu_+$ as $n \to \infty$. Then the fact that x and x' are uniformly continuous (because they are in $W^{2,\infty}$), together with the differential equation (4.1.1) for x, implies that x'' is uniformly continuous. From this and the boundedness of x'' it follows in a standard fashion that, possibly after passing to a subsequence, $x'(\xi_n) \to 0$ and $x''(\xi_n) \to l \leq 0$ for some l. After passing to a further subsequence, we may assume that for $j = 0 \ldots N$, there exist μ_j such that $x(\xi_n + r_j) \to \mu_j \leq \mu_+$ as $n \to \infty$. Upon inserting $\xi = \xi_n$ into (4.1.1) and taking the limit, we have that

$$0 \leq -\gamma l = F(\mu_+, \mu_2, \ldots, \mu_N, \rho) \leq F(\mu_+, \mu_+, \ldots, \mu_+, \rho) = \Phi(\mu_+, \rho), \tag{4.2.10}$$

where condition (b3) has been used together with the observation that $\mu_1 = \mu_+$. This now proves the inequality in (4.2.8) as desired.

Assume now that $\mu_\pm$ are given by (4.2.7). Let $\xi_n \to \infty$ be such that $x(\xi_n) \to \mu_+$ as $n \to \infty$, and set $y_n(\xi) = x(\xi + \xi_n)$. On each interval $[\tau, \infty)$ the sequence y_n is uniformly bounded and equicontinuous (as x is uniformly continuous), so by passing to a subsequence we may assume the limit $y_n(\xi) \to y(\xi)$ holds uniformly on compact intervals. The function $y : \mathbb{R} \to \mathbb{R}$ is a solution to (4.1.1) and satisfies $\mu_- \leq y(\xi) \leq \mu_+$ for all $\xi \in \mathbb{R}$ and in fact $\mu_+ = \sup_{\xi \in \mathbb{R}} y(\xi)$. Therefore, from the first part of this proof, we have that μ_+ is in the set as indicated by (4.2.8), as claimed.

The proof for μ_- is similar, as is the proof for $\xi \to -\infty$. ☐

Corollary 4.2.3. *If $(P, c) \in W_0^{2,\infty} \times \mathbb{R}$ is a connecting solution to (4.1.1), then*

$$-1 < P(\xi) < 1, \quad \xi \in \mathbb{R}. \tag{4.2.11}$$

Proof. Lemma 4.2.2 implies that $-1 \leq P(\xi) \leq 1$ for all $\xi \in \mathbb{R}$. The strict inequalities now follow from an application of Lemma's 3.3.2 and 3.3.1. ☐

Lemma 4.2.4. *Let $(P, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (4.1.1). Then for some quantities $C_\pm > 0$ and $\epsilon > 0$ we have that*

$$P(\xi) = \begin{cases} -1 + C_- e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \\ 1 - C_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \end{cases} \tag{4.2.12}$$

where $\lambda_-^u \in (0,\infty)$ is the unique positive eigenvalue of the linearization of (4.1.1) about $x=-1$ and $\lambda_+^s \in (-\infty,0)$ is the unique negative eigenvalue of the linearization about $x=1$. The formulae for $P'(\xi)$ obtained by formally differentiating (4.2.12) also hold.

Proof. We consider only the limit $\xi \to \infty$, as the proofs of the results for $\xi \to -\infty$ are similar. We note that it follows from the proof of Lemma 3.3.2 that if $y(\xi) = 1 - P(\xi)$, then y satisfies (2.3.3) with coefficients $A_j(\xi)$ given by (3.3.9) with $x_1 = 1$ and $x_2 = P$. Since equation (2.3.3) is asymptotically hyperbolic at $+\infty$, we have $y(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$ for some $a > 0$, by Proposition 2.2.5. It follows from this exponential decay, equation (3.3.9) and the Lipschitz condition (b2) on the derivative of F, that $A_j(\xi)$ approaches the limiting coefficients $A_{j+}(\rho)$ exponentially fast as $\xi \to \infty$. Due to the boundedness of P, we have that $A_j(\xi)$ is bounded for all $0 \le j \le N$. Note that due to the assumption (b2), we have that $A_j(\xi) > 0$ for all $\xi \in \mathbb{R}$ and for $1 \le j \le N$, which now allows us to conclude that $A_j(\xi) \ge \alpha_j > 0$ for $1 \le j \le N$ and some $\alpha_j \in \mathbb{R}$. Now remembering that $A_{\Sigma+}(\rho) < 0$ by (4.2.3) and condition (b5), we see that our linear differential difference equation (2.3.3) satisfies the conditions of Lemma 3.5.2, at least for $\xi \to \infty$. As $y(\xi) > 0$ by Corollary 4.2.3, we conclude by Lemma 3.5.2 the asymptotic formula (4.2.12) at $+\infty$ for some $C_+ > 0$. In addition, the differentiated version of that formula holds. $\square$

Lemma 4.2.5. *If $(P,c) \in W_0^{2,\infty} \times \mathbb{R}$ is a connecting solution to (4.1.1), then $P'(\xi) > 0$ for all $\xi \in \mathbb{R}$.*

Proof. We note that it is sufficient to prove that $P'(\xi) \ge 0$ for all $\xi \in \mathbb{R}$, since Corollary 3.3.3 then immediately implies the strict positivity $P'(\xi) > 0$.

By (4.2.12) we see that there exists $\tau > 0$ such that $P'(\xi) > 0$ whenever $|\xi| \ge \tau$ and such that $P(-\tau) < P(\xi) < P(\tau)$ whenever $\xi < \tau$. From this we have $P(\xi + k) > P(\xi)$ for all $\xi \in \mathbb{R}$, provided that $k \ge 2\tau$. Now suppose that $P'(\xi) < 0$ for some ξ and set

$$k_0 = \inf\{k > 0 \mid P(\xi + k) > P(\xi) \text{ for all } \xi \in \mathbb{R}\}. \tag{4.2.13}$$

Certainly $k_0 > 0$. Also, $k_0 \le 2\tau$ and $P(\xi + k_0) \ge P(\xi)$ for all $\xi \in \mathbb{R}$. If $0 < k < k_0$ then $P(\xi + k) \le P(\xi)$ for some ξ, where necessarily $|\xi| \le \tau$. Therefore, there exists some ξ_0, with $|\xi_0| \le \tau$, for which $P(\xi_0 + k_0) = P(\xi_0)$. We can now define $x_1(\xi) = P(\xi + k_0)$ and $x_2(\xi) = P(\xi)$. Because

$x_1(\xi) \geq x_2(\xi)$ for all $\xi \in \mathbb{R}$ and $x_1(\xi_0) = x_2(\xi_0)$, Lemma 3.3.2 implies that $P(\xi + k_0) = P(\xi)$ for all $\xi \in \mathbb{R}$. This is a contradiction, because $P'(\xi) > 0$ for all large $|\xi|$. $\square$

Proposition 4.2.6. *Let $(P_0, c_0) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (4.1.1) for some $\rho_0 \in V$ and for some $c_0 \in \mathbb{R}$. Consider the linearization (2.3.3) of equation (4.1.1) about P_0 and let $\Lambda_{c_0,\gamma,L}$ denote the associated linear operator from $W^{2,\infty}$ to L^∞. Then the derivative of $\mathcal{G}$,*

$$D_{1,2}\mathcal{G}(P_0, c_0, \rho_0) : W_0^{2,\infty} \times \mathbb{R} \to L^\infty, \tag{4.2.14}$$

at the solution (P_0, c_0), with respect to the first two arguments, is given by

$$D_{1,2}\mathcal{G}(P_0, c_0, \rho_0)(\psi, b)(\xi) = -bP_0'(\xi) + (\Lambda_{c_0,\gamma,L}\psi)(\xi) \tag{4.2.15}$$

and is an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ onto L^∞.

Proof. The fact that $\mathcal{G}$ is C^1-Frechet differentiable follows from the fact that F is a C^1-function. The explicit formula (4.2.15) follows by direct differentiation of (4.1.7). The operator $\Lambda_{c_0,\gamma,L}$ can be seen to satisfy all the conditions of Theorem 3.5.1. In particular, as we have seen before in the proof of Lemma 4.2.4, the conditions (3.5.1) and (3.5.3) are satisfied and the exponential condition (3.5.4) follows from the exponential approach of $P(\xi)$ to ± 1, by Lemma 4.2.4. We also have that $x(\xi) = P_0'(\xi)$ satisfies the linear equation (2.3.3), which by Lemma 4.2.5 gives the strictly positive $p = P_0' \in \mathcal{K}_{c_0,\gamma,L}$ in the statement of Theorem 3.5.1. One also sees $A_{\Sigma+}(\rho) < 0$ by (4.2.3) and condition (b5). Thus, by Theorem 3.5.1, the kernel $\mathcal{K}_{c_0,\gamma,L}$ of $\Lambda_{c_0,\gamma,L}$ is precisely the one-dimensional span of P_0'. The strict positivity $P_0'(0) > 0$ implies that $P_0' \notin W_0^{2,\infty}$, hence the restriction of $\Lambda_{c_0,\gamma,L}$ to $W_0^{2,\infty} \subseteq W^{2,\infty}$ is an isomorphism from $W_0^{2,\infty}$ onto its range $\mathcal{R}_{c_0,\gamma,L} \subseteq L^\infty$, which has codimension one. Also from Theorem 3.5.1, we see that there exists $p^* \in \mathcal{K}_{c_0,\gamma,L^*}$, with $p^*(\xi) > 0$ for all $\xi \in \mathbb{R}$. In particular this means that

$$\int_{-\infty}^{\infty} p^*(\xi)P_0'(\xi)d\xi > 0, \tag{4.2.16}$$

which implies that $P_0'(\xi) \notin \mathcal{R}_{c_0,\gamma,L}$ by the Fredholm Alternative (see Theorem 2.2.1). From this and the explicit formula for $D_{1,2}\mathcal{G}$ the claim immediately follows. $\square$

It is now an easy task to complete the proof of our main proposition.

Proof of Proposition 4.2.1. The local continuation follows from the implicit function theorem, together with Lemma 4.2.6. The limits follow from the observation that the quantities $\mu_{\pm}(\rho)$ in (4.2.7) for $P(\rho)$ vary continuously with ρ, together with $\mu_{\pm}(\rho_0) = 1$ and the identity (4.2.6). □

4.3 Global Continuation

This section is devoted to the proof of Theorem 4.1.1. In Lemma 4.3.1 we prove that for each $\rho \in V$ equation (4.1.1) has at most one connecting solution, establishing the uniqueness claim in Theorem 4.1.1. Theorem 4.3.3 will allow us to extend the local continuation in Proposition 4.2.1 into a global continuation for all $\rho \in V$, by proving that limits of connecting solutions are connecting solutions to the limiting differential difference equation. This means that once we have established the existence of a connecting solution to (4.1.1) for one value of the detuning parameter, $\rho_0 \in V$, we know that (4.1.1) has a connecting solution for all values $\rho \in V$. This is why we give an explicit solution to a prototype differential difference equation in Lemma 4.3.4. By constructing a new family (4.1.1), which mixes the original differential difference equation and the prototype system, we can combine Theorem 4.3.3 and Proposition 4.2.1 to establish the existence of a connecting solution to our original family (4.1.1) at one value of the detuning parameter ρ, as required.

Lemma 4.3.1. *For each $\rho \in V$ there exists at most one value $c \in \mathbb{R}$ such that equation (4.1.1) possesses a monotone increasing solution $x = P(\xi)$, satisfying the boundary conditions*

$$\lim_{\xi \to \pm\infty} x(\xi) = \pm 1. \tag{4.3.1}$$

For each $c \in \mathbb{R}$ and $\rho \in W$ there exists at most one solution $x = P(\xi)$ of (4.1.1), up to translation, satisfying the boundary conditions (4.3.1).

Proof. We start by showing that the wavespeed c is unique for solutions to (4.1.1). Suppose to the contrary that for some $\rho \in V$ there exist $c_1 > c_2$ and monotone solutions $x = P_j(\xi)$ to (4.1.1) with $c = c_j$ for $j = 1, 2$, satisfying (4.3.1). Without loss we may assume that $P_1(\xi) < P_2(\xi)$ for some ξ, as we can always replace P_2 by a translate obtained by shifting the graph of P_2 to the left. Also, by (3.4.10) of Lemma 3.4.2, we have that

$$\lambda_-^u(c_1, \rho) < \lambda_-^u(c_2, \rho), \quad \lambda_+^s(c_1, \rho) < \lambda_+^s(c_2, \rho), \tag{4.3.2}$$

for the real eigenvalues of the linearized equations at $x = \pm 1$. From (4.3.2) and Lemma 4.2.4 it follows that $P_1(\xi) > P_2(\xi)$ for all large $|\xi|$. Thus, there exists $\tau \in \mathbb{R}$ such that $P_1(\xi) > P_2(\xi)$ for all

ξ satisfying $|\xi| > \tau$. Let

$$k_0 = \inf\{k > 0 \mid P_1(\xi + k) > P_2(\xi) \text{ for all } \xi \in \mathbb{R}\}. \tag{4.3.3}$$

We certainly have $k_0 > 0$, as $P_1(\xi_0) < P_2(\xi_0)$ for some $\xi_0 \in \mathbb{R}$. Also it is easy to see that $k_0 \leq 2\tau$, using the monotonicity of P_1 and P_2. We also must have $P_1(\xi + k_0) \geq P_2(\xi)$ for all $\xi \in \mathbb{R}$. If $0 < k < k_0$, then $P_1(\xi + k) \leq P_2(\xi)$ for some ξ, necessarily satisfying $|\xi| \leq \tau$. Therefore, there exists some ξ_0, with $|\xi_0| \leq \tau$, for which $P_1(\xi_0 + k_0) = P_2(\xi_0)$. But now Lemma 3.3.4 implies that these solutions are equal and constant as $\xi \to \infty$, contradicting the monotonicity $P_1'(\xi) > 0$. This contradiction establishes the uniqueness of the wavespeed c.

Now suppose for some $c \in \mathbb{R}$, that (4.1.1) admits two solutions $P_1(\xi)$ and $P_2(\xi)$, which both satisfy the limits (4.3.1), but are not translates of each other. With the same quantities $\lambda_-^u = \lambda_-^u(c, \rho)$ and $\lambda_+^s = \lambda_+^s(c, \rho)$ for both solutions, we have from Lemma 3.5.16 that

$$P_j(\xi) = \begin{cases} -1 + C_{j-} - e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \\ 1 - C_{j+} e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \end{cases} \tag{4.3.4}$$

for (generally different) constants $C_{j\pm} > 0$ and $\epsilon > 0$. We may assume without loss that $C_{1-} > C_{2-}$ and $C_{2+} > C_{1+}$, as we can always replace $P_1(\xi)$ by $P_1(\xi + k)$ for some $k > 0$ if necessary. The asymptotic descriptions (4.3.4) now imply that $P_1(\xi) > P_2(\xi)$ for all large $|\xi|$. By further shifting the graph of P_1 to the left, we can ensure that $P_1(\xi) > P_2(\xi)$ for all $\xi \in \mathbb{R}$. Assuming this, set

$$k_0 = \sup\{k \geq 0 \mid P_1(\xi) \geq P_2(\xi + k) \text{ for all } \xi \in \mathbb{R}\}. \tag{4.3.5}$$

Then we certainly have that $k_0 < \infty$ and $P_1(\xi) \geq P_2(\xi + k_0)$ for all $\xi \in \mathbb{R}$, which gives us the estimates $C_{1-} \geq C_{2-} e^{\lambda_-^u k_0}$ and $C_{2+} e^{\lambda_+^s k_0} \geq C_{1+}$. We claim that we in fact have the strict inequalities

$$C_{1-} > C_{2-} e^{\lambda_-^u k_0}, \quad C_{2+} e^{\lambda_+^s k_0} > C_{1+}, \tag{4.3.6}$$

which imply that $P_1(\xi) > P_2(\xi + k_0)$ for all large $|\xi|$. If this is true, then as before we have that $P_1(\xi_0) = P_2(\xi_0 + k_0)$ for some $\xi_0 \in \mathbb{R}$ and Lemma 3.3.2 yields a contradiction.

