ELEMENTARY PROOFS OF PALEY–WIENER THEOREMS FOR THE DUNKL TRANSFORM ON THE REAL LINE

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Abstract. We give an elementary proof of the Paley–Wiener theorem for smooth functions for the Dunkl transforms on the real line, establish a similar theorem for L^2-functions and prove identities in the spirit of Bang for L^p-functions. The proofs seem to be new also in the special case of the Fourier transform.

1. Introduction and overview

The Paley–Wiener theorem for the Dunkl transform D_k with multiplicity k (where $\operatorname{Re} k \geq 0$) on the real line states that a smooth function f has support in the bounded interval $[-R, R]$ if, and only if, its transform $D_k f$ belongs to the space of rapidly decreasing entire functions f of exponential type R. Various proofs of this result are known, all of which use explicit formulas available in this one-dimensional setting (see Remark 6 for more details). In this paper, however, we present an alternative proof which does not use such explicit expressions, being based almost solely on the formal properties of the transform. Along the same lines, we also obtain a Paley–Wiener theorem for L^2-functions for $k \geq 0$. The case $k = 0$ specializes to the Fourier transform, and to our knowledge the proofs of both Paley–Wiener theorems are new even in this case.

In addition, we establish two identities in the spirit of Bang [4, Theorem 1]. These results could be called real Paley–Wiener theorems (although terminology is not yet well-established), since they relate certain growth rates of a function *on the real line* to the support of its transform. The approach at this point is inspired by similar techniques in [1, 2, 3]. Our results in this direction partially overlap with [6, 7], but the new proofs are considerably simpler, as they are again based almost solely on the formal properties of the transform. We will comment on this in more detail later on, as these results will have been established. For $k = 0$, one retrieves Bang's result; we feel that the present method of proof, which, e.g., does not use the Paley–Wiener theorem for smooth functions, but rather implies it, is then more direct than that in [4].

The rather unspecific and formal structure of the proofs suggests that the methods can perhaps be put to good use for other integral transform with a symmetric kernel, both for the Paley–Wiener theorems and the equalities in the spirit of Bang (cf. Remark 6). The structure of the proof is also such that, if certain combinatorial problems can be surmounted, a proof of the Paley–Wiener theorem for the Dunkl transform for invariant balanced compact convex sets in arbitrary dimension might be possible. This would be further evidence for the validity of this theorem for compact convex sets (cf. [13, Conjecture 4.1]), but at the time of writing this higher dimensional result has not been established.

This paper is organized as follows. In Section 2 the necessary notations and previous results are given. Section 3 contains the Paley–Wiener theorem for smooth functions and can—perhaps—serve as a model for a proof of such a theorem in other contexts. The rest of the paper is independent of this section. Section 4 is concerned with the real Paley–Wiener theorem in the L^p-case and the L^2-case is settled in Section 5.

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2. Dunkl operators and the Dunkl transform on $\mathbb{R}$

The Dunkl operators and the Dunkl transform were introduced for arbitrary root systems by Dunkl [8, 9, 10]. In this section we recall some basic properties for the one-dimensional case of A_1, referring to [17, Section 1 and 2] for a comprehensive overview and to [8, 9, 10, 12, 14, 16] for details. We suppress the various explicit formulas which are known in this one-dimensional context (as these are not necessary for the proofs), thus emphasizing the basic structure of the problem which might lead to generalizations to the case of arbitrary root systems.

Let $k \in \mathbb{C}$, and consider the Dunkl operator T_k

$$T_k f(x) = f'(x) + k\frac{f(x) - f(-x)}{x} \qquad (f \in C^\infty(\mathbb{R}),\ x \in \mathbb{R}).$$

Rewriting this as

$$T_k f(x) = f'(x) + k\int_{-1}^{1} f'(tx)\,dt, \tag{1}$$

it follows that T_k maps $C^\infty(\mathbb{R})$, $C_c^\infty(\mathbb{R})$ and the Schwartz space $\mathcal{S}(\mathbb{R})$ into themselves.

