Stability in linear problems

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Abstract

Using analytic stability we construct quotients associated to linear actions and we give a necessary and sufficient condition for their compactness. These results are applied for quiver factorisation problems, deducing that the compactness of the corresponding quotients is related to a combinatorial property. As an example, we discuss the case of linear systems, constructing compactifications of the spaces of accessible and of minimal linear systems.

0 Introduction

In the present note we will be concerned with *linear problems*, i.e. we will be interested in the study of the action of a connected reductive complex Lie group G on a finite dimensional complex vector space W through a representation ρ and in the construction of associated spaces of orbits.

In the first of the two sections of the paper we describe general results concerning such linear problems. After a preliminary part we recall the corresponding definitions for analytic, symplectic and GIT-stability. Our main tool in the study of linear problems will be analytic stability, as introduced and studied in [6] and [10]. In the general set-up the analytic stability depends on a symplectisation of the G-action and we point out that, for linear problems, it depends only on a parameter τ varying in a finite dimensional real vector space. We also give the explicit description of the (semi)stable loci in a concrete example, which arises in control theory. The stability concepts make possible the construction of orbit spaces since the restriction of the action to the semistable locus yields a geometric quotient. Moreover, such a quotient admits an alternative description using tools from symplectic geometry. An important result of this section is stated in Proposition 1.9 and relates the compactness of these quotients to the study of the properness of a certain map, related to the symplectic approach (namely the canonical moment map μ_{can}).

In the second section we focus our attention to a special class of linear problems. We explain how, starting with a pair (Q, S) consisting of a quiver Q (i.e. a diagram containing points and arrows) and of a subset S of the set Q_0 of points of Q, one obtains in a natural way a linear problem, which is called *quiver factorisation problem* (QFP for shorthand) associated to the combinatorial data (Q, S). The corresponding quotients are called QFP-quotients and their geometry depends on (Q, S), on a dimension vector $\alpha \in \mathbb{N}^{Q_0}$ and on a parameter $\tau \in \mathbb{R}^S$. However, we show in Theorem 2.3 that the compactness of a QFP-quotient is related only to a combinatorial property of the pair (Q, S). In the last part of the paper we return to the example concerning linear systems, which can be represented as a quiver factorisation problem. The corresponding QFP-quotients are not compact and in Theorem 2.7 we prove that they admit natural compactifications which are themselves QFP-quotients. It remains an open question whether it is possible to construct analogous natural compactifications for arbitrary non-compact QFP-quotients.

1 General results

Let G be a connected reductive complex Lie group with Lie algebra $\mathfrak{g}$ (throughout this paper the Lie algebra of a Lie group will be denoted by the corresponding 'german' character). We denote by Z the center of G and by $Z_{\mathbb{R}}$ its unique maximal compact subgroup. We also set:

$$\mathcal{T}_G := \left\{\tau \in \mathfrak{g}^\vee \mid \tau_{|[\mathfrak{g},\mathfrak{g}]} = 0,\ \tau(\mathfrak{z}_{\mathbb{R}}) \subset \mathbb{R}\right\},$$

which is a real vector space, naturally isomorphic to $(\mathfrak{z}_{\mathbb{R}})^\vee$ (the dual of the Lie algebra of $Z_{\mathbb{R}}$). Further let $\rho : G \to \mathrm{GL}(W)$ be a representation of G on a finite dimensional complex vector space W; its kernel will be denoted by H.

1.1 Preliminaries

1.1.1 Elements of Hermitian type and weight spaces

An element s of $\mathfrak{g}$ is called of Hermitian type if there exists a compact subgroup K of G such that $s \in i\mathfrak{k}$; the set of elements of Hermitian types will be denoted by $H(G)$. Other equivalent characterizations of the elements of $H(G)$ can be found in [10, Definition 2.1]. In particular, if $s \in H(G)$, the endomorphism $\rho_*(s)$ has only real eigenvalues and is diagonalizable. For an eigenvalue λ, we denote by $W(\lambda)$ the corresponding subspace of eigenvectors and we set

$$W^{\leq 0}(s) := \bigoplus_{\lambda \leq 0} W(\lambda), \qquad W^{<0}(s) := \bigoplus_{\lambda < 0} W(\lambda).$$

Every vector $w \in W$ can be written uniquely as $w = w_- + w_0 + w_+$, where $w_- \in \bigoplus_{\lambda<0} W(\lambda)$, $w_0 \in W(0)$ and $w_+ \in \bigoplus_{\lambda>0} W(\lambda)$.

Remark 1.1 *If $W = \bigoplus_{i=1}^p W_i$ is the direct sum of the representations W_i of the group G, then for any $s \in H(G)$ it holds*

$$W^{\leq 0}(s) = \bigoplus_{i=1}^p W_i^{\leq 0}(s), \quad W^{<0}(s) = \bigoplus_{i=1}^p W_i^{<0}(s).$$

1.1.2 The case of the general linear group

In this subsection we pay attention to the special case of the group $\mathrm{GL}(V)$ of general linear transformations of a finite dimensional complex vector space V and to some $\mathrm{GL}(V)$-actions. Let $s \in H(\mathrm{GL}(V))$ be an element of Hermitian type. We denote by $\lambda_1 < \ldots < \lambda_q$ the (real) eigenvalues of s and by $V(\lambda_i)$ the corresponding eigenspaces. For any $i = 1, \ldots, q$ we define

$$V_i := \bigoplus_{\lambda_j \leq \lambda_i} V(\lambda_j)$$

and we put $V_0 := \{0\}$, obtaining in this way a filtration on V, denoted by $\mathcal{F}_s$,

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_q = V.$$

We also define

$$\mathcal{F}_s^{\leq 0} := \bigoplus_{\lambda_j \leq 0} V(\lambda_j), \qquad \mathcal{F}_s^{<0} := \bigoplus_{\lambda_j < 0} V(\lambda_j).$$

In particular, there exist j_1, j_2 such that $\mathcal{F}_s^{\leq 0} = V_{j_1}$, $\mathcal{F}_s^{<0} = V_{j_2}$.

Example 1.2 *Let V, V_1 and V_2 be finite dimensional complex vector spaces and let $s \in H(\mathrm{GL}(V))$ be a fixed element of Hermitian type.*

(i) Take the $\mathrm{GL}(V)$-action by conjugation on $W := \mathrm{End}_{\mathbb{C}}(V)$. Then $\varphi \in W^{\leq 0}(s)$ if and only if the filtration $\mathcal{F}_s$ is φ-invariant.

(ii) Take the $\mathrm{GL}(V)$-action on $W_1 := \mathrm{Hom}_{\mathbb{C}}(V_1, V)$ given by $(g, \varphi_1) \mapsto g \circ \varphi_1$. Then $\varphi_1 \in W_1^{\leq 0}(s)$ if and only if $\mathrm{im}(\varphi_1) \subset \mathcal{F}_s^{\leq 0}$.

(iii) Take the $\mathrm{GL}(V)$-action on $W_2 := \mathrm{Hom}_{\mathbb{C}}(V, V_2)$ given by $(g, \varphi_2) \mapsto \varphi_2 \circ g^{-1}$. Then $\varphi_2 \in W_2^{\leq 0}(s)$ if and only if $\mathcal{F}_s^{<0} \subset \ker(\varphi_2)$.

Proof. We prove the assertion (i), the other ones follow using similar arguments. In the case of the action ρ given by conjugation, the eigenvalues of $\rho_*(s)$ are $\{\lambda_a - \lambda_b\}_{a,b}$ and, according to our notations, let $(W(\lambda_a - \lambda_b))_{a,b}$ be the corresponding eigenspaces. As noticed above, for an endomorphism $\varphi \in W$ we have the decomposition

$$\varphi = \varphi_- + \varphi_0 + \varphi_+,$$

where $\varphi_- \in \bigoplus_{\nu<0} W(\nu)$, $\varphi_0 \in W(0)$ and $\varphi_+ \in \bigoplus_{\nu>0} W(\nu)$.

