Center Manifolds for Periodic Functional Differential Equations of Mixed Type

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Abstract. We study the behaviour of solutions to nonlinear functional differential equations of mixed type (MFDEs), that remain sufficiently close to a prescribed periodic solution. Under a discreteness condition on the Floquet spectrum, we show that all such solutions can be captured on a finite dimensional invariant center manifold, that inherits the smoothness of the nonlinearity. This generalizes the results that were obtained previously in [18] for bifurcations around equilibrium solutions to MFDEs.

Keywords. mixed type functional differential equation, Floquet theory, center manifold, Hopf bifurcation, finite dimensional reduction, advanced and retarded arguments

1 Introduction

In this paper we provide a tool to analyze the behaviour of solutions to a functional differential equation of mixed type (MFDE),

$$\dot{x}(\xi) = G(x_\xi), \tag{1.1}$$

that lie in the vicinity of a prescribed periodic solution. Here x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$, the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \leq 0$ and $r_{\max} \geq 0$, hence the operator G may depend on advanced and retarded arguments simultaneously.

Historically, the primary motivation for the study of MFDEs comes from the study of lattice differential equations (LDEs), which are systems of differential equations indexed by points on an (infinite) spatial lattice, e.g. $h\mathbb{Z}^n$ for some integer n and grid size $h > 0$. Such equations allow the incorporation of nonlocal interactions into otherwise local models and exhibit richer behaviour than the limiting PDEs that arise when taking $h \to 0$ [6, 11, 17, 24, 25]. For these reasons, models involving LDEs have been developed in many scientific disciplines, including chemical reaction theory [12, 21], image processing and pattern recognition [7], material science [3, 5] and biology [2, 4, 20]. As a specific example which is interesting in view of our main equation (1.1), we recall a Frenkel-Kontorova type model that was analyzed numerically in [1]. This model was originally developed to describe the motion of dislocations in a crystal [29, 30], but now has numerous other applications in the literature. In particular, consider a chain of particles that have positions x_k, with $k \in \mathbb{Z}$. The dynamics are given by the LDE

$$\ddot{x}_k(t) + \gamma \dot{x}_k(t) = x_{k-1}(t) - 2x_k(t) + x_{k+1}(t) - d \sin x_k(t) + F, \tag{1.2}$$

in which γ and d are parameters and F is an external applied force. In the literature a special class of travelling wave solutions, which have been named uniform sliding states, has been constructed for (1.2). Such solutions can be written in the form $x_k(t) = \phi(k - ct)$ for some waveprofile ϕ and wavespeed c and in addition satisfy the special condition $x_{k+N} = x_k + 2\pi M$, in which N and M are fixed integers. It is not hard to see that (1.2) can be restated in such a way that these states become periodic and hence the study of bifurcations from these solutions can be fitted into the framework developed here.

Recently [18], based upon earlier work by several authors [9, 23, 32], a center manifold approach was developed to capture all solutions of (1.1) that remain sufficiently close to a given equilibrium $\overline{x}$. It was shown that the dimension and linear structure on the center manifold are entirely determined by the holomorphic characteristic matrix $\Delta : \mathbb{C} \to \mathbb{C}^{n \times n}$ associated to the linearized system $\dot{v}(\xi) = DG(\overline{x})v_\xi$. This matrix is explicitly given by $\Delta(z) = zI - DG(\overline{x}) \exp(z\cdot)$ and is thus relatively straightforward to construct and analyze in many practical applications, see e.g. [8, 15]. As an illustration of the strength of this reduction, consider a parameter dependent family of MFDEs,

$$\dot{x}(\xi) = G(x_\xi, \mu), \tag{1.3}$$

that admit a common equilibrium $\overline{x}$ and suppose that a pair of roots of the characteristic equation $\det \Delta(z, \mu) = 0$ crosses the imaginary axis at a certain parameter value μ_0. Under suitable conditions the Hopf bifurcation theorem can be lifted to the infinite dimensional setting of (1.3) and hence one may conclude the existence of a branch of periodic solutions to (1.3) bifurcating from the equilibrium $\overline{x}$ for $\mu \sim \mu_0$. In [16] this approach was used to analyze an economic optimal control problem involving delays. This problem was proposed by Rustichini in order to simplify a model describing the dynamics of a capital market [27], whilst still retaining the periodic orbits that are compulsory for any such model. The existence of these periodic orbits was established by numerically analyzing the resulting characteristic equation and looking for root-crossings through the imaginary axis.

The main goal of this paper is to facilitate a similar bifurcation analysis around periodic solutions p to (1.1). In order to do this, we will set out to capture all sufficiently small solutions to the equation

$$\dot{y}(\xi) = DG(p_\xi)y_\xi + \big(G(p_\xi + y_\xi) - DG(p_\xi)y_\xi - G(p_\xi)\big) \tag{1.4}$$

on a finite dimensional center manifold, hence generalizing the approach in [18] for equilibria $p = \overline{x}$. Our results should be seen in the setting of Floquet theory in infinite dimensions. In particular, the linear dynamics and structure on the center manifold are related to Floquet solutions of the linear part of (1.4), i.e., functions v of the form $v(\xi) = e^{\lambda\xi}q(\xi)$ that satisfy

$$\dot{v}(\xi) = DG(p_\xi)v_\xi, \tag{1.5}$$

in which q is a periodic function that has the same period as p and $\lambda \in \mathbb{C}$ is called a Floquet exponent. In particular, we will be interested in linearized equations that admit Floquet exponents on the imaginary axis.

In contrast to the autonomous case, the construction and subsequent analysis of a characteristic matrix for (1.5) in general poses a significant challenge. In the study of delay equations, at least two approaches have been developed to deal with this problem. The first approach uses the fact that a delay equation may be seen as an initial value problem on the state space $C([r_{\min}, 0], \mathbb{C}^n)$, which allows one to define a monodromy map on this space. It is possible to show that this map is compact, which immediately implies that the set of Floquet exponents is discrete [9]. Applying the theory developed in [19] to the monodromy map, Szalai et al. were able to construct a characteristic matrix for general periodic delay equations (1.5), which in addition can be used efficiently for numerical computations [31]. However, only in very special cases can an explicit form for this matrix be given. In addition, this approach fails whenever $r_{\max} > 0$, since in general MFDEs are ill-defined as initial value problems [14].

If the operator $DG(p_\xi) : C([r_{\min}, r_{\max}], \mathbb{C}^n) \to \mathbb{C}^n$ can be written in the form

$$DG(p_\xi)\phi = \sum_{j=0}^{N} A_j(\xi)\phi(\xi + r_j) \tag{1.6}$$

and if the sizes of the shifts r_j in (1.6) are all rationally related to the period of p, the Floquet exponents can be studied in a more direct fashion. This is done by substituting $q(\xi) = e^{-\lambda\xi}v(\xi)$ into

(1.5) and looking for periodic solutions q. The resulting equation can be transformed into an ODE by introducing new variables $q_k(\xi) = q(\xi + kr_*)$, for some r_* that divides all the shifts r_j. In [28, 33], the authors use this reduction to analyze a scalar delay differential equation with a single delay,

$$\dot{x}(\xi) = -\mu x(\xi) + f(x(\xi - 1)), \tag{1.7}$$

in which f is an odd C^1-smooth nonlinearity. In particular, a characteristic matrix was constructed for the Floquet exponents of a special class of periodic orbits p that satisfy $p(\xi + r) = -p(\xi)$, for some $r > 0$ and all $\xi \in \mathbb{R}$. Under some additional restrictions on f and p it was possible to explicitly verify the presence of Floquet exponents on the imaginary axis. In general however, this will become intractible. One will hence have to resort to numerical calculations in the spirit of [22, 31] to detect Floquet exponents that cross through the imaginary axis as the parameters of a system are varied.

To state our results we will need to assume that the Floquet spectrum of (1.5) is discrete in some sense. In Section 3 we will use the ODE reduction described above to verify this condition in a number of special cases, but at the moment it is unclear if this holds for general MFDEs. Our main results are formulated in Section 2 and the necessary linear machinery is developed in Sections 4 through 6. We remark here that the approach in Section 4 was chiefly motivated by the work of Mielke. In [26], he constructed a center manifold to study bifurcations in the setting of elliptic PDEs and hence also had to cope with the absence of a time evolution map. However, we will need to deviate from his approach considerably, for reasons which should become clear in the sequel. In Section 7 we use the Lyapunov–Perron technique to define the center manifold and derive the associated flow, much along the lines of [9, 18, 15]. Finally, in Section 8 we use techniques developed by Vanderbauwhede and van Gils [32] to address the smoothness of the center manifold.

2 Main Results

Consider the following functional differential equation of mixed type,

$$\dot{x}(\xi) \quad = \quad L(\xi)x_\xi + R(\xi, x_\xi), \quad \xi \in \mathbb{R}, \tag{2.1}$$

in which x is a continuous mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some integer $n \geq 1$ and the operators $L(\xi)$ and $R(\xi, \cdot)$ are a linear respectively nonlinear map from the state space $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$ into $\mathbb{C}^n$. The state $x_\xi \in X$ is defined by $x_\xi(\theta) = x(\xi + \theta)$ for any $r_{\min} \leq \theta \leq r_{\max}$, with $r_{\min} \leq 0 \leq r_{\max}$. Furthermore, we require throughout this paper that L and R are periodic, in the sense that $L(\xi + 2\pi)\phi = L(\xi)\phi$ and $R(\xi + 2\pi, \phi) = R(\xi, \phi)$ for all $\xi \in \mathbb{R}$ and $\phi \in X$. For ease of notation, we will present our results for (2.1) under the assumption that L acts on point delays only, i.e., we assume that for some integer N the operator $L(\xi) : X \to \mathbb{C}^n$ can be written in the form

$$L(\xi)\phi = \sum_{j=0}^{N} A_j(\xi)\phi(r_j), \tag{2.2}$$

for $\mathbb{C}^{n\times n}$-valued functions A_j and shifts $r_{\min} = r_0 < r_1 \ldots < r_N = r_{\max}$. We remark however that the arguments developed here can easily be extended to arbitrary $L(\xi) : X \to \mathbb{C}^n$.