To prove the strict inequalities (4.3.6), let $y(\xi) = P_1(\xi) - P_2(\xi + k_0)$. Then from (4.3.4) we have

$$y(\xi) = (C_{2+} e^{\lambda_+^s k_0} - C_{1+}) e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), \quad \xi \to \infty \tag{4.3.7}$$

and $y(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. We know that y does not vanish identically on any interval $[\tau, \infty)$, as otherwise Lemma 3.3.1 with $x_1 = P_1(\xi)$ and $x_2 = P_2(\xi + k_0)$ would imply that y vanishes identically on $\mathbb{R}$, which means P_1 is a translate of P_2. Now noting as in the proof of Lemma 4.2.4 that $y(\xi)$ satisfies a linear differential difference equation which satisfies the conditions of Lemma 3.5.2, we conclude that the coefficient $C_{2+}e^{\lambda_+^s k_0} - C_{1+}$ in (4.3.7) is positive, which establishes the second inequality in (4.3.6). The first inequality follows from a similar argument at $-\infty$. This proves the uniqueness of solutions to (4.1.1), for each $c \in \mathbb{R}$. □

The following result, concerning the linearization around the (unstable) equilibrium $q(\rho)$, will prove to be useful in establishing the boundary conditions $x(\pm\infty) = \pm 1$ for limits of connecting solutions x_n.

Lemma 4.3.2. *For every $\rho \in V$, $\gamma \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}$ there do not exist two monotone increasing solutions $x_\pm : \mathbb{R} \to \mathbb{R}$ of equation (4.1.1) such that*

$$\begin{aligned} &\lim_{\xi \to -\infty} x_-(\xi) = -1, \quad \lim_{\xi \to \infty} x_-(\xi) = q(\rho), \\ &\lim_{\xi \to -\infty} x_+(\xi) = q(\rho), \quad \lim_{\xi \to \infty} x_+(\xi) = 1. \end{aligned} \tag{4.3.8}$$

Proof. The case where $\gamma = 0$ was considered in [24, Lemma 7.1], so we will assume $\gamma > 0$. First notice that

$$\Delta_{c,\gamma,L_\diamond(\rho)}(0) = -A_{\Sigma\diamond}(\rho) = -D_1\Phi(q(\rho), \rho) < 0, \tag{4.3.9}$$

which by Lemma 3.4.3 implies that there do not simultaneously exist eigenvalues $\Lambda_\diamond^u < 0 < \lambda_\diamond^s$ for the constant coefficient system $L_\diamond$ defined in (4.2.2).

Now assume that there exist monotone increasing x_- and x_+ satisfying conditions (4.3.8). Consider $y(\xi) = q(\rho) - x_-(\xi)$, which is a monotone decreasing function on the real line, satisfying the linear equation (2.3.3) with coefficients given by (3.3.9), with $x_1 = q(\rho)$ and $x_2 = x_-(\xi)$. The coefficients of this linear equation are bounded according to (3.1.2) and Assumption 3.1.1 holds. From the proof of Lemma 3.1.1 we thus see that for all $\xi \in \mathbb{R}$,

$$y'(\xi) \geq -By(\xi) \tag{4.3.10}$$

for some $B > 0$. Now take any sequence $\xi_n \to \infty$, and let $z_n(\xi) = y(\xi + \xi_n)/y(\xi_n)$. Then each z_n also satisfies $z_n'(\xi) \geq -Bz_n(\xi)$ on $\mathbb{R}$. As $z_n(0) = 1$, we conclude that the sequence of functions z_n is uniformly bounded and equicontinuous on each compact interval and so without loss we have that $z_n(\xi) \to z(\xi)$ uniformly on compact intervals. From the differential equation (4.1.1) we see that we can use the uniform bound on z_n' to obtain a uniform bound on $z_n''(\xi)$, thus concluding that also $z_n'(\xi)$ is equicontinuous on each compact interval. One now easily sees that z satisfies the autonomous limiting constant coefficient equation associated to $L_\diamond$. Moreover, $-Bz(\xi) \le z'(\xi) \le 0$ for all $\xi \in \mathbb{R}$, with $z(0) = 1$, so $z(\xi) > 0$ and z does not decay faster than exponentially. We may now apply Proposition 2.2.4 to the solution z. We conclude that $z(\xi) = w(\xi) + O(e^{-(b+\epsilon)\xi})$ as $\xi \to \infty$, where w is a nontrivial sum of eigensolutions corresponding to a set of eigenvalues with $\operatorname{Re}\lambda = -b \le 0$. The positivity of z, together with Lemma 3.3.5, implies that the linearization about $x = q(\rho)$ possesses a nonpositive eigenvalue $\lambda_\diamond^s \le 0$. Since $\Delta_{c,\gamma,L_\diamond(\rho)}(0) < 0$ we have $\lambda_\diamond^s < 0$. We can use similar reasoning applied to $x_+(\xi)$ to conclude that the linearization about $x = q(\rho)$ must also possess a positive eigenvalue $\lambda_\diamond^u > 0$. This yields a contradiction. $\square$

Remark 4.3.1. *In the above proof we could not apply Proposition 2.2.4 directly to the function $y(\xi)$, as it may not be the case that $y(\xi)$ approaches its limits $y(\pm\infty)$ exponentially fast.*

The next theorem enables us to take limits of connecting solutions, which will be crucial in establishing global existence of solutions.

Theorem 4.3.3. *Let $\rho_n \in V$ and $\gamma_n \in \mathbb{R}_{>0}$ be two sequences satisfying $\gamma_n \to \gamma_0$ and $\rho_n \to \rho_0$ as $n \to \infty$. Let $(P_n(\xi), c_n)$ denote any connecting solution to (4.1.1) with $\rho = \rho_n$ and $\gamma = \gamma_n$. Then, after possibly passing to a subsequence, the limit*

$$\lim_{n\to\infty} P_n(\xi) = P_0(\xi) \tag{4.3.11}$$

exists pointwise and also the limit

$$\lim_{n\to\infty} c_n = c_0 \tag{4.3.12}$$

exists, with $|c_0| < \infty$. Furthermore, $P_0(\xi)$ satisfies the limiting differential difference equation

$$-\gamma_0 P_0''(\xi) - c_0 P_0'(\xi) = F\big(P_0(\xi), P_0(\xi + r_1), \dots, P_0(\xi + r_N), \rho_0\big) \tag{4.3.13}$$

almost everywhere. In addition, we have the limits

$$\lim_{x\to\pm\infty} P_0(\xi) = \pm 1. \tag{4.3.14}$$

We remark that the case $\gamma_0 = 0$ can occur.

Proof. Using the fact that the functions $P_n(\xi)$ satisfy $P_n' > 0$, we may argue in a standard fashion that, after passing to a subsequence, the pointwise limit $P_0(\xi) = \lim_{n\to\infty} P_n(\xi)$ exists for all $\xi \in \mathbb{R}$.

Due to the limits $\lim_{n\to\infty} P_n(\xi) = \pm 1$, we have $\int_{-\infty}^{\infty} P_n'(s)ds = 2$. Writing $F(\xi) = \liminf_{n\to\infty} P_n'(\xi)$ we obtain, using Fatou's Lemma,

$$\int_{-\infty}^{\infty} F(s)ds \leq 2. \tag{4.3.15}$$

In particular, this implies that the measure of the set for which $F(s) = \infty$ is zero. Letting β_n be any sequence with $\beta_n \to 0$ as $n \to \infty$, we have that, if we choose ξ_0 appropriately,

$$\liminf_{n\to\infty} \beta_n (P_n'(\xi) - P_n'(\xi_0)) = 0 \quad \text{almost everywhere .} \tag{4.3.16}$$

Now suppose that $\liminf_{n\to\infty} |c_n| = \infty$. Without loss assume $c_n > 0$. Write $q_0 = q(\rho_0)$ and fix a point

$$q_* \in (q_0, 1). \tag{4.3.17}$$

Let $x_n(\xi) = P_n(c_n\xi + \xi_n)$, where $\xi_n \in \mathbb{R}$ is such that $P_n(\xi_n) = q_*$. Then (4.1.1) in integrated form gives us

$$-\gamma_n c_n^{-2}(x_n'(\xi) - x_n'(\xi_0)) - (x_n(\xi) - x_n(\xi_0)) = \int_{\xi_0}^{\xi} F\big(x_n(s), x_n(s + r_1 c_n^{-1}), \ldots, x_n(s + r_N c_n^{-1}), \rho_n\big)ds. \tag{4.3.18}$$

Again, because the x_n are monotonically increasing functions, we can pass to a subsequence for which the pointwise limit $x(\xi) = \lim_{n\to\infty} x_n(\xi)$ exists and is continuous at all but countably many points. We have seen above that $\liminf_{n\to\infty} \beta_n c_n^{-1} x_n'(\xi) = 0$ almost everywhere, for a sequence $\beta_n \to 0$. After taking the limit $\liminf_{n\to\infty}$ we thus obtain, using $\beta_n = c_n^{-1} \to 0$,

$$-\big(x(\xi) - x(\xi_0)\big) = \int_{\xi_0}^{\xi} F\big(x(s), x(s), \ldots, x(s), \rho_0\big)ds, \tag{4.3.19}$$

which holds almost everywhere. By redefining x on a set of measure zero, which does not affect the right hand side of (4.3.19), we can make this identity hold everywhere. From this identity we also see that $x(\xi)$ is differentiable and satisfies

$$-x'(\xi) = \Phi(x(\xi), \rho_0). \tag{4.3.20}$$

Since $x(\xi) \leq q_*$ for almost all $\xi \leq 0$, we cannot have $x(\xi) = 1$ for some ξ, as this would imply $x(\xi) = 1$ for all ξ. Now $x_n(\xi) \geq q_*$ for all $\xi \geq 0$, hence also $1 > x(\xi) \geq q_*$ for $\xi \geq 0$ and thus $x'(\xi) = -\Phi(x(\xi), \rho_0) < 0$ whenever $\xi \geq 0$. On the other hand, $x_n'(\xi) > 0$, hence $x'(\xi) \geq 0$, for all

ξ. This contradiction implies that $\liminf_{n\to\infty} |c_n| < \infty$. Thus, after passing to a subsequence, the limit $c_0 = \lim_{n\to\infty} c_n$ exists.

Integration of (4.1.1) yields

$$-\gamma_n\big(P_n'(\xi) - P_n'(\xi_0)\big) - c_n\big(P_n(\xi) - P_n(\xi_0)\big) = \int_{\xi_0}^{\xi} F\big(P_n(s), P_n(s+r_1), \ldots, P_n(s+r_N), \rho_n\big) ds. \tag{4.3.21}$$

Consider the case where $\gamma_0 \neq 0$. Notice that $y_n(\xi) = 1 - P_n(\xi)$ is a monotone decreasing function on the real line, which satisfies the linear equation (2.3.3) with coefficients given by (3.3.9), with $x_1 = 1$ and $x_2 = P_n$. Referring to these coefficients as $A_{j,n}(\xi)$, we see that they are bounded according to (3.1.2) and that Assumption 3.1.1 holds. From the proof of Lemma 3.1.1 we see that

$$y_n'(\xi) \geq -B_n y_n(\xi), \tag{4.3.22}$$

in which $B_n = \sqrt{\frac{c_n^2}{4\gamma_n^2} - \frac{\alpha_{0,n}}{\gamma_n}} + \frac{c_n}{\gamma_n}$. Now there exists α_0 such that $0 \geq \alpha_{0,n} \geq \alpha_0$, as the functions $y_n(\xi)$ are uniformly bounded and $D_1 F$ is a continuous function, which attains its maxima and minima on compact sets. This means that the constants B_n are bounded, $0 \leq B_n \leq B$ for some B. From (4.3.22) we now see that y_n' and hence P_n' are uniformly bounded. From the differential equation (4.1.1) it now also follows that the functions P_n'' are uniformly bounded. Thus P_n' is an equicontinuous family, allowing us to pass to a subsequence for which $P_n'(\xi) \to P_0'(\xi)$ and $P_n(\xi) \to P_0(\xi)$ uniformly on compact intervals.

Thus, taking the limit $\liminf_{n\to\infty}$ in (4.3.21), we now obtain for all $\gamma_0 \geq 0$

$$-\gamma_0\big(P_0'(\xi) - P_0'(\xi_0)\big) - c_0\big(P_0(\xi) - P_0(\xi_0)\big) = \int_{\xi_0}^{\xi} F\big(P_0(s), P_0(s+r_1), \ldots, P_0(s+r_N), \rho_0\big) ds, \tag{4.3.23}$$

which holds for all $\xi \in \mathbb{R}$ if $\gamma_0 \neq 0$ and almost everywhere if $\gamma_0 = 0$. In case $\gamma_0 = 0$ and $c_0 \neq 0$ we can again by redefining P_0 on a set of measure zero ensure that (4.3.23) holds for all $\xi \in \mathbb{R}$. After differentiation we see that $P_0(\xi)$ satisfies the differential difference equation stated in the theorem.