We first remark that, if $v \in V(\lambda)$ is a vector of the eigenspace $V(\lambda)$ corresponding to the eigenvalue λ and $\psi \in W(\lambda_a - \lambda_b)$, then $\psi(v) \in V(\lambda_a - \lambda_b + \lambda)$, since one has

$$s(\psi(v)) = [s, \psi](v) + \psi(s(v)) = (\lambda_a - \lambda_b + \lambda)\psi(v).$$

We deduce that for $v \in V(\lambda)$ it holds

$$\varphi_-(v) \in \bigoplus_{\mu<\lambda} V(\mu), \quad \varphi_0(v) \in V(\lambda), \quad \varphi_+(v) \in \bigoplus_{\mu>\lambda} V(\mu).$$

Suppose first that $\varphi \in W^{\leq 0}(s)$, that is $\varphi_+ = 0$. In particular, for a vector $v \in V(\lambda)$ one has $\varphi(v) \in \bigoplus_{\mu \leq \lambda} V(\mu)$. We conclude that for any $i = 1, \dots, q$ the following inclusion hold:

$$\varphi(V_i) = \varphi\big(\bigoplus_{\lambda \leq \lambda_i} V(\lambda)\big) \subset \bigoplus_{\lambda \leq \lambda_i} V(\lambda) = V_i,$$

that is the filtration $\mathcal{F}_s$ is φ-invariant.

Conversely, we suppose that $\mathcal{F}_s$ is φ-invariant and we claim that $\varphi_+ = 0$. In fact, it is enough to show that for any eigenvalue λ it holds $\varphi_+|_{V(\lambda)} = 0$. We first notice that, for i suitable chosen one has $V_i = \bigoplus_{\mu \leq \lambda} V(\mu) \supset V(\lambda)$. Let now $v \in V(\lambda)$ be a vector of the space $V(\lambda)$. On one hand, $\varphi(v) \in V_i$ (by assumption) and $\varphi_-(v) \in V_i$ (by the arguments above), hence $\varphi_+(v) \in V_i$. On the other hand, $\varphi_+(v) \in \bigoplus_{\mu > \lambda} V(\mu)$ and we conclude that $\varphi_+(v) = 0$. ■

1.1.3 Symplectisations

We return now to the general set-up of a linear action of a connected complex reductive group G on a finite dimensional complex vector space W. Let K be a maximal compact subgroup of G. Then its Lie algebra $\mathfrak{k}$ can be decomposed as $\mathfrak{k} = \mathfrak{z}_{\mathbb{R}} \oplus [\mathfrak{k}, \mathfrak{k}]$ (recall that $\mathfrak{z}_{\mathbb{R}}$ is the Lie algebra of the torus $Z_{\mathbb{R}}$, which is the unique maximal compact subgroup of the center Z of G). In particular, since K is connected, the following relations hold

$$\{\tau \in \mathfrak{k}^\vee | \langle \tau, \mathrm{ad}_g(\xi) \rangle = \langle \tau, \xi \rangle, \ \forall g \in K, \xi \in \mathfrak{k}\} = \{\tau \in \mathfrak{k}^\vee | \tau_{|[\mathfrak{k},\mathfrak{k}]} = 0\} \simeq \mathcal{T}_G. \quad (1)$$

In the sequel we will tacitly use this identification.

Let further h_W be a Hermitian inner product on W such that $\rho(K) \subset \mathrm{U}(W, h_W)$. In this way K acts in a symplectic fashion with respect to the symplectic structure induced by h_W. Moreover, there exists a standard moment map for the K-action on W, namely

$$\mu_{\mathrm{can}} : W \to \mathfrak{k}^\vee, \quad \mu_{\mathrm{can}}(w) := -\rho_*^\vee \left(\frac{i}{2} (w \otimes \bar{w}) \right).$$

This map depends on the representation, but we skipped the index ρ in order to simplify the notation. Here we denoted by $(w \otimes \bar{w})$ the Hermitian endomorphism given by $w \otimes \bar{w} = h_W(\cdot, w)w$, whereas $\rho_*^\vee$ is the dual of $\rho_* : \mathfrak{k} \to \mathrm{u}(W)$. We used the identification between $\mathrm{u}(W)$ and $\mathrm{u}(W)^\vee$ given by the inner product $(A, B) \mapsto -\mathrm{Tr}(AB)$. Moreover, from the equality (1) it follows that for any $\tau \in \mathcal{T}_G$ the map given by

$$\mu_\tau : W \to \mathfrak{k}^\vee, \qquad \mu_\tau(w) := \mu_{\mathrm{can}}(w) - \tau$$

is also a moment map for the K-action on W.

The triple (K, h_W, μ_τ) yields a *symplectisation* σ_τ of the G-action on W given by ρ. According to the definition given in [6], σ_τ is the equivalence class of (K, h_W, μ_τ) with respect to the following equivalence relation: let (K_i, h_i, μ_i)

$(i = 1, 2)$ be two triples consisting of a maximal compact subgroup K_i of G, a K_i-invariant Hermitian metric h_i on W and a moment map $\mu_i : W \to \mathfrak{k}_i^\vee$ for the K_i-operation on W with respect to the symplectic structure induced by h_i. They are considered to be equivalent if there exists $g \in G$ such that

$$K_2 = \mathrm{Ad}_g(K_1), \quad h_2 = (g^{-1})^* h_1, \quad \mu_2 = \mathrm{ad}^t_{g^{-1}} \circ \mu_1 \circ g^{-1}.$$

1.2 Analytic, symplectic and GIT-stability

In this section we briefly recall the corresponding definitions and we describe the relationship between analytic, symplectic and GIT-stability. For further details we refer the reader to the papers [5], [6] and [10].

I. Our main tool in the study of linear problems will be the *analytic stability*, as introduced and studied in [6] and [10]. For a symplectisation σ of the G-action on W, one constructs a well-defined map

$$\lambda_\sigma : H(G) \times W \to \mathbb{R} \cup \{\infty\}, \qquad (s, w) \mapsto \lambda^s_\sigma(w).$$

A point $w \in W$ is called *analytically σ-semistable* if and only if $\lambda^s_\sigma(w) \geq 0$, for any $s \in H(G)$. It is *analytically σ-stable* if and only if it is σ-semistable and $\lambda^s_\sigma(w) > 0$ for any $s \in H(G) \setminus \mathfrak{h}$.

In the sequel we will take a closer look at analytic stability in the case of linear problems. We begin by recalling the construction of the map λ. Take an element of Hermitian type $s \in H(G)$ and choose a representant (K, h_W, μ_τ) of σ such that $s \in i\mathfrak{k}$. Then, for any $w \in W$ and $t \in \mathbb{R}$ one defines

$$\lambda^{s,t}_\sigma : W \to \mathbb{R}, \qquad \lambda^{s,t}_\sigma(w) := \langle \mu_\tau(e^{ts} \cdot w), -is \rangle.$$

The map λ_σ, which is independent on the choice of the representant of the symplectisation, is given by

$$\lambda^s_\sigma : W \to \mathbb{R} \cup \{\infty\}, \qquad \lambda^s_\sigma(w) := \lim_{t\to\infty} \lambda^{s,t}_\sigma(w).$$

The following Lemma gives an explicit description of this map in the case of linear actions:

Lemma 1.3 *Let σ be a symplectisation of the G-action on W, $s \in H(G)$ an element of Hermitian type and (K, h_W, μ_τ) a representant of σ such that $s \in i\mathfrak{k}$. Then for any $w \in W$ it holds*

$$\lambda^s_\sigma(w) = \begin{cases} \langle \tau, is \rangle, & \text{if } w \in W^{\leq 0}(s) \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. A straightforward computation shows that for any $t \in \mathbb{R}$ one has $\lambda^{s,t}_\sigma(w) = \lambda^{s,t}(w) + \langle \tau, is \rangle$, where

$$\lambda^{s,t}(w) := \frac{1}{2} h_W(e^{ts} \cdot w, \rho_*(s)(e^{ts} \cdot w)).$$