If $\operatorname{Re} k \geq 0$, as we will assume for the remainder of this section, then, for each $\lambda \in \mathbb{C}$, there exists a unique holomorphic solution $\psi_\lambda^k : \mathbb{C} \to \mathbb{C}$ of the differential-reflection problem

$$\begin{cases} T_k f = i\lambda f, \\ f(0) = 1. \end{cases} \tag{2}$$

The map $(z, \lambda) \to \psi_\lambda^k(z)$ is entire on $\mathbb{C}^2$, and we have the estimate

$$\left|\psi_\lambda^k(z)\right| \leq e^{|\operatorname{Im} \lambda z|} \qquad (\lambda, z \in \mathbb{C}). \tag{3}$$

In view of (3) the Dunkl transform $D_k f$ of $f \in L^1(\mathbb{R}, |w_k(x)|dx)$, where the complex-valued weight function w_k is given by $w_k(x) = |x|^{2k}$, is meaningfully defined by

$$D_k f(\lambda) = \frac{1}{c_k}\int_{\mathbb{R}} f(x)\psi_{-\lambda}^k(x) w_k(x)dx \qquad (\lambda \in \mathbb{R}), \tag{4}$$

where

$$c_k = \int_{\mathbb{R}} e^{-\frac{|x|^2}{2}} w_k(x)\,dx \neq 0.$$

We note that D_0 is the Fourier transform on $\mathbb{R}$. The Dunkl transform is a topological isomorphism of $\mathcal{S}(\mathbb{R})$ onto itself, the inverse transform D_k^{-1} being given by

$$D_k^{-1} f(x) = \frac{1}{c_k}\int_{\mathbb{R}} f(\lambda)\psi_\lambda^k(x) w_k(\lambda)d\lambda = D_k f(-x) \qquad (f \in \mathcal{S}(\mathbb{R}),\ x \in \mathbb{R}),$$

The operator T_k is anti-symmetric with respect to the weight function w_k, i.e.,

$$\langle T_k f, g\rangle_k = -\langle f, T_k g\rangle_k, \tag{5}$$

for $f \in \mathcal{S}(\mathbb{R})$ and $g \in C^\infty(\mathbb{R})$ such that both g and $T_k g$ are of at most polynomial growth. Here $\langle f, g\rangle_k$ is defined by

$$\langle f, g\rangle_k = \int_{\mathbb{R}} f(x)g(x)w_k(x)dx,$$

for functions f and g such that $fg \in L^1(\mathbb{R}, |w_k(x)|dx)$. In particular, (5) yields the intertwining identity

$$D_k(T_k f)(\lambda) = i\lambda(D_k f)(\lambda) \quad (f \in \mathcal{S}(\mathbb{R}),\ \lambda \in \mathbb{R}).$$

Furthermore, for $\lambda, z, s \in \mathbb{C}$ the symmetry properties $\psi_\lambda^k(z) = \psi_z^k(\lambda)$, and $\psi_{s\lambda}^k(z) = \psi_\lambda^k(sz)$ are valid. Using the first of these, an application of Fubini gives

$$\langle D_k f, g\rangle_k = \langle f, D_k g\rangle_k \quad (f, g \in L^1(\mathbb{R}, |w_k(x)|dx)). \tag{6}$$

If $k \geq 0$, the Plancherel theorem states that D_k preserves the weighted two-norm on $L^1(\mathbb{R}, w_k(x)dx) \cap L^2(\mathbb{R}, w_k(x)dx)$ and extends to a unitary operator on $L^2(\mathbb{R}, w_k(x)dx)$.

3. Paley–Wiener theorem for smooth functions

The method of proof in this section is inspired by results of Bang [4]. To be more specific, if $k \geq 0$, and f is a rapidly decreasing entire function of exponential type R, we will establish that

$$\sup\{|\lambda| \,:\, \lambda \in \operatorname{supp} D_k f\} \leq \liminf_{n\to\infty} \|T_k^n f\|_\infty^{1/n} \leq \limsup_{n\to\infty} \|T_k^n f\|_\infty^{1/n} \leq R, \tag{7}$$

after which the proof of the Paley–Wiener theorem for smooth functions is a mere formality. Starting towards the third of these inequalities, we first use (1) to gain control over repeated Dunkl derivatives.

