ON EXAMPLES OF DIFFERENCE OPERATORS FOR $\{0,1\}$-VALUED FUNCTIONS OVER FINITE SETS

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1. Introduction and basic definitions

Recently V. I. Arnold have formulated a geometrical concept of monads and apply it to the study of difference operators on the sets of $\{0,1\}$-valued sequences of length n. In [1]–[4] he made first steps in the study of this subject and formulated many nice questions. In [5] A. Garber showed an algorithm that gives a description of the combinatorial structure of monads for difference operators and answered many questions of V. I. Arnold. In the present note we show particular examples of these monads and indicate one question arising here.

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A *monad* by V. I. Arnold is a map of a finite set into itself. Suppose $M : S \to S$ is an arbitrary monad. It is naturally to associate an oriented graph to the monad M. Its vertices coincide with elements of S, and the set of its edges is the set of ordered pairs $(x, M(x))$. We denote such graph by $G(M)$ The idea of V. I. Arnold is to study the combinatorial geometry of graphs for monads. He proposed to start with one important example.

Consider any positive integer n, and take the set $A_n = \{1, \ldots, n\}$. Denote by $\mathcal{F}_2(A_n)$ the vector space of $\mathbb{Z}_2$-valued functions on A_n. Consider a "differential" *difference* operation Δ, defined as follows:

$$(\Delta f)(x) = \begin{cases} f(x) + f(x+1), & \text{if } x \neq n \\ f(n) + f(1), & \text{if } x = n \end{cases}.$$

On Figure 1 we show an example of a monad for the difference operator for the set A_6.

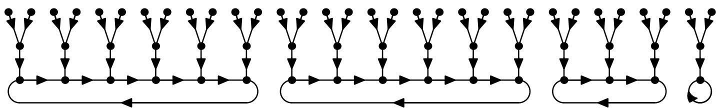

Figure 1. An example of a monad $G(\Delta)$ for the case of A_6.

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2. A few words about Arnold's complexity

Let us briefly describe the concept of Arnold's functional complexity. Further we will need the additivity property of Δ:

$$\Delta(f+g)=\Delta(f)+\Delta(g).$$

The simplest functions on the set A_n are polynomials. The set of all solutions of the functional equation $\Delta^k(f)=0$ is called the *set of polynomials of degree less than* k, we denote it by Pol_{k-1}. Suppose $f\in\mathrm{Pol}_k$ and $f\notin\mathrm{Pol}_l$ for $l=0,\dots,k-1$, then f is called a *polynomial of degree* k. Denote by $\mathrm{Pol}(A_n)$ the set of all polynomials on A_n.

Actually, the set $\mathrm{Pol}(A_n)$ is a vector space. If $n=2^l m$ where m is odd, then the space $\mathrm{Pol}(A_n)$ is 2^l-dimensional and contains 2^{2^l} elements. In particular, if $n=2^l$, then $\mathcal{F}_2(A_n)=\mathrm{Pol}(A_n)$.

Now we define another set of nice functions. Consider the functional equation $\Delta^k(f)=f$. The set of all solutions of such equation is called the *set of special rational exponential polynomials of orders that divide* k, we denote it by Exp_k. Let $\mathrm{Exp}_0=\{0\}$. Suppose $f\in\mathrm{Exp}_k$ and $f\notin\mathrm{Exp}_l$ for $l=0,\dots,k-1$, then f is called a *special rational exponential polynomial of order* k, or *exp-polynomial* for short. Denote by $\mathrm{Exp}(A_n)$ the set of all exp-polynomials on A_n. Note that the set $\mathrm{Exp}(A_n)$ is a vector space.

Proposition 2.1. *Any function f of $\mathcal{F}_2(A_n)$ can be uniquely written in the form $f=p+r$, where p is a polynomial and r is an exp-polynomial, or in other worlds*

$$\mathcal{F}_2(A_n)=\mathrm{Pol}(A_n)\oplus\mathrm{Exp}(A_n).$$

Definition 2.2. Consider an arbitrary function of $\mathcal{F}_2(A_n)$. Let $f=p+r$, where p is a polynomial and r is an exp-polynomial. We say that *degree* of f is degree of p and denote it by $\deg(f)$. We say that *order* of f is order of r and denote it by $\mathrm{ord}(f)$.

V. I. Arnold proposed the following definition the notion of functional complexity.

Definition 2.3. A function f_1 is said to be *more complicated* (*in the sense of Arnold*) than f_2 if either $\mathrm{ord}(f_1)>\mathrm{ord}(f_2)$, or $\mathrm{ord}(f_1)=\mathrm{ord}(f_2)$ and $\deg(f_1)>\deg(f_2)$.
If $\mathrm{ord}(f_1)=\mathrm{ord}(f_2)$ and $\deg(f_1)=\deg(f_2)$ then the functions f_1 and f_2 are said to be of the *same complexity* (*in the sense of Arnold*).