We choose a h_W-orthonormal basis $b_1, \ldots, b_m$ of W consisting of eigenvectors of $\rho_*(s)$, such that any b_i corresponds to an eigenvalue λ_i. Writing $w = \sum_i w_i b_i$, we have

$$e^{ts} \cdot w = \sum_{i=1}^{m} (e^{t\lambda_i} w_i) b_i, \qquad \rho_*(s)(e^{ts} \cdot w) = \sum_{i=1}^{m} (\lambda_i e^{t\lambda_i} w_i) b_i,$$

and hence

$$\lambda^{s,t}(w) = \sum_{i=1}^{m} \lambda_i |e^{t\lambda_i}|^2 |w_i|^2 b_i.$$

Taking the limit we obtain

$$\lim_{t\to\infty} \lambda^{s,t}(w) = \begin{cases} 0, & \text{if } w \in W^{\leq 0}(s) \\ \infty, & \text{otherwise} \end{cases}$$

and our assertion follows. ∎

In particular, Lemma 1.3 shows that the map λ^s_σ and hence the stability concept do not depend on K and h_W, but only on τ. More precisely, if we consider symplectisations $\sigma = [(K, h_W, \mu_\tau)]$ and $\widetilde{\sigma} = [(\widetilde{K}, \widetilde{h}_W, \mu_\tau)]$, corresponding to the pairs (K, h_W), respectively $(\widetilde{K}, \widetilde{h}_W)$ and to the same τ, then a point w is σ-(semi)stable if and only if it is $\widetilde{\sigma}$-(semi)stable. In this case we will say that w is *τ-(semi)stable*. In fact, we obtained the following explicit description of the τ-(semi)stable locus:

Proposition 1.4 *a) The following assertions are equivalent:*
(i) w is τ-semistable,
(ii) for any $s \in H(G)$ such that $w \in W^{\leq 0}(s)$, it holds $\langle \tau, is \rangle \geq 0$.
b) The following assertions are equivalent:
(i) w is τ-stable,
(ii) w is τ-semistable and for any $s \in H(G) \setminus \mathfrak{h}$ such that $w \in W^{\leq 0}(s)$ it holds $\langle \tau, is \rangle > 0$. ∎

Remark 1.5 Let $a > 0$ be a positive constant. Then a point w is τ-(semi)stable if and only if it is $a\tau$-(semi)stable.

II. The second concept of stability is that one of *symplectic stability.*

Let $\sigma = [(K, h_W, \mu_\tau)]$ be a symplectisation of the G-action on W. According to [6] a point $w \in W$ is called *symplectically σ-semistable* if $\overline{G \cdot w} \cap \mu_\tau^{-1}(0) \neq \emptyset$. It is called *symplectically σ-stable* if $G \cdot w \cap \mu_\tau^{-1}(0) \neq \emptyset$ and $\dim G_w = \dim H$, where G_w denotes the stabilizer of w.

III. Last but no least, we briefly recall some definitions related to *GIT-stability.* As noticed in [5], a character $\chi : G \to \mathbb{C}^*$ of G yields a linearisation of the trivial complex line bundle L over W. On the total space of $L^{-1} = W \times \mathbb{C}$, the group G acts by $g \cdot (w, \lambda) := (\rho(g)w, \chi^{-1}(g)\lambda)$. In this context, Mumford's definitions for (semi)stability with respect to this linearisation can be phrased as follows [5]. A point $w \in W$ is called *χ-semistable* if there exist $n \geq 1$ and a

χ^n-equivariant polynomial $f \in \mathbb{C}[W]^{G,\chi^n}$ such that $f(w) \neq 0$. If moreover, the G-operation on $\{w | f(w) \neq 0\}$ is closed and the dimension of the stabilizer G_w of w is equal to the dimension of H, then w will be called *χ-stable.*

These three stability concepts are, essentially, equivalent:

Proposition 1.6 *Let $\sigma = [(K, h_W, \mu_\tau)]$ be a symplectisation of the G-action on W.*

i) [6], [10] *A point w in W is analytically τ-(semi)stable if and only if it is symplectically σ-(semi)stable.*

ii) [5] *Suppose that τ is provided by the derivative of a character χ of G. Then a point is symplectically σ-(semi)stable if and only if it is GIT χ-(semi)stable.*

Notation For a fixed τ we will denote by $W^{s,\tau}$ and $W^{ss,\tau}$ the τ-stable, respectively τ-semistable locus.

1.2.1 An example from control theory: linear systems

In this subsection we will give an explicite description of the (semi)stable locus in a particular case. More precisely, we consider three complex vector spaces V, V_1, V_2 of dimensions $r > 0$, r_1, respectively r_2 such that $(r_1, r_2) \neq (0, 0)$ and we set

$$W := \mathrm{End}_{\mathbb{C}}(V) \oplus \mathrm{Hom}_{\mathbb{C}}(V_1, V) \oplus \mathrm{Hom}_{\mathbb{C}}(V, V_2).$$

The group $G := \mathrm{GL}(V)$ acts in a natural fashion on W by

$$g \cdot (\varphi, \varphi_1, \varphi_2) := (g \circ \varphi \circ g^{-1}, g \circ \varphi_1, \varphi_2 \circ g^{-1}).$$

A triple $(\varphi, \varphi_1, \varphi_2)$ will be called a *linear system.* We notice that, after fixing basis in the spaces V, V_1, V_2, one can represent a linear system by a triple of matrices. Such triples of matrices arise in control theory and one can find more results concerning these systems, for instance, in the monography [9].

We notice that the kernel of the representation described above is trivial, by the assumption that $(r_1, r_2) \neq (0, 0)$. Since $Z(\mathrm{GL}(V)) \simeq \mathbb{C}^*$, the real vector space $\mathcal{T}_{\mathrm{GL}(V)}$ is one-dimensional and, hence, according to Remark 1.5, there are three different stability concepts, corresponding to the following elements of $\mathcal{T}_{\mathrm{GL}(V)}$:

$$\tau_0 := 0, \qquad \tau_1 := i\mathrm{Tr}, \qquad \tau_{-1} := -i\mathrm{Tr}.$$

In order to describe explicitely the corresponding (semi)stable loci, we first fix some notations: for a triple $(\varphi, \varphi_1, \varphi_2) \in W$ we put

$$V_{\varphi,\varphi_1} := \mathrm{im}(\varphi_1) + \mathrm{im}(\varphi \circ \varphi_1) + \ldots + \mathrm{im}(\varphi^{r-1} \circ \varphi_1) \subset V,$$

$$V^{\varphi,\varphi_2} := \ker(\varphi_2) \cap \ker(\varphi_2 \circ \varphi) \cap \ldots \cap \ker(\varphi_2 \circ \varphi^{r-1}) \subset V.$$

Proposition 1.7 *It holds:*

$$(i) \qquad W^{s,\tau_0} = \{(\varphi, \varphi_1, \varphi_2) \mid V_{\varphi,\varphi_1} = V,\ V^{\varphi,\varphi_2} = 0\}, \quad W^{ss,\tau_0} = W;$$

$$(ii) \qquad W^{s,\tau_1} = W^{ss,\tau_1} = \{(\varphi, \varphi_1, \varphi_2) \mid V_{\varphi,\varphi_1} = V\};$$

$$(iii) \qquad W^{s,\tau_{-1}} = W^{ss,\tau_{-1}} = \{(\varphi, \varphi_1, \varphi_2) \mid V^{\varphi,\varphi_2} = 0\}.$$

Proof. Let $s \in H(G)$ be of Hermitian type. According to Remark 1.1 and to Example 1.2, a triple $(\varphi, \varphi_1, \varphi_2)$ is an element of $W^{\leq 0}(s)$ if and only if the filtration $\mathcal{F}_s$ is φ-invariant, $\mathrm{im}(\varphi_1) \subset \mathcal{F}_s^{\leq 0}$ and $\ker(\varphi_2) \supset \mathcal{F}_s^{<0}$.