Lemma 1. *Let $k \in \mathbb{C}$, $f \in C^\infty(\mathbb{R})$ and $n \in \mathbb{N}$. Then*

$$T_k^n f(x) = (T_k^{n-1}(f'))(x) + k \int_{-1}^{1} t^{n-1} (T_k^{n-1}(f'))(tx)dt \qquad (x \in \mathbb{R}).$$

Proof. For $g \in C^\infty(\mathbb{R})$ and $m \in \mathbb{N} \cup \{0\}$, let $I_{g,m} \in C^\infty(\mathbb{R})$ be defined by

$$I_{g,m}(x) = \int_{-1}^{1} t^m g(tx)\, dt \quad (x \in \mathbb{R}).$$

Using (1), we then find

$$\begin{aligned}(T_k I_{g,m})(x) &= \int_{-1}^{1} t_1^{m+1} g'(t_1 x)\, dt_1 + k \int_{-1}^{1}\int_{-1}^{1} t_1^{m+1} g'(t_1 t_2 x)\, dt_1\, dt_2 \\ &= \int_{-1}^{1} t_1^{m+1} \left(g'(t_1 x) + k \int_{-1}^{1} g'(t_1 t_2 x)\, dt_2\right) dt_1 \\ &= \int_{-1}^{1} t_1^{m+1} (T_k g)(t_1 x)\, dt_1 = I_{T_k g, m+1}(x).\end{aligned}$$

We conclude that $T_k I_{g,m} = I_{T_k g, m+1}$. Since (1) can be written as $T_k f = f' + k I_{f',0}$ one has

$$T_k^n f = T_k^{n-1}(f' + k I_{f',0}) = T_k^{n-1}(f') + k I_{T_k^{n-1}(f'), n-1},$$

which is the statement in the lemma. □

It follows from Lemma 1 that

$$|T_k^n f(x)| \leq \left(1 + \frac{2|k|}{n}\right) \sup_{y \in [-|x|,|x|]} |(T^{n-1}(f'))(y)| \qquad (x \in \mathbb{R}),$$

and induction then yields the following basic estimate, which is more explicit than [6, Prop. 2.1].

Corollary 2. *Let $k \in \mathbb{C}$, $f \in C^\infty(\mathbb{R})$ and $n \in \mathbb{N}$. Then*

$$|T_k^n f(x)| \leq \frac{\Gamma(n + 1 + 2|k|)}{n!\, \Gamma(1 + 2|k|)} \sup_{y \in [-|x|,|x|]} |f^{(n)}(y)| \qquad (x \in \mathbb{R}).$$

The third inequality in (7) can now be settled.

Proposition 3. *Let $k \in \mathbb{C}$, and suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $R > 0$, i.e.,*

$$|f(z)| \leq C e^{R|\operatorname{Im} z|} \qquad (z \in \mathbb{C}),$$

for some positive constant C. Then, for all $n \in \mathbb{N}$, $T_k^n f$ is bounded on the real line, and

$$\limsup_{n\to\infty} \|T_k^n f\|_\infty^{1/n} \leq R.$$

Proof. We have, for any $r > 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|\zeta - z| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \qquad (z \in \mathbb{C}).$$

If $|\zeta - z| = r$, then

$$|f(\zeta)| \leq C e^{R(|\operatorname{Im} z| + r)},$$

implying

$$|f^{(n)}(z)| \leq C \frac{n!}{r^n} e^{R(|\operatorname{Im} z| + r)} \qquad (z \in \mathbb{C}).$$

Choosing $r = n/R$, so that $r > 0$ if $n \in \mathbb{N}$, we find

$$|f^{(n)}(z)| \leq C \frac{n! e^n}{n^n} R^n e^{R|\operatorname{Im} z|} \qquad (n \in \mathbb{N}, z \in \mathbb{C}),$$

whence $\|f^{(n)}\|_\infty \leq C n! e^n n^{-n} R^n$, for $n \in \mathbb{N}$. Combining this with Corollary 2 yields

$$\|T_k^n f\|_\infty \leq C \frac{e^n \, \Gamma(n + 1 + 2|k|)}{n^n \, \Gamma(1 + 2|k|)} R^n \quad (n \in \mathbb{N}).$$

The result now follows from Stirling's formula. □

As to the first inequality in (7), it is actually easy to prove that it holds for the norm $\| \cdot \|_{k,p}$ in $L^p(\mathbb{R}, w_k(x)dx)$ for arbitrary $1 \leq p \leq \infty$ (not just for $p = \infty$), as is shown by the following lemma. It should be noted that this result can be generalized—with different proofs—to complex multiplicities (cf. Lemma 7) and to L^p-functions for $k \geq 0$ (cf. Theorem 10), but we present it here separately nevertheless, in order to illustrate that for the case $k \geq 0$, the proof of one of the crucial inequalities (as far as the Paley–Wiener theorem is concerned) is rather elementary and intuitive.