We now set out to prove the limits (4.3.14). Because $P_0(\xi)$ is a bounded monotonically increasing function, the limits $\lim_{\xi\to\pm\infty} P_0(\xi)$ exist. We will refer to these limits as $P_0(\xi)$. When $c_0 \neq 0$, the function $P_0'(\xi)$ decays exponentially, and when $\gamma_0 \neq 0$, the function $P_0''(\xi)$ decays exponentially. Taking the limits $\xi \to \pm\infty$ in equation (4.3.13) we obtain

$$0 = F\big(P_0(\pm\infty), P_0(\pm\infty), \ldots, P_0(\pm\infty), \rho_0\big) = \Phi\big(P_0(\pm\infty), \rho_0\big), \tag{4.3.24}$$

which implies that

$$P_0(\pm\infty) \in \{-1, q(\rho_0), 1\}\,. \tag{4.3.25}$$

Since we know that $P_n(\xi) < 0$ if $\xi < 0$ and $P_n(\xi) > 0$ if $\xi > 0$, we have that $P_0(\xi) \le 0$ if $\xi < 0$ and $P_0(\xi) \ge 0$ if $\xi > 0$ almost everywhere. In particular, if $q(\rho_0) = \pm 1$ then the proof is complete as then necessarily $P_0(\pm\infty) = \pm 1$. Thus assume that $q(\rho_0) \in (-1, 1)$. Fix any points q_1 and q_2 satisfying $-1 < q_1 < q(\rho_0) < q_2 < 1$ and let $\xi_n, \zeta_n \in \mathbb{R}$ be such that

$$\begin{aligned} &P_n(\xi) \le q_1, && \xi < \zeta_n,\\ &q_1 \le P_n(\xi) \le q_2, && \zeta_n < \xi < \xi_n,\\ &P_n(\xi) \ge q_2, && \xi > \xi_n. \end{aligned} \tag{4.3.26}$$

Without loss (we may always pass to a subsequence) we may assume the limits $\xi_n \to \xi_0$ and $\zeta_n \to \zeta_0$ both exist, although possibly are infinite. It is enough to show that the difference $\xi_n - \zeta_n$ is bounded. Indeed, if this is the case, and if ξ_n and hence also ζ_n are themselves bounded, so that ξ_0 and ζ_0 are both finite, then $P_0(\xi) \le q_1$ for all $\xi < \zeta_0$ and $P_0(\xi) \ge q_2$ for all $\xi > \xi_0$, which with (4.3.25) implies the limits (4.3.14). The case $\xi_0 = \zeta_0 = \pm\infty$ cannot occur, since then either $P_0(\xi) \le q_1$ or $P_0(\xi) \ge q_2$, hence $P_0(\xi) = \pm 1$ for all $\xi \in \mathbb{R}$, which is a contradiction.

To prove that $\xi_n - \zeta_n$ is bounded, assume $\xi_n - \zeta_n \to \infty$ and define

$$x_{n+}(\xi) = P_n(\xi + \xi_n), \;\; x_{n-}(\xi) = P_n(\xi + \zeta_n). \tag{4.3.27}$$

Upon passing to a subsequence and taking limits $x_{n\pm} \to x_\pm$ as above, we obtain solutions of (4.3.13) which satisfy the four boundary conditions in (4.3.8) with $q(\rho_0)$ replacing $q(\rho)$. However, this is impossible by Lemma 4.3.2. □

Lemma 4.3.4. *Suppose that the function $q : \overline{V} \to \mathbb{R}$ associated to (4.1.1) satisfies $q(\rho^*) = 0$ for some $\rho^* \in \overline{V}$. Then (4.1.1) with $\rho = \rho^*$ has a connecting solution $(P(\xi), c)$ for some $c \in \mathbb{R}$.*

Proof. First we consider the specific equation for some $k > 0$,

$$-\gamma x''(\xi) - x'(\xi) = \beta^{-1}\big(x(\xi - k) - x(\xi)\big) - f\big(x(\xi)\big), \tag{4.3.28}$$

in which f is given by

$$f(x) = \frac{\beta x(x^2-1)}{1-\beta x} + 2\gamma x(x^2-1), \quad \beta = \tanh k, \tag{4.3.29}$$

for $x \in [-1,1]$. Outside this interval f is modified to be a nonzero C^1 function on the real line. It is routine to check that $x = \tanh(\xi)$ satisfies (4.3.28).

Now let $g : [0,1] \to [0,1]$ be any C^1 smooth function satisfying $g(\frac{1}{4}) = 0$ and $g(\frac{3}{4}) = 1$ and consider the family of equations

$$\begin{aligned} -\gamma x''(\xi) - cx'(\xi) &= (1-g(\rho))\Big(\beta^{-1}\big(x(\xi-k)-x(\xi)\big) - f\big(x(\xi)\big)\Big) \\ &\quad + g(\rho)F\big(x(\xi+r_0),\ldots,x(\xi+r_N),\rho^*\big) \end{aligned} \tag{4.3.30}$$

for $\rho \in [0,1]$. It is easy to see that this family satisfies the conditions (b1) through (b5), with $q(\rho) = 0$ for all $\rho \in [0,1]$. We know that at $\rho = \frac{1}{4}$ the equation (4.3.30) has a connecting solution, namely $c = 1$, $x = \tanh\xi$. Due to Proposition 4.2.1 we see that solutions to (4.3.30) exist in a neighbourhood of $\rho = \frac{1}{4}$ and Theorem 4.3.3 allows us to extend this continuation to the interval $(0,1)$. This proves the claim, as at $\rho = \frac{3}{4}$ the system reduces to the specified equation (4.1.1) with $\rho = \rho^*$. □

In case there is no value ρ^* for which $q(\rho^*) = 0$, the following lemma shows that we can choose an arbitrary value $\rho_0 \in V$ and embed the differential difference equation (4.1.1) with $\rho = \rho_0$ into a new family which does have $q(\rho_*) = 0$ for some ρ^*. We can then again apply the reasoning in the proof of Lemma 4.3.4 to the new family to obtain a connecting solution to our original family at $\rho = \rho_0$.

Lemma 4.3.5 (see [24, Lemma 8.6]). *Consider the system*

$$-\gamma x''(\xi) - cx'(\xi) = F_0\big(x(\xi+r_0),\ldots,x(\xi+r_N)\big) \tag{4.3.31}$$

satisfying the conditions (b1) through (b5) without the parameter ρ. Assume that $q = q_0 \in (-1,1)$ for the quantity in condition (b5). Then there exists a family (4.1.1), with $V = (-1,1)$ and $q(\rho) = \rho$, satisfying the conditions (b1) through (b5), which reduces to (4.3.31) at $\rho = q_0$.

We now have all the ingredients to complete the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. One can use Lemma's 4.3.5 and 4.3.4 to establish the existence of a solution at some parameter $\rho_* \in V$, after which a global continuation for all $\rho \in V$ of this solution can be constructed using Theorem 4.3.3 and Proposition 4.2.1. Uniqueness follows from Lemma 4.3.1. Here we have assumed V is connected, if not, use this construction for each connected component of V. □

Chapter 5

The Algorithm

5.1 The Problem

In this chapter we present a numerical method for solving the nonlinear autonomous differential difference equation

$$-cx'(\xi) - \gamma x''(\xi) = F\big(x(\xi)\big) + G\big(x(\xi + r_1), x(\xi + r_2), \ldots, x(\xi + r_N)\big), \tag{5.1.1}$$

in which $\gamma > 0$. As in the previous chapter, we demand that $r_i \neq r_j$ if $i \neq j$ and $r_i \neq 0$ for $i = 1 \ldots N$, where $N \geq 1$.

The difference between the equation (5.1.1) presented here and the family (4.1.1) used in the previous chapter, is the explicit splitting between the dependence on the shifted arguments $x(\xi + r_i)$ for $1 \leq i \leq N$ and the unshifted argument $x(\xi)$. In the context of the algorithm presented in this section we will see that this splitting is useful, as it allows us to isolate and relax the dependence on the shifted arguments, which of course cause all the numerical difficulties.

Throughout this chapter we will assume that the appropriately modified conditions (b1) through (b5) introduced in the previous chapter apply to the functions F and G. For completeness we have listed the modified conditions below.

(c1) The nonlinearity $F : \mathbb{R} \to \mathbb{R}$ is C^1-smooth. Also, $G : \mathbb{R}^N \to \mathbb{R}$ is C^1-smooth in $\mathbb{R}^N$.

(c2) The first order derivative $D_1 F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in u. Also, the Jacobian $D_1 G : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz in v.

(c3) For $j = 1, \ldots, N$, we have that

$$\frac{\partial G(v)}{\partial v_j} > 0. \tag{5.1.2}$$

(c4) Let $\Gamma : \mathbb{R} \to \mathbb{R}$ be defined as

$$\Gamma(\phi) = F(\phi) + G(\phi, \phi, \dots, \phi). \tag{5.1.3}$$

Then for some quantity $a \in (-1, 1)$ we have that

$$\begin{aligned} &\Gamma(-1) = \Gamma(a) = \Gamma(1) = 0 \\ &\Gamma(\phi) > 0, \quad \phi \in (-\infty, -1) \cup (a, 1) \\ &\Gamma(\phi) < 0, \quad \phi \in (-1, a) \cup (1, \infty) \end{aligned} \tag{5.1.4}$$

(c5) We have that

$$\begin{aligned} &D_1\Gamma(-1, \rho) < 0 \text{ if } a \neq -1, \\ &D_1\Gamma(a, \rho) > 0 \text{ if } a \in (-1, 1), \\ &D_1\Gamma(1, \rho) < 0 \text{ if } a \neq 1, \end{aligned} \tag{5.1.5}$$

for the quantity a introduced in condition (c4).

In the present context the functional $\mathcal{G} : W_0^{2,\infty} \times \mathbb{R} \times V \to L^\infty$ introduced in (4.1.7) is given by

$$\mathcal{G}(\phi, c)(\xi) = -\gamma\phi''(\xi) - c\phi'(\xi) - F\big(\phi(\xi)\big) - G\big(\overline{\phi}(\xi)\big), \tag{5.1.6}$$

in which we have used the notation $\overline{\phi}(\xi) = \big(\phi(\xi + r_1), \phi(\xi + r_2), \dots, \phi(\xi + r_N)\big) \in \mathbb{R}^N$. Following Definition 4.1.1, we define a connecting solution to (5.1.1) to be a pair $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ which satisfies (5.1.1) and has the limits

$$\lim_{\xi \to \pm\infty} \phi(\xi) = \pm 1. \tag{5.1.7}$$

5.2 The Newton iteration

It is clear that solutions to (5.1.1) correspond to zeroes of the operator $\mathcal{G}$ defined in (5.1.6). The numerical method we use to solve the differential difference equation (5.1.1) consists of applying a variant of Newton's method to find a zero of the operator $\mathcal{G}$ which satisfies the boundary conditions (5.1.7). Normally, applying Newton's method to seek a zero of $\mathcal{G}$ would involve an iteration step of the form

$$(\phi_{n+1}, c_{n+1}) = (\phi_n, c_n) - [D_{1,2}\mathcal{G}(\phi_n, c_n)]^{-1}\mathcal{G}(\phi_n, c_n). \tag{5.2.1}$$

To execute this step one would have to solve the linear differential difference equation

$$D_{1,2}\mathcal{G}(\phi_n, c_n)(\phi_{n+1}, c_{n+1}) = D_{1,2}\mathcal{G}(\phi_n, c_n)(\phi_n, c_n) - \mathcal{G}(\phi_n, c_n). \tag{5.2.2}$$

Since this is a difficult procedure due to the presence of the shifted arguments, we want to reduce their contribution as much as possible. To this end, we define the linear operator $\mathcal{F}^\mu : W^{2,\infty} \times \mathbb{R} \to L^\infty$, given by

$$\mathcal{F}^\mu(\phi, c)(\xi) = -\gamma\phi''(\xi) - c\phi'(\xi) - F\big(\phi(\xi)\big) - \mu G\big(\overline{\phi}(\xi)\big), \tag{5.2.3}$$

in which $\mu \in [0,1]$ is a fixed relaxation parameter. Since F and G do not depend on c and since F and G are both C^1-functions, $\mathcal{F}^\mu$ and $\mathcal{G}$ are C^1 Frechet differentiable. In particular, taking the derivative of $\mathcal{F}^\mu$ with respect to the first two variables, we see that $D_{1,2}\mathcal{F}^\mu(\phi, c) : W_0^{2,\infty} \times \mathbb{R} \to L^\infty$ is given by

$$D_{1,2}\mathcal{F}^\mu(\phi,c)(\psi,b)(\xi) = -\gamma\psi''(\xi) - c\psi'(\xi) - D_1F(\phi)\psi(\xi) - \mu D_1G(\overline{\phi})\overline{\psi(\xi)} - b\phi'(\xi). \tag{5.2.4}$$

This operator $D_{1,2}\mathcal{F}^\mu(\phi, c)$ will play an important role in the variant of Newton's method we employ to solve (5.1.1). In particular, the iteration step in our method consists of solving the linear differential difference equation

$$D_{1,2}\mathcal{F}^\mu(\phi_n, c_n)(\phi_{n+1}, c_{n+1}) = D_{1,2}\mathcal{F}^\mu(\phi_n, c_n)(\phi_n, c_n) - \mathcal{G}(\phi_n, c_n). \tag{5.2.5}$$

We note here that when $\mu = 1$, the iteration step (5.2.5) is equivalent to the Newton iteration defined in (5.2.1). However, when $\mu = 0$, (5.2.5) is just an ordinary differential equation, which can be solved using standard techniques.

It will be useful to rewrite (5.2.5) in the form

$$(\phi_{n+1}, c_{n+1}) = (\phi_n, c_n) - [D_{1,2}\mathcal{F}^\mu(\phi_n, c_n)]^{-1}\mathcal{G}(\phi_n, c_n). \tag{5.2.6}$$

At the moment it is not yet clear if this iteration step is well-defined. In particular, we will show that for μ close enough to 1, the operator $D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)$ is invertible for all pairs (ϕ_*, c_*) sufficiently close to the solution (ϕ, c).

5.3 Convergence of the method

The main theorem of this section roughly states that the numerical method introduced in Section 5.2 converges to a solution of (5.1.1). In order to make this precise, we need to define what we mean by a point of attraction of the Newton iteration (5.2.5).

Definition 5.3.1. A pair $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ is a point of attraction of the Newton iteration (5.2.5) if there is an open neighbourhood $S \subseteq W_0^{2,\infty} \times \mathbb{R}$, with $(\phi, c) \in S$, such that for any $(\phi_0, c_0) \in S$, the iterates defined by (5.2.5) all lie in $W_0^{2,\infty} \times \mathbb{R}$ and converge to (ϕ, c). $\square$

Theorem 5.3.1. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to the nonlinear autonomous differential difference equation (5.1.1). Then there exists $\epsilon > 0$ such that (ϕ, c) is a point of attraction for the Newton iteration (5.2.5) for all μ satisfying $|\mu - 1| < \epsilon$.*

Theorem 5.3.1 will be proved in a number of steps. We first prove that the Newton iteration (5.2.5) is well-defined for appropriate choices of the parameter μ and the initial condition (ϕ_0, c_0). Then we will consider the linearization of (5.2.6) around the solution (ϕ, c) and prove that the spectral radius of this linearized operator is smaller than one, which will allow us to complete the proof.

The first two lemma's use the fact that $D_{1,2}\mathcal{G}(\phi, c)$ is an isomorphism to show that this also holds for the operator $D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)$, for pairs (ϕ_*, c_*) sufficiently close to (ϕ, c).