(i) The fact that $W^{ss,\tau_0} = W$ follows at once from Proposition 1.4 a). Let now $(\varphi, \varphi_1, \varphi_2)$ be τ_0-stable and suppose that $V_{\varphi,\varphi_1} \neq V$. We consider the φ-invariant filtration

$$\{0\} \subset V_{\varphi,\varphi_1} \subsetneq V.$$

One can find $s \in H(G) \setminus \{0\}$ such that for the associated filtration $\mathcal{F}_s$ it holds $\mathcal{F}_s^{\leq 0} = V_{\varphi,\varphi_1}$ and $\mathcal{F}_s^{<0} = \{0\}$. In particular, this means that the triple $(\varphi, \varphi_1, \varphi_2)$ is an element of $W^{\leq 0}(s)$. Since $\langle \tau_0, is \rangle = 0$, it follows that $(\varphi, \varphi_1, \varphi_2)$ cannot be τ_0-stable. If $V^{\varphi,\varphi_2} \neq 0$, one obtains an analogous contradiction by considering the filtration

$$\{0\} \subsetneq V^{\varphi,\varphi_2} \subset V.$$

Conversely, let's suppose that it holds $V_{\varphi,\varphi_1} = V$ and $V^{\varphi,\varphi_2} = 0$. We claim that there exists no element $s \in H(G) \setminus \{0\}$ such that $(\varphi, \varphi_1, \varphi_2) \in W^{\leq 0}(s)$. Indeed, let $s \neq 0$ be an element of Hermitian type such that $\mathcal{F}_s$ is φ-invariant, $\mathrm{im}(\varphi_1) \subset \mathcal{F}_s^{\leq 0}$ and $\ker(\varphi_2) \supset \mathcal{F}_s^{<0}$. Since $s \neq 0$, then either $\mathcal{F}_s^{\leq 0} \neq V$ or $\mathcal{F}_s^{<0} \neq 0$. In the first case, it follows that $V_{\varphi,\varphi_1} \neq V$, whereas in the second situation it holds $V^{\varphi,\varphi_2} \neq 0$ and this yields a contradiction.

(ii) For simplicity, we denote by M the set of triples $(\varphi, \varphi_1, \varphi_2)$ such that $V_{\varphi,\varphi_1} = V$.

Let $(\varphi, \varphi_1, \varphi_2)$ be an element of M. If s is an element of Hermitian type such that $(\varphi, \varphi_1, \varphi_2) \in W^{\leq 0}(s)$, it follows that $V_{\varphi,\varphi_1} = \mathcal{F}_s^{\leq 0} = V$. In particular all the eigenvalues of s are non-positive and hence $\langle \tau_1, is \rangle = -\mathrm{Tr}(s) \geq 0$. Moreover, if $s \neq 0$, then $\langle \tau_1, is \rangle > 0$. This proves that $M \subset W^{s,\tau_1}$.

Let now $(\varphi, \varphi_1, \varphi_2)$ be a triple such that $V_{\varphi,\varphi_1} \neq V$. Consider s with eigenvalues 0 and 1 and such that the associated filtration is

$$\{0\} \subset V_{\varphi,\varphi_1} \subsetneq V.$$

Then one has $(\varphi, \varphi_1, \varphi_2) \in W^{\leq 0}(s)$ and it holds

$$\langle \tau_1, is \rangle = -\mathrm{Tr}(s) = -\dim(V/V_{\varphi,\varphi_1}) < 0,$$

which shows that the given triple is not τ_1-semistable. Hence we proved that the semistable locus is included in M. Since the stable locus is included in the semistable one, the required equalities follow.

(iii) This assertion is the 'dual' of (ii) and can be proved using similar arguments. ∎

Remark A triple $(\varphi, \varphi_1, \varphi_2)$ such that $V_{\varphi,\varphi_1} = V$ is called in the control theory *accessible* linear system, whereas a triple for which $V_{\varphi,\varphi_1} = V$ and $V^{\varphi,\varphi_2} = 0$ is called *minimal* system. Hence, the first statement of Proposition 1.7 shows that the set of minimal systems can be described as the set of analytically stable points with respect to a suitable stability concept. The second statement of Proposition 1.7 together with Proposition 1.6 yield an alternative proof to a known result of Byrnes and Hurt [1, Theorem 4.1] which states that the set of accessible linear systems coincides with the GIT-(semi)stable locus corresponding to a suitable linearization.

1.3 Geometry of quotients

We return to the general set-up of an action of a connected reductive complex Lie group G on a complex vector space W. The following result follows from [10, Theorem 1.4] (compare also [2]) and from Proposition 1.6:

Proposition 1.8 *Let $\tau \in \mathcal{T}_G$ be fixed. The sets $W^{ss,\tau}, W^{s,\tau}$ are open in W and there exists a good quotient*

$$q_\tau : W^{ss,\tau} \to W/\!\!/(G,\tau)$$

with the property that two G-orbits have the same image in $W/\!\!/(G,\tau)$ if and only if their closures in $W^{ss,\tau}$ are not disjoint. Moreover, the natural map

$$\mu_\tau^{-1}(0)/K \longrightarrow W/\!\!/(G,\tau)$$

is a homeomorphism. ■

This result shows not only that the restriction to the semistable locus yields a good quotient, but also that this quotient admits, in the category of topological spaces, an alternative description obtained by using symplectic tools (i.e the moment map). Some topological properties of the quotient $W/\!\!/(G,\tau)$ can be easier understood if one combines the two approaches. For instance, we will take in the sequel a closer look at the compactness of this quotient and we will show that it can be related to the properness of the moment map μ_{can} obtained by choosing a maximal compact subgroup K of G and a Hermitian inner product on W such that K acts on W by h_W-unitary transformations. Since the topology of $W/\!\!/(G,\tau)$ depends only on the representation and on τ, the result is independent on the choice of the pair (K, h_W).

Proposition 1.9 *(i) Suppose that μ_{can} is proper. Then for any τ the quotient $W/\!\!/(G,\tau)$ is compact.*

(ii) Suppose that μ_{can} is not proper. Then for any τ such that $W/\!\!/(G,\tau) \neq \emptyset$, this quotient is not compact.

Proof. (i) If μ_{can} is proper, then for any τ the topological space

$$W/\!\!/(G,\tau) \simeq \mu_\tau^{-1}(0)/K = \mu_{\mathrm{can}}^{-1}(\tau)/K$$

is obviously compact.

(ii) Suppose now that μ_{can} is not proper. By Lemmas 1.10 and 1.11 (stated below) one first deduces that the quotient

$$W/\!\!/(G,\tau_0) \simeq \mu_{\tau_0}^{-1}(0)/K = \mu_{\mathrm{can}}^{-1}(0)/K$$

is not compact. Moreover, since $W^{ss,\tau_0} = W$, it is a non-empty connected topological space. For τ arbitrary one obtains, using the properties of the good quotients q_τ and q_{τ_0}, a surjective continuous map from $W/\!\!/(G,\tau)$ to $q_{\tau_0}(W^{ss,\tau})$, which is an open subset of $W/\!\!/(G,\tau_0)$ and our statement follows. ■

Lemma 1.10 [8, Lemma 1.1] *The following assertions are equivalent:*
(i) The map $\mu_{\rm can}$ is proper,
(ii) $\mu_{\rm can}^{-1}(0) = \{0\}$,
(iii) $\mu_{\rm can}^{-1}(0)$ is bounded. ■

Lemma 1.11 *Let (W, h_W) be a finite dimensional Hermitian vector space and let K be a compact group which acts on W by unitary transformations. If $X \subset W$ is K-invariant and unbounded, the quotient space X/K cannot be compact.*

Proof. We will denote by $\pi : X \to X/K$ the canonical projection, $x \mapsto K \cdot x$. Since X is not bounded, we can find a sequence $(x_n)_n \subset X$ for which it holds $\|x_n\|_{h_W} \to \infty$. Let's suppose now that the topological space X/K is compact. Then, by passing at a subsequence, we can find an orbit $\pi(y) \in X/K$ of an element $y \in X$ such that $\pi(x_n) \to \pi(y)$. We claim that we can find a sequence $(k_n)_n \subset K$ and a subsequence $(x_{m_n})_n$ such that $k_n \cdot x_{m_n} \to y$, hence, in particular, $\|k_n \cdot x_{m_n}\|_{h_W} \to \|y\|_{h_W}$. On the other hand, since K acts by unitary transformations, we have

$$\|k_n \cdot x_{m_n}\|_{h_W} = \|x_{m_n}\|_{h_W} \to \infty,$$

which yields a contradiction. It remains to construct the sequence k_n indicated above. Let $(B_i)_{i\in\mathbb{N}}$ be a basis of neighborhouds of y such that $B_{i+1} \subset B_i$. Then $(\pi(B_i))_{i\in\mathbb{N}}$ is a fundamental system of neighborhouds of $\pi(y)$. We fix $n \in \mathbb{N}$. There exists i_n such that, for any $i \geq i_n$ one has $\pi(x_i) \in \pi(B_n)$. Set $m_n := \max\{n, i_n\}$; in particular $\pi(x_{m_n}) \in \pi(B_n)$, i.e. we can find k_n such that $k_n \cdot x_{m_n} \in B_n$ and we conclude that $k_n \cdot x_{n_m} \to y$. ■

