# COMMUTATIVE C*-ALGEBRAS AND SEQUENTIALLY NORMAL MORPHISMS

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Abstract. We show that the image of a commutative monotone sequentially complete $C^*$-algebra, under a sequentially normal morphism, is again a monotone sequentially complete $C^*$-algebra, and also a monotone sequentially closed $C^*$-subalgebra. As a consequence, the image of an algebra of this type, under a sequentially normal representation in a separable Hilbert space, is strongly closed. In the case of a unital representation of $C(X)$ in a separable Hilbert space, where $X$ is a compact Hausdorff space, this implies that the von Neumann algebra generated by the image of $C(X)$ is the image of the Baire functions on $X$ under the extension of the representation to the bounded Borel functions.

## 1. Main result and application

It is well-known that the image of a von Neumann algebra under a normal unital representation is again a von Neumann algebra [3, Theorem 2.5.3]. In this note, we prove a theorem in the same vein, but now in the category of $C^*$-algebras. The domain $C^*$-algebra is in our case a commutative algebra, possessing a sequential order completeness property to be defined below. The result is subsequently applied in the context of a unital separable representation of $C(X)$, where $X$ is a compact Hausdorff space.

We start by recalling the relevant definitions. For the sake of clarity, let us mention explicitly that the $C^*$-algebras in this note are not necessarily unital; neither are (if applicable) the morphisms.

**Definition 1.1.** Cf. [3, 3.9.2].

(1) A $C^*$-algebra $A$ is *monotone sequentially complete* if every bounded increasing sequence of self-adjoint elements of $A$ has a supremum in $A$.
(2) A $C^*$-subalgebra $A$ of a $C^*$-algebra $B$ is a *monotone sequentially closed $C^*$-subalgebra of $B$* if $\sup_{n\geq 1}^{B} a_n \in A$, whenever $a_1 \leq a_2 \leq \ldots$ is a bounded

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increasing sequence of self-adjoint elements of $A$ which has a supremum $\sup^B_{n\geq 1} a_n$ in $B$.

(3) A morphism $\phi : A \mapsto B$ between two $C^*$-algebras is *sequentially normal* if $\phi(\sup_{n\geq 1} a_n) = \sup_{n\geq 1} \phi(a_n)$, for each bounded increasing sequence $a_1 \leq a_2 \leq \ldots$ of self-adjoint elements of $A$ which has a supremum in $A$.

Our main result then reads as follows.

**Theorem 1.2.** *Let $A$ be a commutative monotone sequentially complete $C^*$-algebra. Suppose $\phi : A \mapsto B$ is a sequentially normal morphism into a $C^*$-algebra $B$. Then $\phi(A)$ is a commutative monotone sequentially complete $C^*$-algebra, and a monotone sequentially closed $C^*$-subalgebra of $B$.*

The proof is given in Section 2.

**Corollary 1.3.** *Let $A$ be a commutative monotone sequentially complete $C^*$-algebra. Suppose $\pi : A \mapsto B(H)$ is a sequentially normal representation in a separable Hilbert space $H$. Then $\pi(A)$ is strongly closed.*

Indeed, $\pi(A)$ is a monotone sequentially closed $C^*$-subalgebra of $B(H)$ by Theorem 1.2. Then Pedersen's Up–Down theorem for separable Hilbert spaces [3, Theorem 2.4.3] implies that it is strongly closed.

As an application of Corollary 1.3, let $X$ be a compact Hausdorff space, with associated $C^*$-algebras $C(X)$ of continuous functions and $\mathcal{B}_b(X)$ of bounded Borel measurable functions. Suppose that $\pi : C(X) \mapsto B(H)$ is a unital representation. Then, as is well known (see, e.g., [1]), $\pi$ extends uniquely to a unital representation of $\mathcal{B}_b(X)$, again denoted by $\pi$, such that

$$(\pi(f)\xi, \xi) = \int_X f(x)\, d\mu_\xi(x) \quad (f \in \mathcal{B}_b(X),\, \xi \in H). \tag{1.1}$$

Here, for $\xi \in H$, $\mu_\xi$ denotes the positive and bounded unique regular Borel measure on $X$, which is provided by the Riesz representation theorem when one requires (1.1) to hold for all $f \in C(X)$.