As in [18], we will employ the following families of Banach spaces,

$$\begin{array}{rcl} BC_\eta(\mathbb{R},\mathbb{C}^n) & = & \left\{ x \in C(\mathbb{R},\mathbb{C}^n) \mid \|x\|_\eta := \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}\,|x(\xi)| < \infty \right\}, \\ BC^1_\eta(\mathbb{R},\mathbb{C}^n) & = & \left\{ x \in BC_\eta(\mathbb{R},\mathbb{C}^n) \cap C^1(\mathbb{R},\mathbb{C}^n) \mid \dot{x} \in BC_\eta(\mathbb{R},\mathbb{C}^n) \right\}, \end{array} \tag{2.3}$$

parametrized by $\eta \in \mathbb{R}$, with the standard norm $\|x\|_{BC^1_\eta} = \|x\|_\eta + \|\dot{x}\|_\eta$. Notice that for any pair $\eta_2 \geq \eta_1$, there exist continuous inclusions $\mathcal{J}_{\eta_2\eta_1} : BC_{\eta_1}(\mathbb{R},\mathbb{C}^n) \hookrightarrow BC_{\eta_2}(\mathbb{R},\mathbb{C}^n)$ and $\mathcal{J}^1_{\eta_2\eta_1} : BC^1_{\eta_1}(\mathbb{R},\mathbb{C}^n) \hookrightarrow BC^1_{\eta_2}(\mathbb{R},\mathbb{C}^n)$.

An essential step towards understanding the behaviour of (2.1) is the study of the homogeneous linear equation

$$\dot{x}(\xi) = L(\xi)x_\xi. \tag{2.4}$$

In particular, we are interested in the special class of solutions to (2.4) that can be written in the form $x(\xi) = e^{\lambda\xi}p(\xi)$ with $p \in C^{\mathrm{per}}_{2\pi}(\mathbb{R},\mathbb{C}^n)$, i.e., p is a periodic continuous function with $p(\xi + 2\pi) = p(\xi)$ for all $\xi \in \mathbb{R}$. The parameter $\lambda \in \mathbb{C}$ is called a Floquet exponent for (2.4) if and only if any such solution exists. We need to impose the following restrictions on (2.4).

(HL) The map $\mathbb{R} \to \mathcal{L}(X,\mathbb{C}^n)$ given by $\xi \mapsto L(\xi)$ is of class C^r, for some integer $r \geq 3$.

(HF) There exist $\gamma_- < 0$ and $\gamma_+ > 0$ such that (2.4) has no Floquet exponents $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \in \{\gamma_-, \gamma_+\}$.

When studying delay equations, which in our context means $r_{\max} = 0$, one can show that (HF) is always satisfied [13]. However, the proof requires the existence of an evolution map defined on the entire statespace X and hence fails to work when $r_{\max} > 0$. At the moment, it is unclear if equations (2.4) exist for which (HF) fails. However, in Section 3 we give some criteria which will help establish (HF) in the case where all the shifts r_j appearing in (2.2) are rationally related to the period 2π.

The following proposition, which will be proved throughout Sections 5 and 6, exhibits the finite dimensional space X_0 on which the center manifold will be defined.

Proposition 2.1. *Consider any homogeneous linear equation (2.4) that satisfies the conditions (HL) and (HF) and pick a constant γ with $0 < \gamma < |\gamma_\pm|$, in which $\gamma_\pm$ are as introduced in (HF). Then there exists a finite dimensional linear subspace $X_0 \subset X$, a C^r-smooth operator $\Pi : \mathbb{R} \to \mathcal{L}(X, X_0)$ and a matrix $W \in \mathcal{L}(X_0)$, such that the following properties hold.*

(i) Suppose $x \in \bigcap_{\eta>0} BC_\eta(\mathbb{R},\mathbb{C}^n)$ is a solution of (2.4). Then for any $\xi \in \mathbb{R}$ we have $\Pi(\xi)x_\xi \in X_0$.

(ii) For any $\phi \in X_0$, there is a unique solution $x = E\phi \in BC_\gamma(\mathbb{R},\mathbb{C}^n)$ of (2.4) such that $x_0 = \phi$. Moreover, we have that $x \in BC^1_\eta(\mathbb{R},\mathbb{C}^n)$ for any $\eta > 0$.

(iii) For any $\phi \in X_0$ we have $\Pi(\xi)(E\phi)_\xi = e^{\xi W}\phi$.

We also need to impose the following assumptions on the periodic nonlinearity R, after which we are ready to state our main results.

(HR1) The nonlinearity R is C^k-smooth as a function $\mathbb{R} \times X \to \mathbb{C}^n$, for some integer $k \geq 2$.

(HR2) For all $\xi \in \mathbb{R}$ we have $R(\xi, 0) = 0$ and $D_2 R(\xi, 0) = 0$.

Theorem 2.2. *Consider the nonlinear equation (2.1) and assume that (HL), (HF), (HR1) and (HR2) are satisfied. Then there exists $\gamma > 0$ such that (2.4) has no Floquet exponents λ with $0 < |\mathrm{Re}\,\lambda| < \gamma$. Fix an interval $I = [\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ such that $\eta_{\max} > \min(r, k)\eta_{\min}$, with r and k as introduced in (HL) and (HR2). Then there exists a mapping $u^* : X_0 \times \mathbb{R} \to \bigcap_{\eta > 0} BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ and constants $\varepsilon > 0$, $\varepsilon^* > 0$ such that the following statements hold.*

(i) For any $\eta \in (\min(r, k)\eta_{\min}, \eta_{\max}]$, the function u^* viewed as a map from $X_0 \times \mathbb{R}$ into $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ is $C^{\min(r,k)}$-smooth.

(ii) Suppose for some $\zeta > 0$ that $x \in BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ is a solution of (2.1) with $\sup_{\xi \in \mathbb{R}} |x(\xi)| < \varepsilon^*$. Then we have $x = u^*(\Pi(0)x_0, 0)$. In addition, the function $\Phi : \mathbb{R} \to X_0$ defined by $\Phi(\xi) = \Pi(\xi)x_\xi \in X_0$ is of class $C^{\min(r,k+1)}$ and satisfies the ordinary differential equation

$$\dot{\Phi}(\xi) = W\Phi(\xi) + f(\xi, \Phi(\xi)). \tag{2.5}$$

Here the function $f : \mathbb{R} \times X_0 \to X_0$ is $C^{\min(r-1,k)}$-smooth with $f(\xi, 0) = 0$ and $Df(\xi, 0) = 0$ for all $\xi \in \mathbb{R}$. Furthermore, it is periodic in the first variable, with $f(\xi + 2\pi, \psi) = f(\xi, \psi)$ for all $(\xi, \psi) \in \mathbb{R} \times X_0$ and given explicitly by

$$\begin{array}{rcl} f(\xi, \psi) & = & [D\Pi(\xi)][u^*(\psi, \xi) - Ee^{-\xi W}\psi]_\xi \\ & & +\Pi(\xi)\chi^1(\psi, \xi) \\ & & +\Pi(\xi)\chi^2(\psi, \xi). \end{array} \tag{2.6}$$

Here the states $\chi^i(\psi, \xi) \in X$, for $i = 1, 2$, are defined as

$$\begin{array}{rcl} \chi^1(\psi, \xi)(\sigma) & = & L(\xi + \sigma)[u^*(\psi, \xi) - Ee^{-\xi W}\psi]_{\xi+\sigma} \\ \chi^2(\psi, \xi)(\sigma) & = & R(\xi + \sigma, (u^*(\psi, \xi))_{\xi+\sigma}). \end{array} \tag{2.7}$$

Finally, we have $x_\xi = \big(u^*(\Phi(\xi), \overline{\xi})\big)_{\overline{\xi}}$ for any pair $\xi, \overline{\xi} \in \mathbb{R}$ that satisfies $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$.

(iii) For any $\phi \in X_0$ such that $\sup_{\xi \in \mathbb{R}} |u^*(\phi, 0)(\xi)| < \varepsilon^*$, the function $u^*(\phi, 0)$ satisfies (2.1).

(iv) For any continuous function $\Phi : \mathbb{R} \to X_0$ that satisfies (2.5) with $\|\Phi(\xi)\| < \varepsilon$ for all $\xi \in \mathbb{R}$, we have that $x = u^*(\Phi(0), 0)$ is a solution of (2.1). In addition, we have $x_\xi = \big(u^*(\Phi(\xi), \overline{\xi})\big)_{\overline{\xi}}$ for any pair $\xi, \overline{\xi} \in \mathbb{R}$ that satisfies $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$.

