# Continuity properties of regularly perturbed fundamental matrices*

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**Abstract**

In this note we characterize regular perturbations of finite state Markov chains in terms of continuity properties of its fundamental matrix. A perturbation turns out to be regular if and only if the fundamental matrix can be approximated by the discounted deviation from stationarity for small perturbation parameters. We also give bounds to asses the quality of the approximation.

KEYWORDS: Markov chain, fundamental matrix, regular perturbation, coupling.
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## 1 Introduction

In the theory of Markov chains the fundamental matrix (denoted by $Z$) is a key notion that is often used to establish identities for mean hitting and occupation times (see for instance, [9] and [1]). In recent years a lot of work has been done on the analysis of perturbed Markov chains, a line of research stemming from Schweitzer's seminal paper [11]. In these works, the probability transition matrix $P$ is replaced by $P_\varepsilon$, where $\varepsilon$ is called the *perturbation parameter* and the asymptotic properties (as $\varepsilon \to 0$) of the resulting Markov chain are the subject of investigations. A recent survey of results emerging from these studies - many of which concern the limit of $Z_\varepsilon$, the fundamental matrix of the perturbed process as $\varepsilon \to 0$ - can be found in Avrachenkov et al [2]. However, it is well-known that the fundamental matrix can also be obtained as an "Abel limit" of powers of deviations of the probability transition matrix and the stationary distribution matrix of the underlying Markov chain. Here the *discount factor* $\alpha$ is the parameter of interest and the limit is as $\alpha \to 1$, from below.

In view of the above it is evident that the asymptotic analysis of $Z_\varepsilon$ as $\varepsilon \to 0$ actually constitutes a study of an iterated limit of the matrix $U(\alpha,\varepsilon) = \sum_{t\geq 0} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t$, first as $\alpha \to 1$ and then as $\varepsilon \to 0$, where $P_\varepsilon^*$ is the stationary distribution matrix of the perturbed Markov chain. This immediately raises the question of whether the latter is consistent with the reverse iterated limit of $U(\alpha,\varepsilon)$. To the best of our knowledge, this question has not been studied up to now.

The fundamental matrix of a perturbed Markov chain is also used in applications of stochastic analysis to combinatorial optimisation and control theory. For instance, in [7], [4] the $[1,1]$-element of the fundamental matrix of Markov chains associated with the Hamiltonian cycle problem embedded in Markov decision processes provides the global minimum precisely for the chains corresponding to Hamiltonian cycles. These Markov decision processes, in fact, constitute a family of perturbed Markov chains. The problem is that they may be periodic and in general they allow a multichain

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structure. Thus the question of whether the corresponding fundamental matrices can be adequately approximated by matrices of the form $U(\alpha,\varepsilon)$ becomes quite relevant. Of course, the interchangeability of the limits (with respect to $\alpha$ and $\varepsilon$) is a key requirement in this context.

Consequently, in this note we derive necessary and sufficient conditions for the above interchangeability of limits and, in the process, we characterise certain continuity properties of the fundamental matrix in terms of the perturbation parameter, for perturbed Markov chains with a possibly periodic and/or multichain structure.

Consider a Markov chain $\xi_t$, $t = 0, 1, \ldots$, on a finite state space $\boldsymbol{S}$ with (stationary) transition matrix $P$, which has entries $p_{ij} = \mathsf{P}\{\xi_{t+1} = j \,|\, \xi_t = i\}$. The *Cesaro sum limit* ([15])

$$P^* = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} P^{(t-1)}$$

always exists and is also called the *stationary matrix*. Here $P^{(0)} = \mathbf{I}$ and $P^{(t)}$ is the $t$-th iterate of $P$. The *fundamental matrix*

$$Z = (\mathbf{I} - P + P^*)^{-1}$$

exists as well and is given by

$$Z = \lim_{\alpha\uparrow 1} \sum_{t=0}^{\infty} \alpha^t (P - P^*)^t = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{n=1}^{t} (P - P^*)^{n-1}. \tag{1.1}$$

The stationary matrix $P^*$ is a solution to

$$P^*P = PP^* = P^* \tag{1.2}$$

and the fundamental matrix $Z$ is a solution to

$$\begin{aligned} ZP^* &= P^*Z = P^* \\ PZ &= ZP = Z + P^* - \mathbf{I}. \end{aligned} \tag{1.3}$$

Note that the stationary matrix is, generally, not the unique solution to (1.2). In contrast, the fundamental matrix is the unique solution to (1.3). A less restricted system suffices: suppose that the matrix $A$ solves

$$P^*X = P^* \tag{1.4}$$

$$PX = X + P^* - \mathbf{I}, \tag{1.5}$$

then $A = Z$. Indeed, let $A$ and $Z$ be two solutions of (1.4)-(1.5). Then, by (1.5), $A - Z = P(A - Z)$. Iterating this, we obtain $A - Z = P^{(t)}(A - Z)$, $t = 0, 1, 2, \ldots$. Hence

$$A - Z = \frac{1}{T} \sum_{t=0}^{T-1} P^{(t)}(A - Z).$$

Taking the limit $T \to \infty$ yields that $A - Z = P^*(A - Z) = 0$ by virtue of (1.4). We will need this result later on.

Instead of the matrix $P$ we consider a set of perturbed but stochastic matrices $P_\varepsilon$, $\varepsilon \in (0, \varepsilon_0]$, such that $P_\varepsilon$ is close $P$ for $\varepsilon$ sufficiently small. In other words, we assume that

$$\lim_{\varepsilon\downarrow 0} P_\varepsilon = P.$$

**Example 1.** Take any multichain Markov chain with transition matrix $P$. Define $P_\varepsilon = (1-\varepsilon)P + \varepsilon\mathbf{I}$.

**Example 2.** Take the same multichain Markov chain. Define $\mathbf{E}$ by $\mathbf{E}_{ij} = 1$ for all $i, j \in \boldsymbol{S}$. Define $P_\varepsilon = (1-\varepsilon)P + \frac{\varepsilon}{\#S}\mathbf{E}$.

All quantities associated with an $\varepsilon$-perturbation have a subscript $\varepsilon$. For notational convenience, we indicate all operators associated with the original matrix by subscript 0, so that we write $P_0$ instead of $P$, and so on.

One might expect that $Z_0$ can be approached by $\sum_{t=0}^{\infty} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t$ for $\alpha$ close to 1 and $\varepsilon$ sufficiently small. The latter expression may be computationally more tractable in some contexts. Preferably, one would also wish to have bounds on the difference between the limit and its approximation.

In order to investigate this problem, we recall the definition of a regular perturbation. A collection of $\varepsilon$-perturbations, $(0, \varepsilon_0]$, is called *regular*, if $\lim_{\varepsilon\downarrow 0} P_\varepsilon^* = P_0^*$. Schweitzer ([11], Theorem 5) showed that $\lim_{\varepsilon\downarrow 0} P_\varepsilon^* = P_0^*$ if and only if $\lim_{\varepsilon\downarrow 0} \nu_\varepsilon = \nu_0$, where $\nu_\varepsilon$ is the number of closed classes in the $\varepsilon$-perturbed Markov chain. Since $\nu_\varepsilon$ has only integer values, the latter holds if and only if there exists $\varepsilon_1$, $0 < \varepsilon_1 < \varepsilon_0$, such that $\nu_\varepsilon = \nu_0$ for $\varepsilon \le \varepsilon_1$.