The constructions involved in extending $\pi$ from $C(X)$ to $\mathcal{B}_b(X)$ make it obvious that $\pi$ maps $\mathcal{B}_b(X)$ *into* the strong closure of $\pi(C(X))$, implying that $\pi(\mathcal{B}_b(X))$ and $\pi(C(X))$ have the same strong closure. If $H$ is separable, then $\pi(\mathcal{B}_b(X))$ is actually *equal* to the strong closure of $\pi(C(X))$ [1, Proposition 9.5.3]. This description of the von Neumann algebra generated by $\pi(C(X))$, as being equal to $\pi(\mathcal{B}_b(X))$, is then a starting point for further investigation of the unital separable representation $\pi$ of $C(X)$.

The proof of the equality of $\pi(\mathcal{B}_b(X))$ and the strong closure of $\pi(C(X))$ in the separable case is usually based on measure-theoretical arguments, and tends to be somewhat more involved than others in this circle of ideas, cf. [1, proof of Proposition 9.5.3]. Corollary 1.3, which is based on order properties, provides an alternative approach. Indeed, we note that $\mathcal{B}_b(X)$ is monotone sequentially complete and that, as a consequence of (1.1) and the monotone convergence theorem, $\pi$ is a sequentially normal representation of $\mathcal{B}_b(X)$. By Corollary 1.3, $\pi(\mathcal{B}_b(X))$ is strongly closed, hence equal to the strong closure of $\pi(C(X))$.

In fact, we can now—still for a unital separable representation—also give a more economic description of the von Neumann algebra generated by $\pi(C(X))$. Namely, Corollary 1.3 also implies that the image of the monotone sequential completion of $C(X)$ in $\mathcal{B}_b(X)$ is strongly closed, and is therefore, by the same reasoning, also equal to the von Neumann algebra generated by $\pi(C(X))$.

The monotone sequential completion of $C(X)$ in $\mathcal{B}_b(X)$ is known as the Baire algebra. If $X$ is second countable, then the Baire algebra coincides with $\mathcal{B}_b(X)$, but in other cases it may be a proper $C^*$-subalgebra of $\mathcal{B}_b(X)$ [2, 6.2.10]. The Baire $C^*$-algebra is perhaps the natural pre-image of the von Neumann algebra, generated by the image of $C(X)$ under the unital separable representation $\pi$.

## 2. Proofs

The proof of Theorem 1.2 consists mainly of the combination of an elementary lifting result for functions and the Gelfand–Naimark isomorphism. The resulting Proposition 2.2 is somewhat stronger than needed for our purposes.

**Lemma 2.1.** *Let $X$ be a topological space, and let $C_0(X)$ denote the continuous functions on $X$ vanishing at infinity. Suppose that $Y \subset X$ is a non-empty subset.*

(1) *If $0 \leq f_1 \leq f_2 \leq \ldots$ is a sequence of functions on $Y$, such that each $f_n$ is the restriction of some element of $C_0(X)$ to $Y$, then there exists a sequence $0 \leq g_1 \leq g_2 \leq \ldots$ in $C_0(X)$ such that, for each $n$, $g_n$ restricts to $f_n$ and $\|g_n\|_\infty = \|f_n\|_\infty$.*

(2) *If $f_1 \geq f_2 \geq \ldots \geq 0$ is a sequence of functions on $Y$, such that each $f_n$ is the restriction of some element of $C_0(X)$ to $Y$, then there exists a sequence $g_1 \geq g_2 \geq \ldots \geq 0$ in $C_0(X)$ such that, for each $n$, $g_n$ restricts to $f_n$ and $\|g_n\|_\infty = \|f_n\|_\infty$.*