2 Quiver factorisation problems

2.1 Generalities. Compactness of QFP-quotients

The aim of this section is to discuss a special class of linear problems, called quiver factorisation problems. We begin by giving some definitions concerning quivers and their representations. A *quiver* is a diagram consisting of points and arrows. Formally, a quiver Q is a quartet (Q_0, Q_1, h, t) consisting of the set of *vertices* Q_0, the set of *arrows* Q_1 and the maps $h, t : Q_1 \to Q_0$, which associate to every arrow a the *head*, respectively the *tail* of a. In the sequel we will consider only finite quivers, i.e. the sets Q_0 and Q_1 will be assumed to be finite. If $a_1, \dots, a_n$ are arrows such that $h(a_i) = t(a_{i+1})$ for any $i = 1, \dots, n-1$, they give rise to the *oriented path* $a_n \cdot \ldots \cdot a_1$. If, moreover, one has $h(a_n) = t(a_1)$, this oriented path will be called a *loop*. A vertex v is called a *sink* (respectively a *source*) if all the arrows meeting v are directed to v (respectively to the other vertex). If Q contains no loops, then it has at least one source and one sink.

A *representation* (U, ψ) of a quiver Q (over the field of complex numbers) is given by a family of finite dimensional complex vector spaces $(U_v)_{v \in Q_0}$ and a family of linear maps $(\psi_a)_{a \in Q_1}$, with $\psi_a : U_{t(a)} \to U_{h(a)}$. The *dimension vector* $\alpha \in \mathbb{N}^{Q_0}$ of the representation (U, ψ) is defined by $\alpha_v := \dim_{\mathbb{C}}(U_v)$ for any vertex v. A *morphism* of representations (U, ψ), (U', ψ'), is given by linear maps $(f_a)_{a \in Q_1}$ such that for every a it holds $f_{h(a)} \circ \psi_a = \psi'_a \circ f_{t(a)}$.

For a fixed representation $r = (U, \psi)$ of a quiver $Q = (Q_0, Q_1, h, t)$ we consider the vector space

$$W_r := \bigoplus_{a \in Q_1} \mathrm{Hom}(U_{t(a)}, U_{h(a)}).$$

The group $\prod_{v \in Q_0} \mathrm{GL}(U_v)$ acts on W_r in a natural fashion

$$(g_v)_v \cdot (\psi_a)_a := (g_{h(a)} \circ \psi_a \circ g_{t(a)}^{-1})_a.$$

Since in some situations the action of a smaller symmetry group is needed, for a subset of vertices $S \subset Q_0$ we define

$$G_S := \prod_{v \in S} \mathrm{GL}(U_v) \subset \prod_{v \in Q_0} \mathrm{GL}(U_v).$$

According to the terminology of [7], this is a *quiver factorisation problem* associated to the combinatorial data (Q, S). In the particular case when $S = Q_0$, it is called a *standard quiver factorisation problem.* We notice that the orbits in the space W_r of the symmetry group G_S are in bijective correspondence with the isomorphism classes of representations $(\tilde{U}, \tilde{\psi})$ of Q having the same dimension vector as $r = (U, \psi)$ and such that for any $v \in Q_0 \setminus S$ it holds $\tilde{U}_v = U_v$.

Let now τ be an element of $\mathcal{T}_{G_S} \simeq \mathbb{R}^S$. The categorical quotient $W_r /\!\!/ (G_S, \tau)$ will be called a *QFP-quotient.* According to the results described in the first section, such a quotient admits an alternative symplectic description. If $(h_{U_v})_{v \in Q_0}$ are Hermitian inner products on the vector spaces $(U_v)_{v \in Q_0}$, then we get in a natural fashion an induced Hermitian inner product h_{W_r} on the space W_r. Moreover, the group

$$K_S := \prod_{v \in S} \mathrm{U}(U_v, h_{U_v})$$

acts on W_r by h_{W_r}-unitary transformations. The corresponding canonical moment map $\mu_{\mathrm{can}} : W_r \to \bigoplus_{v \in S} \mathrm{u}(U_v, h_{U_v})$ is given by

$$\mu_{\mathrm{can}}((\psi_a)_a) = \bigoplus_{v \in S} \left(-\frac{i}{2} \sum_{v = h(a)} \psi_a \circ \psi_a^* + \frac{i}{2} \sum_{v = t(a)} \psi_a^* \circ \psi_a \right)$$

and one has a homeomorphism between the symplectic quotient $\mu_\tau^{-1}(0)/K_S$ and the QFP-quotient $W_r /\!\!/ (G_S, \tau)$.

We notice that if (U, ψ) and $(\tilde{U}, \tilde{\psi})$ are representations of Q having the same dimension vector and if we consider the same subset of vertices S and parameters $\tau, \tilde{\tau}$ for which the associated semistable loci coincide, then the corresponding QFP-quotients will be isomorphic. We conclude that, up to isomorphism, a QFP-quotient depends on:

-the combinatorial data (Q, S),

-the dimension vector $\alpha \in \mathbb{N}^{Q_0}$,

-the parameter $\tau \in \mathcal{T}_{G_S}$.

It is therefore a natural problem to understand how the geometry of the QFP-quotients depends on these data. In the sequel we will give some results concerning the relationship between the compactness of these quotients and the following combinatorial property of a pair (Q, S):

Property $(*)$

(i) Q has no loops,

(ii) any oriented path in Q contains at most one vertex lying in $Q_0 \setminus S$.

Lemma 2.1 *Let $(U_i, h_{U_i})_{i=0,\ldots,n}$ be a family of Hermitian vector spaces of positive finite dimension such that $(U_0, h_{U_0}) = (U_n, h_{U_n})$. Then one can find non-zero linear maps $\psi_i : U_i \to U_{i+1}$ $(i = 0, \ldots, n-1)$, such that, putting $\psi_n := \psi_0$, one has for any $i = 1, \ldots, n$*

$$\psi_{i-1} \circ \psi_{i-1}^* = \psi_i^* \circ \psi_i.$$

Proof. For any $i = 0, \ldots, n-1$ we denote by l_i the dimension of the vector space U_i and we fix a h_{U_i}-orthonormal basis $b^i = (b_1^i, \ldots, b_{l_i}^i)$ in U_i; we also put $b^n := b^0$. The linear maps ψ_i $(i = 0, \ldots, n-1)$ acting on the given basis by

$$\psi_i(b_1^i) = b_1^{i+1}, \quad \psi_i(b_j^i) = 0 \ \forall\, j = 2, \ldots, l_i$$

satisfy the required property. ■

Proposition 2.2 *Let $Q = (Q_0, Q_1, h, t)$ be a quiver and $S \subset Q_0$ a fixed subset of vertices. Let $r = (U, \psi)$ be a representation of Q with Hermitian vector spaces $(U_v, h_{U_v})_{v \in Q_0}$.*

(i) If the pair (Q, S) satisfies the Property $(*)$, then the canonical moment map associated to the corresponding quiver factorisation problem is proper.

(ii) If we assume that any vector space U_v has positive dimension, then the converse of (i) is also true.

Proof. (i) We take $\psi \in W_r$ such that $\mu_{\mathrm{can}}(\psi)$ vanishes and we show that ψ is zero. We first notice that, since the pair (Q, S) verifies the Property $(*)$, the quiver Q has at least one sink or one source belonging to S. If Q has a source $v \in S$, then the projection of the canonical moment map on $\mathrm{u}(U_v, h_{U_v})$ is $\left(\frac{i}{2} \sum_{v=t(a)} \psi_a^* \circ \psi_a\right)$, whereas if Q has a sink $v \in S$, this projection is $\left(-\frac{i}{2} \sum_{v=h(a)} \psi_a \circ \psi_a^*\right)$. In both cases the projection on $\mathrm{u}(U_v, h_{U_v})$ vanishes if and only if $\psi_a = 0$ for any a which meets v. We consider now the quiver Q' obtained from Q by eliminating v and those arrows which meet v. Then the pair $(Q', S \setminus \{v\})$ also verifies the Property $(*)$. Applying the same reasoning, after a finite number of steps we eliminate all the vertices lying in S. Since any arrow meets at least one vertex in S, we conclude that $\psi_a = 0$ for any $a \in Q_1$, that is $\psi = 0$.