As a consequence, Example 1 is a regular perturbation, whereas Example 2 is not. The latter is called a *singular* perturbation.

# 2 Main Results

As before, we define

$$U(\alpha, \varepsilon) = \sum_{t\ge 0} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t.$$

By virtue of (1.1), $\lim_{\alpha\uparrow 1} U(\alpha, \varepsilon) = Z_\varepsilon$. The posed problem now reduces to investigating the following two questions:
**(i)** Under what conditions

$$\lim_{\alpha\uparrow 1} \lim_{\varepsilon\downarrow 0} U(\alpha, \varepsilon) = \lim_{\varepsilon\downarrow 0} \lim_{\alpha\uparrow 1} U(\alpha, \varepsilon)? \tag{2.1}$$

**(ii)** Under what conditions is this limit equal to $Z_0$?

The following theorem supplies an essentially complete answer to the above questions.

**Theorem 2.1** *The limit on either side of (2.1) is finite if and only if the $\varepsilon$-perturbation is regular. In that case we have*

$$\lim_{\alpha\uparrow 1} \lim_{\varepsilon\downarrow 0} U(\alpha, \varepsilon) = Z_0 = \lim_{\varepsilon\downarrow 0} \lim_{\alpha\uparrow 1} U(\alpha, \varepsilon).$$

While this result is not unexpected[1] - to the best of our knowledge - it has not been formally stated and proved, hitherto. Furthermore, its proof is not entirely straightforward and hence it has been divided into a sequence of four lemmata that are proved in Sections 3-5, below. The latter also derive some of the tools needed to establish an approximation result that is discussed next.

It is easy to see that (cf. Schweitzer[11], Theorem 5) there is an enumeration of the recurrent classes $R_0^n$ of $P_0$ (respectively, $R_\varepsilon^n$ of $P_\varepsilon$) for $0 < \varepsilon \le \varepsilon_1$, such that $R_0^n \subset R_\varepsilon^n$, $n = 1, \ldots, \nu_0$. Choose $b_n \in R_0^n$, $n = 1, \ldots, \nu_0$, and denote $\mathfrak{B} = \{b_1, \ldots, b_{\nu_0}\}$. We will call $b_n$ a *reference state* and $\mathfrak{B}$ a *set*

[1] It may, indeed, be known to some experts.

*of reference states* (for the chain $\xi_{0,t}$ associated with $P_0$). Note that $\mathfrak{B}$ is a set of reference states for all $\varepsilon$-perturbations, with $\varepsilon \leq \varepsilon_1$. Let $d$ denote the period of the Markov chain associated with $P_0$. One can choose $\varepsilon_1$ small enough, so that the period of all $\varepsilon$-perturbations, $\varepsilon \leq \varepsilon_1$, is at most equal to $d$.

Choose $0 < \delta < 1$, $t_0$ and $\varepsilon_2$, $0 < \varepsilon_2 \leq \varepsilon_1$, such that for all $i \in \boldsymbol{S}$ there exists $k \in \{0, \ldots, d-1\}$ for which

$$\sum_{b \in B} p_{ib,\varepsilon}^{(t_0-k)} \geq \delta, \qquad \text{for } \varepsilon \leq \varepsilon_2. \tag{2.2}$$

This is possible for $\delta$ small enough, that is, for $\delta < \min_{i \in S, b \in \mathfrak{B}} \pi_{ib}$.

Furthermore, let

$$\eta = (1 - \frac{d-1}{t_0})^{t_0}, \quad c = 2 \cdot \frac{1}{(1-\eta^2\delta^2)^{1-t_0^{-1}}}, \quad \beta = (1-\eta^2\delta^2)^{1/t_0}. \tag{2.3}$$

The main approximation result of this paper can now be summarized in the statement of the following theorem.

**Theorem 2.2** *Let a set of regular $\varepsilon$-perturbations, be given, such that $\nu_\varepsilon = \nu_0$, $\varepsilon \leq \varepsilon_1$. Let $0 < \delta < 1$ and $t_0$ be such that (2.2) holds. Choose constants $c$ and $\beta$ as in (2.3). Fix time $N > 1$ and let positive constants $\gamma$ and $\varepsilon_3 \leq \varepsilon_2$, be such that*

$$\sup_i \sum_j |\, p_{ij,\varepsilon}^{(t)} - p_{ij,0}^{(t)}| \leq \gamma, \quad \textit{for} \quad \varepsilon \leq \varepsilon_3, t < N.$$

*Then*

$$\sum_j |U_{ij}(\alpha,\varepsilon) - Z_{ij,0}| \leq 2cN\frac{\beta^{N-1}}{1-\beta} + 2N\gamma + c \cdot \frac{(1-\alpha)}{(1-\beta)^2}, \quad \varepsilon \leq \varepsilon_3.$$

As mentioned earlier, the proofs of these results are broken up into a number of components. One reason is that the proof of Theorem 2.1 requires distinguishing between aperiodic and periodic perturbations.

# 3 Characterizing regularity

In this section we prove a lemma that contains some useful conditions characterizing regular perturbations. Note that to establish Theorem 2.1, we shall need to show that the existence of the limit $\lim_{\alpha\uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha,\varepsilon)$ implies the existence of the limit $\lim_{\varepsilon\downarrow 0} Z_\varepsilon$.

**Lemma 3.1** **i)** *Suppose that either limit*

$$\lim_{\varepsilon\downarrow 0} Z_\varepsilon = A$$

*or*

$$\lim_{\alpha\uparrow 1}\lim_{\varepsilon\downarrow 0} U(\alpha,\varepsilon) = A$$

*exists for some finite $\boldsymbol{S} \times \boldsymbol{S}$-matrix $A$. Then, $A = Z_0$ and the set of $\varepsilon$-perturbations is regular, that is, $\lim_{\varepsilon\downarrow 0} P_\varepsilon^* = P_0^*$.*

**ii)** *If the set of $\varepsilon$-perturbations is regular, then $\lim_{\alpha\uparrow 1}\lim_{\varepsilon\downarrow 0} U(\alpha,\varepsilon) = Z_0$.*

*Proof.* Assume that the limit $\lim_{\varepsilon\downarrow 0} Z_\varepsilon = A$ exists for some finite matrix $A$. We adapt an argument from Spieksma ([13], p.72). Consider the set of matrix limit points of $P^*_\varepsilon$ as $\varepsilon \downarrow 0$: these are also stochastic matrices. Take any limit point $B$ and a sequence $\{\varepsilon_n\}_n \downarrow 0$ such that $P^*_{\varepsilon_n} \to B$, $n \to \infty$.

