Renormalization of interacting diffusions: a program and four examples

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Abstract

Systems of hierarchically interacting diffusions allow for a rigorous *renormalization analysis*. By bringing into play the powerful machinery of stochastic analysis, it is possible to obtain a complete classification of the large space-time behavior of these systems into *universality classes*. The present paper outlines a general renormalization program that is being pursued since ten years and describes four examples where this program has been successfully carried through. The systems under consideration model the evolution of multi-type populations subject to migration and resampling.

Keywords: Interacting diffusions, hierarchical group, multi-type populations, multi-scale block averages, renormalization transformation, fixed points and fixed shapes, attracting orbits, universality classes.
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1 Overview

1.1 Interacting diffusions

The systems that will be considered in this paper evolve according to the following set of coupled one-dimensional SDE's:

$$dX_\eta(t) = \sum_{\zeta\in\Omega_N} a_N(\eta,\zeta)\,[X_\zeta(t) - X_\eta(t)]\,dt + \sqrt{g(X_\eta(t))}\,dW_\eta(t), \qquad \eta\in\Omega_N,\, t\geq 0, \tag{1.1}$$

where

(1) $X_\eta(t) \in S \subseteq \mathbb{R}$ is the single-component *state space*.

(2) Ω_N is the *hierarchical group* of order $N\in\mathbb{N}$.

(3) $a_N(\cdot,\cdot)$ is the *interaction kernel* on $\Omega_N\times\Omega_N$.

(4) $g(\cdot)$ is the $[0,\infty)$-valued *diffusion function* on S.

(5) $\{W_\eta(\cdot)\}_{\eta\in\Omega_N}$ are independent standard Brownian motions on $\mathbb{R}$.

We will also look at higher-dimensional versions of (1.1). In what follows we will focus on one particular choice for $a_N(\cdot,\cdot)$, but we will consider several choices of S and g. As *initial condition* we take

$$X_\eta(0) = \theta \in \mathrm{int}(S) \quad \forall\,\eta\in\Omega_N. \tag{1.2}$$

Equation (1.1) arises as the continuum limit of discrete models in *population dynamics*. In these models, individuals of different types live in large *colonies*, labelled by the hierarchical group. The state of a colony describes the *composition* of the population at that colony (such as the fractions or the total masses of the different types of individuals). Individuals *migrate* between colonies, i.e., they move from one colony to another according to a random mechanism that depends on the locations of the two colonies. This is described by the first term in the right-hand side of (1.1). Moreover, individuals are subject to *resampling* within each colony, i.e., they are replaced by new individuals according to a random mechanism that depends on the state of the colony. This is described by the second term in the right-hand side of (1.1). The system in (1.1) arises after letting the number of individuals per colony tend to infinity and normalizing both the state and the rate of evolution of the colony appropriately. For more background, the reader is referred to Sawyer and Felsenstein [20] and Ethier and Kurtz [16], Chapter 10.

1.2 Hierarchical group and multi-scale block averages

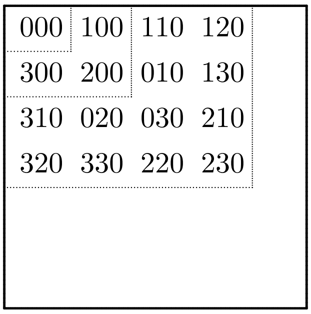
The hierarchical group of order N is the set

$$\Omega_N = \left\{\xi = (\xi_i)_{i\in\mathbb{N}} \in \{0,1,\ldots,N-1\}^{\mathbb{N}}\colon\ \sum_{i\in\mathbb{N}} \xi_i < \infty\right\} \tag{1.3}$$

with addition modulo N. On Ω_N, the hierarchical distance is defined as

$$d(\eta,\zeta) = \min\{i\in\mathbb{N}\cup\{0\}\colon\ \eta_j = \zeta_j\ \forall\, j > i\}, \tag{1.4}$$

which is an *ultrametric.*



Schematic picture of Ω_4. Blocks of radius 0, 1 and 2 around the origin (first three digits indicated only).