Lemma 4. *Let $k \geq 0$, $1 \leq p \leq \infty$ and $f \in \mathcal{S}(\mathbb{R})$. Then in the extended positive real numbers we have*

$$\liminf_{n \to \infty} \|T_k^n f\|_{k,p}^{1/n} \geq \sup\{|\lambda| \, : \, \lambda \in \operatorname{supp} D_k f\}. \tag{8}$$

Proof. Suppose $0 \neq \lambda_0 \in \operatorname{supp} D_k f$ and let $0 < \varepsilon < |\lambda_0|$. Define

$$\phi(\lambda) = \overline{D_k f(\lambda)} \qquad (\lambda \in \mathbb{R}).$$

Then, with q denoting the conjugate exponent and using (6), we find

$$\begin{aligned} \left\|T_k^{2n} f\right\|_{k,p} \|D_k \phi\|_{k,q} &\geq |\langle T_k^{2n} f, D_k \phi \rangle_k| = |\langle D_k(T_k^{2n} f), \phi \rangle_k| \\ &= \left| \int_{\mathbb{R}} (i\lambda)^{2n} D_k f(\lambda) \phi(\lambda) w_k(\lambda) d\lambda \right| \\ &= \int_{\mathbb{R}} \lambda^{2n} |D_k f(\lambda)|^2 w_k(\lambda) d\lambda \\ &\geq (|\lambda_0| - \varepsilon)^{2n} \int_{|\lambda| \geq |\lambda_0| - \varepsilon} |D_k f(\lambda)|^2 w_k(\lambda) d\lambda. \end{aligned}$$

With $\psi(\lambda) = \overline{D_k(T_k f)(\lambda)}$ we similarly get

$$\left\|T_k^{2n+1} f\right\|_{k,p} \|D_k \psi\|_{k,q} \geq (|\lambda_0| - \varepsilon)^{2n+1} \int_{|\lambda| \geq |\lambda_0| - \varepsilon} |\lambda| |D_k f(\lambda)|^2 w_k(\lambda) d\lambda.$$

These two estimates together yield

$$\liminf_{n \to \infty} \|T_k^n f\|_{k,p}^{1/n} \geq |\lambda_0| - \varepsilon,$$

and the lemma follows. □

Now that (7) has been established, we come to the Paley–Wiener theorem for smooth functions. Introducing notation, for $R > 0$, we let $C_R^\infty(\mathbb{R})$ denote the space of smooth functions on $\mathbb{R}$ with support in $[-R, R]$. Let $\mathcal{H}_R(\mathbb{C})$ denote the space of rapidly decreasing entire functions of exponential type R, i.e., those entire functions f with the property that, for all $n \in \mathbb{N} \cup \{0\}$, there exists a constant $C_n > 0$ such that

$$|f(z)| \leq C_n(1 + |z|)^{-n} e^{R|\operatorname{Im} z|} \qquad (z \in \mathbb{C}).$$

Theorem 5 (Paley–Wiener theorem for smooth functions)**.** *Let $R > 0$ and $k \geq 0$. Then the Dunkl transform D_k is a bijection from $C_R^\infty(\mathbb{R})$ onto $\mathcal{H}_R(\mathbb{C})$.*

Proof. If $f \in C_R^\infty(\mathbb{R})$, then it is easy to see that $D_k f \in \mathcal{H}_R(\mathbb{C})$ [12, Corollary 4.10]. Now assume that $f \in \mathcal{H}_R(\mathbb{C})$. Using Cauchy's integral representation as in the proof of Proposition 3, we retrieve the well-known fact that $f \in \mathcal{S}(\mathbb{R})$. From (7) we infer that $D_k f$ has support in $[-R, R]$. Since $D_k^{-1} f(x) = D_k f(-x)$, for $x \in \mathbb{R}$, the same is true for $D_k^{-1} f$, as was to be proved. □

Remark 6.