First note that $P_{\varepsilon_n} P^*_{\varepsilon_n} = P^*_{\varepsilon_n} P_{\varepsilon_n} = P^*_{\varepsilon_n}$ (1.2) implies

$$P_0 B = B P_0 = B, \tag{3.1}$$

by taking the limit $n \to \infty$ and using that all matrices are finite matrices. Similarly, (1.4) for $\varepsilon_n$ implies that

$$BA = B. \tag{3.2}$$

By (1.5),

$$A = \lim_{n\to\infty} Z_{\varepsilon_n} = \lim_{n\to\infty} (P_{\varepsilon_n} Z_{\varepsilon_n} + \mathbf{I} - P^*_{\varepsilon_n}) = P_0 A + \mathbf{I} - B.$$

Hence,

$$A = \mathbf{I} - B + P_0 A, \tag{3.3}$$

whence by subtracting $zP_0A$ from both sides, $z \in \mathbf{C}$,

$$(\mathbf{I} - zP_0)A = \mathbf{I} - B + (1-z)P_0 A. \tag{3.4}$$

For $z$, with $|z| < 1$, the inverse matrix $(\mathbf{I} - zP_0)^{-1}$ exists and equals

$$(\mathbf{I} - zP_0)^{-1} = \sum_{t\geq 0} z^t P_0^{(t)}.$$

Multiplying both sides of (3.4) by this inverse matrix yields

$$\begin{aligned} A &= \left[\sum_{t\geq 0} z^t P_0^{(t)}\right] (\mathbf{I} - B) + (1-z) \left[\sum_{t\geq 0} z^t P_0^{(t)}\right] P_0 A \\ &= \sum_{t\geq 0} z^t (P_0^{(t)} - B) + (1-z) \left[\sum_{t\geq 0} z^t P_0^{(t)}\right] P_0 A, \end{aligned} \tag{3.5}$$

since $P_0^{(t)} B = B$, for each $t = 0, 1, \cdots$, by virtue of (3.1). Now, putting $z = x \in \mathbf{R}_+$ and taking the limit as $x \uparrow 1$ in (3.5) implies

$$A = \lim_{x\uparrow 1} \sum_{t\geq 1} x^t (P_0^{(t)} - B) + P_0^* A,$$

because

$$\lim_{x\uparrow 1} (1-x) \left[\sum_{t\geq 0} x^t P_0^{(t)}\right] P_0 A = P_0^* P_0 A = P_0^* A,$$

where the first equality above follows from finiteness of the state space and the fact that $\lim_{x\uparrow 1}(1 - x)\sum_{t\geq 0} x^t P_0^{(t)} = P_0^*$ by a now classical result due to Blackwell [5]. Next, for $x < 1$

$$\sum_{t\geq 0} x^t (P_0^{(t)} - B) = \sum_{t\geq 0} x^t (P_0^{(t)} - P_0^*) + \sum_{t\geq 0} x^t (P_0^* - B), \tag{3.6}$$

because $\sum_t x^t$ converges. Since $\lim_{x\uparrow 1}\sum_{t\geq 0} x^t(P_0^{(t)} - B)$ and $\lim_{x\uparrow 1}\sum_{t\geq 0} x^t(P_0^{(t)} - P_0^*)$ both exist as finite limits, necessarily the limit $\lim_{x\uparrow 1}\sum_{t\geq 0} x^t(P_0^* - B)$ exists. However, $P_0^* - B$ is independent of $t$, and so this limit can only exist if $P_0^* = B$.

Any limit point of the sequence $P_{\varepsilon_n}^*$ must henceforth be equal to $P_0^*$ and so we have $\lim_{\varepsilon\downarrow 0} P_\varepsilon^* = P_0^*$.

As a consequence, the limit matrix $A$ is a solution to (3.3) and (3.2) for $B = P_0^*$. This is the determining set of equations (1.4), (1.5) for the fundamental matrix $Z_0$, so that $A = Z_0$.

Next, we assume that

$$\lim_{\alpha\uparrow 1}\lim_{\varepsilon\downarrow 0} U(\alpha,\varepsilon) = A$$

for some finite matrix $A$.

Again we consider the set of matrix limit points of the sequence $\{P_\varepsilon^*\}_{\varepsilon\downarrow 0}$. Take any limit point $B$ and a sequence $\varepsilon_n$ such that $\lim_{n\to\infty} P_{\varepsilon_n}^* = B$. As before, $BP_0 = P_0B = B$.

By dominated convergence

$$\begin{aligned}\lim_{n\to\infty} U(\alpha,\varepsilon_n) &= \lim_{n\to\infty}\sum_{t\geq 0}\alpha^t(P_{\varepsilon_n} - P_{\varepsilon_n}^*)^t\\ &= \sum_{t\geq 0}\alpha^t(P_0 - B)^t\\ &= \sum_{t\geq 0}\alpha^t(P_0^{(t)} - B) + B. \qquad (3.7)\end{aligned}$$

One can now follow the argument from (3.6) onwards to obtain that $B = P_0^* = \lim_{\varepsilon\downarrow 0} P_\varepsilon^*$. Taking the limit $\alpha\uparrow 1$, yields $\lim_{\alpha\uparrow 1}\lim_{n\to\infty} U(\alpha,\varepsilon_n) = Z_0$.

In order to prove (ii), we assume that the set of $\varepsilon$-perturbations is regular. Then

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} U(\alpha,\varepsilon) &= \lim_{\varepsilon\downarrow 0}\sum_{t\geq 0}\alpha^t(P_\varepsilon - P_\varepsilon^*)^t\\ &= \sum_{t\geq 0}\alpha^t(P_0 - P_0^*)^t,\end{aligned}$$

by dominated convergence, regularity and the fact that $|(P_0 - P_0^*)^t_{ij}| \leq 2$ for $i,j\in\boldsymbol{S}$ and $t\geq 0$. Taking the limit $\alpha\uparrow 1$ yields the required result. QED

# 4 Analysis of aperiodic perturbations

For completing the proof of Theorem 2.1, it suffices to show that regularity of the $\varepsilon$-perturbations implies the existence of the limit $\lim_{\varepsilon\downarrow 0} Z_\varepsilon$ or, equivalently, of the second iterated limit $\lim_{\varepsilon\downarrow 0}\lim_{\alpha\uparrow 1} U(\alpha,\varepsilon)$.

This follows largely from results by Dekker and Hordijk [6] on denumerable state Markov decision chains with compact action spaces. One can view the perturbation parameter $\varepsilon$ as a control parameter. Their conditions translated to the finite state space case, reduce to: (i) continuity of the one-step transition probabilities and the one-step rewards as a function of the control parameter, and (ii) continuity of the number of closed classes as a function of the stationary, deterministic policies. A consequence of their theory is, that the fundamental matrix is a continuous matrix function of the stationary, deterministic policies.

Here we would have the same situation, if we would allow to 'mix' the perturbation parameters across initial states. However, this is merely a formal matter. The proofs in Dekker and Hordijk for deriving continuity of the fundamental matrix, apply in the general setting of a compact parameter space, the parameter indexing policies or a perturbation parameter, etc.

