

COMMUTATIVE C^* -ALGEBRAS AND σ -NORMAL MORPHISMS

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ABSTRACT. We prove in an elementary fashion that the image of a commutative monotone σ -complete C^* -algebra under a σ -normal morphism is again monotone σ -complete and give an application of this result in spectral theory.

1. INTRODUCTION

The image of a monotone σ -complete C^* -algebra under a σ -normal morphism is again monotone σ -complete, as can be deduced from somewhat technical results in the literature [1, 7]. In Section 2 of the present paper, however, we give a straightforward proof when the algebra is commutative, using only the commutative Gelfand–Naimark theorem. We also indicate how the general statement can be inferred. Section 3 contains an application in spectral theory which justifies special interest in the commutative case.

2. MAIN RESULT

The C^* -algebras in this paper are not necessarily unital and neither are (if applicable) the morphisms. This being said, we recall the relevant order definitions.

Definition 2.1. Cf. [6, 3.9.2].

- (1) A C^* -algebra A is *monotone σ -complete* if every bounded increasing sequence of self-adjoint elements of A has a supremum in A .
- (2) A C^* -subalgebra A of a C^* -algebra B is a *monotone σ -closed C^* -subalgebra of B* if $\sup_{n \geq 1}^B a_n \in A$, whenever $a_1 \leq a_2 \leq \dots$ is a bounded increasing sequence of self-adjoint elements of A which has a supremum $\sup_{n \geq 1}^B a_n$ in B .
- (3) A morphism $\phi : A \mapsto B$ between two C^* -algebras is a *σ -normal morphism* if $\phi(\sup_{n \geq 1} a_n) = \sup_{n \geq 1} \phi(a_n)$, for each bounded increasing sequence $a_1 \leq a_2 \leq \dots$ of self-adjoint elements of A which has a supremum in A .

Our main result then reads as follows.

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Theorem 2.2. *Let A be a commutative monotone σ -complete C^* -algebra. Suppose $\phi : A \mapsto B$ is a σ -normal morphism into a C^* -algebra B . Then $\phi(A)$ is a commutative monotone σ -complete C^* -algebra, and a monotone σ -closed C^* -subalgebra of B .*

The proof of Theorem 2.2 which we will now give uses no result beyond the commutative Gelfand–Naimark theorem. We start with an isometric σ -monotone lifting result for functions.

Lemma 2.3. *Let X be a topological space, and let $C_0(X)$ denote the continuous functions on X vanishing at infinity. Suppose that $Y \subset X$ is a nonempty subset.*

- (1) *If $0 \leq f_1 \leq f_2 \leq \dots$ is a sequence of functions on Y , such that each f_n is the restriction of some element of $C_0(X)$ to Y , then there exists a sequence $0 \leq g_1 \leq g_2 \leq \dots$ in $C_0(X)$ such that, for each n , g_n restricts to f_n and $\|g_n\|_\infty = \|f_n\|_\infty$.*
- (2) *If $f_1 \geq f_2 \geq \dots \geq 0$ is a sequence of functions on Y , such that each f_n is the restriction of some element of $C_0(X)$ to Y , then there exists a sequence $g_1 \geq g_2 \geq \dots \geq 0$ in $C_0(X)$ such that, for each n , g_n restricts to f_n and $\|g_n\|_\infty = \|f_n\|_\infty$.*

Proof. Suppose that $h_n \in C_0(X)$ restricts to f_n . Replacing h_n with $|h_n|$, we may assume that $h_n \geq 0$. After a subsequent replacement of h_n with $\min(h_n, \|f_n\|_\infty) \in C_0(X)$, we may assume that $h_n \geq 0$ and that $\|h_n\|_\infty = \|f_n\|_\infty$. In the case of an increasing sequence, define $g_n = \max_{1 \leq i \leq n} h_i$. In the case of a decreasing sequence, define $g_n = \min_{1 \leq i \leq n} h_i$. Then the g_n have the required properties. \square

Combining this with the commutative Gelfand–Naimark theorem, we obtain the following generalization of the above result. It is somewhat stronger than needed for our purpose, as we will use only the first part without the isometric property in the sequel.