(v) Consider the interval $I = (\xi_-, \xi_+)$, where $\xi_- = -\infty$ and $\xi_+ = \infty$ are allowed. Let $\Phi : I \to X_0$ be a continuous function that satisfies (2.5) for every $\xi \in I$ and in addition has $\|\Phi(\xi)\| < \varepsilon$ for all such ξ. Then for any $\zeta \in (\xi_-, \xi_+)$ we have that $x = u^*(\Phi(\zeta), \zeta)$ satisfies (2.1) for all $\xi \in I$. In addition, we have $x_\xi = \big(u^*(\Phi(\xi), \overline{\xi})\big)_{\overline{\xi}}$ for any pair $(\xi, \overline{\xi}) \in I \times \mathbb{R}$ that satisfies $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$.

3 Preliminaries

In addition to the spaces $BC_\eta(\mathbb{R},\mathbb{C}^n)$ that contain continuous functions, we introduce two extra families of Banach spaces, parametrized by $\mu,\nu\in\mathbb{R}$, that contain distributions that have controlled exponential growth at $\pm\infty$,

$$\begin{array}{lcl} BX_{\mu,\nu}(\mathbb{R},\mathbb{C}^n) & = & \Big\{x\in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n)\mid \|x\|_{BX_{\mu,\nu}}:=\sup_{\xi<0}e^{-\mu\xi}\,|x(\xi)|+\sup_{\xi\geq 0}e^{-\nu\xi}\,|x(\xi)|<\infty\Big\}, \\ BX^1_{\mu,\nu}(\mathbb{R},\mathbb{C}^n) & = & \Big\{x\in W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n)\cap C(\mathbb{R},\mathbb{C}^n)\mid \|x\|_{BX^1_{\mu,\nu}}:=\|x\|_{BX_{\mu,\nu}}+\|\dot{x}\|_{BX_{\mu,\nu}}<\infty\Big\}. \end{array} \tag{3.1}$$

In order to improve the readability of our arguments, we also introduce the notation $\mathrm{ev}_\xi x=x_\xi\in X$ for any $x\in C(\mathbb{R},\mathbb{C}^n)$ and $\xi\in\mathbb{R}$, together with the shift operators T_ξ defined by $T_\xi f(\cdot)=f(\cdot+\xi)$, for any $f\in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n)$.

Recall here the definitions of the Fourier transform $\mathcal{F}^+(f)(k)=\widehat{f}(k)$ of a function $f\in L^2(\mathbb{R},\mathbb{C}^n)$ and the inverse Fourier transform $\mathcal{F}^-(g)(\xi)=\check{g}(\xi)$ for any $g\in L^2(\mathbb{R},\mathbb{C}^n)$, given by

$$\widehat{f}(k)=\textstyle\int_{-\infty}^{\infty}e^{-ik\xi}f(\xi)d\xi, \qquad \check{g}(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik\xi}g(k)dk. \tag{3.2}$$

We remark here that the integrals above are well-defined only if $f,g\in L^1(\mathbb{R},\mathbb{C}^n)$. If this is not the case, the integrals have to be understood as integrals in the Fourier sense, i.e., the functions

$$h_n(k)=\int_{-n}^{n}e^{-ik\xi}f(\xi)d\xi \tag{3.3}$$

satisfy $h_n(k)\to\hat{f}$ in $L^2(\mathbb{R},\mathbb{C}^n)$ and in addition there is a subsequence $\{n'\}$ such that $h_{n'}(k)\to\widehat{f}(k)$ almost everywhere. We recall that the Fourier transform takes convolutions into products, i.e., $(\widehat{f*g})(k)=\widehat{f}(k)\widehat{g}(k)$ for almost every k.

Now suppose $f:\mathbb{R}\to\mathbb{C}^n$ satisfies $f(\xi)=O(e^{-a\xi})$ as $\xi\to\infty$. Then for any z with $\operatorname{Re}z>-a$, define the Laplace transform

$$\widetilde{f}_+(z)=\int_0^\infty e^{-z\xi}f(\xi)d\xi. \tag{3.4}$$

Similarly, if $f(\xi)=O(e^{b\xi})$ as $\xi\to-\infty$, then for any z with $\operatorname{Re}z<b$, define

$$\widetilde{f}_-(z)=\int_0^\infty e^{z\xi}f(-\xi)d\xi. \tag{3.5}$$

The inverse transformation is described in the next result, which can be found in the standard Laplace transform literature [34, 7.3-5].

Lemma 3.1. *Let $f:\mathbb{R}\to\mathbb{C}^n$ satisfy a growth condition $f(\xi)=O(e^{-a\xi})$ as $\xi\to\infty$ and suppose that f is of bounded variation on bounded intervals. Then for any $\gamma>-a$ and $\xi>0$ we have the inversion formula*

$$\frac{f(\xi+)+f(\xi-)}{2}=\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}e^{z\xi}\widetilde{f}_+(z)dz, \tag{3.6}$$

whereas for $\xi=0$ we have

$$\frac{f(0+)}{2}=\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}e^{z\xi}\widetilde{f}_+(z)dz. \tag{3.7}$$

Consider the linear operator $L(\xi) : X \to \mathbb{C}^n$ appearing in (2.1). One may split this operator into an autonomous part and a periodic part, i.e., write $L(\xi) = L_{\mathrm{aut}} + L_{\mathrm{per}}(\xi)$ with

$$\begin{array}{rcl} L_{\mathrm{aut}}\phi & = & \sum_{j=0}^{N} A^j_{\mathrm{aut}}\phi(r_j), \\ L_{\mathrm{per}}(\xi)\phi & = & \sum_{j=0}^{N} B^j(\xi)\phi(r_j). \end{array} \tag{3.8}$$

We recall the characteristic matrix $\Delta(z) = zI - \sum_{j=0}^{N} A^j_{\mathrm{aut}} e^{zr_j}$ associated to L_{aut} and repeat some useful properties of Δ that were established in [18].

Lemma 3.2. *Consider any closed vertical strip $S = \{z \in \mathbb{C} \mid a_- \leq \operatorname{Re} z \leq a_+\}$ and for any $\rho > 0$ define $S_\rho = \{z \in S \mid |\operatorname{Im} z| > \rho\}$. Then there exist $C, \rho > 0$ such that* $\det \Delta(z) \neq 0$ *for all $z \in S_\rho$ and in addition $\left|\Delta(z)^{-1}\right| < \frac{C}{|\operatorname{Im} z|}$ for each such z. In particular, there are only finitely many zeroes of* $\det \Delta(z)$ *in S.*

Notice that the splitting (3.8) is obviously ambiguous, in the sense that L_{aut} can be chosen freely. We will use this freedom to ensure that the characteristic equation $\det \Delta(z) = 0$ has no roots in a small strip around the imaginary axis, which will allow us to solve linear systems of the form

$$\dot{x}(\xi) = L(\xi)x_\xi + f(\xi), \tag{3.9}$$

for suitable classes of inhomogeneities f. As a final matter of notation, for any function x we will write Lx to represent the function $\xi \mapsto L(\xi)x_\xi$.

We conclude this section by discussing the assumption (HF) concerning the Floquet exponents for the system (2.4). We provide a number of results with which this criterion can be verified.

Lemma 3.3. *Consider any system of the form (2.4) that has the property that all the shifts are rationally related to the period, i.e., we have $r_j \in \pi\mathbb{Q}$ for all $0 \leq j \leq N$. Then either every $\lambda \in \mathbb{C}$ is a Floquet exponent, or (HF) is satisfied.*

Proof. Choose $r^* = \frac{2\pi}{M} \in \mathbb{R}$ such that for some numbers $m_j \in \mathbb{Z}$ we have $r_j = m_j r^*$ for all $0 \leq j \leq N$. Suppose that $\lambda \in \mathbb{C}$ is Floquet exponent and let $p \in C^{\mathrm{per}}_{2\pi}(\mathbb{R}, \mathbb{C}^n)$ be the corresponding nontrivial periodic function, such that $\dot{u} = Lu$. Associated to p we introduce the $\mathbb{C}^{Mn}$-valued function $\mathbf{p}$, the components of which are defined by $\mathbf{p}_i(\xi) = p(\xi + ir^*)$ for $0 \leq i \leq M-1$. Since p is periodic, it is clear that $\mathbf{p}_i(r_*) = \mathbf{p}_{i+1 \bmod M}(0)$ for all $0 \leq i \leq M-1$, which can be reformulated as $\mathbf{p}(r_*) = I_n \otimes \mathcal{T}\mathbf{p}(0)$, in which I_n is the $n \times n$ identity matrix and the $M \times M$-matrix $\mathcal{T}$ is defined by $\mathcal{T}_{i,j} = \delta_{j,i+1 \bmod M}$. After an appropriate shift one may assume $\mathbf{p}(0) \neq 0$. Furthermore, a quick calculation shows that $\mathbf{p}$ satisfies the ODE

$$\dot{\mathbf{p}}(\xi) \quad = \quad \mathcal{F}(\xi, \lambda)\mathbf{p}(\xi), \tag{3.10}$$

in which $\mathcal{F}$ is given by

$$(\mathcal{F}(\xi, \lambda)\mathbf{v})_i \quad = \quad -\lambda \mathbf{v}_i + \sum_{j=0}^{N} e^{\lambda r_j} A_j(\xi + ir^*)\mathbf{v}_{i+m_j \bmod M}. \tag{3.11}$$

Writing $\Omega(\xi, \lambda)$ for the fundamental matrix for the ODE (3.10), we have $\Omega(r_*, \lambda)\mathbf{p}(0) = I_n \otimes \mathcal{T}\mathbf{p}(0)$ and hence

$$\det[\Omega(r_*, \lambda) - I_n \otimes \mathcal{T}] = 0. \tag{3.12}$$

Since the coefficients of the ODE (3.10) depend analytically on $\lambda \in \mathbb{C}$, it follows that for any fixed $\xi \in \mathbb{R}$ the matrix $\Omega(\xi, \cdot)$ is an entire function in the second variable [10, Section 10.7]. This however implies that either (3.12) is satisfied for all $\lambda \in \mathbb{C}$, or the set of solutions is discrete. To complete the proof, observe that $\lambda \in \mathbb{C}^n$ is a Floquet exponent if and only if $\lambda + i$ is a Floquet exponent, which means that the set of real parts of Floquet exponents is discrete whenever the set of Floquet exponents is discrete. □

In some special cases we can get extra information on the fundamental matrix Ω and show that not all $\lambda \in \mathbb{C}$ can be Floquet exponents.