However, these proofs do not yield any bounds. We prefer to give an alternative proof using coupling techniques, that does allow us to obtain the bounds as in Theorem 2.2. For the bounds we do not use any prior knowledge of the stationary distributions at play.

However, to achieve this we have to reduce the problem to the case of aperiodic $\varepsilon$-perturbations, that is, that all transition matrices $P_\varepsilon$ correspond to aperiodic Markov chains, for $\varepsilon$ sufficiently small. Assuming this, one has (cf. [1])

$$Z_\varepsilon = \sum_{t\geq 0}(P_\varepsilon - P_\varepsilon^*)^t = \sum_{t\geq 0}(P_\varepsilon^{(t)} - P_\varepsilon^*) + P_\varepsilon^*. \tag{4.1}$$

The proof for the general case (allowing periodicity) will follow from a transformation trick.

In order to obtain bounds, we will need a result on uniform exponentially quick convergence of the marginal distributions to the stationary matrix. To this end we will use a standard coupling argument (cf. Thorisson [14]), that we will describe first.

Let $\{X_t, X_t'\}_{t=0,1,\ldots}$ be a bivariate stochastic process on $\boldsymbol{S}\times\boldsymbol{S}$ and let $\tau$ be a stopping time, such that $\mathsf{P}\{X_t = X_t', t\geq\tau\} = 1$, that is, from stage $\tau$ on the chains coincide. Then we have the following bound:

$$\begin{aligned}
\sum_j |\mathsf{P}\{X_t = j\} - \mathsf{P}\{X_t' = j\}| &\leq \sum_j \mathsf{E}|\{\mathbf{1}_{\{X_t=j\}} - \mathbf{1}_{\{X_t'=j\}}| \\
&\leq \sum_j \mathsf{E}|\mathbf{1}_{\{X_t=j,\tau\leq t\}} - \mathbf{1}_{\{X_t'=j,\tau\leq t\}}| + \sum_j \mathsf{E}|\mathbf{1}_{\{X_t=j,\tau>t\}} - \mathbf{1}_{\{X_t'=j,\tau>t\}}| \\
&= \sum_j \mathsf{E}|\mathbf{1}_{\{X_t=j,\tau>t\}} - \mathbf{1}_{\{X_t'=j,\tau>t\}}| \\
&\leq 2\mathsf{E}\mathbf{1}_{\{\tau>t\}} = 2\mathsf{P}\{\tau>t\}.
\end{aligned} \tag{4.2}$$

We aim to apply this by constructing a bivariate stochastic process in such a way that $\mathsf{P}\{X_t = j\} = p_{ij,\varepsilon}^{(t)}$ and $\mathsf{P}\{X_t' = j\} = p_{ij,\varepsilon}^*$. Then the distance of marginal and stationary probabilities will be bounded by tail distribution of the coupling time.

Let us now be given a Markov chain with transition matrix $P$. Let $R_1,\ldots,R_\nu$ be the recurrent classes of this chain, $R = \cup_{n=1}^\nu R_n$ and $T = \boldsymbol{S}\setminus R$ is the set of transient states. Choose a set of reference states $B = \{b_1,\ldots,b_\nu\}$, $b_n\in R_n$.

Fix $i\in R_n$. Let $\{X_t\}_t$ and $\{\tilde{X}_t\}_t$ be two independent Markov chains with the same transition probabilities, but different, independent, initial distributions: $\mathsf{P}\{X_0 = j\} = \delta_{ij}$ and $\mathsf{P}\{\tilde{X}_0 = j\} = p_{b_n j}^*$, $j\in\boldsymbol{S}$. Let $\tau = \min\{t\geq 1\,|\,X_t = \tilde{X}_t = b_n\}$. Define a third Markov chain $\{X_t'\}_t$ by

$$X_t' = \begin{cases} \tilde{X}_t, & t\leq\tau \\ X_t, & t>\tau. \end{cases} \tag{4.3}$$

This is essentially the usual coupling described in Thorisson ([14]), Chapter 2.4. Now, (4.2) implies that

$$\sum_j |\,p_{ij}^{(t)} - p_{ij}^*| \leq 2\mathsf{P}\{\tau>t\}. \tag{4.4}$$

In the case of initial states that are transient, we have to couple the two processes somewhat differently. One has to ensure that the two processes are absorbed, with probability 1, in the same recurrent class.

So let $i \in T$. Let $a_i^n$ denote the absorption probability in the class $R^n$:

$$a_i^n = \mathsf{P}\{\cup_{t\geq 1}\{X_t \in R_n\} \mid X_0 = i\}.$$

Then $p^*_{ij} = a_i^n \, p^*_{b_n j}$, $j \in \boldsymbol{S}$, $\sum_{j\in R_n} p^*_{ij} = a_i^n$ and $\sum_{n=1}^{\nu} a_i^n = 1$.

Divide the unit interval $(0,1]$ up into $\nu$ left open, right closed intervals $I_i(n)$ of length $a_i^n$. Each of these intervals we further partition into denumerably many left open, right closed intervals $I(t,j)$ of length $|I(t,j)| = \mathsf{P}\{X_t = j, X_k \notin R_n, k < t | X_0 = i\}$, $t = 1,\ldots,$ $j \in R_n$. Additionally, we partition each interval $I(t,j)$ into $(\#T)^{t-1}$ left open, right closed intervals $I(t,j,i_1,\ldots,i_{t-1})$, with $i_1,\ldots,i_{t-1} \in T$, of length

$$|I(t,j,i_1,\ldots,i_{t-1})| = \mathsf{P}\{X_1 = i_1,\ldots,X_{t-1} = i_{t-1} \mid X_0 = i, X_t = j\}|I(t,j)|.$$

Finally, we partition each interval $I(t,j,i_1,\ldots,i_{t-1})$ into left open, right closed intervals $I(t,j,i_1,\ldots,i_{t-1},j')$ of length $p^*_{b_n j'}|I(t,j,i_1,\ldots,i_{t-1})|$, $j' \in R_n$.

At time 0, we select a random number $u$ from the unit interval $(0,1]$. The interpretation is that - if $u \in I_i(n)$ - this selects the recurrent class that the two chains to be constructed, will be absorbed in. Furthermore, if $u$ is also in $I(t,j)$, then this selects the time $t$ and the particular entrance state $j$ into that recurrent class, for the chain starting at $i \in T$. If, additionally, $u \in I(t,j,i_1,\ldots,i_{t-1})$, then this selects the sequence of transient states that leads to hitting $R_n$ at time $t$ in state $j$. Finally, if $u \in I(t,j,i_1,\ldots,i_{t-1},j')$, then $j'$ is the initial state for the chain starting with initial distribution $p^*_{i\cdot}$, with $j,j'$ belonging to to the same recurrent class.

Suppose now that the randomly sampled number from $(0,1]$ is $u \in I(t_0,j_0,i_1,\ldots,i_{t_0-1},j_0')$, with $j_0,j_0' \in R_n$. Run two independent Markov chains $\{X_{t_0+t}\}_{t\geq 0}$ and $\{\tilde{X}_t\}_{t\geq 0}$, with the same transition matrix $P$, and initial states $\mathsf{P}\{X_{t_0} = j\} = \delta_{j_0 j}$ and $\mathsf{P}\{\tilde{X}_0 = j\} = \delta_{j_0' j}$.