(ii) Suppose first that Q contains a loop with vertices $v_1, \ldots, v_n = v_0$ and with arrows $a_0, a_1, \ldots, a_{n-1}, a_n = a_0$ such that for any $i = 0, \ldots, n-1$ one has $t(a_i) = v_i, h(a_i) = v_{i+1}$. We construct non-zero linear maps $\psi_{a_i} : U_{v_i} \to U_{v_{i+1}}$ as in Lemma 2.1. For any $a \in Q_1 \setminus \{a_0, \ldots, a_{n-1}\}$ we set $\psi_a := 0$, obtaining in this way a non-zero element $\psi \in W_r$. We claim that it holds $\mu_{\text{can}}(\psi) = 0$. Indeed, if $v = v_i$ $(i = 1, \ldots, n)$ one has

$$\sum_{v=h(a)} \psi_a \circ \psi_a^* - \sum_{v=t(a)} \psi_a^* \circ \psi_a = \psi_{a_{i-1}} \circ \psi_{a_{i-1}}^* - \psi_{a_i}^* \circ \psi_{a_i} = 0$$

and for all other v this sum vanishes, since the corresponding linear maps are zero. Hence, by Lemma 1.10, it follows that μ_{can} is not a proper map.

Let's now assume that Q contains an oriented path with vertices $v_1, \ldots, v_n$ such that v_1 and v_n are in $Q_0 \setminus S$ and with arrows $a_1, \ldots, a_{n-1}$ such that $t(a_i) = v_i, h(a_i) = v_{i+1}$ $(i = 1, \ldots, n-1)$. Applying again Lemma 2.1, we construct non-zero linear maps $\psi_{a_i} : U_{v_i} \to U_{v_{i+1}}$ for $i = 1, \ldots, n-1$. Setting $\psi_a := 0$ for any $a \in Q_1 \setminus \{a_0, \ldots, a_{n-1}\}$, we get a non-zero element $\psi \in W_r$. Then:

$$\sum_{v=h(a)} \psi_a \circ \psi_a^* - \sum_{v=t(a)} \psi_a^* \circ \psi_a = \begin{cases} -\psi_{a_1}^* \circ \psi_{a_1}, & \text{if } v = v_1, \\ \psi_{a_{n-1}} \circ \psi_{a_{n-1}}^*, & \text{if } v = v_n, \\ 0, & \text{otherwise.} \end{cases}$$

But v_1 and v_n are not elements of S and hence $\mu_{\text{can}}(\psi) = 0$. Again by Lemma 1.10, it follows that μ_{can} cannot be a proper map. ■

Using Propositions 2.2 and 1.9 we deduce at once:

Theorem 2.3 *(i) Any QFP-quotient associated to a pair (Q, S) satisfying the Property $(*)$ is compact.*

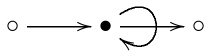
(ii) If a non-empty QFP-quotient associated to a pair (Q, S) and corresponding to a dimension vector $\alpha \in \mathbb{N}_+^{Q_0}$ is compact, then the pair (Q, S) must fulfil the Property $(*)$. ■

2.2 Compactification of QFP-quotients

In the remaining of the paper we will focus our attention to the particular case concerning linear systems. As in section 1.2.1, we consider complex vector spaces of positive finite dimension V, V_1, V_2 and the natural action of the group $G := \text{GL}(V)$ on the space

$$W := \text{End}_{\mathbb{C}}(V) \oplus \text{Hom}_{\mathbb{C}}(V_1, V) \oplus \text{Hom}_{\mathbb{C}}(V, V_2).$$

This linear problem is, in fact, a quiver factorisation problem associated to the quiver Q



and to the set S containing just one vertex, namely $\bullet$. The pair (Q,S) does not verify the Property $(*)$ and hence, by Theorem 2.3, the corresponding quotients are not compact. This yields an alternative proof to the fact that the moduli space of accessible linear systems is not compact, as shown for instance in [9]. Our main aim is to construct natural compactifications for these spaces which are themselves QFP-quotients associated to a suitable pair $(\widetilde{Q},\widetilde{S})$ and will be achieved in Theorem 2.7. Our construction is motivated by Helmke's compactification of the space of accessible linear systems by using generalized linear systems, which is described in [3] and [4]. Hence, in the paper [3] one considers the group

$$\widetilde{G} := \mathrm{GL}(V) \times \mathrm{GL}(V),$$

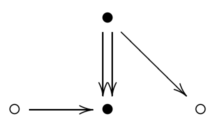
which acts on the space

$$\widetilde{W} := \mathrm{Hom}_{\mathbb{C}}(V,V)^{\oplus 2} \oplus \mathrm{Hom}_{\mathbb{C}}(V_1,V) \oplus \mathrm{Hom}_{\mathbb{C}}(V,V_2)$$

in a natural fashion

$$(g,h)\cdot(\psi,\varphi,\varphi_1,\varphi_2) := (g\circ\psi\circ h^{-1}, g\circ\varphi\circ h^{-1}, g\circ\varphi_1, \varphi_2\circ h^{-1}).$$

The study of the $\widetilde{G}$-action on $\widetilde{W}$ can also be represented as a quiver factorisation problem associated to the quiver $\widetilde{Q}$



and for which the set $\widetilde{S}$ contains two vertices, both drawn with $\bullet$. Since the pair $(\widetilde{Q},\widetilde{S})$ verifies the Property $(*)$, it follows by Theorem 2.3 that the associated QFP-quotients are compact. In remains to get a natural relationship between a QFP-quotient corresponding to (Q,S) and a suitable QFP-quotient associated to $(\widetilde{Q},\widetilde{S})$. We first notice that one has a natural map

$$i : W \to \widetilde{W}, \quad i(\varphi,\varphi_1,\varphi_2) := (\mathrm{id},\varphi,\varphi_1,\varphi_2).$$

On the the other hand, the geometry of QFP-quotients also depends on a parameter τ, i.e. on a stability concept. In section 1.2.1, we explained that there are three stability concepts associated to this first quiver factorisation problem, which were denoted by $\tau_\varepsilon \in \mathcal{T}_G$ ($\varepsilon \in \{0,\pm 1\}$). Our aim is to show that the images of the corresponding semistable loci under the map i are also included in some sets of semistable points, with respect to suitable elements in $\mathcal{T}_{\widetilde{G}}$. More precisely, we consider $\widetilde{\tau}_0, \widetilde{\tau}_1, \widetilde{\tau}_{-1} \in \mathcal{T}_{\widetilde{G}}$ given by

$$\langle \widetilde{\tau}_\varepsilon, (s',s'')\rangle := \begin{cases} i(\mathrm{Tr}(s') - \mathrm{Tr}(s'')), & \text{if } \varepsilon = 0 \\ i((r+1)\mathrm{Tr}(s') - r\mathrm{Tr}(s'')), & \text{if } \varepsilon = 1 \\ i(-r\mathrm{Tr}(s') + (r+1)\mathrm{Tr}(s'')), & \text{if } \varepsilon = -1 \end{cases}$$

and we claim that the following result holds

Proposition 2.4 *Take any $\varepsilon \in \{0, \pm 1\}$. A triple $(\varphi, \varphi_1, \varphi_2) \in W$ is τ_ε-(semi)stable if and only if $i(\varphi, \varphi_1, \varphi_2) \in \widetilde{W}$ is $\widetilde{\tau}_\varepsilon$-(semi)stable.*

In order to prove the proposition, we need some preparations. Let (s', s'') be an element of $H(\widetilde{G})$ (this space can be identified with $H(\mathrm{GL}(V)) \times H(\mathrm{GL}(V))$). Let $\lambda'_1 < \lambda'_2 < \ldots < \lambda'_a$, respectively $\lambda''_1 < \lambda''_2 < \ldots < \lambda''_b$ be the eigenvalues of s', respectively s'' and let

$$\{0\} = V'_0 \subset V'_1 \subset \ldots \subset V'_a, \qquad \{0\} = V''_0 \subset V''_1 \subset \ldots \subset V''_b$$

be the corresponding filtrations. We denote by $d'_i := \dim_{\mathbb{C}}(V'_i)$ $(i = 1, \ldots, a)$, respectively $d''_i := \dim_{\mathbb{C}}(V''_i)$ $(i = 1, \ldots, b)$ the dimensions of the vector spaces arising in the filtrations $\mathcal{F}_{s'}$, respectively $\mathcal{F}_{s''}$. For any $k = 1, \ldots, b$ we put

$$j(k) := \begin{cases} \max\{l \mid \lambda'_l \leq \lambda''_k\}, & \text{if } \{l \mid \lambda'_l \leq \lambda''_k\} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}.$$