Corollary 3.4. *Consider any scalar system of the form (2.4) that has the property that all shifts are integer multiples of the period, i.e., we have $r_j \in 2\pi\mathbb{Z}$ for all $0 \le j \le N$. Then (HF) is satisfied.*

Proof. In this case (3.10) is scalar and the fundamental matrix reduces to $\Omega(2\pi, \lambda) = \exp[-2\pi\lambda + \sum_{j=0}^{N} e^{\lambda r_j} \int_0^{2\pi} A_j(\sigma) d\sigma]$, hence the set of roots of (3.12) is discrete. □

Corollary 3.5. *Consider any system of the form (2.4) that has the property that all shifts are rationally related to the period, i.e., we have $r_j \in \pi\mathbb{Q}$ for all $0 \le j \le N$. Suppose that there exists a vector $\mathbf{v} \in \mathbb{C}^{nM}$ that is an eigenvector for $\mathcal{F}^*(\xi, \lambda)$ for all ξ and all λ, with $\mathcal{F}$ and r_* as given in (3.11). Then (HF) is satisfied.*

Proof. Observe that the complex conjugate of the eigenvalue $\mu = \mu(\xi, \lambda)$ corresponding to the eigenvector $\mathbf{v}$ of $\mathcal{F}^*(\xi, \lambda)$ is given by

$$\mu^* = -\lambda + P(\xi)(\exp(\lambda r_*)) + Q(\xi)(\exp(-\lambda r_*)), \tag{3.13}$$

in which $P(\xi)(\cdot)$ and $Q(\xi)(\cdot)$ are polynomials for every $\xi \in \mathbb{R}$, with $P(\xi+2\pi) = P(\xi)$ and $Q(\xi+2\pi) = Q(\xi)$. Introducing the scalar function $q(\xi) = \mathbf{v}^*\mathbf{p}$, we may now calculate

$$\dot{q}(\xi) = \mathbf{v}^*\mathcal{F}(\xi, \lambda)\mathbf{p} = \mu^* v^* \mathbf{p} = -\lambda q(\xi) + P(\xi)(\exp(\lambda r_*)) + Q(\xi)(\exp(-\lambda r_*))q(\xi). \tag{3.14}$$

This means that $q(2\pi) = \exp[-2\pi\lambda + \int_0^{2\pi} P(\sigma)(\exp(\lambda r_*)) d\sigma + \int_0^{2\pi} Q(\sigma)(\exp(-\lambda r_*)) d\sigma] q(0) = q(0)$, which concludes the proof. □

As an example to illustrate the result above, consider the equation

$$\dot{x}(\xi) = \sin(\xi)x(\xi - \pi) + \sin(\xi)x(\xi + \pi). \tag{3.15}$$

If $\lambda \in \mathbb{C}$ is a Floquet exponent for (3.15) with corresponding scalar $p \in C_{2\pi}^{\mathrm{per}}(\mathbb{R}, \mathbb{C}^n)$, then the $\mathbb{R}^2$-valued function $\mathbf{p}(\xi) = (p_0(\xi), p_1(\xi)) = (p(\xi), p(\xi + \pi))$ satisfies the system

$$\begin{array}{rcl} \dot{p}_0(\xi) & = & -\lambda p_0(\xi) + \sin(\xi)[e^{-\pi\lambda} + e^{\pi\lambda}]p_1(\xi), \\ \dot{p}_1(\xi) & = & -\lambda p_1(\xi) - \sin(\xi)[e^{-\pi\lambda} + e^{\pi\lambda}]p_0(\xi). \end{array} \tag{3.16}$$

Writing $q(\xi) = p_0(\xi) + ip_1(\xi)$, we find that q solves the scalar ODE

$$\dot{q}(\xi) = -\lambda q(\xi) - i\sin(\xi)[e^{-\pi\lambda} + e^{\pi\lambda}]q(\xi) \tag{3.17}$$

and satisfies $q(0) = q(2\pi)$. Using the variation-of-constants formula for q it is clear that (HF) must be satisfied.

4 Linear inhomogeneous equations

We introduce the linear operator $\Lambda : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$, given by

$$(\Lambda x)(\xi) = \dot{x}(\xi) - L(\xi)x_\xi. \tag{4.1}$$

In this section we set out to solve equations of the form $\Lambda x = f$ and hence define an inverse for Λ. Using Fourier transform techniques, we will first show that Λ is invertible when considered as an operator from $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ into $L^2(\mathbb{R}, \mathbb{C}^n)$. This result can then be extended to compute $\Lambda^{-1} f$ for $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$.

Due to the periodicity of $L(\xi)$, the transform $\mathcal{F}^+[Lx](\eta_0)$ will only involve $\hat{x}(\eta_1)$ when $\eta_1 - \eta_0 \in \mathbb{Z}$. It will hence be fruitful to follow the approach employed by Mielke [26] and introduce the sequence space

$$\ell_2 = \big\{ w = (w_k)_{k\in\mathbb{Z}} \mid w_k \in \mathbb{C}^n \text{ and } \|w\|_2^2 := \sum_{k\in\mathbb{Z}} |w_k|^2 < \infty \big\}. \tag{4.2}$$

Recalling the splitting (3.8), we need to solve

$$\dot{x}(\xi) = \sum_{j=0}^{N} A^j_{\mathrm{aut}} x(\xi + r_j) + \sum_{j=0}^{N} B^j(\xi)x(\xi + r_j) + f(\xi). \tag{4.3}$$

Since $B^j \in C^r(\mathbb{R}, \mathbb{C}^{n\times n})$ with $B^j(\xi + 2\pi) = B^j(\xi)$, we may write

$$B^j(\xi) \quad = \quad \textstyle\sum_{k=-\infty}^{\infty} B^j_k e^{ik\xi}, \tag{4.4}$$

in which the coefficients satisfy the estimate

$$\left| B^j_k \right| \leq C/(1 + |k|)^r. \tag{4.5}$$

for some $C > 0$. For any $0 \leq j \leq N$, define the convolution operator $\widetilde{B}^j : \ell_2 \to \ell_2$ by

$$(\widetilde{B}^j w)_n \quad = \quad \textstyle\sum_{k\in\mathbb{Z}} e^{i(n-k)r_j} B^j_k w_{n-k}, \tag{4.6}$$

together with $B_z : \ell_2 \to \ell_2$ given by $B_z = \sum_{j=0}^N e^{zr_j} \widetilde{B}^j$. To see that $\widetilde{B}^j$ is well-defined and bounded, use the Cauchy-Schwartz inequality and the estimate (4.5) to compute

$$\begin{array}{rcl} \left\| \widetilde{B}^j w \right\|_2^2 & = & \sum_{n\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} e^{i(n-m)r_j} B^j_m w_{n-m} \right|^2 \leq \sum_{n\in\mathbb{Z}} \big[\sum_{m\in\mathbb{Z}} \left| B^j_m \right|^{\frac{1}{2}} \left| B^j_m \right|^{\frac{1}{2}} |w_{n-m}| \big]^2 \\ & \leq & \sum_{n\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} \left| B^j_m \right|)(\sum_{m\in\mathbb{Z}} \left| B^j_m \right| |w_{n-m}|^2) = (\sum_{m\in\mathbb{Z}} \left| B^j_m \right|)^2 \|w\|_2^2 . \end{array} \tag{4.7}$$

Note that it is possible to choose L_{aut} in such a way that $\det \Delta(z) = 0$ has no roots in a strip $|\mathrm{Re}\, z| < \varepsilon$. For any such z, we can hence define a multiplication operator $\Delta_z : \ell_2 \to \ell_2$ by

$$(\Delta_z w)_n \quad = \quad \Delta(z + in)^{-1} w_n. \tag{4.8}$$

We claim that Δ_z is compact. To see this, consider any bounded sequence $\{w^n\}_{n\in\mathbb{N}} \subset \ell_2$, write $v^n = \Delta_z w^n$ and use a diagonal argument to pass to a subsequence for which each component v^n_k converges as $n \to \infty$. For any $K > 0$ we find

$$\|v^n - v^m\|_2^2 \le \sum_{|k|<K} |v^n_k - v^m_k|^2 + (1+K)^{-1} \sum_{|k|\ge K} (1+|k|)\, |v^n_k - v^m_k|^2 . \tag{4.9}$$

Fixing any $\varepsilon > 0$ and noting that the estimate in Lemma 3.2 implies that the second sum can be bounded independently of K, n and m, we can choose $K > 0$ sufficiently large to ensure that the entire second term on the righthand side of (4.9) is bounded by $\varepsilon/2$. Similarly, for such a choice of K we can choose a $M > 0$ such that the first term is bounded by $\varepsilon/2$ for any $n \ge M$ and $m \ge M$, which shows that Δ_z is indeed compact.