This whole construction forces both chains $\{X_t\}_{t\geq 0}$ and $\{\tilde{X}_t\}_{t\geq 0}$ to absorb in the same recurrent class with probability 1, while preserving the marginal probabilities of the states.

Moreover, given that both processes enter recurrent class $R_n$, the distributions of the two processes upto the entrance time, are independent. And last, but not least, conditioning on $X_t = i'$, yields a bivariate stochastic process, the (conditional) distribution of which at time $t+s$ is the same as the distribution at time $s$ of the bivariate process starting with initial state $X_0 = i'$. This is important in deriving a bound on the tail distribution of the coupling time.

Define $\tau = \min\{t \geq 0 \mid X_t = \tilde{X}_t, X_t \in \{b_1,\ldots,b_\nu\}\}$. For the processes to couple, it is necessary for the first process $X_t$ to have entered the set of recurrent states. Again, define a third chain $\{X_t'\}_{t\geq 0}$ by (4.3). Then (4.4) holds. In the aperiodic case $d = 1$, so that (2.2) reduces to

$$\inf_i \sum_{b\in B} p^{(t_0)}_{ib,\varepsilon} \geq \delta, \quad \varepsilon \leq \varepsilon_2,$$

and in (2.3) we have $\eta = 1$.

**Lemma 4.1** *Assume the existence of $\varepsilon' \leq \varepsilon_0$, such that the collection of $\varepsilon$-perturbations, $0 < \varepsilon \leq \varepsilon'$, is regular. Assume that all transition matrices $P_\varepsilon$, $0 \leq \varepsilon' \leq \varepsilon$ are aperiodic. Let $\varepsilon_1 \leq \varepsilon'$ be such $\nu_\varepsilon = \nu_0$ for $\varepsilon \leq \varepsilon_1$. Then for the constants $\delta$, $\varepsilon_2$, $c$ and $\beta$ given by (2.2) and (2.3) with constant $\eta = 1$, we have*

$$\sup_i \sum_j |\, p^{(t)}_{ij,\varepsilon} - \, p^*_{ij,\varepsilon}| \leq c \cdot \beta^t, \quad t = 1,2,\ldots, \quad \varepsilon \leq \varepsilon_2. \tag{4.5}$$

*Proof.* Fix $\varepsilon \leq \varepsilon_2$. We consider the 'embedded' transition matrices $P_\varepsilon^{(tt_0)}$, at times $0, t_0, 2t_0, \ldots$. The embedded chain $\{X_t\}_t$ has the same stationary matrix as the original one. Define a coupling as described above, for the *embedded* chains. Note that given that $X_0 \in R^n_\varepsilon$, the stopping time $\tau$ is a coupling time only since $\nu_\varepsilon = \nu_0$!

For $i, i' \in R^n_\varepsilon$, $n = 1, \dots, \nu$, one can show by a standard induction argument that

$$\mathsf{P}\{\tau > t \,|\, X_0 = i, X'_0 = i'\} \leq (1 - \delta^2)^t. \tag{4.6}$$

Indeed,

$$\mathsf{P}\{\tau \leq 1 \,|\, X_0 = i, X'_0 = i'\} = p^{(t_0)}_{ib_n,\varepsilon}\, p^*_{i'b_n,\varepsilon} \geq \delta^2,$$

since $p^*_{i'b_n,\varepsilon} = \sum_{j \in R^n_\varepsilon} p^*_{b_n j,\varepsilon}\, p^{(t_0)}_{jb_n,\varepsilon} \geq \delta$. Now assume that (4.6) holds for $t = 1, \dots, t_0$. Then

$$\begin{aligned}
&\mathsf{P}\{\tau > t_0 + 1 \,|\, X_0 = i, X'_0 = i'\} \\
&\quad = \sum_{j,j' \in R^n_\varepsilon, j \neq b_n} \mathsf{P}\{\tau > t_0 \,|\, X_0 = j, X'_0 = j'\}\mathsf{P}\{X_1 = j, X'_1 = j' \,|\, X_0 = i, X'_0 = i'\} \\
&\qquad + \sum_{j' \in R^n_\varepsilon, j' \neq b_n} \mathsf{P}\{\tau > t_0 \,|\, X_1 = b_n, X'_1 = j'\}\mathsf{P}\{X_1 = b_n, X'_1 = j' \,|\, X_0 = i, X'_0 = i'\} \\
&\quad \leq (1 - \delta^2)^{t_0}\mathsf{P}\{(X_1, X'_1) \neq (b_n, b_n) \,|\, X_0 = i, X'_0 = i'\} \\
&\quad = (1 - \delta^2)^{t_0}\mathsf{P}\{\tau > 1 \,|\, X_0 = i, X'_0 = i'\} \leq (1 - \delta^2)^{t_0+1}.
\end{aligned}$$

Hence, for $i \in R^n_\varepsilon$, $n = 1, \dots, \nu$,

$$\mathsf{P}\{\tau > t \,|\, X_0 = i\} \leq (1 - \delta^2)^t. \tag{4.7}$$

If $i$ is transient for the $\varepsilon$-perturbed chain, one has

$$\begin{aligned}
\mathsf{P}\{\tau \leq 1 \,|\, X_0 = i\} &= \sum_{n=1}^{\nu} \mathsf{P}\{I(1, b_n, b_n) \,|\, X_0 = i\} \\
&= \sum_{n=1}^{\nu} p^{(t_0)}_{ib_n,\varepsilon}\, p^*_{b_n b_n,\varepsilon} \\
&\geq \delta \sum_{n=1}^{\nu} p^{(t_0)}_{ib_n,\varepsilon} \geq \delta^2.
\end{aligned}$$

This implies (4.7) for $t = 1$.

Assume that (4.7) holds for $t = 1, \dots, t_0$ and all states $i$. Then

$$\begin{aligned}
&\mathsf{P}\{\tau > t_0 + 1 \,|\, X_0 = i\} \\
&\quad = \sum_{i' \notin \mathcal{B}} \mathsf{P}\{\tau > t_0 + 1 \,|\, X_0 = i, X_1 = i'\}\mathsf{P}\{X_1 = i' \,|\, X_0 = i\} \\
&\qquad + \sum_{n=1}^{\nu} \sum_{j' \in R^n_\varepsilon, \neq b_n} \mathsf{P}\{\tau > t_0 + 1 \,|\, X_0 = i, X_1 = b_n, X'_1 = j'\}\mathsf{P}\{X_1 = b_n, X'_1 = j' \,|\, X_0 = i\} \\
&\quad = \sum_{i' \notin \mathcal{B}} \mathsf{P}\{\tau > t_0 \,|\, X_1 = i'\}\mathsf{P}\{X_1 = i' \,|\, X_0 = i\} \\
&\qquad + \sum_{n=1}^{\nu} \sum_{j' \in R^n_\varepsilon, \neq b_n} \mathsf{P}\{\tau > t_0 \,|\, X_0 = b_n, X'_0 = j'\}\mathsf{P}\{X_1 = b_n, X'_1 = j' \,|\, X_0 = i\} \\
&\quad \leq (1 - \delta^2)^{t_0}\mathsf{P}\{\tau > 1 \,|\, X_0 = i\} \leq (1 - \delta^2)^{t_0+1}.
\end{aligned}$$