Lemma 5.3.2. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.1.1). Then there exists $\epsilon > 0$ such that $D_{1,2}\mathcal{F}^{\mu}(\phi, c)$ is an isomorphism for all μ satisfying $|\mu - 1| < \epsilon$.*

Proof. We start out by noting that $D_{1,2}\mathcal{F}^1(\phi, c) = D_{1,2}\mathcal{G}(\phi, c)$, which is an isomorphism from $W_0^{2,\infty} \times \mathbb{R}$ onto L^∞. It follows from [27, Theorem 5.10] that $[D_{1,2}\mathcal{G}(\phi, c)]^{-1}$ is a bounded linear operator. We can thus write $v = \left\|[D_{1,2}\mathcal{G}(\phi, c)]^{-1}\right\|$ and since $D_{1,2}\mathcal{G}(\phi, c)$ is a nontrivial operator, $0 < v < \infty$ must hold. Noticing that

$$\|[D_{1,2}\mathcal{F}^{\mu_1}(\phi, c)] - [D_{1,2}\mathcal{F}^{\mu_2}(\phi, c)]\| = |\mu_1 - \mu_2| \left\|D_1 G(\overline{\phi})\right\| \tag{5.3.1}$$

and using the fact that $\left\|D_1 G(\overline{\phi})\right\| < \infty$ as ϕ is bounded, we see that we can choose $\epsilon > 0$ such that

$$\|[D_{1,2}\mathcal{F}^{\mu}(\phi, c)] - [D_{1,2}\mathcal{G}(\phi, c)]\| < \frac{1}{2v} \tag{5.3.2}$$

whenever $|\mu - 1| < \epsilon$. Now fix $\mu \in (1 - \epsilon, 1 + \epsilon)$ and let I be the identity operator on $W_0^{2,\infty} \times \mathbb{R}$. Since

$$\begin{aligned} &\left\|I - [D_{1,2}\mathcal{G}(\phi, c)]^{-1}[D_{1,2}\mathcal{F}^{\mu}(\phi, c)]\right\| \\ &= \left\|[D_{1,2}\mathcal{G}(\phi, c)]^{-1}([D_{1,2}\mathcal{G}(\phi, c)] - [D_{1,2}\mathcal{F}^{\mu}(\phi, c)])\right\| \le \tfrac{1}{2v} v = \tfrac{1}{2} < 1, \end{aligned} \tag{5.3.3}$$

Neumann's Lemma implies that $[D_{1,2}\mathcal{G}(\phi, c)]^{-1}[D_{1,2}\mathcal{F}^{\mu}(\phi, c)]$ is invertible and hence $D_{1,2}\mathcal{F}^{\mu}(\phi, c)$ has a left inverse. Because $D_{1,2}\mathcal{G}(\phi, c)$ is an isomorphism, it has a left and right inverse and so by an analogous argument involving the identity operator on L^∞ the existence of a right inverse for $D_{1,2}\mathcal{F}^{\mu}(\phi, c)$ can be established. This completes the proof that $D_{1,2}\mathcal{F}^{\mu}(\phi, c)$ is an isomorphism. $\square$

For convenience, we define open balls $B_{\psi,b,\delta}$ in $W_0^{2,\infty} \times \mathbb{R}$ given by

$$B_{\psi,b,\delta} = \left\{(\phi_*, c_*) \in W^{2,\infty} \times \mathbb{R} \mid \|(\psi, b) - (\phi_*, c_*)\| < \delta\right\}. \tag{5.3.4}$$

Lemma 5.3.3. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.1.1). Then there exists $\epsilon > 0$, such that for all $\mu \in \mathbb{R}$ with $|\mu - 1| < \epsilon$, there is an open ball $B = B_{\phi,c,\delta}$, for some $\delta > 0$, with the property that the linear operator $D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)$ is an isomorphism for all $(\phi_*, c_*) \in B$.*

Proof. The proof is analogous to the proof of Lemma 5.3.2. One uses the fact that $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is invertible and the observation that $\left\| D_{1,2}\mathcal{F}^\mu(\tilde{\phi}, \tilde{c}) \right\|$ is continuous with respect to $(\tilde{\phi}, \tilde{c})$ in the norm on $W_0^{2,\infty} \times \mathbb{R}$. To establish this, one needs the local Lipschitz condition (c2) on the derivatives of F and G. Together with Lemma A.3 and the boundedness of all $\phi_* \in W_0^{2,\infty}$, one establishes that for fixed ϕ_* and for all ϕ_{**} with $\|\phi_{**} - \phi_*\| \leq C$, we have $|D_1 F(\phi_*)(\xi) - D_1 F(\phi_{**})(\xi)| \leq D \|\phi_* - \phi_{**}\|$ for some $D < \infty$. With this estimate, the continuity is easily established. □

We remark that Lemma 5.3.3 guarantees that for μ close enough to 1, there exists $\delta > 0$ such that the Newton iteration step given by (5.2.5) is well-defined whenever $(\phi_n, c_n) \in B_{\phi,c,\delta}$. We can now define the operator $H^\mu : B_{\phi,c,\delta} \to W_0^{2,\infty} \times \mathbb{R}$ given by

$$H^\mu(\phi_*, c_*) = (\phi_*, c_*) - [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}\mathcal{G}(\phi_*, c_*). \tag{5.3.5}$$

Lemma 5.3.4. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.1.1). Then there exists $\epsilon > 0$ such that for all μ satisfying $|\mu - 1| < \epsilon$, the operator H^μ defined by (5.3.5) is Frechet differentiable at (ϕ, c). For these values of μ, the corresponding derivative with respect to ϕ_* and c_* at this point is given by*

$$D_{1,2}H^\mu(\phi, c) = I - [D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1} D_{1,2}\mathcal{G}(\phi, c). \tag{5.3.6}$$

Proof. From Lemma 5.3.2 we know that there exists $\epsilon > 0$ such that for all μ satisfying $|\mu - 1| < \epsilon$, $D_{1,2}\mathcal{F}^\mu(\phi, c)$ is an isomorphism. From the proof of Lemma 5.3.2 we also know that for such μ we have $\left\| 1 - [D_{1,2}\mathcal{G}(\phi, c)]^{-1} D_{1,2}\mathcal{F}^\mu(\phi, c) \right\| < 1$. Now fix μ satisfying $|\mu - 1| < \epsilon$.

Fix $\beta > 0$. We know that G is Frechet-differentiable at (ϕ, c), hence there exists δ_1 such that

$$\|\mathcal{G}(\phi_*, c_*) - \mathcal{G}(\phi, c) - D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)]\| \leq \beta \|(\phi_*, c_*) - (\phi, c)\| \tag{5.3.7}$$

for all $(\phi_*, c_*) \in B_{\phi,c,\delta_1}$. From Lemma 5.3.3 we know that there exists δ_2 such that $D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)$ is an isomorphism for all $(\phi_*, c_*) \in B_{\phi,c,\delta_2}$. In the proof of Lemma 5.3.3 we have seen that $\|D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)\|$ is continuous in ϕ_* and c_*. Using this, we see from Lemma A.2 that there exists $\delta_3 > 0$ such that

$$\left\| [D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1} - [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}][D_{1,2}\mathcal{G}(\phi, c)] \right\| \leq \beta \tag{5.3.8}$$

whenever $\|(\phi, c) - (\phi_*, c_*)\| < \delta_3$. From (5.3.8) it also follows that when $\|(\phi, c) - (\phi_*, c_*)\| < \delta_3$ we have

$$\left\| [D_{1,2}\mathcal{F}^\mu(\phi_*, c_*)]^{-1}[D_{1,2}\mathcal{G}(\phi, c)] \right\| \leq \beta + \left\| [D_{1,2}\mathcal{F}^\mu(\phi, c)]^{-1}[D_{1,2}\mathcal{G}(\phi, c)] \right\| = \beta + C \tag{5.3.9}$$

for some finite constant C. Using the identity

$$[D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)]^{-1} = [D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)]^{-1}[D_{1,2}\mathcal{G}(\phi, c)][D_{1,2}\mathcal{G}(\phi, c)]^{-1}, \tag{5.3.10}$$

we see that $\left\|[D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)]^{-1}\right\| \leq D(\beta{+}C)$ for some finite constant D, whenever $\|(\phi, c) - (\phi_*, c_*)\| < \delta_3$.

Now choose $\delta = \min(\delta_1, \delta_2, \delta_3)$. Using the fact that $(\phi, c) = H^{\mu}(\phi, c)$ we obtain for all $(\phi_*, c_*) \in B_{\phi,c,\delta}$

$$\begin{aligned}
&\left\|H^{\mu}(\phi_*, c_*) - H^{\mu}(\phi, c) - [I - [D_{1,2}\mathcal{F}^{\mu}(\phi, c)]^{-1}D_{1,2}\mathcal{G}(\phi, c)][(\phi_*, c_*) - (\phi, c)]\right\| \\
&= \left\|[D_{1,2}\mathcal{F}^{\mu}(\phi, c)]^{-1}D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)] - [D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)]^{-1}\mathcal{G}(\phi_*, c_*)\right\| \\
&\leq \left\|-[D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)]^{-1}[\mathcal{G}(\phi_*, c_*) - \mathcal{G}(\phi, c) - D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)]]\right\| \\
&\quad + \left\|[[D_{1,2}\mathcal{F}^{\mu}(\phi, c)]^{-1} - [D_{1,2}\mathcal{F}^{\mu}(\phi_*, c_*)]^{-1}][D_{1,2}\mathcal{G}(\phi, c)[(\phi_*, c_*) - (\phi, c)]]\right\| \\
&\leq (D(\beta + C)\beta + \beta)\left\|(\phi_*, c_*) - (\phi, c)\right\|.
\end{aligned} \tag{5.3.11}$$

This completes the proof that H^{μ} is Frechet differentiable. □

We can now use the fact that $D_{1,2}\mathcal{F}^{\mu}(\phi, c)$ is an isomorphism to establish the crucial fact that the spectral radius of the linear operator $D_{1,2}H^{\mu}(\phi, c)$ is less than one.

Lemma 5.3.5. *Let $(\phi, c) \in W_0^{2,\infty} \times \mathbb{R}$ be a connecting solution to (5.1.1). Let $\hat{\sigma}^{\mu}$ denote the spectral radius of $D_{1,2}H^{\mu}(\phi, c)$. Then there exists $\epsilon > 0$, such that for all μ satisfying $|\mu - 1| < \epsilon$, we have $\hat{\sigma}^{\mu} < 1$.*

Proof. Writing out the eigenvalue problem for $D_{1,2}H^{\mu}(\phi, c)$, we obtain the equation

$$(1 - \mu)[D_{1,2}\mathcal{F}^{\mu}(\phi, c)]^{-1}[D_1G(\overline{\phi})\overline{\psi}] - \lambda(\psi, b) = (0, 0), \tag{5.3.12}$$

where λ is the eigenvalue and (ψ, b) are the eigenfunctions. After applying $D_{1,2}\mathcal{F}^{\mu}(\phi, c)$ and using the explicit form of $D_{1,2}\mathcal{F}^{\mu}$ this is equivalent to

$$-D_{1,2}\mathcal{F}^{\tilde{\mu}(\lambda)}(\phi, c)(\psi, b) = 0, \tag{5.3.13}$$

in which

$$\tilde{\mu}(\lambda) = \mu + \frac{1 - \mu}{\lambda}. \tag{5.3.14}$$

We know from Lemma 5.3.2 that there exists $\delta > 0$ such that $D_{1,2}\mathcal{F}^{\mu}(\phi, c)$ is an isomorphism for all μ satisfying $|\mu - 1| < \delta$. If we now choose $\epsilon = \frac{\delta}{2}$, we see that for all μ satisfying $|\mu - 1| < \epsilon$ and for all $|\lambda| \geq 1$,

$$|\tilde{\mu}(\lambda) - 1| \leq \frac{\delta}{2} + \frac{\delta}{2}|\lambda|^{-1} \leq \delta. \tag{5.3.15}$$

In particular, this means that for these μ and λ equation (5.3.13) has only the zero solution, as $D_{1,2}\mathcal{F}^{\tilde{\mu}(\lambda)}(\phi, c)$ is an isomorphism. Thus for these μ there cannot be any eigenvalues λ with $|\lambda| \geq 1$, proving that $\hat{\sigma}^{\mu} < 1$. $\square$

We are now ready to complete the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1. Fix $\beta > 0$ such that for all μ satisfying $|\mu - 1| < \beta$, we have that the operator H^{μ} is well-defined in a neighbourhood of (ϕ, c) and Frechet differentiable at (ϕ, c), together with the inequality $\hat{\sigma}^{\mu} < 1$, where $\hat{\sigma}^{\mu}$ is the spectral radius of $D_{1,2}H^{\mu}(\phi, c)$. Now fix μ satisfying $|\mu - 1| < \beta$, write $H = H^{\mu}$ and $\hat{\sigma} = \hat{\sigma}^{\mu}$ and choose $\epsilon > 0$ such that $\hat{\sigma} + \epsilon < 1$. Let H^p be the p-fold iterate of H. Since H is Frechet-differentiable at (ϕ, c), so is H^p. From the chain rule it follows that $D_{1,2}H^p(\phi, c) = [D_{1,2}H(\phi, c)]^p$.

From Lemma A.4 it follows that we may choose p such that

$$\begin{aligned} &\|[D_{1,2}H(\phi, c)]^p\| \leq (\hat{\sigma} + \epsilon)^p, \\ &(\hat{\sigma} + \epsilon)^p + \epsilon < 1. \end{aligned} \tag{5.3.16}$$

Let s be an integer. From the Frechet-differentiability of H^s we know that there exists $\delta > 0$, such that for all $(\phi_*, c_*) \in B_{\phi,c,\delta}$ and for all $1 \leq s \leq p$,

$$\|H^s(\phi_*, c_*) - H^s(\phi, c) - [D_{1,2}H(\phi, c)]^s[(\phi_*, c_*) - (\phi, c)]\| \leq \epsilon \, \|(\phi_*, c_*) - (\phi, c)\| . \tag{5.3.17}$$

With this we can compute

$$\begin{aligned} \|H^s(\phi_*, c_*) - (\phi, c)\| &\leq \|H^s(\phi_*, c_*) - H^s(\phi, c) - [D_{1,2}H(\phi, c)]^s[(\phi_*, c_*) - (\phi, c)]\| \\ &\qquad + \|[D_{1,2}H(\phi, c)]^s\| \, \|(\phi_*, c_*) - (\phi, c)\| \\ &\leq (\|[D_{1,2}H(\phi, c)]^s\| + \epsilon) \, \|(\phi_*, c_*) - (\phi, c)\| . \end{aligned} \tag{5.3.18}$$

Writing

$$w = \max\left(\epsilon, \max\left\{\|[D_{1,2}H(\phi, c)]^s\| \mid s = 1 \ldots p\right\}\right), \tag{5.3.19}$$

we see that we can ensure $H^s(\phi_0, c_0) \in B_{\phi,c,\delta_*}$ for $s = 1 \ldots p$ by choosing $(\phi_0, c_0) \in B_{\phi,c,\delta_*/2w}$. For $s = p$ equation (5.3.18) reduces to

$$\|H^p(\phi_*, c_*) - (\phi, c)\| \leq [(\hat{\sigma} + \epsilon)^p + \epsilon] \, \|(\phi_*, c_*) - (\phi, c)\| . \tag{5.3.20}$$

Combining everything, we see that by choosing $(\phi_0, c_0) \in B_{\phi,c,\delta/2w}$ all the Newton iterates lie in the ball $B_{\phi,c,\delta}$. Now choosing $\delta > 0$ so small that H is well-defined on $B_{\phi,c,\delta}$, we see that the Newton process is well-defined and satisfies

$$\lim_{n\to\infty} \|(\phi_n, c_n) - (\phi, c)\| \leq \lim_{n\to\infty} (2w)\big((\hat{\sigma} + \epsilon)^p + \epsilon\big)^{\left\lfloor \frac{n}{p} \right\rfloor} \|(\phi_0, c_0) - (\phi, c)\| = 0. \tag{5.3.21}$$

This concludes the proof of the theorem. $\square$

Remark 5.3.1. *It is not clear if Theorem 5.3.1 holds for $\mu = 0$. The equations (5.3.1) and (5.3.2) from the proof of Lemma 5.3.2 give us information about the values of ϵ which satisfy the claim in Theorem 5.3.1. In particular, smaller values of $\|D_1 G\|$ give us larger possible values of ϵ. Referring back to (1.7), we see there that $\|D_1 G\|$ is proportional to the parameter α. Since we are interested in solutions to (1.7) far from the continuous limit, i.e., for small values of the parameter α, this observation leads us to believe we can take $\mu = 0$ in many cases of interest. See Section 6.5 for a further discussion and some numerical examples.*

One can also try to find conditions on the system (5.1.1) for which $D_{1,2}\mathcal{F}^s(\phi, c)$ is an isomorphism from $W_0^{2,\infty}$ onto L^∞ for all $-1 \leq s \leq 1$. The theory developed in Chapter 4 however does not cover these cases, but some of the results in Chapter 3 have been stated in a more general setting than the one in which they were used in light of such a deeper investigation.