At first sight, the choice for Ω_N as the index set for the colonies may seem a bit artificial. However, it is completely natural within a genetics context. Indeed, what Ω_N does is organize the colonies according to their *genetic location.* The population is divided into families, clans, neighborhoods, villages, regions, etc. Site ξ contains all individuals that are in family ξ_1, clan ξ_2, neighborhood ξ_3, village ξ_4, region ξ_5, etc. The hierarchical distance between two colonies measures the smallest level in the hierarchy to which both colonies belong.

Our goal will be to study the evolution of the system in (1.1–1.2) on large space-time scales in the limit as $N \to \infty$, the so-called *hierarchical mean-field limit.* To that end, we define *block averages on space-time scale* $k \in \mathbb{N} \cup \{0\}$ by putting

$$Y^{[k]}_{\eta,N}(t) = \frac{1}{N^k} \sum_{\substack{\zeta \in \Omega_N \\ d(\eta,\zeta) \leq k}} X_\zeta(N^k t), \qquad \eta \in \Omega_N,\, t \geq 0, \tag{1.5}$$

where we average over all components in a block of radius k around a given site and speed up time proportionally to the size of the block.

In what follows, we will make a particular choice for the interaction kernel in (1.1), namely,

$$a_N(\eta,\zeta) = \sum_{k \geq d(\eta,\zeta)} c_k N^{1-2k} \quad \forall\, \eta \neq \zeta, \tag{1.6}$$

where $(c_k)_{k\in\mathbb{N}}$ is a given sequence of positive constants such that the sum is finite for all N large enough. One reason for this choice is that (1.1) takes on a particularly suitable form,

$$dX_\eta(t) = \sum_{k\geq 1} c_k N^{1-k} \left[Y^{[k]}_{\eta,N}(tN^{-k}) - X_\eta(t) \right] dt + \sqrt{g(X_\eta(t))}\, dW_\eta(t), \tag{1.7}$$

where single components are attracted towards successive block averages. Another reason is that, modulo normalization, (1.6) is the transition kernel of a random walk on Ω_N whose potential-theoretic properties are independent of N, for N sufficiently large, and are completely determined by the sequence $(c_k)_{k\in\mathbb{N}}$.

The interpetation of (1.6) is that the random walk picks an integer k with probability proportional to c_k/N^{k-1} and jumps to a site within the k-block around its current position according to the uniform distribution on this k-block. Throughout the paper we will assume that

$$\sum_{k\in\mathbb{N}} \frac{1}{c_k} = \infty, \tag{1.8}$$

which makes the random walk on Ω_N *critically recurrent*. This is crucial for the universal behavior to be described later on.

The key point about (1.7) is that it is susceptible to a *renormalization analysis*.

1.3 Renormalization transformation

Let us consider the blocks around the origin. If we let $N \to \infty$ in (1.7), then only the term with $k = 1$ survives. Moreover, the term $Y^{[1]}_{0,N}(tN^{-1})$ converges to θ for all $t \geq 0$, because of (1.2). Therefore we find that ($\Longrightarrow$ denotes convergence in law)

$$\{X_0(t)\colon\ t \geq 0\} \Longrightarrow \{Z_{\theta,g,c_1}(t)\colon\ t \geq 0\} \quad \text{as } N \to \infty, \tag{1.9}$$

where $Z_{\theta,g,c_1}(t)$ is the solution of the *autonomous* SDE

$$dZ(t) = c_1\,[\theta - Z(t)]\,dt + \sqrt{g(Z(t))}\,dW(t), \qquad Z(0) = \theta. \tag{1.10}$$

In other words, in the limit as $N \to \infty$ the single components *decouple* and follow a simple diffusion equation parameterized by θ, g and c_1. (The behavior expressed by (1.9–1.10) is often referred to as “McKean-Vlasov limit” and “propagation of chaos”.) Under mild restrictions on S and g, this diffusion equation has an *equilibrium distribution*, which we denote by ν_{θ,g,c_1} and which lives on S.