Hence, by (4.4)

$$\sum_j |\, p_{ij,\varepsilon}^{(tt_0)} - p_{ij,\varepsilon}^*| \le 2(1-\delta^2)^t.$$

Moreover, for $k = tt_0 + r$, $0 < r < t_0$,

$$\begin{aligned}\sum_j |\, p_{ij,\varepsilon}^{(k)} - p_{ij,\varepsilon}^*| &= \sum_j |\sum_v p_{iv,\varepsilon}^{(r)}(\, p_{vj,\varepsilon}^{(tt_0)} - p_{vj,\varepsilon}^*)| \\ &\le \sum_v p_{iv,\varepsilon}^{(r)} \sum_j |\, p_{vj,\varepsilon}^{(tt_0)} - p_{vj,\varepsilon}^*| \\ &\le 2(1-\delta^2)^t.\end{aligned}$$

Combining yields

$$\sum_j |\, p_{ij,\varepsilon}^{(t)} - p_{ij,\varepsilon}^*| \le 2 \cdot (1-\delta^2)^{\lfloor t/t_0 \rfloor} \le c \cdot \beta^t.$$

QED

**Lemma 4.2** *Assume the existence of $\varepsilon' \le \varepsilon_0$, such that the collection of $\varepsilon$-perturbations, $0 < \varepsilon \le \varepsilon'$, is regular. Assume that all transition matrices $P_\varepsilon$, $0 \le \varepsilon \le \varepsilon'$ are aperiodic. Then*

$$\lim_{\varepsilon \downarrow 0} Z_\varepsilon = Z_0.$$

*Proof.* By Lemma 4.1, there exist $\varepsilon_2 > 0$, $\varepsilon_2 < \varepsilon'$, and positive constants $\beta < 1$ and $c$ such that (4.5) holds for $\varepsilon \le \varepsilon_2$. Hence, by dominated convergence and regularity,

$$\sum_t (\, p_{ij,\varepsilon_n}^{(t)} - p_{ij,\varepsilon_n}^*) \to \sum_t (\, p_{ij,0}^{(t)} - p_{ij,0}^*), \quad n \to \infty,$$

for any sequence $\varepsilon_n \to 0$, $n \to \infty$. By regularity and (4.1), we obtain that $\lim_{n\to\infty} Z_{\varepsilon_n} = Z_0$. QED

# 5 The aperiodic transformation

This section discusses a data-transformation (cf. Schweitzer [12]), showing that is sufficient to restrict to the aperiodic case.

Following the construction of Example 1, let $\lambda \in (0,1)$ and put $P_{\varepsilon,\lambda} = \lambda \mathbf{I} + (1-\lambda)P_\varepsilon$. All previous notation with subscript $\lambda$ added, denotes the corresponding quantity for the transformed chains. By inspection one has $P_{\varepsilon,\lambda}^* = P_\varepsilon^*$. First, we note that

$$\begin{aligned}U_\lambda(\alpha,\varepsilon) &= \sum_{t=0}^{\infty} \alpha^t (P_{\varepsilon,\lambda} - P_{\varepsilon,\lambda}^*)^t \\ &= \sum_{t=0}^{\infty} \alpha^t P_{\varepsilon,\lambda}^{(t)} - \frac{\alpha}{1-\alpha} P_{\varepsilon,\lambda}^*.\end{aligned}$$

Next, we rewrite the first term on the right side of the last equality in terms powers of $P_\varepsilon$:

$$\sum_{t=0}^{\infty} \alpha^t P_{\varepsilon,\lambda}^{(t)} = \sum_{t=0}^{\infty} \alpha^t (\lambda \mathbf{I} + (1-\lambda)P_\varepsilon)^t$$

$$= \sum_{t=0}^{\infty} \Big( \sum_{n=0}^{t} \binom{t}{n} \lambda^{t-n} (1-\lambda)^n P_\varepsilon^{(n)} \Big) \alpha^t$$

$$= \sum_{n=0}^{\infty} \alpha^n (1-\lambda)^n P_\varepsilon^{(n)} \sum_{t \geq n} \binom{t}{n} (\lambda\alpha)^{t-n}$$

$$= \frac{1}{1-\lambda\alpha} \sum_{n=0}^{\infty} \Big[ \frac{(1-\lambda)\alpha}{1-\lambda\alpha} \Big]^n P_\varepsilon^{(n)}.$$

Putting $\alpha' = (1-\lambda)\alpha/(1-\lambda\alpha)$, yields the relation

$$U_\lambda(\alpha,\varepsilon) = \frac{1}{1-\lambda\alpha} U(\alpha',\varepsilon) - \frac{\lambda\alpha}{1-\lambda\alpha} P_\varepsilon^*. \tag{5.1}$$

Note that $\alpha \to 1 \iff \alpha' \to 1$. In view of (5.1), the fundamental matrices $\tilde{Z}_\varepsilon$ and $Z_\varepsilon$ are related by

$$Z_{\varepsilon,\lambda} = \frac{1}{1-\lambda} Z_\varepsilon - \frac{\lambda}{1-\lambda} P_\varepsilon^*. \tag{5.2}$$

We may now prove Lemma 4.2 for a general collection of $\varepsilon$-perturbations, $n = 1, 2, \ldots$.

**Lemma 5.1** *Assume the existence of $\varepsilon' \leq \varepsilon_0$, such that the collection of $\varepsilon$-perturbations is regular for $\varepsilon \leq \varepsilon'$. Then* $\lim_{\varepsilon \downarrow 0} Z_\varepsilon = Z_0$.

*Proof.* Regularity of the $\varepsilon$-perturbations implies regularity of the transformed $\varepsilon$-perturbations. Hence, Lemma 4.2 applies to the transformed set of $\varepsilon$-perturbations, so that

$$\lim_{\varepsilon \downarrow 0} Z_{\varepsilon,\lambda} = Z_{0,\lambda} = \frac{1}{1-\lambda} Z_0 - \frac{\lambda}{1-\lambda} P_0^*.$$

By (5.2) and regularity of the set of $\varepsilon$-perturbations we obtain

$$\lim_{\varepsilon \downarrow 0} Z_\varepsilon = \lim_{\varepsilon \downarrow 0} (1-\lambda) Z_{\varepsilon,\lambda} + \lambda P_\varepsilon^* = (1-\lambda) Z_{0,\lambda} + \lambda P_0^* = Z_0.$$

QED

Theorem 2.1 now follows immediately from Lemmas 3.1 and 5.1.

# 6 Bounds

To establish a bound on the difference

$$U(\alpha,\varepsilon) - Z_0,$$

as in Theorem 2.2, we can now employ the result of Lemma 4.1.