Remark 2.4. All the above easily can be generalized to the case of $\mathbb{Z}_N$-valued functions for an arbitrary positive integer $N>2$.

3. Some examples of $G(\Delta)$

3.1. **A list of examples.** Suppose $n=2^l m$. The set of polynomials is a 2-valent symmetric tree of $r=2^{2^l}$ elements. Denote this tree by T_r. Denote also a cycle of s elements by O_s.

Each connected component of the graph $G(\Delta)$ contains a cycle. Denote its length by s. To each vertex of the cycle it is attached a tree equivalent to T_r (as on Fig. 1 for the case $n=6$). Denote such component by $O_s{*}T_r$.

Let us enumerate connected components for the graphs of the sets $\mathcal{F}_2(A_n)$ where $n\le 25$. Expression $k(O_s*T_r)$ means that there are k components of the type O_s*T_r. The graph $G(\Delta)$ for the set $\mathcal{F}_2(A_n)$ contains
in the case of $n=1$: $O_1{*}T_2$;
in the case of $n=2$: $O_1{*}T_4$;
in the case of $n=3$: $O_1{*}T_2$, $O_3{*}T_2$;

in the case of $n=4$: $O_1{*}T_{16}$;
in the case of $n=5$: $O_1{*}T_2$, $O_{15}{*}T_2$;
in the case of $n=6$: $O_1{*}T_4$, $O_3{*}T_4$, $2(O_6{*}T_4)$;
in the case of $n=7$: $O_1{*}T_2$, $9(O_7{*}T_2)$;
in the case of $n=8$: $O_1{*}T_{256}$;
in the case of $n=9$: $O_1{*}T_2$, $O_3{*}T_2$, $4(O_{63}{*}T_2)$;
in the case of $n=10$: $O_1{*}T_4$, $O_{15}{*}T_4$, $8(O_{30}{*}T_4)$;
in the case of $n=11$: $O_1{*}T_2$, $3(O_{341}{*}T_2)$;
in the case of $n=12$: $O_1{*}T_{16}$, $O_3{*}T_{16}$, $2(O_6{*}T_{16})$, $20(O_{12}{*}T_{16})$;
in the case of $n=13$: $O_1{*}T_2$, $5(O_{819}{*}T_2)$;
in the case of $n=14$: $O_1{*}T_4$, $9(O_7{*}T_4)$, $288(O_{14}{*}T_4)$;
in the case of $n=15$: $O_1{*}T_2$, $O_3{*}T_2$, $30(O_5{*}T_2)$, $1082(O_{15}{*}T_2)$;
in the case of $n=16$: $O_1{*}T_{2^{16}}$;
in the case of $n=17$: $O_1{*}T_2$, $51(O_{85}{*}T_2)$, $240(O_{255}{*}T_2)$;
in the case of $n=18$: $O_1{*}T_4$, $O_3{*}T_4$, $2(O_6{*}T_4)$, $4(O_{63}{*}T_4)$, $518(O_{126}{*}T_4)$;
in the case of $n=19$: $O_1{*}T_2$, $27(O_{9709}{*}T_2)$;
in the case of $n=20$: $O_1{*}T_{16}$, $O_{15}{*}T_{16}$, $8(O_{30}{*}T_{16})$, $1088(O_{60}{*}T_{16})$;
in the case of $n=21$: $O_1{*}T_2$, $O_3{*}T_2$, $9(O_7{*}T_2)$, $9(O_{21}{*}T_2)$, $16640(O_{63}{*}T_2)$;
in the case of $n=22$: $O_1{*}T_4$, $3(O_{341}{*}T_4)$, $1536(O_{682}{*}T_4)$;
in the case of $n=23$: $O_1{*}T_2$, $2049(O_{2047}{*}T_2)$;
in the case of $n=24$: $O_1{*}T_{256}$, $O_3{*}T_{256}$, $2(O_6 * T_{256})$, $20(O_{12}{*}T_{256})$, $2720(O_{24}{*}T_{256})$;
in the case of $n=25$: $O_1{*}T_2$, $O_{15}{*}T_2$, $656(O_{25575}{*}T_2)$.

Denote by $s(n)$ the order of the maximal possible length of cycles for the n-elements sequences. Actually the listed examples gives the negative answer to the following question of V. I. Arnold: *is it true that* $(s(n)/n)+1$ *is some power of* 2? It is not true, for example, for $n=23$ where $s(23)=2047$. Here $s(23)$ is $2^{11}-1$ itself.