We next move up one step in the hierarchy. By summing (1.7) over the components in a 1-block, we get

$$dY^{[1]}_{\eta,N}(t) = \sum_{k\geq 2} c_k N^{2-k} \left[Y^{[k]}_{\eta,N}(tN^{1-k}) - Y^{[1]}_{\eta,N}(t)\right]\,dt + \frac{1}{\sqrt{N}} \sum_{\substack{\zeta\in\Omega_N \\ d(\eta,\zeta)\leq 1}} \sqrt{g(X_\zeta(Nt))}\,dW_\zeta(t). \tag{1.11}$$

Here, time is scaled up by a factor N, both in the 1-block and in the Brownian motions (hence the factor $1/\sqrt{N}$), and the term with $k = 1$ cancels out. If we let $N \to \infty$ in (1.11), then only the term with $k = 2$ survives. Moreover, the term $Y^{[2]}_{0,N}(tN^{-1})$ converges to θ for all $t \geq 0$ because of (1.2). Furthermore, if $Y^{[1]}_{0,N}(t) = y$, then each of the N components in this block is linearly attracted towards a value that is approximately y, since the attraction towards the values of the k-blocks with $k > 2$ is weak when N is large (recall (1.7)). Therefore, at time Nt each of these components is close in distribution to the equilibrium ν_{y,g,c_1} (associated with (1.10) after replacing θ by y). Thus, it is *reasonable* to expect that

$$\left\{Y^{[1]}_{0,N}(t)\colon\ t \geq 0\right\} \Longrightarrow \left\{Z_{\theta,F_{c_1}g,c_2}(t)\colon\ t \geq 0\right\} \quad \text{as } N \to \infty, \tag{1.12}$$

where $Z_{\theta,F_{c_1}g,c_2}(t)$ is the solution of the SDE

$$dZ(t) = c_2\,[\theta - Z(t)]\,dt + \sqrt{(F_{c_1}g)(Z(t))}\,dW(t), \qquad Z(0) = \theta, \tag{1.13}$$

where $F_{c_1}g$ is the *diffusion function on scale* 1 obtained from the diffusion function g on scale 0 by *averaging* it w.r.t. the equilibrium distribution associated with (1.10) on scale 0:

$$(F_{c_1}g)(y) = \int_S g(x)\nu_{y,g,c_1}(dx). \tag{1.14}$$

This formula defines a *renormalization transformation* F_{c_1} acting on the function g.

The above renormalization procedure can be iterated. Indeed, it is *reasonable* to expect that

$$\left\{Y^{[k]}_{0,N}(t)\colon\ t \geq 0\right\} \Longrightarrow \left\{Z_{\theta,F^{[k]}g,c_{k+1}}(t)\colon\ t \geq 0\right\} \qquad \text{as } N \to \infty, \tag{1.15}$$

where $Z_{\theta,F^{[k]}g,c_{k+1}}(t)$ is the solution of the SDE

$$dZ(t) = c_{k+1}\,[\theta - Z(t)]\,dt + \sqrt{(F^{[k]}g)(Z(t))}\,dW(t), \qquad Z(0) = \theta, \tag{1.16}$$

where $F^{[k]}g$ is the *diffusion function on scale* k obtained from the diffusion function g on scale 0 by applying k times the renormalization transformation:

$$F^{[k]} = F_{c_k} \circ \cdots \circ F_{c_1}. \tag{1.17}$$

The intuition behind this claim is as follows:

(a) On time scale $N^k t$, the block averages on scale k *fluctuate* while the block averages on scales $> k$ *almost stand still* and therefore remain close to the initial vlaue θ.

(b) Given that the block averages on scale k have value y, the block averages on scale $k-1$ reach *equilibrium* with drift towards y almost instantly on time scale $N^k t$.

(c) Consequently, the diffusion function on scale k is the *average* of the diffusion function on scale $k-1$ under this equilibrium.

The limit $N \to \infty$ provides the *separation of successive space-time scales.*

1.4 Renormalization program

The above heuristic observations naturally lead to a two-step programme for renormalization:

(I) Stochastic part: Show that (1.15–1.16) indeed arise from (1.1–1.2) in the limit as $N \to \infty$ for all scales $k \in \mathbb{N}$.

(II) Analytic part: Study the orbits of $(F^{[k]})_{k\in\mathbb{N}}$, determine their fixed points, and classify their domains of attraction.