Chapter 6

Numerical Results

In this chapter we give a detailed discussion concerning our implementation of the method discussed in Chapter 5. Some numerical results obtained by our algorithm are presented in order to illustrate some of the key phenomena encountered in the qualitative study of lattice differential equations, together with some of the technical difficulties involved with the numerical computation of solutions to such equations.

6.1 The test problem

All the results in this section were obtained by solving the differential difference equation given by

$$-\gamma\phi''(\xi) - c\phi'(\xi) = \epsilon\big(\phi(\xi+1) - \phi(\xi-1) - 2\phi(\xi)\big) - f(\phi(\xi), a), \tag{6.1.1}$$

in which f is a cubic nonlinearity given by

$$f(x,a) = x(x-1)(x-a), \tag{6.1.2}$$

where $a \in (0,1)$ is a continuation parameter. The solutions of (6.1.1) were required to satisfy the limits

$$\phi(-\infty) = 0, \quad \phi(\infty) = 1 \tag{6.1.3}$$

and were normalized to have $\phi(0) = a$.

We remark here that this choice of limits and normalization condition differs from the one used in the theory developed in Chapters 4 and 5. This has been done to be consistent with the experimental work in [13], so that the results in this section can be easily compared to the results in [13, Section

5]. The generic behaviour of the solutions is of course unaffected by this choice, as the solutions corresponding to different choices can be related to each other by linear transformations and shifts.

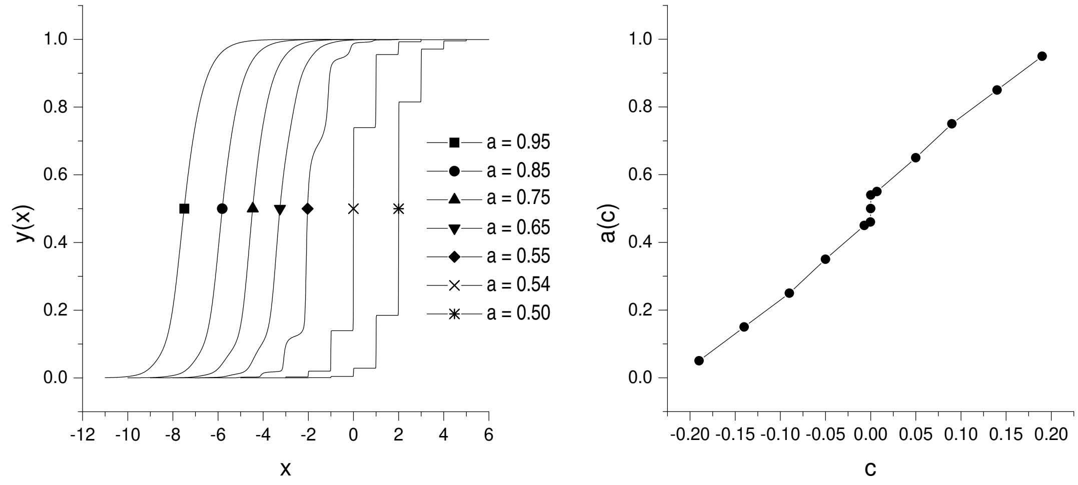
It is easy to verify that the family (6.1.1) satisfies all the requirements (c1) through (c5) from the beginning of Chapter 5, where a plays the role of the detuning parameter ρ. Of course, the precise statement of these requirements will have to be modified in an obvious manner to account for the different limits and normalization condition. Also note that if $\phi(x)$ is a solution to the problem (6.1.1) satisfying the limits (6.1.3) at some parameter $a = a_0$ with wavespeed $c = c_0$, then $\psi(x) = 1 - \phi(-x)$ is a solution to the same problem with $a = 1 - a_0$ and wavespeed $c = -c_0$ and also satisfies the limits (6.1.3).

6.2 Implementation details

Performing the iteration step defined in (5.2.5) with $\mu = 0$ amounts to solving a boundary value problem on the real line. This observation in principle allows one to perform the Newton iterations requiring the help of a boundary value problem solver for ordinary differential equations only. Of course, in practice, one has to truncate the problem to some finite interval $[L, R]$. As we have seen that the solutions approach their limits exponentially, one expects that if the interval is chosen to be large enough, the solutions will not be affected much. In Section 6.6 we will present some numerical data to illustrate the importance of the choice of interval.

At each iteration step the boundary value solver COLMOD [6] was used in our C++ implementation to solve the boundary value problem at hand. The boundary conditions imposed were chosen to be $\phi(L) = 0$, $\phi(R) = 1$ and $\phi(0) = a$, to pick out the unique translate. As the boundary value problem which has to be solved has degree three, these boundary conditions are sufficient. This is one of the reasons why we included the $\gamma\phi''(\xi)$ term in the differential difference equation, as otherwise we would have too many boundary conditions. This second order term also has a smoothening effect on the solutions, since they always are twice differentiable as long as $\gamma > 0$, even if $c \to 0$ and the solutions with $\gamma = 0$ may become discontinuous. We will study the $\gamma \to 0$ limiting behaviour in Section 6.4.

If the wavespeed satisfies $c \neq 0$, then in principle it is also possible to perform the Newton iterations (5.2.5) with $\gamma = 0$ using a boundary value problem solver. One chooses the boundary conditions $\phi(L) = 0$ and $\phi(R) = 1$, but omits the normalization condition $\phi(0) = a$. This normalization can be restored by shifting the resulting solution, as it is a monotone increasing function. Abell, Elmer and Van Vleck take this approach in [1]. However, in the singular perturbation limit $c \to 0$, this will no longer work, as the boundary value problem which has to be solved becomes singular and COLMOD can no longer handle it. This is the main reason why we included the $\gamma\phi''(\xi)$ term.



(a) (b)

Figure 6.1: *In (a) the waveprofiles $y(x)$ have been plotted for solutions to the differential difference equation (6.1.1) with $\gamma = 10^{-6}$ and $\epsilon = 0.1$, at different values of the detuning parameter a. Please note that for presentation purposes the curves have been shifted by different amounts along the x-axis. In (b) the $a(c)$ relation has been plotted, i.e., for each value of the detuning parameter a the corresponding wavespeed c is given. The solid dots represent the wavespeeds corresponding to the curves in (a), which have been continued to $a < 0.5$, using the observation that $\psi(\xi) = 1 - \phi(-\xi)$ is a connecting solution with wavespeed $-c$ if $\phi(\xi)$ is a connecting solution with wavespeed c. From (b) it is easily seen that there exists a nontrivial interval of a in which $c \sim 0$, hence propagation failure occurs.*

We shall see that we can choose γ to be small enough to prevent this extra term from having a large effect on the solutions, at least when studying the family (6.1.1).

One of course also has to supply a starting value (ϕ_0, c_0) for the Newton iterations. It turns out that this is very hard in general: very often the algorithm requires a very accurate initial guess to converge. One has to use the technique of continuation to arrive at a suitable starting value. In general, this means that one starts by solving an "easy" problem to a certain degree of accuracy and gradually moves toward the "hard" problem, using the solution of one problem as the starting value for the next problem which lies "nearby". It is here that the continuity in parameter space established in Proposition 4.2.1 comes into the picture. More precisely, identifying a specific equation from the family (4.1.1) with the corresponding parameter $\rho \in V$, a continuation scheme from an "easy" problem ρ_0 to a "hard" problem ρ_1 is a continuous function $\Pi : [0, 1] \to V$ satisfying $\Pi(0) = \rho_0$ and $\Pi(1) = \rho_1$. Under appropriate conditions Proposition 4.2.1 tells us that the solutions $\phi\big(\Pi(s)\big)$ will vary continuously with s as s increases from 0 to 1. A continuation scheme for our family (6.1.1) could, for example, vary the detuning parameter a, the size of the delay term ϵ, the size of γ or combinations of these parameters.

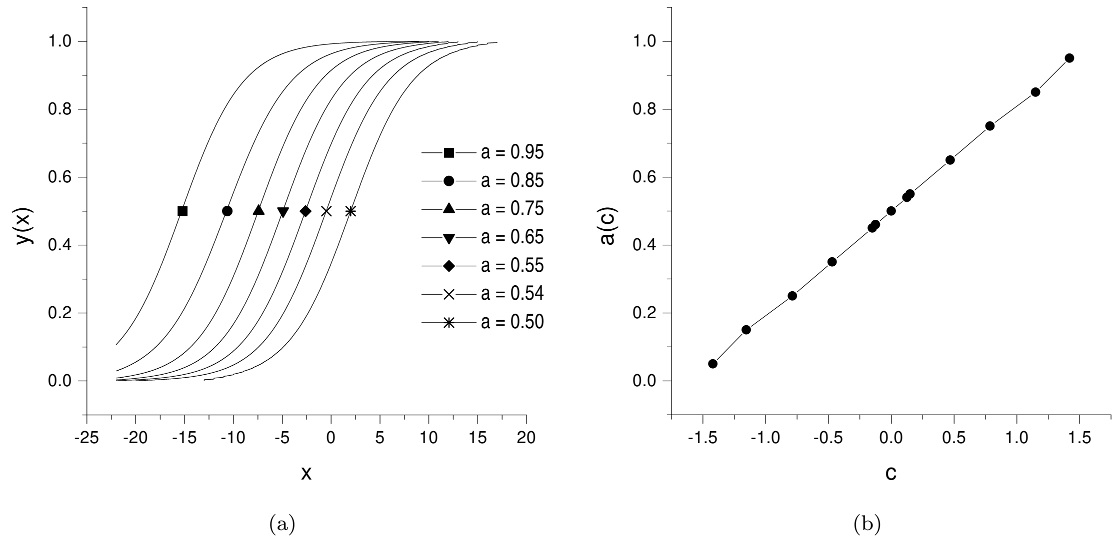
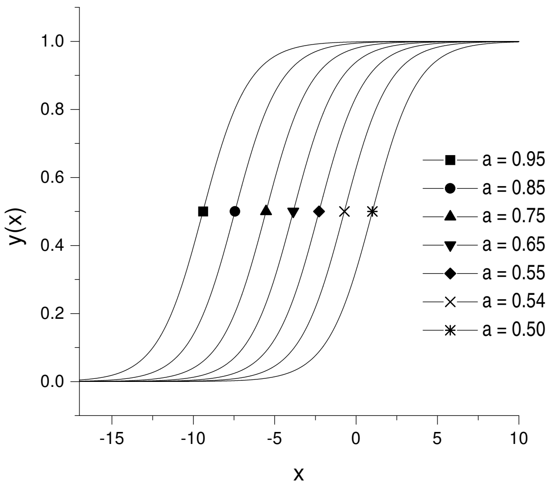


Figure 6.2: *In (a) the waveprofiles $y(x)$ have been plotted for solutions to (6.1.1) with $\gamma = 10^{-4}$ and $\epsilon = 5$, at different values of the detuning parameter a. The wavespeeds for these solutions are given in (b). The calculations to obtain the solution curves in this figure were performed on the finite interval $[-15, 15]$. Notice that in (b) there is no nontrivial interval of a for which $c = 0$. Indeed, the solution curves in (a) remain continuous as $a \to 0.5$.*

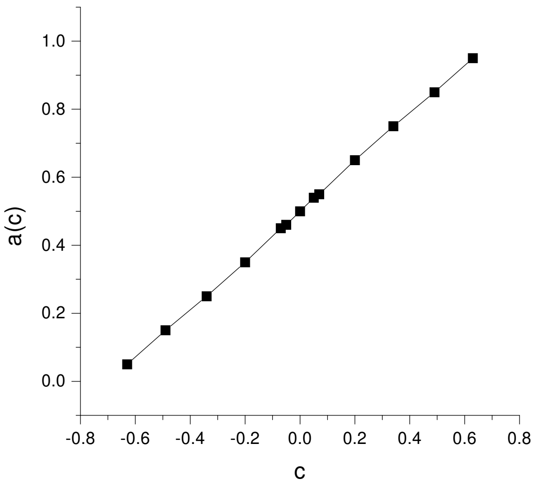
Our implementation requires an initial stepsize Δs_0 and starts out by setting $\Delta s = \Delta s_0$. The variable Δs is used to control the speed in which the continuation scheme is applied: the algorithm will try to solve the problem corresponding to $\rho = \Pi(s_* + \Delta s)$ once the problem $\Pi(s_*)$ has been solved, using the solution to $\Pi(s_*)$ as an initial value. If unsuccessful, Δs is decreased and another attempt is made. The number of iterations needed to solve an intermediate problem can be used to adjust the stepsize Δs. For example, if the convergence was fast, then a bigger leap in parameter space can be attempted, while if the convergence was very slow, Δs can be decreased. One can also control the rate of convergence by specifying to what degree of accuracy the intermediate and final problems should be solved. If one relaxes the intermediate tolerance, the time needed to solve each individual problem will of course decrease, but the total number of problems which have to be solved might increase due to the fact that the initial guesses will be worse. A good deal of fine tuning is required here.

6.3 Propagation failure

The phenomenon of propagation failure has been studied extensively in [24]. In particular, in Corollary 2.5 of [24] it is shown that for our family (6.1.1) with $\gamma = 0$, there exist quantities



(a)



(b)

Figure 6.3: *Waveprofiles $y(x)$ for solutions to (6.1.1) with $\gamma = 10^{-4}$ and $\epsilon = 1$ have been plotted in (a), while (b) gives the corresponding wavespeeds. We again see from (b) that there is no nontrivial interval of a for which $c = 0$.*

$0 \leq a_- \leq a_+ \leq 1$, such that (6.1.1) only has connecting solutions with wavespeed $c = 0$ for $a_- \leq a \leq a_+$. It may happen that $a_- < a_+$, that is, that there is a nontrivial interval of the detuning parameter a for which the wavespeed vanishes. This phenomenon is called propagation failure. One generally expects discontinuous solutions when $c = 0$ and $\gamma = 0$. Of course, since all the numerical computations were performed with $\gamma > 0$, which forces the solutions to remain continuous, it is not clear if one can accurately reproduce the solution profiles at $\gamma = 0$ and thus actually uncover the propagation failure.