**Proof of Theorem 2.2.** First assume that for some $\varepsilon' \leq \varepsilon_3$ all $\varepsilon$-perturbations are aperiodic. Note that

$$\sum_j \sum_{t \geq N} \alpha^t |(P_\varepsilon - P_\varepsilon^*)_{ij}^t| \leq c \cdot \frac{\beta^N}{1-\beta},$$

where $\beta = (1-\delta^2)^{1/t_0}$ and $c = 2 \cdot (1-\delta^2)^{t_0^{-1}-1}$, i.e. the constants $\beta$ and $c$ are given by (2.2) for $\eta = 1$. Hence,

$$
\begin{aligned}
&\sum_j |U_{ij}(\alpha,\varepsilon) - Z_{ij,0}| \\
&\leq \sum_j \Big\{ |\sum_{t\geq N} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t_{ij}| + |\sum_{t\geq N} (P_0 - P_0^*)^t_{ij}| + |\sum_{t=1}^{N} (\alpha^t ( p^{(t)}_{ij,\varepsilon} - p^*_{ij,\varepsilon}) - p^{(t)}_{ij,0} + p^*_{ij,0})| \Big\} \\
&\leq 2c \cdot \frac{\beta^N}{1-\beta} + \sum_j \Big\{ |\sum_{t=1}^{N} \Big( ( p^{(t)}_{ij,\varepsilon} - p^{(t)}_{ij,0}) - ( p^*_{ij,\varepsilon} - p^*_{ij,0}) - (1-\alpha^t)( p^{(t)}_{ij,\varepsilon} - p^*_{ij,\varepsilon}) \Big)| \Big\} \\
&\leq 2c \cdot \frac{\beta^N}{1-\beta} + \sum_{t=1}^{N-1} \sum_j | p^{(t)}_{ij,\varepsilon} - p^{(t)}_{ij,0}| + (N-1) \sum_j | p^*_{ij,\varepsilon} - p^*_{ij,0}| + \\
&\qquad + \sum_{t=1}^{N-1} (1-\alpha^t) \sum_j | p^{(t)}_{ij,\varepsilon} - p^*_{ij,\varepsilon}|. \qquad (6.1)
\end{aligned}
$$

By assumption, the first summation is at most $(N-1)\gamma$, and the last one is at most

$$
\begin{aligned}
\sum_{t=1}^{N-1} (1-\alpha^t) c\beta^t &= c(1-\alpha) \sum_{t=1}^{N-1} \sum_{k=0}^{t-1} \alpha^k \beta^t \\
&= c(1-\alpha) \sum_{k=0}^{N-2} \alpha^k \sum_{t=k+1}^{N-1} \beta^t \leq \frac{c(1-\alpha)}{1-\beta} \sum_{k\geq 1} \alpha^k \beta^{k+1} \\
&\leq \frac{c(1-\alpha)}{(1-\beta)^2}.
\end{aligned}
$$

We have to bound the difference $\sum_j | p^*_{ij,\varepsilon} - p^*_{ij,0}|$:

$$
\begin{aligned}
\sum_j | p^*_{ij,\varepsilon} - p^*_{ij,0}| &\leq \sum_j | p^*_{ij,\varepsilon} - p^{(N-1)}_{ij,\varepsilon}| + \sum_j | p^*_{ij,0} - p^{(N-1)}_{ij,0}| + \sum_j | p^{(N-1)}_{ij,\varepsilon} - p^{(N-1)}_{ij,0}| \\
&\leq 2c\beta^{N-1} + \gamma.
\end{aligned}
$$

Combination with (6.1) yields

$$
\sum_j |U_{ij}(\alpha,\varepsilon) - Z_{ij,0}| \leq 2cN \frac{\beta^{N-1}}{1-\beta} + 2N\gamma + c\frac{1-\alpha}{(1-\beta)^2}. \qquad (6.2)
$$

This proves the theorem for the aperiodic case.

Next we turn to the general case. Fix $\lambda \in (0,1)$. We use the notation from Section 5, but we interchange the roles of $\alpha$ and $\alpha'$, so that now

$$
\alpha = \frac{(1-\lambda)\alpha'}{1-\lambda\alpha'} \quad \text{and} \quad \alpha' = \frac{\alpha}{\lambda\alpha + 1 - \lambda} > \alpha.
$$

Rewriting yields

$$
U(\alpha,\varepsilon) - Z_0 = (1-\lambda)(U_\lambda(\alpha',\varepsilon) - Z_{0,\lambda}) + \lambda(P^*_\varepsilon - P^*_0) + \lambda(1-\alpha')(U_\lambda(\alpha',\varepsilon) - P^*_\varepsilon).
$$

Fix initial state $i \in \boldsymbol{S}$. Then $p^{(t_0-k)}_{iB,\varepsilon} \geq \delta$ implies that

$$p^{(t_0)}_{iB,\varepsilon,\lambda} \geq \binom{t_0}{t_0-k}(1-\lambda)^{t_0-k}\lambda^k \cdot \delta.$$

Put $\lambda = (d-1)/t_0$, then

$$\begin{aligned} p^{(t_0)}_{iB,\varepsilon,\lambda} &\geq \binom{t_0}{t_0-k}\Big(1-\frac{d-1}{t_0}\Big)^{t_0}\Big(\frac{d-1}{t_0-d+1}\Big)^k \cdot \delta \\ &\geq \Big(1-\frac{d-1}{t_0}\Big)^{t_0} \cdot \delta = \eta\delta, \end{aligned}$$

where $\eta$ is defined in (2.3). Moreover, $\sum_j |\, p^{(t)}_{ij,\varepsilon} - p^{(t)}_{ij,0}| \leq \gamma$, $t \leq N-1$, implies

$$\begin{aligned} \sum_j |\, p^{(t)}_{ij,\varepsilon,\lambda} - p^{(t)}_{ij,0,\lambda}| &\leq \sum_{n=0}^{t}\binom{t}{n}\lambda^n(1-\lambda)^{t-n}\sum_j |\, p^{(n)}_{ij,\varepsilon} - p^{(n)}_{ij,0}| \\ &\leq \gamma, \quad t \leq N-1. \end{aligned}$$

Now, the assertion for the aperiodic case, applies to the transformed $\varepsilon$-perturbation, $\varepsilon \leq \varepsilon_3$, with constants

$$\tilde{\delta} = \eta\delta,$$

$\tilde{c} = 2 \cdot (1-\eta^2\delta^2)^{t_0^{-1}-1}$, $\tilde{\beta} = (1-\eta^2\delta^2)^{1/t_0}$ and discount factor $\alpha' > \alpha$. Note that $\tilde{\delta}$, $\tilde{\beta}$ and $\tilde{c}$ are precisely the constants from (2.3).