For any $\tau \in \mathbb{R}$, consider a function $f : \tau + i\mathbb{R} \to \mathbb{C}^n$ such that $\eta \mapsto f(\tau + i\eta) \in L^2(\mathbb{R}, \mathbb{C}^n)$. For any complex z with $\mathrm{Re}\, z = \tau$, define the sequence $(\mathcal{J}_z f)_k = f(z + ik)$. Notice that for almost all such z, we have $\mathcal{J}_z f \in \ell_2$. Finally, for any $w \in \ell_2$, define $\mathrm{ev}_n w = w_n \in \mathbb{C}^n$ and $(T_n w)_k = w_{k+n}$. With these preparations we are ready to provide the inverse $\Lambda^{-1} f$ for $f \in L^2(\mathbb{R}, \mathbb{C}^n)$.

Proposition 4.1. *Suppose that (4.3) admits no Floquet exponents λ with $\mathrm{Re}\,\lambda = 0$. Then Λ is an isomorphism from $W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ onto $L^2(\mathbb{R}, \mathbb{C}^n)$, with inverse given by*

$$\Lambda^{-1} f = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{z\xi} \mathrm{ev}_0 [I - \Delta_z B_z]^{-1} \Delta_z \mathcal{J}_z [\widetilde{f}_+(\cdot) + \widetilde{f}_-(\cdot)] dz. \tag{4.10}$$

In addition, there exists a Greens function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{n\times n}$ such that for every $\xi \in \mathbb{R}$, the function $G(\xi, \cdot) \in L^2(\mathbb{R}, \mathbb{C}^{n\times n})$ satisfies (4.3) in the sense of distributions, with $f(\xi') = \delta(\xi' - \xi) I$. In addition, $G(\xi, \cdot)$ is bounded, admits a jump $G(\xi, \xi+) - G(\xi, \xi-) = I$ and is C^1-smooth on $\mathbb{R} \setminus \{\xi\}$.

Proof. First consider any sequence $w \in \ell_2$ such that $w = \Delta_z B_z w$. Then the function $u(\xi) = e^{z\xi} \sum_{n\in\mathbb{Z}} e^{in\xi} w_n$ satisfies $\Lambda u = 0$. In addition, since $(n w_n) \in \ell_2$, we have that the periodic function $p(\xi) = e^{-z\xi} u(\xi)$ satisfies $p \in W^{1,2}([0, 2\pi], \mathbb{C}^n) \subset C([0, 2\pi], \mathbb{C}^n)$. We hence conclude that z must be a Floquet exponent. Due to the absence of Floquet exponents on the imaginary axis, the Fredholm alternative now implies that $\mathbf{1} - \Delta_z B_z$ is invertible as a map from ℓ_2 onto ℓ_2 for all $z \in i\mathbb{R}$. Since both $z \mapsto \Delta_z$ and $z \mapsto B_z$ are continuous, the same holds for $z \mapsto [\mathbf{1} - \Delta_z B_z]^{-1}$. Notice in addition that one has $B_{z+i} = T_1 B_z T_{-1}$, together with a similar identity for Δ_z. This implies that the norm $\big\|[\mathbf{1} - \Delta_z B_z]^{-1}\big\|$ can be bounded independently of z for $z \in i\mathbb{R}$.

Taking the Fourier transform of (4.3), we arrive at

$$\Delta(i\eta)\hat{x}(\eta) = \hat{f}(\eta) + \sum_{j=0}^{N} \sum_{k=-\infty}^{\infty} e^{i(\eta-k)r_j} B^j_k \hat{x}(\eta - k). \tag{4.11}$$

This identity can be inverted by introducing the sequence $\hat{f}^\theta \in \ell_2$ via $\hat{f}^\theta_n = \hat{f}(\theta + n)$ where this is well-defined and choosing, for $\theta \in [0,1)$ and $n \in \mathbb{Z}$,

$$\hat{x}(\theta + n) = \mathrm{ev}_n[\mathbf{1} - \Delta_{i\theta} B_{i\theta}]^{-1} \Delta_{i\theta} \hat{f}^\theta. \tag{4.12}$$

It remains to show that $\hat{x}$ thus constructed is in fact an L^2 function. We calculate

$$\begin{array}{lcl} \int_{-\infty}^{\infty} |\hat{x}(\eta)|^2 \, d\eta & = & \int_0^1 \sum_{n\in\mathbb{Z}} |\hat{x}(\theta+n)|^2 \, d\theta = \int_0^1 \left\| [\mathbf{1} - \Delta_{i\theta} B_{i\theta}]^{-1} \Delta_{i\theta} \hat{f}^\theta \right\|_2^2 \\ & \leq & C \int_0^1 \left\| \Delta_{i\theta} \hat{f}^\theta \right\|_2^2 = C \int_{-\infty}^{\infty} \left| \Delta(i\eta)^{-1} \hat{f}(\eta) \right|^2 \leq C' \left\| \hat{f} \right\|_2. \end{array} \tag{4.13}$$

In addition, using (4.11) together with the estimate (4.5) it follows that $\eta \mapsto \eta \hat{x}(\eta)$ is an L^2 function, from which we conclude $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$, as desired. To show that Λ is injective, consider any $x \in W^{1,2}(\mathbb{R}, \mathbb{C}^n)$ with $\Lambda x = 0$. There exists a $\theta \in \mathbb{R}$ such that $\hat{x}^\theta \in \ell_2$ with $\hat{x}^\theta \neq 0$ and using (4.11) it follows that $i\theta$ must be Floquet exponent, which yields a contradiction.

Without loss of generality, we will prove the statements concerning the Greens function G only for $\xi = 0$. To this end, note that the construction above remains valid if we take $\widehat{f} = 1$ and $G(0, \cdot) = \Lambda^{-1}\delta(\cdot)$. The only modification that is required is the last inequality in (4.13), which can be replaced by

$$\int_{-\infty}^{\infty} \left| \widehat{G}(\eta) \right|^2 d\eta \leq C \int_{-\infty}^{\infty} |\Delta(i\eta)|^2 \, d\eta \leq C \, \|\Delta(i\cdot)\|_{L^2} \leq C'. \tag{4.14}$$

In view of this, we merely have $G(0, \cdot) \in L^2(\mathbb{R}, \mathbb{C}^{n\times n})$, but using the differential equation we find $G(0, \cdot) \in W^{1,2}((0,\infty), \mathbb{C}^{n\times n}) \cup W^{1,2}((-\infty, 0), \mathbb{C}^{n\times n})$ and hence $G(0, \cdot)$ is C^1-smooth on $\mathbb{R} \setminus \{0\}$, as required. The remaining properties also follow from the distributional differential equation that G satisfies. $\square$

Since we are specially interested in situations where (4.3) does admit Floquet exponents λ with $\mathrm{Re}\,\lambda = 0$, we will need a tool to shift such exponents off the imaginary axis. To this end, we introduce the notation $e_\nu f = e^{\nu \cdot} f(\cdot)$ for any $\nu \in \mathbb{R}$ and any $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$. In addition, for $\eta \in \mathbb{R}$ we define the shifted linear operator $\Lambda_\eta : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C(\mathbb{R}, \mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$, by

$$(\Lambda_\eta x)(\xi) = \dot{x}(\xi) - \eta x(\xi) - \sum_{j=0}^{N} [A^j_{\mathrm{aut}} + B^j(\xi)] e^{-\eta r_j} x(\xi + r_j). \tag{4.15}$$

One may check that $e_\eta \Lambda e_{-\eta} x = \Lambda_\eta x$ and hence for any Floquet exponent λ associated to Λ_η, one has that $\lambda + \eta$ is a Floquet exponent associated to Λ.

In view of these observations, we introduce, for any $\eta \in \mathbb{R}$ and $p \in \{2, \infty\}$, the Banach spaces

$$\begin{array}{lcl} L^p_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{ x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in L^p(\mathbb{R}, \mathbb{C}^n) \right\}, \\ W^{1,p}_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{ x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in W^{1,p}(\mathbb{R}, \mathbb{C}^n) \right\}, \end{array} \tag{4.16}$$

with norms given by $\|x\|_{L^p_\eta} = \|e_{-\eta} x\|_{L^p}$ and similarly $\|x\|_{W^{1,p}_\eta} = \|e_{-\eta} x\|_{W^{1,p}}$. The following result now follows immediately from Proposition 4.1.