The goal is to try and carry out this programme for relevant choices of S for appropriate classes $\mathcal{H} = \mathcal{H}(S)$ of diffusion functions. For tutorial overviews on this programme, see Greven [17] and den Hollander [18].

In what follows we will describe two *one-dimensional* examples where the above renormalization program has been *fully* carried through (Section 2) and two *higher-dimensional* examples where it has been *partially* carried through (Section 3). It will turn out that, in each of these four examples, $(F^{[k]})_{k\in\mathbb{N}}$ has an interesting structure of fixed points and domains of attraction. The fixed points correspond to *special choices* of g that play the role of *universal attractors* for the dynamics (1.1–1.2) on the *macroscopic scale* corresponding to $k \to \infty$. We close with listing some open problems (Section 4).

For ease of exposition, we will henceforth restrict to the case where $c_k = 1$ for all k. We then have $F^{[k]} = F^k$ with $F = F_1$. It is straightforward to extend the results to the case (1.8).

2 One dimension

2.1 $S = [0, 1]$

In this example, our choice for the state space and the class of diffusion functions is:

> $S = [0, 1]$ and $g \in \mathcal{H}$, the class of functions satisfying:
>
> 1. g is Lipschitz on $[0, 1]$.
> 2. $g(x) > 0$ for $x \in (0, 1)$.
> 3. $g(0) = g(1) = 0$.

The *stochastic part* of the renormalization program was carried out by Dawson and Greven [9], [10] and is given in Theorem 2.1.

Theorem 2.1 *In the limit as $N \to \infty$, (1.15–1.16) arise from (1.1–1.2) with F given by*

$$(Fg)(y) = \int_{[0,1]} g(x)\nu_{y,g}(dx), \tag{2.1}$$

where $\nu_{y,g}$ is the equilibrium distribution of

$$dZ(t) = [y - Z(t)]\,dt + \sqrt{g(Z(t))}\,dW(t), \tag{2.2}$$

which is given by

$$\nu_{y,g}(dx) = \frac{1}{Z_{y,g}}\,\frac{1}{g(x)}\,\exp\left[-\int_y^x \frac{z-y}{g(z)}\,dz\right]\,dx \tag{2.3}$$

with $Z_{y,g}$ the normalizing constant.

Note that F is an integral operator. Since $\nu_{y,g}$ depends on g itself, F is non-linear.

The *analytic part* of the renormaliztion program was carried out by Baillon, Clément, Greven and den Hollander [2] and is given in Theorems 2.2–2.4.

Theorem 2.2 *(a) $F\mathcal{H} \subset \mathcal{H}$.*
(b) $\forall\, g \in \mathcal{H}$: $y \mapsto (Fg)(y)$ is C^∞ on $(0, 1)$.

Theorem 2.3 *The solution of the eigenvalue problem $Fg = \lambda g$, $g \in \mathcal{H}$, $\lambda > 0$, is the 1-parameter family*

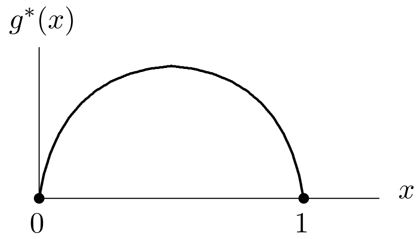
$$g = dg^* \text{ and } \lambda = \frac{1}{1+d}, \qquad d > 0, \tag{2.4}$$

where $g^*(x) = x(1-x)$.

Theorem 2.4 *For all $g \in \mathcal{H}$,*

$$\lim_{k\to\infty} kF^k g = g^* \tag{2.5}$$

uniformly on $[0, 1]$.



Fisher-Wright diffusion function.

These results show that F is well-defined on the class $\mathcal{H}$, is smoothing, and has a *fixed shape* g^* that is *globally attracting* after proper normalization. Thus, (1.1–1.2) exhibits *full universality*: No matter what the diffusion function g on scale 0 is, the diffusion function $F^k g$ on scale k is close to $(1/k)g^*$. The case $g = g^*$ is called the *Fisher-Wright diffusion.*

2.2 $S = [0, \infty)$

In this example, our choice for the state space and the class of diffusion functions is:

$S = [0, \infty)$ and $g \in \mathcal{H}$, the class of functions satisfying:

1. g is locally Lipschitz on $[0, \infty)$.
2. $g(x) > 0$ for $x > 0$.
3. $g(0) = 0$.
4. $\lim_{x \to \infty} g(x)/x^2 = 0$.