Putting this together, by virtue of (6.2), we get for $\lambda = (d-1)/t_0$

$$\begin{aligned} \sum_j |U_{ij}(\alpha,\varepsilon) - Z_{ij,0}| &\leq (1-\lambda)\Big(2\tilde{c}N\frac{\tilde{\beta}^{N-1}}{1-\tilde{\beta}} + 2N\gamma + \tilde{c}\frac{1-\alpha'}{(1-\tilde{\beta})^2}\Big) \\ &\quad +\lambda\Big(2\tilde{c}\tilde{\beta}^{N-1} + \gamma + (1-\alpha')\tilde{c}\frac{1}{1-\tilde{\beta}}\Big) \\ &\leq 2\tilde{c}N\frac{\tilde{\beta}^{N-1}}{1-\tilde{\beta}} + 2N\gamma + \tilde{c}\frac{1-\alpha'}{(1-\tilde{\beta})^2}. \end{aligned}$$

The result follows since $\alpha' > \alpha$. QED

# 7 Conclusion and open problems

In this paper we have characterised regular perturbations by means of the existence of finite limits in both sides of (2.1). For singular perturbations it is not clear whether either limit exists and if so, under which conditions they are equal.

Notice however, that the limit

$$\begin{aligned} \lim_{\alpha\uparrow 1}\lim_{\varepsilon\downarrow 0} U(\alpha,\varepsilon) &= \lim_{\alpha\uparrow 1}\sum_{t\geq 0}\alpha^t(P_0 - P_0^l)^t \\ &= Z_0 + \lim_{\alpha\uparrow 1}\frac{\alpha}{1-\alpha}(P_0^* - P_0^l) \end{aligned} \qquad (7.1)$$

always exists componentwise in the extended reals, provided that the limit $P_0^l := \lim_{\varepsilon\downarrow 0} P_\varepsilon^*$ exists. Haviv and Ritov [10] (see also Avrachenkov and Lasserre [3]) studied the the case of *analytic perturbations*, i.e.

$$P_\varepsilon = P_0 + \sum_{k=1}^{\infty} \varepsilon^k G_k,$$

where $G_k$ are $\boldsymbol{S}\times\boldsymbol{S}$ matrices. Under the assumption that all perturbed Markov chains are unichain, they showed that the fundamental matrix $Z_\varepsilon$ admits a Laurent expansion in a neighbourhood of $\varepsilon = 0$. This implies that also $\lim_{\varepsilon\downarrow 0} Z_\varepsilon$ exists in the extended reals.

The question is whether $\lim_{\varepsilon\downarrow 0} Z_\varepsilon = \lim_{\alpha\uparrow 1}\lim_{\varepsilon\downarrow 0} U(\alpha,\varepsilon)$. We will give two examples, one of a linearly perturbed Markov chain and one of an analytically perturbed one, where this is not the case. In fact, the examples show that in general there does not seem to be much of a connection between the values of either limit for singularly perturbed Markov chains.

**Example 1** The first example is a three state, linearly perturbed Markov chain. More particularly, $\boldsymbol{S} = \{1,2,3\}$ and

$$P_\varepsilon = \mathbf{I} + \varepsilon \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then $P_0^* = \mathbf{I}$ and $Z_0 = \mathbf{I}$, but

$$P_\varepsilon^* = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad \varepsilon > 0.$$

Denote $P_0^l = \lim_{\varepsilon\downarrow 0} P_\varepsilon^*$. By virtue of (7.1) we have

$$\lim_{\varepsilon\downarrow 0} U(\alpha,\varepsilon) = \mathbf{I} + \frac{\alpha}{1-\alpha}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \to \begin{pmatrix} \infty & -\infty & -\infty \\ -\infty & \infty & -\infty \\ -\infty & -\infty & \infty \end{pmatrix}, \quad \alpha \uparrow 1.$$

Note that $\pm\infty$ occurs precisely in those entries where $P_0^l$ and $P_0^*$ differ! This is consistent with the bounds in the previous section.

However,

$$Z_\varepsilon = P_\varepsilon^* + \begin{pmatrix} \frac{1}{3\varepsilon} & 0 & -\frac{1}{3\varepsilon} \\ -\frac{1}{3\varepsilon} & \frac{1}{3\varepsilon} & 0 \\ 0 & -\frac{1}{3\varepsilon} & \frac{1}{3\varepsilon} \end{pmatrix} \to \begin{pmatrix} \infty & \frac{1}{3} & -\infty \\ -\infty & \infty & \frac{1}{3} \\ \frac{1}{3} & -\infty & \infty \end{pmatrix}, \quad \varepsilon \downarrow 0.$$

Here we have the situation where $\lim_{\varepsilon\downarrow 0} Z_{12,\varepsilon} = 1/3$ is finite and positive, whereas $\lim_{\alpha\uparrow 1}\lim_{\varepsilon\downarrow 0} U_{12}(\alpha,\varepsilon) = -\infty$. The next example shows, that the reverse situation may happen as well.

**Example 2** Again we consider a three state Markov chain, but now we have an *analytically* perturbed one of the following form

$$P_\varepsilon = \mathbf{I} + \varepsilon \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

Then, again, $P_0^* = \mathbf{I} = Z_0$. However,

$$P_\varepsilon^* = \frac{1}{1+\varepsilon+\varepsilon^2}\begin{pmatrix} \varepsilon^2 & \varepsilon & 1 \\ \varepsilon^2 & \varepsilon & 1 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix} \to P_0^l = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon \downarrow 0.$$

By (7.1), one has

$$\lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) = \mathbf{I} + \frac{\alpha}{1-\alpha}(P_0^* - P_0^l) = \mathbf{I} + \frac{\alpha}{1-\alpha} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \infty & 0 & -\infty \\ 0 & \infty & -\infty \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha \uparrow 1.$$

On the other hand,

$$Z_\varepsilon = \frac{1}{\varepsilon^2(1+\varepsilon+\varepsilon^2)} \begin{pmatrix} \varepsilon + \varepsilon^4 & 1-\varepsilon+\varepsilon^3 & -1+\varepsilon^2 \\ -\varepsilon^2+\varepsilon^4 & 1-\varepsilon+\varepsilon^2+\varepsilon^3 & -1+\varepsilon+\varepsilon^2 \\ \varepsilon^4 & -\varepsilon+\varepsilon^3 & \varepsilon+\varepsilon^2 \end{pmatrix} \to \begin{pmatrix} \infty & \infty & -\infty \\ -1 & \infty & -\infty \\ 0 & -\infty & \infty \end{pmatrix}, \quad \varepsilon \downarrow 0.$$

Structured cases, when the discount factor is of the form $\alpha = 1 - \epsilon^k$, for a rational number $k$ and $\epsilon \downarrow 0$, may yield further insights as they did in a related analysis of the Cesaro sum limit reported in Filar et al [8]. This is potentially important for the application to the Hamiltonian cycle problem considered in [4] and [7].

One can also consider to extend our results to the denumerable state space case. In this case the fundamental matrix need not always exist, even if the stationary matrix exists. However, even when it exists, it is not clear under what conditions it can be approximated by a discounted deviation from stationarity of a perturbation for a small perturbation parameter. It would be interesting to find sufficient regularity conditions for the validity of such an approximation.

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