In Figure 6.1 the calculated solutions to (6.1.1) are presented, together with their wavespeeds. One sees clearly from Figure 6.1(b) that there is a nontrivial interval of the detuning parameter a for which the wavespeed c vanishes. Looking at Figure 6.1(a), one sees that the solutions for these values of a exhibit step-like behaviour. In the calculations we used $\gamma = 10^{-6}$, which thus indicates that for γ small enough, one can hope to see the effects of propagation failure and make accurate predictions about the parameter values at which it will occur. Propagation failure does not occur at each value of ϵ, as the $a(c)$ curves in Figures 6.3(b) and 6.2(b) show. Notice that the solutions in Figures 6.3(a) and 6.2(a) indeed remain smooth as $a \to 0.5$. We remark here that the wavespeed necessarily satisfies $c = 0$ when $a = 0.5$, but it seems as if for this specific system (6.1.1), the solutions only exhibit discontinuous behaviour when the wavespeed vanishes for a nontrivial interval of the detuning parameter a.

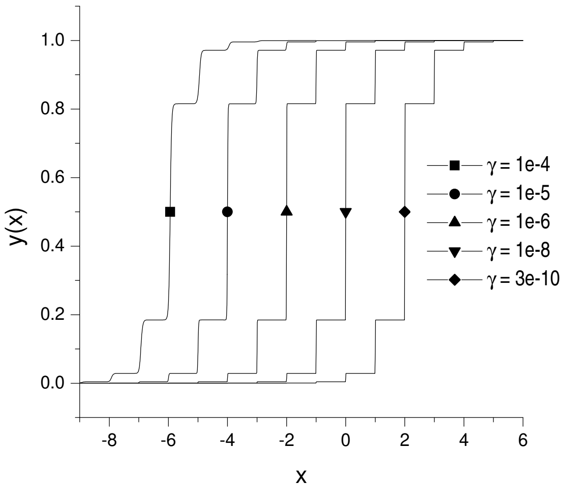
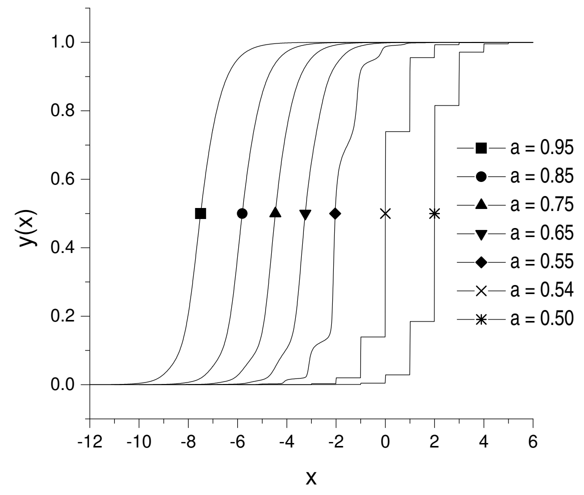


Figure 6.4: *Waveprofiles $y(\xi)$ for solutions to (6.1.1) at different values of γ, while $a = 0.50$ and $\epsilon = 0.1$ remain fixed. This figure demonstrates the robust convergence in the $\gamma \to 0$ limit, showing that already at $\gamma = 10^{-5}$ the waveform has attained its limiting profile.*

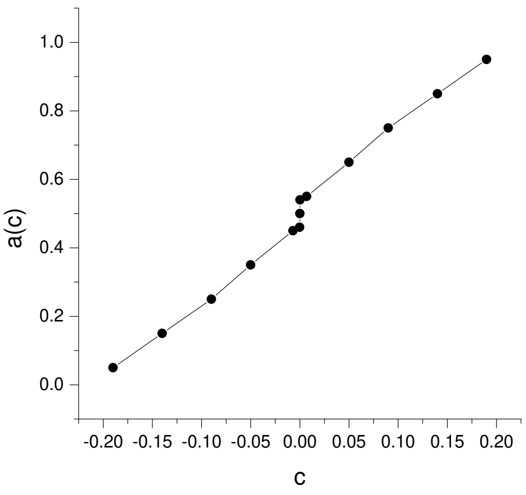
6.4 Convergence in $\gamma \to 0$ limit

As we are often interested in the behaviour of solutions to (4.1.1) at $\gamma = 0$ and we can only compute solutions for $\gamma > 0$, one hopes that, if one chooses γ to be small, the calculated solution will be close to the actual solution at $\gamma = 0$. Theorem 4.3.3 indeed establishes that if we have a sequence of solutions $P_n(\xi)$ to (4.1.1) with $\gamma = \gamma_n$, where $\gamma_n \to 0$, a subsequence of the functions P_n will convergence to a solution at $\gamma = 0$. However, this by no means guarantees that this convergence will be numerically useful. Ideally, one would want the solution curves to remain stable below some value of γ, which is not too small.

In Figure 6.4 the solution curves to (6.1.1) have been plotted for a number of different values of γ, ranging from $\gamma = 10^{-4}$ to $\gamma = 3 \times 10^{-10}$. Notice that the solution curves stay the same for $\gamma = 10^{-5}$ to $\gamma = 3 \times 10^{-10}$, while the curve for $\gamma = 10^{-4}$ does not differ too much. One sees here that in this example computations with $\gamma \sim 10^{-5}$ will probably provide a good approximation to the actual solutions with $\gamma = 0$. In particular, the computations indicate that the discontinuous behaviour due to propagation failure, which occurs at $\gamma = 0$ and $c = 0$, is already visible at $\gamma = 10^{-5}$. This is further illustrated in Figure 6.5, where the results in Figure 6.1 are recalculated at $\gamma = 10^{-8}$. Comparison of the two figures show that the solutions are exactly the same.



(a)



(b)

Figure 6.5: *The same results as in Figure 6.1, recalculated at* $\gamma = 10^{-8}$*. Notice that the solutions remain identical to the ones in Figure 6.1, again illustrating the robust convergence in the* $\gamma \to 0$ *limit.*

6.5 The large delay limit

When we take $\mu = 0$ in the Newton iteration (5.2.5), we are neglecting the presence of the shifted term G. In particular, referring to (5.3.1) in the proof of Lemma 5.3.2, one expects that when the norm of the shifted term G becomes large, problems will arise with the invertibility of the operator $D_{1,2}\mathcal{F}^{\mu}$ and hence with the convergence of the algorithm. In our case, the importance of the shifted term is given by the parameter ϵ. For large ϵ, the term $\epsilon\big(\phi(\xi+1) - \phi(\xi-1) - 2\phi(\xi)\big)$ in (6.1.1) becomes increasingly important. Nevertheless, by using a suitable continuation scheme, we were able to obtain solutions to (6.1.1) for $\epsilon = 5$ and $\epsilon = 10$ at $\gamma = 10^{-4}$ and $a = 0.5$. These solutions have been plotted in Figure 6.6. At these levels of ϵ the delay term has become the dominant term. If one wishes to increase ϵ even further, it no longer suffices to take $\mu = 0$ in (5.2.5). It is however quite satisfactory that the algorithm can be used for practical purposes up to these levels of ϵ, as it is not at all clear from the convergence proof presented in Chapter 5 that this is the case.

6.6 Relevance of choice of Interval

The theory has been developed for solutions to (6.1.1) on the real line, while of course the computations were performed on a finite interval $[L, R]$. Since the solutions approach their limits exponentially fast, it is intuitively clear that a sufficiently large choice of interval will not affect the solutions

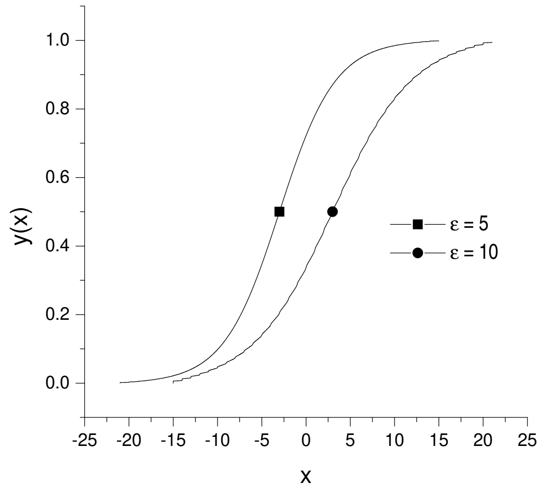


Figure 6.6: *Solutions $y(\xi)$ to (6.1.1) at $\epsilon = 5$ and $\epsilon = 10$ in the critical case $a = 0.50$. The parameter γ was fixed at 10^{-4}. Here the shifted term in (6.1.1) has become the dominating term and thus one expects the algorithm to have trouble converging.*

too badly. In [25, Section 7], functional differential equations on finite intervals are considered. Not much is known in general concerning the truncation of such equations to finite intervals, however [25, Theorem 7.3] gives an interesting specific result on equations of the form

$$x'(\xi) = a(t)x(t) + b(t)x(t-1) + c(t)x(t+1), \tag{6.6.1}$$

where $b(t) > 0$ and $c(t) > 0$ for all $t \in \mathbb{R}$. The theorem states that there are no solutions to (6.6.1) on a truncated interval $[L, R]$, satisfying $x(L) = x(R) = 0$. However, we see from [24, Theorem 2.1] that in general (6.6.1) does have a solution on $\mathbb{R}$, since the derivatives of solutions to (4.1.1) with $\gamma = 0$ satisfy a linear equation of the form (6.6.1) when $N = 2$, $r_1 = -1$ and $r_2 = 1$. Practically, this absence of solutions on a truncated interval does not matter much, since the solution on the line will approach its limits exponentially fast. In the presence of roundoff errors this solution can thus be treated as a truncated solution, if we choose our interval to be large enough.

In Figure 6.7 one clearly sees the effect of truncating the problem to an interval which is too small. The strange behaviour near the boundary for $a > 0.65$ suggests that the solution has been influenced by the truncation. However, at $a = 0.50$ we see that the solution exhibits step-like behaviour. In this case one cannot see directly that boundary effects are responsible, as the solution curve does not behave differently near the boundary. However, looking at the $a(c)$ curve in Figure 6.7(b), one does not expect propagation failure and thus step-like behaviour to occur. Thus the $a(c)$ curve gives us a convenient check to see if the interval has been chosen to small. Indeed, comparing Figure 6.7(a) to Figure 6.2, we see that the latter does not exhibit step-like behaviour at $a = 0.5$. The only difference between the two figures is that the latter was calculated using a larger interval $[L, R]$.

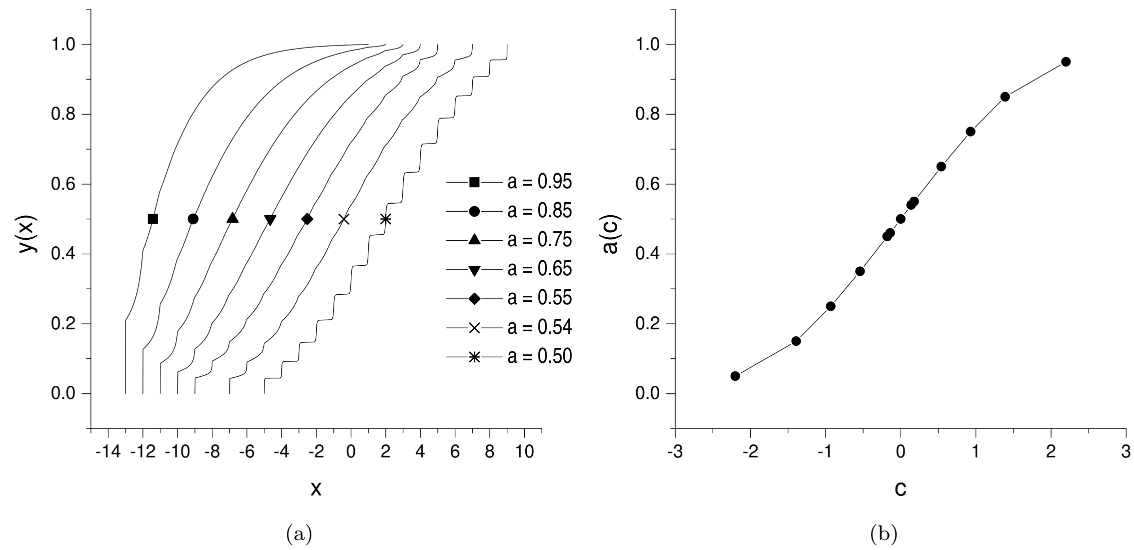


Figure 6.7: *The same results as in Figure 6.2, but now calculated on the smaller interval* $[-7, 7]$. *One clearly sees that for* $a > 0.65$ *the curves are effected by the boundary. At* $a = 0.50$ *we have a step-like solution, and by itself there is no evidence that this is due to the choice of interval. When looking at the wavespeeds in (b) one however does not expect step-like behaviour, which leads one to suspect that the interval is too small.*

Chapter 7

Higher Dimensional Systems

Up to now all the theory has been developed for one dimensional differential difference equations of the form (4.1.1). The question immediately arises if the results can be extended to higher dimensional systems and how the algorithm should be adapted to handle these cases. In this chapter we briefly discuss these issues, using a natural extension of the reaction-diffusion equation which was investigated in the previous chapter.

7.1 Periodic Diffusion

We consider the reaction-diffusion equation on a one dimensional integer lattice,

$$\dot{u}_j(t) = \alpha_{j+1}[u_{j+1}(t) - u_j(t)] + \alpha_j[u_{j-1}(t) - u_j(t)] - f(u_j(t), a), \tag{7.1.1}$$

in which a dot marks differentiation with respect to time and $j \in \mathbb{Z}$ parametrizes the integer lattice. The diffusion constants α_j are required to be positive, i.e., $\alpha_j > 0$, but are allowed to vary spatially. We will assume the function f to be a nonlinearity of bistable type, such as the cubic

$$f(u, a) = du(u - a)(u - 1) \tag{7.1.2}$$

for some constant d.

In the introduction we have seen that one naturally arrives at an equation of the form (7.1.1) with constant diffusion $\alpha_j = \alpha$ when studying the discretization of the continuous reaction diffusion equation (1.5) on some lattice, for example when one is interested in crystals. The diffusion constant α often depends on physical parameters like the lattice spacing, temperature and concentration, which are not necessarily constant in space. To incorporate this in the model, one adds a spatial dependence to the diffusion constant α and arrives at (7.1.1). For a specific example from material science, see [4].

Often the spatial dependence of the diffusion constant turns out to be periodic, i.e. $\alpha_{j+P} = \alpha_j$ for all $j \in \mathbb{Z}$ for some integer P, which we call the period. For example, a crystal containing two different types of ions, arranged alternately, will have period two diffusion.