The *stochastic part* of the renormalization program was carried out by Dawson and Greven [11]. The same formulas as in Theorem 2.1 apply, but now on $[0, \infty)$ instead of $[0, 1]$.

The *analytic part* of the renormalization program was carried out by Baillon, Clément, Greven and den Hollander [3] and is given in Theorems 2.5–2.8.

Theorem 2.5 *(a)* $F\mathcal{H} \subset \mathcal{H}$.
(b) $\forall\, g \in \mathcal{H}$: $y \mapsto (Fg)(y)$ *is* C^∞ *on* $(0, \infty)$.

Theorem 2.6 *The solution of the eigenvalue problem* $Fg = \lambda g$, $g \in \mathcal{H}$, $\lambda > 0$, *is the* 1-*parameter family*

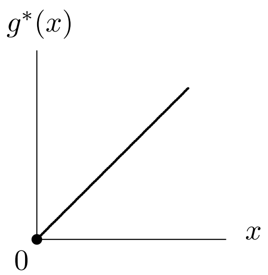
$$g = dg^* \text{ and } \lambda = 1, \qquad d > 0, \tag{2.6}$$

where $g^*(x) = x$.

Theorem 2.7 *If* $\lim_{x \to \infty} x^{-1} g(x) = d$, *then*

$$\lim_{k \to \infty} F^k g = dg^* \tag{2.7}$$

uniformly on compact subsets of $[0, \infty)$.



Feller diffusion function.

Theorem 2.8 *Suppose that* $g(x) \sim x^\alpha L(x)$ *as* $x \to \infty$ *with* $\alpha \in (0,2)\backslash\{1\}$ *and* L *slowly varying at infinity. Let* $(e_k)_{k\in\mathbb{N}}$ *be defined by*

$$\frac{1}{k} = \frac{g(e_k)}{e_k^2}. \tag{2.8}$$

Then there exist constants $0 < K_1(\alpha) \le K_2(\alpha) < \infty$ *such that*

$$K_1(\alpha) g^* \le \liminf_{k\to\infty} \frac{k}{e_k} F^k g \le \limsup_{k\to\infty} \frac{k}{e_k} F^k g \le K_2(\alpha) g^* \tag{2.9}$$

uniformly on compact subsets of $[0,\infty)$.

REMARKS: (i) Theorem 2.7 (which is Theorem 2.8 for $\alpha = 1$ and $L \equiv d$) says that all g that are asymptotically linear are iterated towards a linear.
(ii) In Theorem 2.8, if $L \equiv 1$, then $e_k \sim k^{1/(2-\alpha)}$ as $k \to \infty$, and so

$$F^k g \asymp k^{-(1-\alpha)/(2-\alpha)} g^*. \tag{2.10}$$

Thus, concave g are iterated downwards, while convex g are iterated upwards.
(iii) It is shown in [3] that all solutions of (2.8) have the same asymptotic behavior as $k \to \infty$.
(iv) It is conjectured in [3] that

$$K_1(\alpha) = K_2(\alpha) = (\alpha!)^{1/(2-\alpha)} 2^{(1-\alpha)/(2-\alpha)}. \tag{2.11}$$

The above results show that g^* again acts as a *globally attracting fixed point*, except that now the *normalization* depends on the behavior of g at infinity. Thus, once again (1.1–1.2) exhibits *universality*: No matter what the diffusion function g on scale 0 is, the diffusion function $F^k g$ on scale k is close to $(e_k/k)g^*$. The case $g = g^*$ is called the *Feller diffusion*.