*Proof.* Suppose that $h_n \in C_0(X)$ restricts to $f_n$. Replacing $h_n$ with $|h_n|$, we may assume that $h_n \geq 0$. After a subsequent replacement of $h_n$ with $\min(h_n, \|f_n\|_\infty) \in C_0(X)$, we may assume that $h_n \geq 0$ and that $\|h_n\|_\infty = \|f_n\|_\infty$. In the case of an increasing sequence, define $g_n = \max_{1\leq i\leq n} h_i$. In the case of a decreasing sequence, define $g_n = \min_{1\leq i\leq n} h_i$. Then the $g_n$ have the required properties. □

More generally, we have the following result on isometric monotone lifts of positive monotone sequences.

**Proposition 2.2.** *Let $A$ and $B$ be commutative $C^*$-algebras. Suppose that $\phi : A \mapsto B$ is a surjective morphism.*

(1) *If* $0 \leq b_1 \leq b_2 \leq \ldots$ *is a sequence in* $B$*, then there exists a sequence* $0 \leq a_1 \leq a_2 \leq \ldots$ *in* $A$*, such that, for each* $n$*,* $\phi(a_n) = b_n$ *and* $\|a_n\| = \|b_n\|$*.*

(2) *If* $b_1 \geq b_2 \geq \ldots \geq 0$ *is a sequence in* $B$*, then there exists a sequence* $a_1 \geq a_2 \geq \ldots \geq 0$ *in* $A$*, such that, for each* $n$*,* $\phi(a_n) = b_n$ *and* $\|a_n\| = \|b_n\|$*.*

*Proof.* We may assume that $B \neq 0$. In that case, there is a canonical map $\phi^* : \widehat{B} \mapsto \widehat{A}$ between the maximal ideal spaces, defined by $\phi^*(\beta) = \beta \circ \phi$ for $\beta \in \widehat{B}$. Then $\phi^*$ is injective, as a consequence of the surjectivity of $\phi$. Using $\phi^*$, we identify $\widehat{B}$ as a set with its image in $\widehat{A}$, and we view $C_0(\widehat{B})$ as functions on $\widehat{B} \subset \widehat{A}$. (Actually, $\phi^*$ is a homeomorphic embedding of $\widehat{B}$ into $\widehat{A}$, but this fact is not needed.) Using these two identifications, the following diagram is then commutative:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \Gamma_A}\downarrow & & \downarrow{\scriptstyle \Gamma_B} \\ C_0(\widehat{A}) & \xrightarrow[\text{Res}]{} & C_0(\widehat{B}) \end{array}$$

Here $\Gamma_A$ and $\Gamma_B$ denote the Gelfand–Naimark isomorphisms, and Res is the restriction mapping. The proposition now follows from an application of Lemma 2.1 at the bottom part of the diagram. □

**Proposition 2.3.** *Let* $A$ *be a commutative monotone sequentially complete* $C^*$*-algebra. Suppose* $\phi : A \mapsto B$ *is a sequentially normal morphism into a* $C^*$*-algebra* $B$*. Let* $b_1 \leq b_2 \leq \ldots$ *be a bounded sequence of self-adjoint elements of* $\phi(A)$*. Then this sequence has a supremum in* $B$*, and this supremum in* $B$ *is an element of* $\phi(A)$*.*

*Proof.* We may assume that $b_1 \geq 0$. Using Proposition 2.2 for the morphism $\phi : A \mapsto \phi(A)$, we find a bounded sequence $a_1 \leq a_2 \leq \ldots$ of self-adjoint elements of $A$, such that $\phi(a_n) = b_n$ for all $n$. Since $A$ is monotone sequentially complete, $\sup_{n \geq 1} a_n$ exists in $A$. By the sequential normality of $\phi$, $\sup_{n \geq 1} b_n$ exists in $B$, since it is equal to $\phi(\sup_{n \geq 1} a_n)$. This also shows that this supremum in $B$ is an element of $\phi(A)$. □

Proposition 2.3 implies Theorem 1.2, as the reader will easily verify.

## References

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