(1) By holomorphic continuation and continuity, cf. [13], one sees that Theorem 5 also holds in the more general case $\operatorname{Re} k \geq 0$. Alternatively, one can use Lemma 7 below instead of Lemma 4, which establishes (7) also in the case $\operatorname{Re} k \geq 0$, and then the above direct proof is again valid.

(2) We emphasize that the present proof does not use any explicit formulas for the Dunkl kernel in one dimension, contrary to the alternative methods of proof in [20] (where Weyl fractional integral operators are used), [13] (where asymptotic results for Bessel functions are needed) and [6] (where various integral operators, Dunkl's intertwining operator and the Paley–Wiener theorem for the Fourier transform all play a role). Also, the fact that a contour shifting argument for the transform is usually not possible (since w_k generically has no entire extension) is no obstruction. Given this unspecific nature, it is possible that the present method can be applied to other transforms as well, although the symmetry of the kernel—as reflected in (6), which was used in the proof of Lemma 4 and which will again be used in the proof of the alternative Lemma 7 below—is perhaps necessary. The same suggestion applies to the results in the remaining sections of this paper.

(3) For even functions, Theorem 5 specializes to the Paley–Wiener theorem for the Hankel transform [11].

4. Real Paley–Wiener theorem for L^p-functions

We will now consider the real Paley–Wiener theorem for L^p-functions in the spirit of Bang [4]. The result is first proved for Schwartz functions in Theorem 8 and subsequently for the general case in Theorem 10.

Let $\|\cdot\|_{\operatorname{Re} k, p}$ denote the $L^p(\mathbb{R}, |w_k(x)|dx)$-norm, for $1 \leq p \leq \infty$. Then we have the following generalization of Lemma 4 to complex multiplicities.

Lemma 7. *Let $\operatorname{Re} k \geq 0$, $1 \leq p \leq \infty$ and $f \in \mathcal{S}(\mathbb{R})$. Then in the extended positive real numbers we have*

$$\liminf_{n\to\infty} \|T_k^n f\|_{\operatorname{Re} k, p}^{1/n} \geq \sup\{|\lambda| \,:\, \lambda \in \operatorname{supp} D_k f\}. \tag{9}$$

Proof. Let $0 \neq \lambda_0 \in \operatorname{supp} D_k f$ and choose $\epsilon > 0$ such that $0 < 2\varepsilon < |\lambda_0|$. Also choose $\phi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\phi \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, and $\langle D_k f, \phi\rangle_k \neq 0$. Define $\phi_n(\lambda) = \lambda^{-n}\phi(\lambda)$ and $P_n(x) = x^n$ for $n \in \mathbb{N} \cup \{0\}$. Then

$$(1 + P_N(x))(D_k\phi_n)(x) = \frac{1}{c_k}\int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} \left(1 + (iT_k)^N\right)(\lambda^{-n}\phi(\lambda))\psi_x^k(\lambda)w_k(\lambda)d\lambda \quad (N \in \mathbb{N} \cup \{0\}).$$

We fix N such that N is even and $N > 2\operatorname{Re} k + 1$.

Corollary 2 and the binomial formula imply that

$$\left|\left(1+(iT_k)^N\right)\left(\lambda^{-n}\phi(\lambda)\right)\right| \le C_1 n^N (|\lambda_0|-\varepsilon)^{-n} \qquad (n\in\mathbb{N}\cup\{0\},\ \lambda\in\mathbb{R}),$$

where C_1 is a positive constant. This yields the estimates

$$\begin{aligned}\|D_k\phi_n\|_{\mathrm{Re}\,k,q} &\le \|(1+P_N)^{-1}\|_{\mathrm{Re}\,k,q}\|(1+P_N)D_k\phi_n\|_\infty\\ &\le \frac{2\varepsilon}{|c_k|}C_1 n^N\left(|\lambda_0|-\varepsilon\right)^{-n}\|(1+P_N)^{-1}\|_{\mathrm{Re}\,k,q}\\ &\le C_2 n^N\left(|\lambda_0|-\varepsilon\right)^{-n}\end{aligned}$$

for all $n>N$, where C_2 is a positive constant and q is the conjugate exponent.