3 Higher dimension

In higher dimension our system in (1.1) must be written in vector form:

$$dX_\eta^i(t) = \sum_{\zeta\in\Omega_N} a_N(\eta,\zeta)\,\left[X_\zeta^i(t) - X_\eta^i(t)\right]\,dt + \sqrt{g^i\left(\vec{X}_\eta(t)\right)}\,dW_\eta^i(t), \quad 1 \le i \le d. \tag{3.1}$$

Here, the scalar component $X_\eta(t)$ in (1.1) is replaced by the vector

$$\vec{X}_\eta(t) = \left(X_\eta^1(t), \ldots, X_\eta^d(t)\right) \in S \subseteq \mathbb{R}^d,$$

while the scalar diffucion function $g(X_\eta(t))$ in (1.1) is replaced by the vector

$$\vec{g}\left(\vec{X}_\eta(t)\right) = \left(g^1\left(\vec{X}_\eta(t)\right), \ldots, g^d\left(\vec{X}_\eta(t)\right)\right) \in [0,\infty)^d.$$

The dynamics of the d components are *coupled* because the argument of each g^i is the full vector $\vec{X}_\eta(t)$. The analogue of (1.16) reads

$$dZ^i(t) = c_{k+1}\left[\theta^i - Z^i(t)\right]\,dt + \sqrt{(F^{[k]}\vec{g})^i(\vec{Z}(t))}\,dW^i(t), \quad 1 \le i \le d, \quad \vec{Z}(0) = \vec{\theta}. \tag{3.2}$$

The step from one to more dimensions brings about *major* mathematical complications:

- Only for a *limited* class of choices of S and $\vec{g}$ has it been proved that (3.1) has a unique weak solution. A similar problem occurs for (3.2).
- The *stochastic part* of the renormalization program has been carried through *only for special choices* of S and g (on the basis of so-called duality arguments).
- The *analytic part* of the renormalization program is hampered by the fact that it is in general *not* possible to write down an *explicit formula* for the equilibrium distribution of (3.2) and hence for the renormalization transformation F.

We will look at two cases where the *analytic part* of the renormalization program can be completed.

3.1 $S \subset \mathbb{R}^d$ compact convex

Den Hollander and Swart [19] considered the *isotropic* case

$$g^1 = \ldots = g^d = g, \tag{3.3}$$

chose S to be a compact convex subset of $\mathbb{R}^d$, $d \geq 2$, and took for $\mathcal{H}$ the class of functions satisfying:

1. g is locally Lipschitz on S.
2. $g > 0$ on $\mathrm{int}(S)$.
3. $g = 0$ on ∂S.

Define

(a) $\mathcal{H}'$ is the largest subclass of $\mathcal{H}$ for which (1.16) has a unique weak solution and a unique equilibrium.

(b) $\mathcal{H}''$ is the largest subclass of $\mathcal{H}'$ that is closed under F.

Then $\mathcal{H}'$ is the class on which F is *well-defined* and $\mathcal{H}''$ is the class on which F can be *iterated*.

In what follows we will *assume* that

$$\mathcal{H}'' = \mathcal{H}' = \mathcal{H}. \tag{3.4}$$

Theorems 3.1–3.2 are taken from [19] and rely on (3.4).

Theorem 3.1 *The solution of the eigenvalue problem $Fg = \lambda g$, $g \in \mathcal{H}$, $\lambda > 0$, is the 1-parameter family*

$$g = dg^* \text{ and } \lambda = \frac{1}{1+d}, \qquad d > 0, \tag{3.5}$$

where g^* is the unique continuous solution of

$$\begin{array}{rcll} \Delta g^* & = & -2 & \text{on } \mathrm{int}(S), \\ g^* & = & 0 & \text{on } \partial S. \end{array} \tag{3.6}$$

Theorem 3.2 *For all $g \in \mathcal{H}$,*

$$\lim_{k\to\infty} kF^k g = g^* \tag{3.7}$$

uniformly on S.

The above g^* takes over the role of the Fisher-Wright diffusion function on the unit interval (recall Section 2.1).

It is believed that the assumption in (3.4) is valid in full generality for the choice of S and $\mathcal{H}$ indicated above, but so far this remains a challenge. Swart [22] has proved that $\mathcal{H}'$ contains all those $g \in \mathcal{H}$ for which there exists an $\epsilon > 0$ such that the level sets $\{x \in S\colon\ g(x) \geq r\}$ are convex for all $0 < r < \epsilon$. The *stochastic part* of the renormalization program is largely open.