Proposition 2.4. *Let A and B be commutative C^* -algebras. Suppose that $\phi : A \mapsto B$ is a surjective morphism.*

- (1) *If $0 \leq b_1 \leq b_2 \leq \dots$ is a sequence in B , then there exists a sequence $0 \leq a_1 \leq a_2 \leq \dots$ in A , such that, for each n , $\phi(a_n) = b_n$ and $\|a_n\| = \|b_n\|$.*
- (2) *If $b_1 \geq b_2 \geq \dots \geq 0$ is a sequence in B , then there exists a sequence $a_1 \geq a_2 \geq \dots \geq 0$ in A , such that, for each n , $\phi(a_n) = b_n$ and $\|a_n\| = \|b_n\|$.*

Proof. We may assume that $B \neq 0$. In that case, define the canonical map $\phi^* : \widehat{B} \mapsto \widehat{A}$ between the spectra by $\phi^*(\beta) = \beta \circ \phi$ for $\beta \in \widehat{B}$. Since ϕ is surjective, ϕ^* is injective. We identify \widehat{B} as a set with its image in \widehat{A} using ϕ^* , and we thus view $C_0(\widehat{B})$ as functions on $\widehat{B} \subset \widehat{A}$. Actually, ϕ^* is a homeomorphic embedding of \widehat{B} into \widehat{A} , but this is not needed. Using these two identifications, the following

diagram is then commutative:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \Gamma_A \downarrow & & \downarrow \Gamma_B \\ C_0(\widehat{A}) & \xrightarrow{\text{Res}} & C_0(\widehat{B}) \end{array}$$

Here Γ_A and Γ_B denote the Gelfand–Naimark isomorphisms, and Res is the restriction mapping. The proposition now follows from an application of Lemma 2.3 at the bottom part of the diagram. \square

Proposition 2.5. *Let A be a commutative monotone σ -complete C^* -algebra. Suppose $\phi : A \mapsto B$ is a σ -normal morphism into a C^* -algebra B . Let $b_1 \leq b_2 \leq \dots$ be a bounded sequence of self-adjoint elements of $\phi(A)$. Then this sequence has a supremum in B , and this supremum in B is an element of $\phi(A)$.*

Proof. We may assume that $b_1 \geq 0$. Using Proposition 2.4 for the morphism $\phi : A \mapsto \phi(A)$, we find a bounded sequence $a_1 \leq a_2 \leq \dots$ of self-adjoint elements of A , such that $\phi(a_n) = b_n$ for all n . Since A is monotone σ -complete, $\sup_{n \geq 1} a_n$ exists in A . By the σ -normality of ϕ , $\sup_{n \geq 1} b_n$ exists in B , since it is equal to $\phi(\sup_{n \geq 1} a_n)$. This also shows that this supremum in B is an element of $\phi(A)$. \square

Proposition 2.5 implies Theorem 2.2, as the reader will easily verify.

Remark 2.6. As mentioned in the Introduction, Theorem 2.2 also holds in the noncommutative case. In fact, the noncommutative version of Proposition 2.5 — which implies the noncommutative version of Theorem 2.2 — is valid, as can be seen from [7, Lemma 1.1], which is in turn based on [4, Proposition 5]. The proof of the latter basic result relies on the general Gelfand–Naimark theorem and involves some well chosen use of the Borel functional calculus.

As an alternative route, in the unital noncommutative case the monotone σ -completeness of $\phi(A)$ in Theorem 2.2 also follows from [7, Proposition 1.3], which is in turn based on [1, Lemma 2.13]. A closer inspection of the proof of the latter result shows that it in fact yields Proposition 2.5 in the unital noncommutative setting, and therefore Theorem 2.2 in the unital noncommutative case again. This key proof of [1, Lemma 2.13] relies itself on the fact that a unital monotone σ -complete C^* -algebra is a Rickart C^* -algebra in which the projections form a lattice. As remarked in the introduction of [1], this follows from [2].