Using (6), the identity $D_k(P_n\phi_n)=(-i)^n T_k^n D_k\phi_n$ and Hölder's inequality, we therefore get

$$\begin{aligned}|\langle D_kf,\phi\rangle_k| &= |\langle D_kf,P_n\phi_n\rangle_k| = |\langle f,D_k(P_n\phi_n)\rangle_k| = |\langle f,T_k^n(D_k\phi_n)\rangle_k|\\ &= |\langle T_k^nf,D_k\phi_n\rangle_k| \le \|T_k^nf\|_{\mathrm{Re}\,k,p}\,\|D_k\phi_n\|_{\mathrm{Re}\,k,q}\\ &\le C_2n^N\left(|\lambda_0|-\varepsilon\right)^{-n}\|T_k^nf\|_{\mathrm{Re}\,k,p}\,,\end{aligned}$$

whence

$$\liminf_{n\to\infty}\|T_k^nf\|_{\mathrm{Re}\,k,p}^{1/n} \ge \liminf_{n\to\infty}(C_2n^N)^{-1/n}\left(|\lambda_0|-\varepsilon\right)|\langle D_kf,\phi\rangle_k|^{1/n} = \left(|\lambda_0|-\varepsilon\right),$$

establishing the lemma. □

Using the Paley–Wiener theorem, we can extend the inequality in Proposition 3 to the norms $\|\cdot\|_{\mathrm{Re}\,k,p}$, $1\le p\le\infty$, for $\mathrm{Re}\,k\ge 0$, and we thus have

Theorem 8 (Real Paley–Wiener theorem for Schwartz functions)**.** *Let* $\mathrm{Re}\,k\ge 0$, $1\le p\le\infty$ *and* $f\in\mathcal{S}(\mathbb{R})$. *Then in the extended positive real numbers we have*

$$\lim_{n\to\infty}\|T_k^nf\|_{\mathrm{Re}\,k,p}^{1/n} = \sup\{|\lambda| \,:\, \lambda\in\operatorname{supp}D_kf\}.$$

Proof. In view of Lemma 7 it only remains to be shown that

$$\limsup_{n\to\infty}\|T_k^nf\|_{\mathrm{Re}\,k,p}^{1/n}\le R,$$

if $f\in\mathcal{S}(\mathbb{R})$ is such that $\operatorname{supp}D_kf\subset[-R,R]$ for some finite $R>0$. Using the inversion formula and the intertwining properties of the transform we have

$$x^NT_k^nf(x) = \frac{i^{N+n}}{c_k}\int_{-R}^{R}T_k^N(P_nD_kf)(\lambda)\psi_\lambda^k(x)w_k(\lambda)\,d\lambda \quad (n,N\in\mathbb{N}\cup\{0\}), \tag{10}$$

where again $P_n(x)=x^n$. Now Corollary 2 and the binomial formula imply that

$$\|T_k^N(P_nD_kf)\|_\infty \le C_1n^NR^n,$$

where C_1 is a constant depending on f and N. Therefore (10) yields that

$$\|(1+P_N)T_k^nf\|_\infty \le C_2n^NR^{n+1}, \tag{11}$$

where C_2 is again a constant depending on f and N. We fix N such that N is even and $N>2\mathrm{Re}\,k{+}1$. Then the observation

$$\|T_k^nf\|_{\mathrm{Re}\,k,p}\le\|(1+P_N)^{-1}\|_{\mathrm{Re}\,k,p}\|(1+P_N)T_k^nf\|_\infty$$

and (11) establish the result. □

Remark 9. Theorem 8 is new for complex k. For real k, the result can be found in [6], where it is proved using the Plancherel theorem for the Dunkl transform, the Riesz–Thorin convexity theorem and the theory of Sobolev spaces for Dunkl operators.

For $k \geq 0$, we will now generalize Theorem 8 to the L^p-case in Theorem 10, using the structure of $\mathcal{S}(\mathbb{R})$ as an associative algebra under the Dunkl convolution $*_k$. We refer to [19] for details on this subject.