From now on we will assume equation (7.1.1) has period two diffusion and seek travelling wave solutions to (7.1.1). In this case it is natural to split the integer lattice into an even part and an odd part with a separate waveform and wavespeed for each part. We also allow the wavespeed to vary with time. We thus make the ansatz

$$u_j(t) = \begin{cases} \phi_o(j - \int_0^t c_o(s)ds), & j \text{ odd}, \\ \phi_e(j - \int_0^t c_e(s)ds), & j \text{ even}. \end{cases} \tag{7.1.3}$$

For normalization purposes we require $\phi_o(0) = \phi_e(0) = 0$.

If one fixes the time $t = 0$ one can solve for the waveforms ϕ_o and ϕ_e and the initial wavespeeds $c_o = c_o(0)$ and $c_e = c_e(0)$. Writing $\alpha_j = \alpha_o$ for odd j and $\alpha_j = \alpha_e$ for even j and substituting the ansatz (7.1.3) into (7.1.1), we arrive at

$$\begin{cases} -c_o\phi_o'(\xi) = \alpha_e(\phi_e(\xi+1) - \phi_o(\xi)) + \alpha_o(\phi_e(\xi-1) - \phi_o(\xi)) - f(\phi_o, a), \\ -c_e\phi_e'(\xi) = \alpha_o(\phi_o(\xi+1) - \phi_e(\xi) + \alpha_e(\phi_o(\xi-1) - \phi_e(\xi)) - f(\phi_e, a). \end{cases} \tag{7.1.4}$$

We have to remark here that in general it is not possible to find a solution to (7.1.1) of the form (7.1.3), as the wavespeeds c_e and c_o are not necessarily equal, which implies that for each time t the functions ϕ_o and ϕ_e will have to satisfy different differential difference equations. Taking the above approach for each fixed time t yields a different solution $(c_o(t), c_e(t), (\phi_o)_t, (\phi_e)_t)$. However, when we consider small time intervals, the approximation made by fixing $(\phi_o)_t = (\phi_o)_0$ and $(\phi_e)_t = (\phi_e)_t$ will be quite accurate, especially if $c_e \sim c_o$. See [12] for further details.

Upon defining $\widetilde{\phi}_o(\xi) = \phi_o(\xi+1)$ and $\widetilde{\phi}_e(\xi) = \phi_e(\xi)$, the system (7.1.4) becomes

$$\begin{cases} -c_o\widetilde{\phi}_o'(\xi) = \alpha_e(\widetilde{\phi}_e(\xi+2) - \widetilde{\phi}_o(\xi)) + \alpha_o(\widetilde{\phi}_e(\xi) - \widetilde{\phi}_o(\xi)) - f(\widetilde{\phi}_o, a), \\ -c_e\widetilde{\phi}_e'(\xi) = \alpha_o(\widetilde{\phi}_o(\xi) - \widetilde{\phi}_e(\xi) + \alpha_e(\widetilde{\phi}_o(\xi-2) - \widetilde{\phi}_e(\xi)) - f(\widetilde{\phi}_e, a), \end{cases} \tag{7.1.5}$$

with normalization given by $\widetilde{\phi}_o(-1) = a$ and $\widetilde{\phi}_e(0) = a$. Rewritten in the current form (7.1.5), we can compare our results for the period two diffusion problem to the results in [12].

7.2 General Problem

In general, we wish to solve systems of differential difference equations of the form

$$-\gamma\phi_j''(\xi) - c_j\phi_j'(\xi) = F_j\big(\phi_1(\xi), \phi_2(\xi), \ldots, \phi_M(\xi)\big) + G_j\big(\overline{\phi_1}(\xi), \ldots \overline{\phi_M}(\xi)\big), \tag{7.2.1}$$

for $j = 1 \ldots M$, in which M is the dimension of the system. As usual, we have used

$$\overline{\phi}(\xi) = \big(\phi(\xi + r_1), \ldots, \phi(\xi + r_N)\big) \in \mathbb{R}^N. \tag{7.2.2}$$

We require $r_i \neq 0$ for $1 \leq i \leq N$ and $r_i \neq r_j$ whenever $i \neq j$. We also demand $\gamma > 0$ and $N \geq 1$.

Defining $\kappa(x) = (x, \ldots, x) \in \mathbb{R}^M$, we speculate that one has to replace the conditions (c1) through (c5) from Chapter 5 by their generalizations (d1) through (d5) below in order to obtain convergence results for the Newton iteration.

(d1) $F : \mathbb{R}^M \to \mathbb{R}^M$ is C^1-smooth in $\mathbb{R}^M$. Also, $G : \mathbb{R}^{MN} \to \mathbb{R}^M$ is C^1-smooth in $\mathbb{R}^{MN}$.

(d2) The Jacobians $D_1 F_j : \mathbb{R}^M \to \mathbb{R}^M$ and $D_1 G_j : \mathbb{R}^{MN} \to \mathbb{R}^{MN}$ are locally Lipschitz for all $1 \leq j \leq M$.

(d3) For $i = 1, \ldots M$ and $j = 1, \ldots, MN$ we have that

$$\frac{\partial G_i(v, \rho)}{\partial v_j} > 0. \tag{7.2.3}$$

(d4) Let $\Gamma : \mathbb{R} \to \mathbb{R}^M$ be defined as

$$\Gamma(\phi) = F\big(\kappa(\phi)\big) + G\big(\kappa(\phi), \kappa(\phi), \ldots, \kappa(\phi)\big). \tag{7.2.4}$$

Then for some quantity $a \in [-1, 1]$ we have that

$$\begin{aligned} &\Gamma(-1) = \Gamma(a) = \Gamma(1) = \kappa(0), \\ &\Gamma_i(\phi) > 0, \quad \phi \in (-\infty, -1) \cup (a, 1), \qquad 1 \leq i \leq M, \\ &\Gamma_i(\phi) < 0, \quad \phi \in (-1, a) \cup (1, \infty), \qquad 1 \leq i \leq M. \end{aligned} \tag{7.2.5}$$

(d5) We have that

$$\begin{aligned} &D_1\Gamma_i(-1) < 0 \text{ if } a \neq -1, \\ &D_1\Gamma_i(a) > 0 \text{ if } a \in (-1, 1), \\ &D_1\Gamma_i(1) < 0 \text{ if } a \neq 1, \end{aligned} \tag{7.2.6}$$

for $i = 1, \ldots M$, where a is the quantity introduced in condition (d4).

In analogy with previous chapters we define a connecting solution to (7.2.1) to be a collection $\{(P_j, c_j)\}_{j=1}^M$ of function-wavespeed pairs, with $P_j \in W_0^{2,\infty}$ and $c_j \in \mathbb{R}$, satisfying (7.2.1) and possessing the limits $\lim_{\xi \to \pm\infty} P_j(\xi) = \pm 1$.

The operator $\mathcal{F}^\mu$ from Chapter 5 in this case is an operator from $(W_0^{2,\infty})^M \times \mathbb{R}^M$ to $(L^\infty)^M$, given by

$$\mathcal{F}^\mu\big(\{(\phi_i, c_i)\}\big)_j(\xi) = -\gamma\phi_j''(\xi) - c\phi_j'(\xi) - F_j\big(\phi_1(\xi), \ldots \phi_M(\xi)\big) - \mu G_j\big(\overline{\phi_1}(\xi), \ldots, \overline{\phi_M}(\xi)\big). \tag{7.2.7}$$

The derivative $D_{1,2}\mathcal{F}^\mu(\phi, c) : (W_0^{2,\infty})^M \times \mathbb{R}^M \to (L^\infty)^M$ is given by

$$\begin{array}{rcl} D_{1,2}\mathcal{F}^\mu(\{(\phi_i, c_i)\})(\{(\psi_i, b_i)\})_j(\xi) & = & -\gamma\psi_j''(\xi) - c\psi_j'(\xi) - D_1F_j(\phi_1, \ldots, \phi_M) \cdot \big(\psi_1(\xi), \ldots \psi_M(\xi)\big) \\ & & -\mu D_1 G(\overline{\phi_1}, \ldots, \overline{\phi_M}) \cdot \big(\overline{\psi_1}(\xi), \ldots, \overline{\psi_M}(\xi)\big) - b_j\phi_j'(\xi). \end{array} \tag{7.2.8}$$

With these definitions it is clear how the Newton iterations (5.2.5) should be performed in the higher dimensional case. However, at the moment it is entirely unclear if the Newton method is well-defined, yet alone if it will converge. For example, the results on superexponential decay derived in Section 3.1 and all the comparison principles obtained in Section 3.2 required detailed arguments, which were all one dimensional in nature. We thus remark that it is not entirely trivial to extend the convergence proof of the Newton method to higher dimensions. Indeed, in Section 7.3.2, where we study bifurcation of the $a(c)$ curves, we shall see that we no longer have uniqueness of solutions. This already gives us an indication that higher dimensional systems have a richer structure than their one dimensional counterparts.

7.3 Numerical Results

7.3.1 Period Two Diffusion

In this section, we use the Newton iteration defined in the previous section to numerically solve the system

$$\left\{ \begin{array}{l} -\gamma\phi_e''(\xi) - c_e\phi_e'(\xi) = \alpha_o\big(\phi_o(\xi) - \phi_e(\xi)\big) + \alpha_e\big(\phi_o(\xi-2) - \phi_e(\xi)\big) - 15\big(\phi_e(\xi)\big)\big(\phi_e(\xi) - 1\big)\big(\phi_e(\xi) - a\big), \\ -\gamma\phi_o''(\xi) - c_o\phi_o'(\xi) = \alpha_e\big(\phi_e(\xi+2) - \phi_o(\xi)\big) + \alpha_o\big(\phi_e(\xi) - \phi_o(\xi)\big) - 15\big(\phi_o(\xi)\big)\big(\phi_o(\xi) - 1\big)\big(\phi_o(\xi) - a\big), \end{array} \right. \tag{7.3.1}$$

for different values of the detuning parameter a. We fixed $\gamma = 10^{-3}$ and used two different diffusion constants $\alpha_e = 1.3$ and $\alpha_o = 1.9$. The solutions were normalized to have $\phi_e(0) = a$ and $\phi_o(-1) = a$.

In Figure 7.1 the wavespeed curves $a(c_e)$ and $a(c_o)$ have been calculated, while in Figure 7.2 solution curves (y_e, y_o) have been plotted for different values of the detuning parameter a. From Figure 7.1 we see that there is a nontrivial interval of the detuning parameter a for which $c_e \sim 0$, while c_o stays away from zero. This implies that propagation failure can occur for the waveform y_e and hence for every other point in the lattice. Indeed, looking at the solution curves in Figure 7.2, one sees that the function $y_e(x)$ has discontinuous behaviour at $x = 0$ when $c_e \sim 0$.

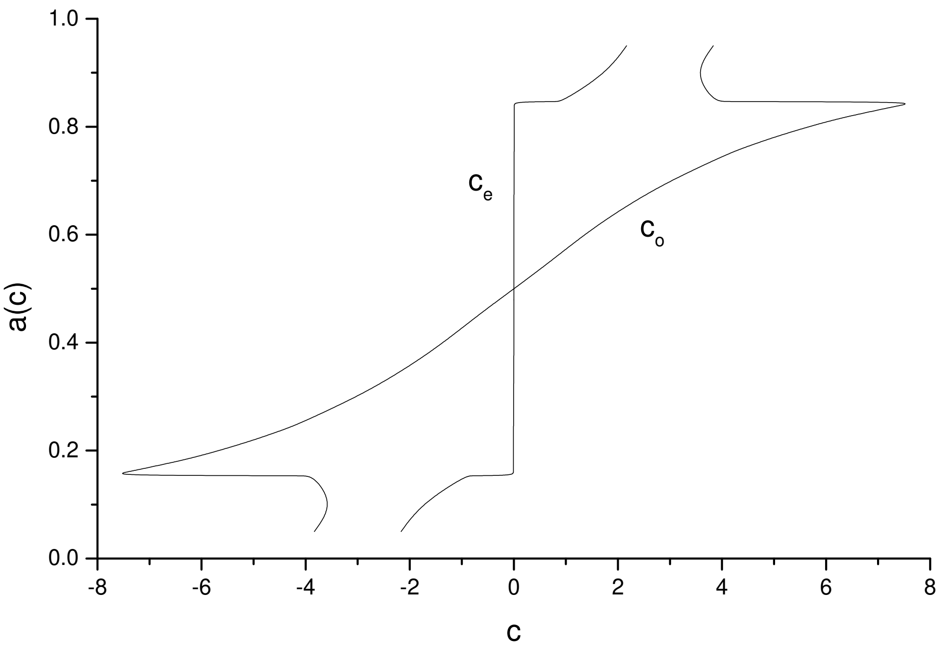


Figure 7.1: *Wavespeed plot for the constant coefficient system (7.3.1). The wavespeeds c_e and c_o have been plotted against the detuning parameter a. Notice the nontrivial interval of the detuning parameter a for which $c_e = 0$.*

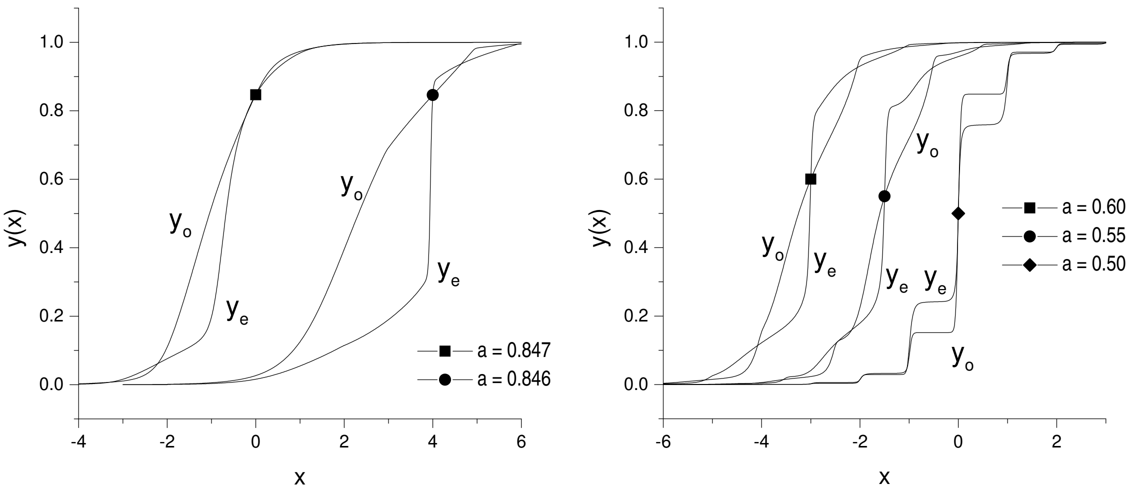
7.3.2 Bifurcation

In this section we study the behaviour of solutions to (7.3.1) in the case of constant diffusion $\alpha_e = \alpha_o = \alpha$. In this simplification, (7.3.1) reduces to

$$\begin{cases} -\gamma\phi_e''(\xi) - c_e\phi_e'(\xi) = \alpha\big(\phi_o(\xi) - 2\phi_e(\xi) + \phi_o(\xi-2)\big) - 15\big(\phi_e(\xi)\big)\big(\phi_e(\xi)-1\big)\big(\phi_e(\xi)-a\big), \\ -\gamma\phi_o''(\xi) - c_o\phi_o'(\xi) = \alpha\big(\phi_e(\xi+2) - 2\phi_o(\xi) + \phi_e(\xi)\big) - 15\big(\phi_o(\xi)\big)\big(\phi_o(\xi)-1\big)\big(\phi_o(\xi)-a\big). \end{cases} \tag{7.3.2}$$

In our calculations we fixed $\gamma = 10^{-5}$ and set $\alpha = 1.6$. The solutions were normalized to have $\phi_e(0) = a$ and $\phi_o(-1) = a$. Notice that if we choose $c_e = c_o$ and $\phi_o(\xi) = \phi_e(\xi+1)$, the system (7.3.2) reduces to a one dimensional problem covered by the theory developed in Chapter 4. In particular, this implies that for each value of the detuning parameter a there is always a solution to (7.3.2) satisfying $c_e = c_o$ and $\phi_o(\xi) = \phi_e(\xi+1)$.