Denote by $]n[$ the set of connected components of graphs $G(\Delta)$, corresponding to the set $\mathcal{F}_2(A_n)$. The work [5] of A. Garber immediately implies the following identities:

a) $]3\cdot 2^m[=\{O_1{*}T_{2^{2^m}}, O_3{*}T_{2^{2^m}}\}\bigcup\left\{\frac{2^{3\cdot 2^k}-2^{4\cdot 2^{k-1}}}{3\cdot 2^k\cdot 2^{2^k}}(O_{3\cdot 2^k}{*}T_{2^{2^m}})\Big| k=2,\dots,m\right\}$;

b) $]5\cdot 2^m[=\{O_1{*}T_{2^{2^m}}, O_{3\cdot 5}{*}T_{2^{2^m}}\}\bigcup\left\{\frac{1}{3}\cdot\frac{2^{5\cdot 2^k}-2^{6\cdot 2^{k-1}}}{5\cdot 2^k\cdot 2^{2^k}}(O_{3\cdot 5\cdot 2^k}{*}T_{2^{2^m}})\Big| k=2,\dots,m\right\}$;

c) $]7\cdot 2^m[=\{O_1{*}T_{2^{2^m}}, 9(O_7{*}T_{2^{2^m}})\}\bigcup\left\{\frac{2^{7\cdot 2^k}-2^{8\cdot 2^{k-1}}}{7\cdot 2^k\cdot 2^{2^k}}(O_{7\cdot 2^k}{*}T_{2^{2^m}})\Big| k=2,\dots,m\right\}$;

d) $]11\cdot 2^m[=\{O_1{*}T_{2^{2^m}}, 3(O_{31\cdot 11}{*}T_{2^{2^m}})\}\bigcup\left\{\frac{1}{31}\frac{2^{11\cdot 2^k}-2^{12\cdot 2^{k-1}}}{11\cdot 2^k\cdot 2^{2^k}}(O_{31\cdot 11\cdot 2^k}{*}T_{2^{2^m}})\Big| k=2,\dots,m\right\}$

...

3.2. **Particular case of δ-functions.** Let us now study the structure of the piece-wise connected components of the graph G containing so-called *δ-function.* Denote by δ_k the following function of $\mathcal{F}_2(A_n)$:

$$\delta_k(x)=\begin{cases}0, & \text{if } x\neq k\\ 1, & \text{if } x=k\end{cases}.$$

In [5] A. Garber showed that the order of δ_k coincide with $s(n)$. So the piece-wise connected component of the graph G containing δ_k is $O_{s(k)}{*}T_{2^{2^l}}$, and it does not depend on the choice of k. We now write down the values of $s(n)$ for $n\leq 50$ in the following list.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$s(n)$	1	1	3	1	15	6	7	1	63	30	341	12	819	14	15	1	255	126	9709	60

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
$s(n)$	63	682	2047	24	25575	1638	13797	28	475107	30	31	1	1023	510	4095

n	36	37	38	39	40	41	42	43	44	45	46	47
$s(n)$	252	3233097	19418	4095	120	41943	126	5461	1364	4095	4095	8388607

n	48	49	50
$s(n)$	48	2097151	51150

There is a regularity in this sequence for primes $n > 2$. We use the following notation. Denote by $\gamma_2(n)$ the minimal solution t of the equation $2^t \equiv 1 (\mathrm{mod}\, n)$. Then

n	3	5	7	11	13	17	19
$\gamma_2(n)$	2	4	3	10	12	8	18
$s(n)$	$3(2^{\frac{2}{2}}-1)$	$5(2^{\frac{4}{2}}-1)$	2^3-1	$11(2^{\frac{10}{2}}-1)$	$13(2^{\frac{12}{2}}-1)$	$17(2^{\frac{8}{2}}-1)$	$19(2^{\frac{18}{2}}-1)$

n	23	29	31	37	41	43	47
$\gamma_2(n)$	11	28	5	36	20	14	23
$s(n)$	$2^{11}-1$	$29(2^{\frac{28}{2}}-1)$	2^5-1	$37(2^{\frac{36}{2}}-1)/3$	$41(2^{\frac{20}{2}}-1)$	$43(2^{\frac{14}{2}}-1)$	$2^{23}-1$

Denote by $q(n)$ the following function

$$q(n) := \begin{cases} n(2^{\frac{\gamma_2(n)}{2}} - 1), & \text{if } \gamma_2(n) \text{ is even} \\ 2^{\gamma_2(n)} - 1, & \text{if } \gamma_2(n) \text{ is odd} \end{cases}.$$

Note that for all observed primes (except 37) we have $s(n) = q(n)$.

Problem 1. Study the behaviour of the maximal length of the cycle. How often does it coincide with $q(n)$? Is it true that $q(n)$ is always divisible by $s(n)$?

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