Let $k \geq 0$, and define a distributional Dunkl transform $D_k^d f$ of $f \in L^p(\mathbb{R}, w_k(x)dx)$ by transposition

$$\langle D_k^d f, \phi \rangle = \langle f, D_k \phi \rangle_k \qquad (\phi \in \mathcal{S}(\mathbb{R})).$$

Clearly D_k^d is injective on $L^p(\mathbb{R}, w_k(x)dx)$. Furthermore, from

$$|\langle D_k^d f, \phi \rangle| \leq \|f\|_p \|D_k \phi\|_\infty \qquad (\phi \in \mathcal{S}(\mathbb{R})),$$

we see that $D_k^d f$ is a tempered distribution. Note also that for $f \in L^p(\mathbb{R}, w_k(x)dx)$ and $1 \leq p \leq 2$ we have

$$\langle D_k^d f, \phi \rangle = \langle D_k f, \phi \rangle_k \qquad (\phi \in \mathcal{S}(\mathbb{R})).$$

by (6) and density of $\mathcal{S}(\mathbb{R})$ in $L^p(\mathbb{R}, w_k(x)dx)$. Thus, for $f \in L^p(\mathbb{R}, w_k(x)dx)$ and $1 \leq p \leq 2$, $D_k^d f$ as defined above corresponds to the distribution $D_k f \, w_k$, implying that in this case $\operatorname{supp} D_k^d f = \operatorname{supp} D_k f$.

Theorem 10 (Real Paley–Wiener theorem for L^p-functions)**.** *Let $k \geq 0$, $1 \leq p \leq \infty$ and $f \in C^\infty(\mathbb{R})$ be such that $T_k^n f \in L^p(\mathbb{R}, w_k(x)dx)$, for all $n \in \mathbb{N} \cup \{0\}$. Then in the extended positive real numbers we have*

$$\lim_{n \to \infty} \|T_k^n f\|_{k,p}^{1/n} = \sup\{|\lambda| \, : \, \lambda \in \operatorname{supp} D_k^d f\}.$$

If in addition $f \in L^s(\mathbb{R}, w_k(x)dx)$, for some $1 \leq s \leq 2$, then the distribution $D_k^d f$ corresponds to the function $D_k f \, w_k$, and the support of $D_k^d f$ in the right hand side is equal to the support of $D_k f$ as a distribution.

Proof. First we note that (9) also holds for f as above: in the proof of Lemma 7 we just have to change $\langle D_k f, \cdot \rangle_k$ into $\langle D_k^d f, \cdot \rangle$. Therefore, it only remains to be shown that

$$\limsup_{n \to \infty} \|T_k^n f\|_{k,p}^{1/n} \leq R, \tag{12}$$

if f is as in the theorem and such that $\operatorname{supp} D_k^d f \subset [-R, R]$ for some finite $R > 0$. To this end, choose $\varepsilon > 0$, and fix a function $\phi_\varepsilon \in \mathcal{S}(\mathbb{R})$ such that $D_k^{-1} \phi_\varepsilon = 1$ on $[-R, R]$ and $D_k^{-1} \phi_\varepsilon = 0$ outside $[-R - \varepsilon, R + \varepsilon]$. We have from [19, Proposition 3] that $D_k^{-1}(\phi *_k \psi) = (D_k^{-1} \phi)\,(D_k^{-1} \psi)$, for all $\phi, \psi \in \mathcal{S}(\mathbb{R})$. With $P_n(x) = x^n$, we thus find for arbitrary $\phi \in \mathcal{S}(\mathbb{R})$ that

$$\begin{aligned} \langle f, \phi *_k T_k^n \phi_\varepsilon \rangle_k &= \langle D_k^d f, D_k^{-1}(\phi *_k T_k^n \phi_\varepsilon) \rangle = (-i)^n \langle D_k^d f, P_n (D_k^{-1} \phi)\,(D_k^{-1} \phi_\varepsilon) \rangle \\ &= (-i)^n \langle D_k^d f, P_n D_k^{-1} \phi \rangle = \langle f, T_k^n \phi \rangle_k. \end{aligned}$$