We will need the following

Lemma 2.5 *(i) Let $\psi \in (\mathrm{Hom}_{\mathbb{C}}(V, V))^{\leq 0}(s', s'')$. Then for any real number m one has*

$$\psi\Big(\bigoplus_{\lambda''_k \leq m} V(\lambda''_k)\Big) \subset \bigoplus_{\lambda'_j \leq m} V(\lambda'_j).$$

(ii) Suppose that $\mathrm{id} \in (\mathrm{Hom}_{\mathbb{C}}(V, V))^{\leq 0}(s', s'')$. *Then for any $k = 1, \ldots, b$ it holds $d''_k \leq d'_{j(k)}$. Moreover, if s' and s'' have the same eigenvalues with the same multiplicities, then s' and s'' have the same associated filtrations, i.e. $\mathcal{F}_{s'} = \mathcal{F}_{s''}$.*

Proof. (i) Since the eigenvalues of (s', s'') are $\{\lambda'_p - \lambda''_q | p, q = 1, \ldots, r\}$, we can decompose any $\psi \in \mathrm{Hom}_{\mathbb{C}}(V, V)$ as

$$\psi = \sum_{p,q} \psi_{\lambda'_p - \lambda''_q},$$

where for any p, q it holds

$$s' \circ \psi_{\lambda'_p - \lambda''_q} - \psi_{\lambda'_p - \lambda''_q} \circ s'' = (\lambda'_p - \lambda''_q)\psi_{\lambda'_p - \lambda''_q}.$$

Let now $v \in V$ be an eigenvector of s'' corresponding to an eigenvalue λ''. Then for any component of ψ, the vector $\psi_{\lambda'_p - \lambda''_q}(v)$ is an eigenvector of s', corresponding to the eigenvalue $(\lambda'' + \lambda'_p - \lambda''_q)$. The assumption that ψ is an element of $(\mathrm{Hom}_{\mathbb{C}}(V, V))^{\leq 0}(s', s'')$ means that in the decomposition of ψ occur only components corresponding to nonpositive eigenvalues. In particular, we deduce that for an eigenvector $v \in V(\lambda'')$, we have $\psi(v) \in \oplus_{\lambda'_j \leq \lambda''} V(\lambda'_j)$ and this yields our assertion.

(ii) Using (i), we deduce that for any k it holds

$$V''_k = \mathrm{id}\Big(\bigoplus_{\lambda''_j \le \lambda''_k} V(\lambda''_j)\Big) \subset \bigoplus_{\lambda'_j \le \lambda''_k} V(\lambda'_j) = \bigoplus_{\lambda'_j \le \lambda'_{j(k)}} V(\lambda'_j) = V'_{j(k)}$$

and hence $d''_k \le d'_{j(k)}$. We notice that the assertion remains true if we replace id with $\psi \in \mathrm{Hom}_{\mathbb{C}}(V,V)$ such that $\det\psi \neq 0$.

Let's now suppose that $a = b$, that for any $k = 1,\dots,a$ we have $\lambda'_k = \lambda''_k$ and that the corresponding multiplicities are equal. In particular we deduce that for any k one has $j(k) = k$ and $\dim(V'_k) = \dim(V''_k)$. Since $V''_k \subset V'_{j(k)} = V'_k$, we conclude that the two filtrations coincide. ■

Remark 2.6 For any $l \in \{1,\dots,r\}$ we put

$$\lambda'(l) := \lambda'_{\min\{m \,|\, d'_m \ge l\}}, \quad \lambda''(l) := \lambda''_{\min\{m \,|\, d''_m \ge l\}}.$$

The sequence of real numbers $(\lambda'(1),\dots,\lambda'(r))$ is nothing else but the sequence of eigenvalues of s' in which each eigenvalue occurs as many times as its multiplicity is; an analogous statement holds for the sequence $(\lambda''(1),\dots,\lambda''(r))$. Then the condition that for any k one has $d''_k \le d'_{j(k)}$ (stated in Lemma 2.5 ii)) is equivalent to the condition that for any $l = 1,\dots,r$ it holds $\lambda'(l) \le \lambda''(l)$. Moreover, it holds $d''_k = d'_{j(k)}$ if and only if $\lambda''(d''_k) < \lambda'(d''_k + 1)$.

Proof of Proposition 2.4.

We first notice that for any $\varepsilon \in \{0,\pm1\}$ it holds: if $(\varphi,\varphi_1,\varphi_2) \in W^{\le 0}(s)$ and $\langle \tau_\varepsilon, is\rangle < (\le) 0$, then $(\mathrm{id},\varphi,\varphi_1,\varphi_2) \in \widetilde{W}^{\le 0}(s,s)$ and $\langle \widetilde{\tau}_\varepsilon, i(s,s)\rangle < (\le) 0$. This means that if $(\varphi,\varphi_1,\varphi_2)$ is not τ_ε-(semi)stable, then $(\mathrm{id},\varphi,\varphi_1,\varphi_2)$ cannot be $\widetilde{\tau}_\varepsilon$-(semi)stable.

We now prove the converse assertion: if $(\varphi,\varphi_1,\varphi_2)$ is τ_ε-(semi)stable, then $(\mathrm{id},\varphi,\varphi_1,\varphi_2)$ is $\widetilde{\tau}_\varepsilon$-(semi)stable.

• **The case** $\varepsilon = 0$. Our first claim is that every element $(\mathrm{id},\varphi,\varphi_1,\varphi_2)$ is $\widetilde{\tau}_0$-semistable. Indeed, if (s',s'') is a pair such that $(\mathrm{id},\varphi,\varphi_1,\varphi_2) \in \widetilde{W}^{\le 0}(s',s'')$, then, by Remark 2.6, we deduce that for any $i = 1,\dots,r$ it holds $\lambda'(i) \le \lambda''(i)$ and hence

$$\langle \widetilde{\tau}_0, i(s',s'')\rangle = -(\mathrm{Tr}(s') - \mathrm{Tr}(s'')) = \sum_{i=1}^{r} (-\lambda'(i) + \lambda''(i)) \ge 0.$$

Suppose now that $(\mathrm{id},\varphi,\varphi_1,\varphi_2)$ is not $\widetilde{\tau}_0$-stable. This means that we have (s',s'') such that $(\mathrm{id},\varphi,\varphi_1,\varphi_2) \in \widetilde{W}^{\le 0}(s',s'')$ and that it holds

$$\sum_{i=1}^{r} (-\lambda'(i) + \lambda''(i)) = \langle \widetilde{\tau}_0, i(s',s'')\rangle \le 0.$$

On the other hand, again by Remark 2.6, we have $\lambda'(i) \le \lambda''(i)$ for any $i = 1,\dots,r$. We conclude that $\lambda'(i) = \lambda''(i)$ for any i, that is that s' and s'' have the same eigenvalues with the same multiplicities. By Lemma 2.5 ii), we deduce that the filtrations $\mathcal{F}_{s'}$ and $\mathcal{F}_{s''}$ coincide. In particular, it follows that $\mathcal{F}_{s'}$ is φ-invariant, $\mathrm{im}(\varphi_1) \subset \mathcal{F}_{s'}^{\leq 0}$ and that $\ker(\varphi_2) \supset \mathcal{F}_{s''}^{<0} = \mathcal{F}_{s'}^{<0}$, that is one has $(\varphi, \varphi_1, \varphi_2) \in W^{\leq 0}(s')$. Since $\langle \tau_0, is' \rangle = 0$, we conclude that $(\varphi, \varphi_1, \varphi_2)$ is not τ_0-stable.