Corollary 4.2. *Suppose that (4.3) admits no Floquet exponents λ with $\operatorname{Re}\lambda = \eta$. Then Λ is an isomorphism from $W^{1,2}_\eta(\mathbb{R}, \mathbb{C}^n)$ onto $L^2_\eta(\mathbb{R}, \mathbb{C}^n)$, with inverse given by*

$$\Lambda^{-1} f = \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} e^{z\xi} \mathrm{ev}_0 [I - \Delta_z B_z]^{-1} \Delta_z \mathcal{J}_z [\widetilde{f}_+(\cdot) + \widetilde{f}_-(\cdot)] dz. \tag{4.17}$$

In addition, there exists a Greens function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{n \times n}$ such that for every $\xi \in \mathbb{R}$, the function $G(\xi, \cdot) \in L^2_\eta(\mathbb{R}, \mathbb{C}^{n\times n})$ satisfies (4.3) in the sense of distributions, with $f(\xi') = \delta(\xi' - \xi)I$. In addition, $e_{-\eta} G(\xi, \cdot)$ is bounded, while $G(\xi, \cdot)$ admits a jump $G(\xi, \xi+) - G(\xi, \xi-) = I$ and is C^1-smooth on $\mathbb{R} \setminus \{\xi\}$.

In order to avoid confusion, we will write $\Lambda^{-1}_{(\eta)}$ for the inverse of Λ when considered as a map from $W^{1,2}_\eta(\mathbb{R}, \mathbb{C}^n)$ onto $L^2_\eta(\mathbb{R}, \mathbb{C}^n)$ and similarly $G_{(\eta)}$ for the corresponding Greens function. In the next section we will use these inverses to construct $\Lambda^{-1} f$ for $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$, by writing f as a sum of two functions in $L^2_{\pm\eta}(\mathbb{R}, \mathbb{C}^n)$ for appropriate $\eta \in \mathbb{R}$. The next result paves the road for this approach, by showing that $\Lambda^{-1} f$ respects the growth rate of f. As a preparation, we emphasize that on the space $W^{1,2}_\eta(\mathbb{R}, \mathbb{C}^n)$ one can also define a norm $\|x\|^2_{\widetilde{W}^{1,2}_\eta} := \|e_{-\eta} x\|_2^2 + \|e_{-\eta} \dot{x}\|_2^2$, which is equivalent to the norm $\|x\|_{W^{1,2}_\eta}$ defined above.

Proposition 4.3. *Consider any $\eta \in \mathbb{R}$ and $\varepsilon > 0$ such that (4.3) admits no Floquet exponents λ with $\operatorname{Re}\lambda \in \{\eta - \varepsilon, \eta + \varepsilon\}$ and assume that $\Lambda^{-1}_{(\eta+\varepsilon)} g = \Lambda^{-1}_{(\eta - \varepsilon)} g$ for all $g \in L^2_{\eta+\varepsilon}(\mathbb{R}, \mathbb{C}^n) \cap L^2_{\eta-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$. Then for any $f \in L^\infty_\eta(\mathbb{R}, \mathbb{C}^n) \cap L^2_{\eta+\varepsilon}(\mathbb{R}, \mathbb{C}^n)$, we have $\Lambda^{-1}_{(\eta+\varepsilon)} f \in W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n)$, with a similar result for $f \in L^\infty_\eta(\mathbb{R}, \mathbb{C}^n) \cap L^2_{\eta-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$.*

Proof. Our arguments here are an adaptation of those presented by Mielke in [26] for elliptic PDEs. Without loss of generality, we will assume that $\eta = 0$ and that time has been rescaled to ensure that $L(\xi)$ has period one. Now consider any $f \in L^\infty(\mathbb{R}, \mathbb{C}^n) \cap L^2_\varepsilon(\mathbb{R}, \mathbb{C}^n)$ and define $x = \Lambda^{-1} f \in W^{1,2}_\varepsilon(\mathbb{R}, \mathbb{C}^n)$.

For any $n \in \mathbb{Z}$, let χ_n denote the indicator function for the interval $[n, n+1]$. Writing $f_n = \chi_n f$, we see that $f_n \in L^2_\varepsilon(\mathbb{R}, \mathbb{C}^n) \cap L^2_{-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$, with $\sum_{n\in\mathbb{Z}} f_n \to f$ in $L^2_\varepsilon(\mathbb{R}, \mathbb{C}^n)$. We can hence define $x_n = \Lambda^{-1} f_n$ and observe that $x_n \in W^{1,2}_\varepsilon(\mathbb{R}, \mathbb{C}^n) \cap W^{1,2}_{-\varepsilon}(\mathbb{R}, \mathbb{C}^n)$, again with $\sum_{n\in\mathbb{Z}} x_n \to x$ in $W^{1,2}_\varepsilon(\mathbb{R}, \mathbb{C}^n)$. The periodicity of the system (4.3) and the rescaling of time ensure that T_n and Λ^{-1} commute, i.e., $T_n \Lambda^{-1}_{(\pm\varepsilon)} = \Lambda^{-1}_{(\pm\varepsilon)} T_n$. We can exploit this fact to compute

$$\begin{array}{lcl}
\|x_n\|_{W^{1,2}([m,m+1])} & = & [\int_m^{m+1} x_n^2(\xi) + \dot{x}_n^2(\xi) d\xi]^{1/2} = [\int_{m-n}^{m-n+1} x_n^2(\xi + n) + \dot{x}_n^2(\xi+n) d\xi]^{1/2} \\
& \leq & [\int_{m-n}^{m-n+1} (x_n(\xi+n) e^{\varepsilon\xi})^2 + (e^{\varepsilon\xi} \dot{x}_n(\xi+n))^2 d\xi]^{1/2} e^{-\varepsilon(m-n)} \\
& \leq & e^{-\varepsilon(m-n)} \|T_n x_n\|_{\widetilde{W}^{1,2}_{-\varepsilon}} \leq C_\varepsilon e^{-\varepsilon(m-n)} \|T_n x_n\|_{W^{1,2}_{-\varepsilon}} \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} \|T_n f_n\|_{L^2_{-\varepsilon}} \leq C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} e^\varepsilon \|T_n f_n\|_{L^2} \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} e^\varepsilon \|\chi_n f\|_\infty \\
& \leq & C_\varepsilon \left\| \Lambda^{-1}_{(-\varepsilon)} \right\| e^{-\varepsilon(m-n)} e^\varepsilon \|f\|_\infty .
\end{array} \tag{4.18}$$

In a similar fashion, we obtain

$$\begin{array}{lll} \|x_n\|_{W^{1,2}([m,m+1])} & \leq & [\int_{m-n}^{m-n+1} (x_n(\xi+n)e^{-\varepsilon\xi})^2 + (e^{-\varepsilon\xi}\dot{x}_n(\xi+n))^2 d\xi]^{1/2} e^{+\varepsilon(m-n+1)} \\ & \leq & e^{\varepsilon(m-n+1)} \|T_n x_n\|_{\widetilde{W}^{1,2}_\varepsilon} \\ & \leq & C_\varepsilon \left\|\Lambda^{-1}_{(\varepsilon)}\right\| e^{\varepsilon(m-n)} e^{\varepsilon} \|f\|_\infty . \end{array} \tag{4.19}$$

Using a Sobolev embedding it now follows that there exists a constant $C > 0$, independent of n and m, such that $\|\chi_m x_n\|_\infty \leq Ce^{\pm\varepsilon(m-n)} \|f\|_\infty$. Summing this identity over $n \in \mathbb{Z}$, we obtain

$$\begin{array}{lll} \|\chi_m x\|_\infty & \leq & C \|f\|_\infty \left[\sum_{n \geq m} e^{\varepsilon(m-n)} + \sum_{n<m} e^{\varepsilon(n-m)}\right] \\ & \leq & \frac{2C}{1-e^{-\varepsilon}} \|f\|_\infty . \end{array} \tag{4.20}$$

This bound does not depend on m, hence $x \in L^\infty(\mathbb{R}, \mathbb{C}^n)$, as desired. The differential equation now implies that in fact $x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$. □

5 The state space

The main goal of this section is to analyze solutions to the homogeneous equation $\Lambda x = 0$ and to provide a pseudo-inverse for Λ that projects out these solutions in some sense. We start by using the Laplace transform to characterize any solution x that satisfies $\Lambda x = f$, even though x may no longer be unique. As a preparation, we introduce the cutoff operators $\Phi_\pm : L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$, defined via $[\Phi_+ f](\xi) = 0$ for all $\xi < 0$, $[\Phi_- f](\xi) = 0$ for all $\xi \geq 0$ and $\Phi_+ f + \Phi_- f = f$.