Furthermore, from loc.cit., one knows that $\|\phi *_k \psi\|_{k,q} \leq 4 \|\phi\|_{k,q} \|\psi\|_{k,1}$, for all $\phi, \psi \in \mathcal{S}(\mathbb{R})$, where q is the conjugate exponent of p. If we combine these two results with (5) and Hölder's reverse inequality, we infer that

$$\|T_k^n f\|_{k,p} = \sup_\phi |\langle T_k^n f, \phi \rangle_k| = \sup_\phi |\langle f, T_k^n \phi \rangle_k| = \sup_\phi |\langle f, \phi *_k T_k^n \phi_\varepsilon \rangle_k| \leq 4 \|f\|_{k,p} \|T_k^n \phi_\varepsilon\|_{k,1},$$

where the supremum is over all functions $\phi \in \mathcal{S}(\mathbb{R})$ with $\|\phi\|_{k,q} = 1$. From Theorem 8 we therefore conclude that

$$\limsup_{n \to \infty} \|T_k^n f\|_{k,p}^{1/n} \leq R + \varepsilon,$$

proving (12).

The statement on supports was established in the discussion preceding the theorem. □

Remark 11. For even functions, when the Dunkl transform reduces to the Hankel transform, the previous result can already be found in [2, 3]. Also using Dunkl convolution, and closely following the approach in [1, 2, 3], Theorem 10 has previously been established in [7]. Our proof is considerably shorter than the proof in loc.cit.

5. Paley–Wiener theorem for L^2-functions (for $k \geq 0$)

In this section we assume that $k \geq 0$. For $R > 0$, we define $L^2_R(\mathbb{R}, w_k(x)dx)$ to be the subspace of $L^2(\mathbb{R}, w_k(x)dx)$ consisting of those functions with distributional support in $[-R, R]$. Let $\mathcal{H}^{2,k}_R(\mathbb{C})$ denote the space of entire functions of exponential type R (cf. Proposition 3) that belong to $L^2(\mathbb{R}, w_k(x)dx)$ when restricted to the real line.

Theorem 12 (Paley–Wiener theorem for L^2-functions)**.** *Let $R > 0$ and $k \geq 0$. Then the Dunkl transform D_k is a bijection from $L^2_R(\mathbb{R}, w_k(x)dx)$ onto $\mathcal{H}^{2,k}_R(\mathbb{C})$.*

Proof. Let $f \in L^2_R(\mathbb{R}, w_k(x)dx)$. Then $f \in L^1(\mathbb{R}, w_k(x)dx)$, and (3) and (4) imply that $D_k f$ is entire and of exponential type R. Conversely, let $f \in \mathcal{H}^{2,k}_R(\mathbb{C})$. By the Plancherel theorem one has $D_k^{-1} f \in L^2(\mathbb{R}, w_k(x)dx)$. In addition, Proposition 3 and Theorem 10 (with $p = \infty$) show that $\operatorname{supp} D_k f \subset [-R, R]$. The same is then true for $D_k^{-1} f$, and the result follows. □

Remark 13.

(1) In the Fourier transform case, for $k = 0$, this is the original Paley–Wiener theorem, see [15] or [18, Theorem 19.3]; cf. also [5, 6.2.4 and 6.7.1]. The present proof seems to be more in terms of general principles than other proofs seen in the literature.

(2) Let $1 \leq p \leq \infty$, and define $\mathcal{H}^{p,k}_R(\mathbb{C})$ as the space of entire functions of exponential type R that belong to $L^p(\mathbb{R}, w_k(x)dx)$ when restricted to the real line. As above, using Proposition 3 and Theorem 10, one sees that $D^d_k f$ has support in $[-R, R]$ when $f \in \mathcal{H}^{p,k}_R(\mathbb{C})$. In particular, for $1 \leq p \leq 2$, with conjugate exponent q, we have that D_k maps $\mathcal{H}^{p,k}_R(\mathbb{C})$ into $L^q_R(\mathbb{R}, w_k(x)dx)$, the subspace of $L^q(\mathbb{R}, w_k(x)dx)$ consisting of those functions with distributional support in $[-R, R]$, cf. [5, 6.8.13] for the Fourier case $k = 0$.

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