• **The case** $\varepsilon = 1$. We will prove that if $(\mathrm{id}, \varphi, \varphi_1, \varphi_2)$ is not $\widetilde{\tau}_1$-stable, then $(\varphi, \varphi_1, \varphi_2)$ is not τ_1-semistable. Let $(s', s'') \notin \widetilde{\mathfrak{g}}_0$ be a pair such that $(\mathrm{id}, \varphi, \varphi_1, \varphi_2) \in \widetilde{W}^{\leq 0}(s', s'')$ and such that $\langle \widetilde{\tau}_1, i(s', s'') \rangle \leq 0$. This inequality can be written as

$$\sum_{i=1}^{r} \lambda'(i) + r(\sum_{i=1}^{r} (\lambda'(i) - \lambda''(i))) \geq 0. \tag{2}$$

We claim that:

(a) s' has at least one positive eigenvalue.

(b) let q be such that $\mathcal{F}_{s'}^{\leq 0} = V'_q$; in particular, if $q \neq 0$, λ'_q is the greatest nonpositive eigenvalue of s'. There exists t such that $q \leq j(t) < r$ and such that $d''_t = d'_{j(t)}$.

Let's suppose that we proved the assertions (a) and (b). We showed in the proof of Lemma 2.5 that $V''_t \subset V'_{j(t)}$ and since their dimensions are equal ($d''_t = d'_{j(t)}$), it follows that these subspaces of V coincide. Moreover, by Lemma 2.5 i) we have

$$\varphi(V'_{j(t)}) = \varphi(V''_t) = \varphi\Big(\bigoplus_{\lambda''_i \leq \lambda''_t} V(\lambda''_i) \Big) \subset \bigoplus_{\lambda'_i \leq \lambda''_t} V(\lambda'_i) = \bigoplus_{\lambda'_i \leq \lambda'_{j(t)}} V(\lambda'_i) = V'_{j(t)}$$

and it holds

$$\mathrm{im}(\varphi_1) \subset \mathcal{F}_{s'}^{\leq 0} = V'_q \subset V'_{j(t)}.$$

We now consider s with eigenvalues 0 and 1 and with associated filtration $\mathcal{F}_s$

$$\{0\} \subset V'_{j(t)} \subsetneq V.$$

We already proved that $\mathcal{F}_s$ is φ-invariant and that $\mathrm{im}(\varphi_1) \subset \mathcal{F}_s^{\leq 0}$; we obviously have $\ker(\varphi_2) \supset \mathcal{F}_s^{<0} = \{0\}$ and we conclude that $(\varphi, \varphi_1, \varphi_2) \in W^{\leq 0}(s)$. On the other hand, one has

$$\langle \tau_1, is \rangle = -\mathrm{Tr}(s) = -\dim(V/V'_{j(t)}) < 0$$

and we conclude that $(\varphi, \varphi_1, \varphi_2)$ is not τ_1-stable.

We now prove the assertions (a) and (b). If all the eigenvalues of s' were nonpositive, then we would have $\lambda'(i) \leq 0$ for any i. On the other hand, by Remark 2.6, for any i we have $\lambda'(i) - \lambda''(i) \leq 0$. Using the inequality (2), we deduce that all the numbers $\lambda'(i), \lambda''(i)$ must be zero, which contradicts our assumption that $(s', s'') \notin \widetilde{\mathfrak{g}}_0$.

By Remark 2.6, it is enough to show that there exists $d'_q \leq l < r$ such that $\lambda''(l) < \lambda'(l+1)$ (if $d'_q = 0$, we set $\lambda'(0) = \lambda''(0) := 0$). Let's suppose that this property is not fulfiled, that is for any $d'_q \leq l \leq r-1$ we have $\lambda''(l) \geq \lambda'(l+1)$. By multiplicating each inequality with a suitable nonnegative number (more precisely me multiply the inequality between $\lambda''(l)$ and $\lambda'(l+1)$ with $r-l+1$) and by using the fact that $\lambda'(d'_q) \leq 0 < \lambda'(r) \leq \lambda''(r)$, we obtain the following inequality

$$\sum_{l=d'_q+1}^{r} \lambda'(l) + \sum_{l=d'_q}^{r} (r-l+1)(\lambda'(l) - \lambda''(l)) < 0.$$

Since $\lambda'(1), \ldots, \lambda'(d'_q), \lambda'(1) - \lambda''(1), \ldots, \lambda'(r) - \lambda''(r)$ are nonpositive, this would imply

$$\sum_{i=1}^{r} \lambda'(i) + r \sum_{i=1}^{r} (\lambda'(i) - \lambda''(i)) < 0,$$

and this contradicts the relation (2).

• **The case** $\varepsilon = -1$. The proof of this case is analogous to the proof of the second one. Using similar arguments it can be shown that s'' has at least one negative eigenvalue. Let q such that $\mathcal{F}^{<0}_{s''} = V''_q$. One proves that there exists $0 < t \leq q$ such that $d''_t = d'_{j(t)}$. Then, considering s with eigenvalues -1 and 0 and with associated filtration

$$0 \subsetneq V''_t \subset V,$$

one gets the desired conclusion. ■

We are now able to prove the main result of this part:

Theorem 2.7 *For any $\tau \in \mathcal{T}_G$, there exists an element $\widetilde{\tau} \in \mathcal{T}_{\widetilde{G}}$, such that the QFP-quotient $\widetilde{W} /\!\!/ (\widetilde{G}, \widetilde{\tau})$ is a compactification of the QFP-quotient $W /\!\!/ (G, \tau)$.*

Proof. We fix an arbitrary $\varepsilon \in \{0, \pm 1\}$. Then, according to Proposition 2.4, the image of the τ_ε-(semi)stable locus $i(W^{(s)s, \tau_\varepsilon})$ is included in the $\widetilde{\tau}_\varepsilon$-(semi)stable locus $\widetilde{W}^{(s)s, \widetilde{\tau}_\varepsilon}$. In particular, this yields an induced map between the corresponding categorical quotients, denoted by ι_ε and one gets a commutative diagram

$$\begin{array}{ccc} W^{ss,\tau_\varepsilon} & \xrightarrow{\ i\ } & \widetilde{W}^{ss,\widetilde{\tau}_\varepsilon} \\ \downarrow & & \downarrow \\ W /\!\!/ (G, \tau_\varepsilon) & \xrightarrow{\ \iota_\varepsilon\ } & \widetilde{W} /\!\!/ (\widetilde{G}, \widetilde{\tau}_\varepsilon) \end{array}$$

It is easy to check that ι_ε is an injective map and, by Proposition 2.4, its image is

$$\mathrm{im}(\iota_\varepsilon) = \{[\psi, \varphi, \varphi_1, \varphi_2] \in \widetilde{W} /\!\!/ (\widetilde{G}, \widetilde{\tau}_\varepsilon) \mid \det \psi \neq 0\},$$

which is an open subset of $\widetilde{W} /\!\!/ (\widetilde{G}, \widetilde{\tau}_\varepsilon)$. We claim that this set is dense in $\widetilde{W} /\!\!/ (\widetilde{G}, \widetilde{\tau}_\varepsilon)$. Indeed, if one considers an arbitrary $\widetilde{\tau}_\varepsilon$-semistable element in $\widetilde{W}$, say $x = (\psi, \varphi, \varphi_1, \varphi_2)$, then one can find a sequence $(x_n)_n$ of $\widetilde{\tau}_\varepsilon$-semistable elements converging to x and such that, writing $x_n = (\psi_n, \varphi_n, \varphi_{1,n}, \varphi_{2,n})$, one has $\det(\psi_n) \neq 0$. Then the sequence $([x_n])_n$ is included in $\mathrm{im}(\iota_\varepsilon)$ and converges to $[x]$. We conclude that $\widetilde{W}/\!\!/(\widetilde{G}, \widetilde{\tau}_\varepsilon)$ is a compactification of $W/\!\!/(G, \tau_\varepsilon)$, getting the desired result. ■

The following question, which would yield a generalization to Theorem 2.7, remains open: given a pair (Q, S) which does not fulfil the Property $(*)$, is it possible to find $(\widetilde{Q}, \widetilde{S})$ verifying $(*)$ such that any QFP-quotient corresponding to (Q, S) admits a compactification which is a QFP-quotient associated to $(\widetilde{Q}, \widetilde{S})$?

Acknowledgements. This work was supported by a post-doctoral research fellowship at the Mathematical Institute of Leiden University. I am grateful to the members of the staff for their hospitality and for the stimulating mathematical environment. I would also like to thank Dr. M. Lübke for very useful discussions.

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