Without the isotropy assumption in (3.3), little is known so far. Dawson and March [14] proved that for S the simplex, $\mathcal{H}'$ contains a small neighborhood of the d-dimensional analogue of the Fisher-Wright diffusion. Cerrai and Clément [6], [7] proved that for S the simplex and the hypercube, respectively, $\mathcal{H}'$ contains a large subset of $\mathcal{H}$. This work represents very important progress on the difficult weak uniqueness issue in the anisotropic case. The *stochastic part* of the renormalization program in the anisotropic case is also largely open, with partial progress in Dawson, Greven and Vaillancourt [13] for S the simplex.

3.2 $S = [0,\infty)^2$

In Dawson, Greven, den Hollander, Sun and Swart [12] (work in progress) the two-dimensional version of (3.2) is considred:

$$\begin{aligned} dX^1_\eta(t) &= \sum_{\zeta\in\Omega_N} a_N(\eta,\zeta)\,[X^1_\zeta(t) - X^1_\eta(t)]\,dt + \sqrt{g_1(X^1_\eta(t),X^2_\eta(t))}\,dW^1_\eta(t),\\ dX^2_\eta(t) &= \sum_{\zeta\in\Omega_N} a_N(\eta,\zeta)\,[X^2_\zeta(t) - X^2_\eta(t)]\,dt + \sqrt{g_2(X^1_\eta(t),X^2_\eta(t))}\,dW^2_\eta(t). \end{aligned} \tag{3.8}$$

Here, $\vec{g} = (g_1, g_2)$ is a pair of diffusion functions driving the pair of components $\vec{X}_\eta(t) = (X^1_\eta(t), X^2_\eta(t))$. Unlike in Section 3.1, we will allow the *anisotropic* case $g_1 \neq g_2$. The two-dimensional version of (3.2) (for $c_k \equiv 1$) reads

$$\begin{aligned} dZ^1(t) &= [\theta^1 - Z^1(t)]\,dt + \sqrt{g_1(Z^1(t),Z^2(t))}\,dW^1(t),\\ dZ^2(t) &= [\theta^2 - Z^2(t)]\,dt + \sqrt{g_2(Z^1(t),Z^2(t))}\,dW^2(t). \end{aligned} \tag{3.9}$$

The *stochastic part* of the renormalization program is addressed in Cox, Dawson and Greven [8] for a special case, and is largely open. The *analytic part* is being addressed in [12]. For this part, $\vec{g} = (g_1, g_2)$ is taken from the following class, which we denote by $\mathcal{H}$:

1. $g_1, g_2 > 0$ on $(0,\infty)^2$.
2. $g_1(x_1,x_2) = x_1 h_1(x_1,x_2)$ with either:
   - 2.a. $h_1 > 0$ on $[0,\infty)^2$ and h_1 Hölder on compact subsets of $[0,\infty)^2$.
   - 2.b. $h_1(x_1,x_2) = x_2\gamma_1(x_1,x_2)$, $\gamma_1 > 0$ on $[0,\infty)^2$ and γ_1 Hölder on compact subsets of $[0,\infty)^2$.
3. $g_2(x_1,x_2) = x_2 h_2(x_1,x_2)$ with either:

3.a. $h_2 > 0$ on $[0,\infty)^2$ and h_2 Hölder on compact subsets of $[0,\infty)^2$.

3.b. $h_2(x_1,x_2) = x_1\gamma_2(x_1,x_2)$, $\gamma_2 > 0$ on $[0,\infty)^2$ and γ_2 Hölder on compact subsets of $[0,\infty)^2$.

4. $g_1(x_1,x_2), g_2(x_1,x_2) \le C(x_1+1)(x_2+1)$ for some $C = C(g_1,g_2) < \infty$.

