SOME REMARKS ON A PROOF OF GEOMETRICAL PALEY–WIENER THEOREMS FOR THE DUNKL TRANSFORM

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Abstract. We argue that a proof of the geometrical form of the Paley–Wiener theorems for the Dunkl transform in the literature is not correct.

1. Introduction

In [2] a proof of the geometrical form of the Paley–Wiener theorems for the Dunkl transform is presented. In our opinion, however, this proof is not correct. We have informed the author of the details underlying this opinion in November, 2003, but at the time of writing he has not disputed our remarks, or agreed with them, or given an alternative correct proof.

The material which was communicated to the author is presented below. It is our opinion that at this moment the geometrical forms of the Paley–Wiener theorems for the Dunkl transform are still unproven.

2. Arguments

In [2] the geometric form of the Paley-Wiener theorem for the Dunkl transform is stated for functions and for distributions, as Theorems 6.2 and 6.3, respectively. The crucial ingredient in the proof of these results is Proposition 6.3. Our arguments concern the proof of this proposition. In formulating them, we will use the notation and definitions of [2].

In the proof of Proposition 6.3, a W-invariant compact convex subset E of $\mathbb{R}^d$ is considered. For $x \in E$ fixed, the function f_x on $\mathbb{R}^d$ is defined as

$$f_x(y) = \frac{e^{-i(x,y)}}{(1+\|y\|^2)^p} \quad (y \in \mathbb{R}^d), \tag{2.1}$$

where p is an integer such that $p \geq \gamma + d/2 + 1$. Since the constant γ is assumed to be strictly positive in line 3 on page 29, we see that $p \geq 1$.

The function F_x is defined on $\mathbb{R}^d$ in equation (69) as essentially the inverse Dunkl transform of f_x, namely

$$F_x(t) = \int_{\mathbb{R}^d} f_x(y) K(iy,t)\omega_k(y)dy \quad (t \in \mathbb{R}^d).$$

Following this definition, it is observed that F_x is continuous.

After a computation involving Riemann sums and contour integration, it is then concluded on line -3 on page 31 that F_x has support in the set E. We thus see that F_x is a compactly supported continuous function.

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The last line on page 31 is an application of the inversion theorem for the Dunkl transform, reconstructing f_x from F_x:

$$f_x(y) = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} F_x(t) K(-iy,t)\omega_k(t)\,dt \quad (y \in \mathbb{R}^d). \tag{2.2}$$

Now, as with the ordinary Fourier transform, the fact that F_x is a continuous function with compact support implies [1, Part 3 of Lemma 4.4] that its Dunkl transform has an entire extension to $\mathbb{C}^d$, i.e., that f_x has an entire extension to $\mathbb{C}^d$. But it follows from (2.1), written as

$$f_x(y) = \frac{e^{-i(x,y)}}{(1+(y,y))^p},$$

where $(\,.\,,\,.\,)$ is the holomorphic standard bilinear form on $\mathbb{C}^d$, that f_x has no such extension from $\mathbb{R}^d$ to $\mathbb{C}^d$, since it would have a pole along the divisor $\{y \in \mathbb{C}^d \mid (y,y) = -1\}$, as a consequence of the fact that $p \geq 1$. This is a contradiction.

Another contradiction occurs in the case where $E = \{0\}$ and $x = 0 \in E$. Then the fact that the continuous function F_0 has support in E implies that $F_0 = 0$. But in that case (2.2) implies that $f_0 = 0$, which contradicts (2.1).

The above arguments are not based on the details of the computation with Riemann sums and contour integration, but they are concerned with the impossibility of the support of F_x being contained in E. These arguments therefore show not only that this computation contains a technical inaccuracy, but they also show that the technique of this computation can not be corrected, since the conclusion which the computation is supposed to yield does not hold.

References

[1] M.F.E. de Jeu, *The Dunkl transform*, Invent. math. **113** (1993), 147-162.

[2] K. Trimèche, *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators*, Integral Transforms Spec. Funct. **13** (2002), 17–38.

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