Proposition 5.1. *Consider a linear equation of the form (4.3) that satisfies the assumption (HL) and admits no Floquet exponents λ on the lines $\operatorname{Re}\lambda = \gamma_\pm$, for some constants $\gamma_\pm$. Fix a pair $\gamma_- < \mu < \nu < \gamma_+$, consider any function $x \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$ and write $\Lambda x = f$. Then the following identity holds,*

$$x = \Lambda^{-1}_{(\gamma_+)} \Phi_+ f + \Lambda^{-1}_{(\gamma_-)} \Phi_- f + \mathcal{P}_{\gamma_-,\gamma_+} x_0, \tag{5.1}$$

in which $\mathcal{P}_{\gamma_-,\gamma_+} : X \to BX^1_{\gamma_-,\gamma_+}(\mathbb{R}, \mathbb{C}^n)$ is given by

$$\begin{array}{lll} (\mathcal{P}_{\gamma_-,\gamma_+}\phi)(\xi) & = & \frac{1}{2\pi i} \int_{\gamma_+ - i\infty}^{\gamma_- + i\infty} e^{z\xi} \mathrm{ev}_0 (I - \Delta_z B_z)^{-1} \Delta_z \mathcal{J}_z h_\phi(\cdot) dz \\ & & + \frac{1}{2\pi i} \int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{z\xi} \mathrm{ev}_0 (I - \Delta_z B_z)^{-1} \Delta_z \mathcal{J}_z h_\phi(\cdot) dz, \end{array} \tag{5.2}$$

with

$$h_\phi(z') = \phi(0) + \sum_{j=0}^{N} e^{z' r_j} \int_{r_j}^{0} e^{-\sigma z'} \left(A^j_{\text{aut}} + B^j(\sigma - r_j)\right) \phi(\sigma) d\sigma. \tag{5.3}$$

In addition, we have the representation

$$(\mathcal{P}_{\gamma_-,\gamma_+}\phi)(\xi) \quad = \quad [\Lambda^{-1}_{(\gamma_+)} g_\phi - \Lambda^{-1}_{(\gamma_-)} g_\phi](\xi) + [G_{(\gamma_+)}(0, \xi) - G_{(\gamma_-)}(0, \xi)]\phi(0), \tag{5.4}$$

in which $g_\phi \in L^2(\mathbb{R}, \mathbb{C}^n)$ has compact support, is continuous on $[-r_{\max}, -r_{\min}]$ and is given by

$$\begin{array}{lll} g_\phi(\xi) & = & \sum_{r_j > 0} \left(A^j_{\text{aut}} + B^j(\xi)\right) \phi(\xi + r_j) \chi_{[-r_j,0]}(\xi) \\ & & + \sum_{r_j < 0} \left(A^j_{\text{aut}} + B^j(\xi)\right) \phi(\xi + r_j) \chi_{[0,-r_j]}(\xi). \end{array} \tag{5.5}$$

Proof. First note that $\Phi_\pm f \in L^2_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n) \cap L^\infty_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n)$. Hence defining $x_\pm = \Lambda^{-1}_{(\gamma_\pm)} \Phi_\pm f \in W^{1,2}_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n) \subset L^\infty_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n)$, the differential equation implies $x_+ + x_- \in BX^1_{\gamma_-,\gamma_+}(\mathbb{R}, \mathbb{C}^n)$. However, this means that also $\mathcal{P}x_0 = x - x_+ - x_- \in BX^1_{\gamma_-,\gamma_+}(\mathbb{R}, \mathbb{C}^n)$, hence $\mathcal{P}$ is well-defined. Taking the Laplace transform of $\Lambda x = f$ yields

$$\begin{array}{rcl} z\widetilde{x}_+(z) & = & x(0) + \sum_{j=0}^N A^j_{\mathrm{aut}} \int_0^\infty e^{-zu} x(u + r_j) du \\ & & \quad + \sum_{j=0}^N \int_0^\infty e^{-zu} B^j(u) x(u + r_j) du + \widetilde{f}_+(z) \\ & = & x(0) + \sum_{j=0}^N A^j_{\mathrm{aut}} e^{zr_j} \big(\widetilde{x}_+(z) + \int_{r_j}^0 e^{-z\sigma} x(\sigma) d\sigma \big) + \widetilde{f}_+(z) \\ & & \quad + \sum_{j=0}^N e^{zr_j} \big(\int_0^\infty e^{-zu} B^j(u - r_j) x(u) du + \int_{r_j}^0 e^{-z\sigma} B^j(\sigma - r_j) x(\sigma) d\sigma \big) \\ & = & x(0) + \sum_{j=0}^N A^j_{\mathrm{aut}} e^{zr_j} \big(\widetilde{x}_+(z) + \int_{r_j}^0 e^{-z\sigma} x(\sigma) d\sigma \big) + \widetilde{f}_+(z) \\ & & \quad + \sum_{j=0}^N e^{zr_j} \big[\sum_{k\in\mathbb{Z}} e^{-ikr_j} B^j_k \widetilde{x}_+(z - ik) + \int_{r_j}^0 e^{-z\sigma} B^j(\sigma - r_j) x(\sigma) d\sigma \big] \end{array} \tag{5.6}$$

and thus after rearrangement we have

$$\begin{array}{rcl} \Delta(z)\widetilde{x}_+(z) & = & x(0) + \sum_{k\in\mathbb{Z}} \sum_{j=0}^N e^{(z-ik)r_j} B^j_k \widetilde{x}_+(z - ik) + \widetilde{f}_+(z) \\ & & \quad + \sum_{j=0}^N e^{zr_j} \int_{r_j}^0 e^{-z\sigma} \big(A^j_{\mathrm{aut}} + B^j(\sigma - r_j) \big) x(\sigma) d\sigma. \end{array} \tag{5.7}$$

Upon defining $y(\xi) = x(-\xi)$ a similar identity may be obtained for $\widetilde{y}_+(z)$. Similarly as in [18], an application of the inversion formula (3.7) now yields the desired result (5.2), upon observing that

$$\int_{\gamma_+ - i\infty}^{\gamma_+ + i\infty} e^{\xi z} \int_\xi^0 e^{-z\sigma} x(\sigma) d\sigma dz + \int_{\gamma_- + i\infty}^{\gamma_- - i\infty} e^{\xi z} \int_\xi^0 e^{-z\sigma} x(\sigma) d\sigma dz = 0. \tag{5.8}$$

We now establish the representation (5.4), by writing $g = g_{x_0}$ and computing $\widetilde{g}_+$ and $\widetilde{g}_-$. This yields

$$\begin{array}{rcl} \widetilde{g}_+(z) & = & \int_0^\infty e^{-z\xi} g(\xi) d\xi = \sum_{r_j<0} \int_0^{-r_j} e^{-z\xi} \big(A^j_{\mathrm{aut}} + B^j(\xi) \big) x(\xi + r_j) d\xi \\ & = & \sum_{r_j<0} e^{zr_j} \int_{r_j}^0 e^{-z\xi'} \big(A^j_{\mathrm{aut}} + B^j(\xi' - r_j) \big) x(\xi') d\xi', \\ \widetilde{g}_-(z) & = & \int_0^\infty e^{z\xi} g(-\xi) d\xi = \sum_{r_j>0} \int_0^{r_j} e^{z\xi} \big(A^j_{\mathrm{aut}} + B^j(-\xi) \big) x(-\xi + r_j) d\xi \\ & = & \sum_{r_j>0} e^{zr_j} \int_{r_j}^0 e^{-z\xi''} \big(A^j_{\mathrm{aut}} + B^j(\xi'' - r_j) \big) x(\xi'') d\xi'', \end{array} \tag{5.9}$$

in which we used the substitutions $\xi' = \xi + r_j$ and $\xi'' = -\xi + r_j$. The result follows using Corollary 4.2, together with the observation that the bounded function g_{x_0} has compact support, which means $g_{x_0} \in L^2_{\gamma_\pm}(\mathbb{R}, \mathbb{C}^n)$. □

We now study the set of solutions to the homogeneous equation (2.4) that have controlled exponential growth. We will therefore consider the spaces

$$\begin{array}{rcl} \mathcal{N}_{\mu,\nu} & = & \big\{ x \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n) \mid \Lambda x = 0 \big\}, \\ X_{\mu,\nu} & = & \{ \phi \in X \mid \phi = x_0 \text{ for some } x \in \mathcal{N}_{\mu,\nu} \}. \end{array} \tag{5.10}$$

From the representation (5.1) it follows immediately that for every $\phi \in X_{\mu,\nu}$ there is a unique $x \in \mathcal{N}_{\mu,\nu}$ with $x_0 = \phi$, which we will denote as $x = E\phi$. Using a standard shifting argument, it is clear that for all $x \in \mathcal{N}_{\mu,\nu}$ and any $k \in \mathbb{Z}$, we also have $T_{2k\pi} x \in \mathcal{N}_{\mu,\nu}$. We can hence define the monodromy operators $M_{\pm 2\pi} : X_{\mu,\nu} \to X_{\mu,\nu}$ by $\phi \mapsto \mathrm{ev}_{\pm 2\pi} E\phi$, which satisfy $M_{2\pi} M_{-2\pi} = M_{-2\pi} M_{2\pi} = I$.

Lemma 5.2. *Consider a homogeneous linear equation (2.4) that satisfies (HL). Suppose further that for two constants $\gamma_- < \gamma_+$, this equation (2.4) admits no Floquet exponents λ on the lines $\mathrm{Re}\,\lambda = \gamma_\pm$. Then for any pair $\gamma_- < \mu < \nu < \gamma_+$, we have that $M_{2\pi}$ is a compact operator on $X_{\mu,\nu}$ and $\mathcal{N}_{\mu,\nu}$ is finite dimensional.*

Proof. The representation (5.4) implies that for some $C > 0$ we have a bound $\|M_{2\pi}\phi\| \le C\,\|\phi\|$ for all $\phi \in X_{\mu,\nu}$, which using the differential equation implies that also $\|DM_{2\pi}\phi\| \le C'\,\|\phi\|$. An application of the Ascoli-Arzela theorem shows that $M_{2\pi}$ is compact. However, since $M_{2\pi}$ has a bounded inverse, the unit ball in $X_{\mu,\nu}$ is compact and hence this space is finite dimensional. $\square$

Since $M_{2\pi}$ is invertible, we can define a matrix W such that $e^{2\pi W} = M_{2\pi}$. Consider any $\psi \in X_{\mu,\nu}$, then the continuous function $P_\psi : \mathbb{R} \to \mathbb{C}^n$ given by $P_\psi = Ee^{-\xi W}\psi$ is periodic, since

$$\begin{aligned} P_\psi(\xi + 2\pi) &= [Ee^{-(\xi+2\pi)W}\psi](\xi + 2\pi) = [E\mathrm{ev}_{2\pi}EM_{-2\pi}e^{-\xi W}\psi](\xi) \\ &= [EM_{2\pi}M_{-2\pi}e^{-\xi W}\psi](\xi) = P_\psi(\xi). \end{aligned} \tag{5.11}$$