Dawson and Perkins [15] (work in progress) show that (3.9) has a unique weak solution under properties 1-3 above. Earlier results in this direction, under stronger restrictions on g, were obtained by Athreya, Barlow, Bass and Perkins [1] and by Bass and Perkins [4], [5]. In [12] it is shown that properties 1-3 are enough to also have a unique equilibrium. Thus, if we define subclasses $\mathcal{H}'' \subset \mathcal{H}' \subset \mathcal{H}$ as in Section 3.1, then we have

$$\mathcal{H}' = \mathcal{H}. \tag{3.10}$$

It is believed that under properties 1-4 above,

$$\mathcal{H}'' = \mathcal{H}, \tag{3.11}$$

although this is still open.

A number of results are derived in [12] subject to (3.11). We cite two results in Theorems 3.3–3.4, subject to the condition that $\vec{g}$ be "sufficiently regular near the boundary and at infinity". The precise regularity conditions are technical. Regularity near the boundary means that g^1 is either everywhere zero or everywhere positive on $\{x_2 = 0\}$, and similarly for g^2 on $\{x_1 = 0\}$. Regularity at infinity means that g^1, g^2 are regularly varying along rays in $[0,\infty)^2$ in a certain uniform sense.

The renormalization transformation F, which acts on the *pair* of diffusion functions $\vec{g} = (g_1, g_2)$, is given by

$$(F\vec{g})(\vec{y}) = \int_{[0,\infty)^2} \vec{g}(\vec{x})\nu_{\vec{y},\vec{g}}(d\vec{x}), \tag{3.12}$$

where $\nu_{\vec{y},\vec{g}}$ is the equilibrium distribution of (3.9).

Theorem 3.3 *The solution of the eigenvalue problem $F\vec{g} = \lambda\vec{g}$, $\vec{g} \in \mathcal{H}$, $\lambda > 0$, subject to g being "sufficiently regular near the boundary and at infinity", is the 4-parameter family*

$$\vec{g} = \vec{g}^*_d \text{ and } \lambda = Id, \qquad d = (d_1,d_2,d_3,d_4) \ge 0, \tag{3.13}$$

where $\vec{g}^*_d = (g^{*,1}_d, g^{*,2}_d)$ has the form

$$\begin{aligned} g^{*,1}_d(x_1,x_2) &= (d_1 + d_2x_2)x_1, \\ g^{*,2}_d(x_1,x_2) &= (d_3 + d_4x_1)x_2. \end{aligned} \tag{3.14}$$

Theorem 3.4 *For all $\vec{g} \in \mathcal{H}$ that are "sufficiently regular near the boundary and at infinity",*

$$\frac{k}{e_k} F^k\vec{g} \asymp \vec{g}^*_d \quad \text{as } k \to \infty, \tag{3.15}$$

where d is determined by the behavior of $\vec{g}$ near the boundary and (e_k) by the behavior of $\vec{g}$ at infinity.

For instance, if $g_1 = g_2 = 0$ on $\{x_1 = 0\} \cup \{x_2 = 0\}$ and $g_1(x_1, x_2) = g_2(x_1, x_2) \asymp x_1^\alpha x_2^\beta$ as $x_1, x_2 \to \infty$ with $0 < \alpha \vee \beta \leq 1$, then

$$d = (0, 1, 0, 1) \quad \text{and} \quad e_k \asymp k^{1/(2-\alpha\vee\beta)}. \tag{3.16}$$

The 4 universality classes in Theorem 3.3 correspond to special diffusions:

$d = (>0, 0, >0, 0)$: *non-catalytic* branching.
$d = (0, >0, >0, 0)$: *catalytic* branching.
$d = (>0, 0, 0, >0)$: *catalytic* branching.
$d = (0, >0, 0, >0)$: *mutually catalytic* branching.

These take over the role of the Feller diffusion function on the halfline (recall Section 2.2).

4 Open problems

The main open problems are:

(a) Carry out the stochastic part of the renormalization program in higher dimension (for the examples in, respectively, Sections 3.1 and 3.2).

(b) Prove (3.4) for S the simplex, respectively, the hypercube. Attempt to extend the proof for S an arbitrary compact convex subset of $\mathbb{R}^d$.

(c) Prove (3.11) for $S = [0, \infty)^2$.

It clearly is a challenge to push the renormalization analysis forward to even richer examples.

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