The above makes it clear that in the commutative case the proof of Theorem 2.2 as given in this section is considerably more elementary than the alternative approach of establishing and subsequently specializing the noncommutative version.

3. APPLICATION IN SPECTRAL THEORY

To conclude, we give an application of Theorem 2.2 in spectral theory that justifies special interest in the commutative case. It is based on the following.

Corollary 3.1. *Let A be a commutative monotone σ -complete C^* -algebra and suppose $\pi : A \mapsto B(H)$ is a σ -normal representation in a separable Hilbert space H . Then $\pi(A)$ is strongly closed. If A and π are unital, then $\pi(A)$ is a von Neumann algebra.*

Indeed, $\pi(A)$ is a monotone σ -closed C^* -subalgebra of $B(H)$ by Theorem 2.2. Hence the Up–Down theorem for separable Hilbert spaces [6, Theorem 2.4.3] implies that it is strongly closed, establishing the corollary (which is also valid in the noncommutative case in view of Remark 2.6).

As an application of Corollary 3.1 in spectral theory, let X be a compact Hausdorff space with associated C^* -algebras $C(X)$ of continuous functions and $\mathcal{B}_b(X)$ of bounded Borel measurable functions. Suppose that $\pi : C(X) \mapsto B(H)$ is a unital representation. Then π extends uniquely to a unital representation of $\mathcal{B}_b(X)$, again denoted by π , such that

$$(3.1) \quad (\pi(f)\xi, \xi) = \int_X f(x) d\mu_\xi(x) \quad (f \in \mathcal{B}_b(X), \xi \in H).$$

Here μ_ξ denotes the positive and bounded unique regular Borel measure on X which is provided by the Riesz representation theorem when one requires (3.1) to hold for all $f \in C(X)$.

It is easy to see that π maps $\mathcal{B}_b(X)$ into the strong closure of $\pi(C(X))$, implying that $\pi(\mathcal{B}_b(X))$ and $\pi(C(X))$ have the same strong closure. If H is separable, then $\pi(\mathcal{B}_b(X))$ is actually equal to the strong closure of $\pi(C(X))$ [3, proof of Proposition 9.5.3]. This description of the von Neumann algebra generated by $\pi(C(X))$ as being equal to $\pi(\mathcal{B}_b(X))$ is a basic ingredient for further investigation of the unital separable representation π of $C(X)$.

The proof of the equality of $\pi(\mathcal{B}_b(X))$ and the strong closure of $\pi(C(X))$ in the separable case is usually based on measure-theoretical arguments and is somewhat more involved than others in this circle of ideas, cf. [3, proof of Proposition 9.5.3], but Corollary 3.1, which is based on order properties, provides an alternative approach. Indeed, $\mathcal{B}_b(X)$ is monotone σ -complete and, as a consequence of (3.1) and the monotone convergence theorem, π is a σ -normal representation of $\mathcal{B}_b(X)$. Therefore, by Corollary 3.1, $\pi(\mathcal{B}_b(X))$ is strongly closed, hence equal to the strong closure of $\pi(C(X))$.

In fact, by the same reasoning Corollary 3.1 also implies — still for a unital separable representation — that the image of the monotone σ -completion of $C(X)$ in $\mathcal{B}_b(X)$ is strongly closed, hence already equal to the von Neumann algebra generated by $\pi(C(X))$. If X is second countable then this monotone σ -completion of $C(X)$ in $\mathcal{B}_b(X)$ — the Baire algebra — coincides with $\mathcal{B}_b(X)$, but in other cases it may be a proper C^* -subalgebra of $\mathcal{B}_b(X)$ [5, Remarks 6.2.10].

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