In Figure 7.3 solution curves (y_e, y_o) to (7.3.2) have been plotted for different values of the detuning parameter a. In Figure 7.3(a) the solutions satisfy $y_o(\xi) = y_e(\xi+1)$ and thus the corresponding wavespeeds are equal, $c_e = c_o$. However, in Figure 7.3(b) this is no longer the case. Here $c_e \neq c_o$ and the waveforms y_e and y_o are clearly not the same. We have just seen that for these values of the

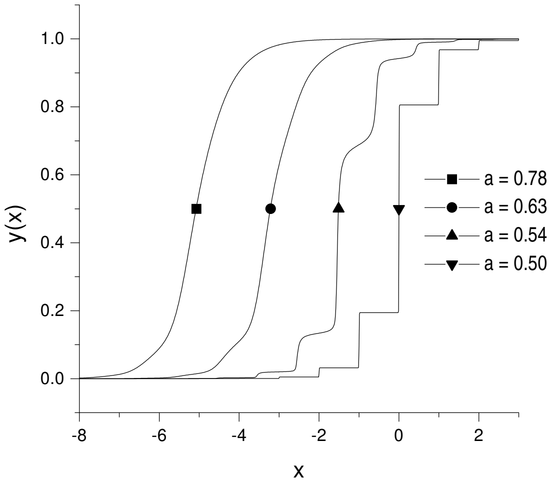


(a) (b)

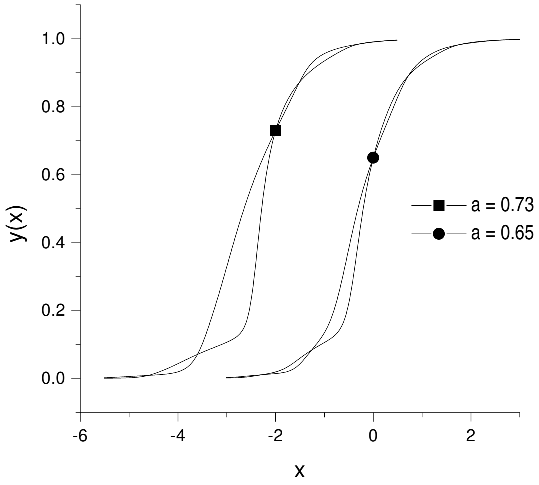
Figure 7.2: *In these figures some solution plots for solution pairs* (y_e, y_o) *to the constant coefficient system (7.3.1) are presented. Please note that for each value of* a*, the curve* y_e *has been shifted one to the right to ease comparison with the curve* y_o*. Also the curve pairs for different values of* a *have been shifted for presentation purposes. We see that in the region where* $c_e \sim 0$*, the solution curve* $y_e(x)$ *has discontinuous behaviour around* $x = 0$*, while the curve* y_o *remains smooth. Notice the sudden change in waveforms when* a *varies from* $a = 0.847$ *to* $a = 0.846$*, where propagation failure sets in.*

detuning parameter a there also exists a solution satisfying $c_e = c_o$ and $\phi_o(\xi) = \phi_e(\xi + 1)$. We thus have to conclude that we no longer have uniqueness of solutions as we had in the one dimensional case.

In Figure 7.4 the wavespeed curves $a(c_e)$ and $a(c_o)$ have been calculated. Here we see that there are two intervals of the detuning parameter a for which $c_e \neq c_o$. We call these regions period two bifurcation regions. In these regions the solution with $c_e = c_o$ is no longer stable and the Newton algorithm cannot find it. At present, it is unclear how to extend the one dimensional theory to cover this case, but from this example it is already clear that higher dimensional differential difference equations have a much richer structure.



(a)



(b)

Figure 7.3: *In these figures some solution plots for solutions* (y_e, y_o) *to the constant diffusion system (7.3.2) are presented. In (a) the curves for* y_e *and* y_o *are identical (and thus overlap) for each shown value of the detuning parameter* a*, while in (b) these curves are no longer identical.*

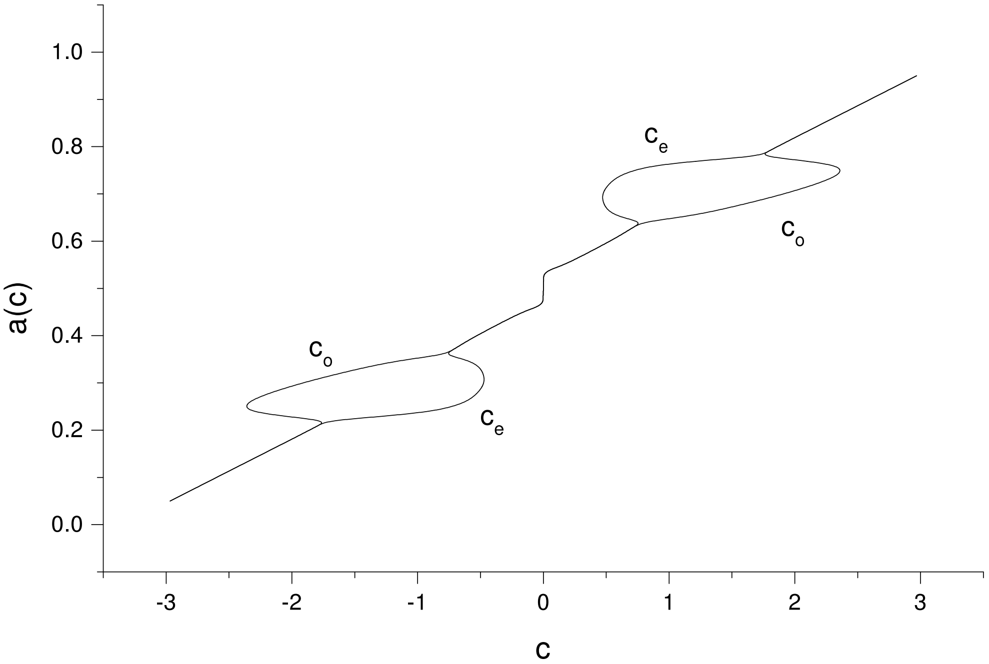


Figure 7.4: *Wavespeed plot for the constant diffusion system (7.3.2). The wavespeeds* c_e *and* c_o *have been plotted against the detuning parameter* a*. Notice the existence of two regions for which* $c_e \neq c_o$*. We call these regions period two bifurcation regions. The presence of these regions demonstrates that, unlike one dimensional systems, higher dimensional systems do not necessarily have unique solutions.*

Appendix A

Technical details

Lemma A.1. *Suppose* $x, y : [\tau_0, \tau_1] \to I \subseteq \mathbb{R}$ *satisfy the differential (in)equalities*

$$\begin{aligned} x''(\xi) &\leq Bx(\xi), \\ y''(\xi) &= By(\xi), \end{aligned} \tag{A.1}$$

on the interval $[\tau_0, \tau_1]$ *for some* $B > 0$. *Suppose furthermore that*

$$\begin{aligned} x(\tau_0) &\leq y(\tau_0), \\ x'(\tau_0) &\leq y'(\tau_0). \end{aligned} \tag{A.2}$$

Then $x(\xi) \leq y(\xi)$ *for all* $\xi \in [\tau_0, \tau_1]$. *The above result also holds if each* $\leq$ *is replaced by* $\geq$.

Proof. We first prove the result in the case where the $\leq$ signs in (A.2) are replaced by $<$ signs. Assuming the claim is false, define

$$\tau^* = \inf \{\xi \mid \xi \in [\tau_0, \tau_1] \text{ and } x'(\xi) \geq y'(\xi)\} \tag{A.3}$$

Then certainly $\tau^* > \tau_0$ and $x'(\tau^*) = y'(\tau^*)$. We also know that $x(\tau^*) < y(\tau^*)$, as $x'(\xi) < y'(\xi)$ for $\xi \in [\tau_0, \tau_*)$. However, this means $x''(\tau^*) < y''(\tau^*)$, which is a contradiction. Now consider the case with the $\leq$ signs in (A.2) and assume the claim is false. Then there exists $\xi_0 \in (\tau_0, \tau_1]$ such that $x(\xi_0) > y(\xi_0)$. Since $y(\xi_0)$ depends continuously on the initial data $y(\tau_0)$ and $y'(\tau_0)$, we can choose initial conditions for y such that $x'(\tau_0) < y'(\tau_0)$ and $x(\tau_0) < y(\tau_0)$ and still $x(\xi_0) > y(\xi_0)$, which gives the desired contradiction. The last statement in the lemma can be proved analogously. $\square$

Lemma A.2. *Let $X : B \to B$ be a bounded linear operator on a Banach space B and suppose that $\|I - X\| < 1$. Then X has an inverse X^{-1} and for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\left\|Y^{-1} - X^{-1}\right\| < \epsilon$ for every $Y : B \to B$ with $\|X - Y\| < \delta$.*

Proof. Notice that the operator

$$A = \sum_{n=0}^{\infty} (I - X)^n \tag{A.4}$$

is well-defined, as the sum convergences in norm and the space $\mathcal{B}(B, B)$ of bounded linear operators from B to B is a Banach space. Noticing that $(I - (I - X))A = A(I - (I - X)) = I$ we see that $A = X^{-1}$, hence X is invertible. Now write $\Delta = Y - X$. If $\|I - Y\| < 1$, we also have

$$X^{-1} - Y^{-1} = \sum_{n=0}^{\infty} (I - X)^n - (I - X - \Delta)^n. \tag{A.5}$$

Now assume that $\|I - X\| + \|\Delta\| < 1$. Then we see

$$\left\|X^{-1} - Y^{-1}\right\| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \|\Delta\|^k \|I - X\|^{n-k} \binom{n}{n-k}. \tag{A.6}$$

This is a power series in $\|\Delta\|$. Using the estimate

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \|\Delta\|^k \|1 - X\|^{n-k} \binom{n}{n-k} \leq \sum_{n=0}^{\infty} (\|I - X\| + \|\Delta\|)^n = (1 - (\|I - X\| + \|\Delta\|))^{-1} < \infty, \tag{A.7}$$

we see that the power series converges absolutely for $\|\Delta\| < 1 - \|I - X\|$. Since a power series is continuous within its radius of convergence and the constant term vanishes, the continuity claim in the statement of the Lemma holds. ☐

Lemma A.3. *Suppose $f : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz, that is, for each $x \in \mathbb{R}^N$ there exists an open neighbourhood $\Omega_x \subseteq \mathbb{R}^N$ and a constant L_x such that for each pair $y, z \in \Omega_x$ we have $|f(y) - f(z)| \leq L_x |z - y|$. Then if $B \subseteq \mathbb{R}^N$ is bounded, closed and convex, f is Lipschitz on B, i.e., there exists a constant $L \in \mathbb{R}$ such that for all $x, y \in B$,*

$$|f(x) - f(y)| \leq L |x - y|. \tag{A.8}$$

Proof. We have $B \subseteq \bigcup_{x \in B} \Omega_x$. Using the compactness of B, we see that there exist a finite number M of points $\{x_i\}_{i=1}^M$ such that $B \subseteq \bigcup_{i=1}^M \Omega_{x_i}$. Define $L^* = \max\{L_{x_i} \mid 1 \leq i \leq M\}$. Fix $y, z \in B$ and let Γ be the line from y to z. Due to the convexity of B we have $\Gamma \subseteq B$. Defining $\widetilde{\Omega}_i = \Omega_{x_i} \cap \Gamma$, we can write $\Gamma \subseteq \bigcup_{i=1}^{M} \widetilde{\Omega}_i$. There exists a sequence $\{v_i\}_{i=1}^{M^*}$ with $M^* \leq M$ of points such that $v_1 = y \in \widetilde{\Omega}_{i_1}$, $v_{M^*} = z \in \widetilde{\Omega}_{i_{M^*-1}}$ and $v_k \in \widetilde{\Omega}_{i_{k-1}} \cap \widetilde{\Omega}_{i_k}$ for $2 \leq k \leq M^* - 1$, with $i_j \neq i_k$ whenever $j \neq k$. Indeed, assuming to the contrary that this is not possible, let l be the supremum of $|w - y|$ for points w which can be reached in this way, and let v be the point on Γ satisfying $|v - y| = l$. Then $v \in \widetilde{\Omega}_k$ for some k. Now choose $\epsilon > 0$ such that $B_\epsilon(v) \subseteq \Omega_{x_k}$ and choose a valid sequence $\{v_i\}_{i=1}^{M^*}$ with $v_{M^*} \in B_\epsilon(v)$. It is clear by definition of v that $k \neq i_j$ for all $1 \leq j \leq M^*$. This means we can validly extend the sequence $\{v_i\}_{i=1}^{M^*}$ to z if $v = z$ or to a point beyond v if $v \neq z$. This is a contradiction. We may now write

$$|f(y) - f(z)| \leq \sum_{j=2}^{M^*} |f(v_j) - f(v_{j-1})| \leq ML^* |y - z|, \tag{A.9}$$

which proves the claim. □

Lemma A.4 ([28, Theorem 10.13]). *Let $X : B \to B$ be a bounded linear operator on some Banach space B. Then*

$$\lim_{m\to\infty} \|X^m\|^{1/m} = \hat{\sigma}, \tag{A.10}$$

i.e., the limit exists and is equal to the spectral radius $\hat{\sigma}$ of X.

Notes and Comments

Here we give a chapter by chapter overview concerning the relation between the present work and the current literature on lattice differential equations.

Chapter 2
The main results stated in Section 2.2 were obtained by Mallet-Paret in [23].

Chapter 3
The main part of this chapter is the generalization of the work in Chapters 3 and 4 from [24]. In particular, compare Section 3.5 to [24, Chapter 4] and Section 3.3 to [24, Chapter 3].

Chapter 4
Section 4.2 extends [24, Chapter 6], while Section 4.3 should be compared to [24, Chapter 8].

Chapter 5
Our analysis in Section 5.3 further develops the ideas in [13] and provides complete proofs for the claims in [13].

Chapter 6
Some of the figures and examples here should be compared to those in [13]. The subject of propagation failure is treated in depth in [26]. For more on the subject of continuation, see [6].

Chapter 7
See [12] for more examples and results concerning periodic diffusion.

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