Consider a Jordan chain $\phi^0, \ldots, \phi^\ell$ of length $\ell + 1$ for W at some eigenvalue λ, i.e., $W\phi^0 = \lambda\phi^0$ and $W\phi^i = \lambda\phi^i + \phi^{i-1}$ for $1 \le i \le \ell$. Recall that $e^{W\xi}\phi^i = \sum_{j=0}^{i} \frac{1}{j!}\xi^j e^{\lambda\xi}\phi^{i-j}$. Writing $x^i = E\phi^i$, we now obtain that

$$e^{\lambda\xi}P_{\phi^i}(\xi) = \sum_{j=0}^{i} \frac{1}{j!}(-\xi)^j x^{i-j}(\xi). \tag{5.12}$$

This can be inverted, yielding $x^0 = e^{\lambda\xi}P_{\phi^0}$, which implies that λ is a Floquet multiplier. Similarly, we have

$$x^i(\xi) = e^{\lambda\xi}P_{\phi^i}(\xi) - \sum_{j=1}^{i} \frac{1}{j!}(-\xi)^j x^{i-j}(\xi). \tag{5.13}$$

We hence conclude that $\mathcal{N}_{\mu,\nu}$ is spanned by functions of the form $e^{\lambda\xi}\xi^j p(\xi)$, with $p \in C_{2\pi}^{\mathrm{per}}(\mathbb{R}, \mathbb{C}^n)$ and λ a Floquet exponent with $\mu \le \mathrm{Re}\,\lambda \le \nu$. This important observation gives a criterion for the existence of an inverse for $\Lambda : W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L_\eta^\infty(\mathbb{R}, \mathbb{C}^n)$, merely in terms of Floquet exponents.

Proposition 5.3. *Consider an equation of the form (4.3) that satisfies (HL). Consider any $\eta \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that (4.3) has no Floquet exponents λ in the strip $\eta - \varepsilon_0 < \mathrm{Re}\,\lambda < \eta + \varepsilon_0$. Then the operator Λ is an isomorphism from $W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ onto $L_\eta^\infty(\mathbb{R}, \mathbb{C}^n)$. For any $0 < \varepsilon < \varepsilon_0$, the inverse is given by*

$$\Lambda^{-1}f = \Lambda_{(\eta+\varepsilon)}^{-1}\Phi_+ f + \Lambda_{(\eta-\varepsilon)}^{-1}\Phi_- f. \tag{5.14}$$

Proof. Notice first that the assumptions of Proposition 4.3 are satisfied. Indeed, for any function $g \in L_{\eta+\varepsilon}^2(\mathbb{R}, \mathbb{C}^n) \cap L_{\eta-\varepsilon}^2(\mathbb{R}, \mathbb{C}^n)$, write $x = \Lambda_{(\eta+\varepsilon)}^{-1}g - \Lambda_{(\eta-\varepsilon)}^{-1}g$, then $x \in BX_{\eta-\varepsilon,\eta+\varepsilon}^1(\mathbb{R}, \mathbb{C}^n)$ with $\Lambda x = 0$, i.e., $x \in \mathcal{N}_{\eta-\varepsilon,\eta+\varepsilon}$. However, the condition on the Floquet exponents implies that $\mathcal{N}_{\eta-\varepsilon,\eta+\varepsilon} = \{0\}$, hence $x = 0$ as desired. Proposition 4.3 now shows that Λ^{-1} defined above indeed maps into $W_\eta^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$. The injectivity of Λ again follows from the condition on the Floquet multipliers. $\square$

The finite dimensionality of $X_{\mu,\nu}$ can be exploited to define a projection from X onto this subspace, using the operator $\mathcal{P}$ appearing in (5.1).

Lemma 5.4. *Consider any set of constants $\widetilde{\gamma}_- < \gamma_- < \gamma_+ < \widetilde{\gamma}_+$ such that the equation (4.3) has no Floquet exponents λ with $\operatorname{Re}\lambda \in \{\gamma_\pm, \widetilde{\gamma}_\pm\}$. Suppose further that (HL) is satisfied. Then the operator $P = P_{\gamma_-,\gamma_+} : X \to X$ defined by $P\phi = \mathrm{ev}_0 \mathcal{P}_{\gamma_-,\gamma_+}\phi$ is a projection, with $\mathcal{R}(P_{\gamma_-,\gamma_+}) = X_{\gamma_-,\gamma_+}$.*

Proof. Notice first that the set of real parts of Floquet exponents between $\widetilde{\gamma}_-$ and $\widetilde{\gamma}_+$ is discrete, hence there exist $\gamma_- < \mu < \nu < \gamma_+$ such that $X_{\gamma_-,\gamma_+} = X_{\mu,\nu}$. Now (5.4) implies that $\mathcal{R}(\mathcal{P}_{\gamma_-,\gamma_+}) \subset \mathcal{N}_{\gamma_-,\gamma_+} = \mathcal{N}_{\mu,\nu}$, hence $\mathcal{R}(P_{\gamma_-,\gamma_+}) \subset X_{\mu,\nu}$. In addition, for any $\phi \in X_{\mu,\nu}$ write $x = E\phi \in BX^1_{\mu,\nu}(\mathbb{R}, \mathbb{C}^n)$ and notice that (5.1) implies $x = \mathcal{P}_{\gamma_-,\gamma_+}\phi$, yielding

$$\phi = x_0 = \mathrm{ev}_0 \mathcal{P}_{\gamma_-,\gamma_+}\phi = P_{\gamma_-,\gamma_+}\phi. \tag{5.15}$$

This shows that indeed $\mathcal{R}(P_{\gamma_-,\gamma_+}) = X_{\mu,\nu} = X_{\gamma_-,\gamma_+}$ and hence also $P^2 = P$. □

From now on fix $\gamma > 0$ such that there are no Floquet exponents with $0 < |\operatorname{Re}\lambda| < \gamma$. For any $0 < \mu < \gamma$, define $X_0 = X_{-\mu,\mu}$, $\mathcal{N}_0 = \mathcal{N}_{-\mu,\mu}$ and $Q_0 = P_{-\mu,\mu}$. Note that these definitions are independent of the particular choice of μ. In addition, for any $0 < \eta < \gamma$, define the pseudo-inverse $\mathcal{K} = \mathcal{K}_\eta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ by

$$\mathcal{K}_\eta f = \Lambda^{-1}_{(\eta)}\Phi_+ f + \Lambda^{-1}_{(-\eta)}\Phi_- f. \tag{5.16}$$

Notice that if $0 < \eta_0 < \eta_1 < \gamma$, then $(\mathcal{K}_{\eta_1})_{|BC_{\eta_0}(\mathbb{R},\mathbb{C}^n)} = \mathcal{K}_{\eta_0}$. This can be verified by means of the same reasoning used to established Proposition 5.3. In combination with (5.1), this allows us to compute

$$\mathcal{K}_{\eta_0} f = \mathcal{K}_{\eta_1}\Lambda\mathcal{K}_{\eta_0} f + \mathcal{P}_{-\eta_1,\eta_1}\mathrm{ev}_0\mathcal{K}_{\eta_0} f = \mathcal{K}_{\eta_1} f + \mathcal{P}_{-\eta_1,\eta_1}\mathrm{ev}_0\mathcal{K}_{\eta_0} f = \mathcal{K}_{\eta_0} f + \mathcal{P}_{-\eta_1,\eta_1}\mathrm{ev}_0\mathcal{K}_{\eta_0} f \tag{5.17}$$

and hence

$$Q_0 ev_0 \mathcal{K}_{\eta_0} f = 0. \tag{5.18}$$

6 Time dependence

For any $\tau \in \mathbb{R}$, consider the shifted mixed type functional differential equation

$$\dot{x}_\xi = L^\tau(\xi)x(\xi) + f(\xi) = L(\xi + \tau)x_\xi + f(\xi) \tag{6.1}$$

and write X^τ_0, $\mathcal{N}^\tau_0$, Λ^τ, Q^τ_0 and $\mathcal{K}^\tau$ for the spaces and operators associated to (6.1) that are the counterparts of those defined for the original unshifted equation (4.3).

Lemma 6.1. *Consider the homogeneous equation (2.4), suppose that (HL) is satisfied and fix two constants $\tau_0, \tau_1 \in \mathbb{R}$. Then for any $\phi \in X^{\tau_0}_0$, we have that $T_{\tau_1-\tau_0}E\phi \in \mathcal{N}^{\tau_1}_0$, i.e.,*

$$Q^{\tau_1}_0 \mathrm{ev}_{\tau_1-\tau_0} E\phi = \mathrm{ev}_{\tau_1-\tau_0} E\phi. \tag{6.2}$$

In addition, for any function $f \in BC_\eta(\mathbb{R}, \mathbb{C}^n)$, the following identity holds,

$$\mathcal{K}_\eta^{\tau_0+\tau_1} T_{\tau_1} f \quad = \quad T_{\tau_1} \mathcal{K}_\eta^{\tau_0} f - E Q_0^{\tau_0+\tau_1} \mathrm{ev}_{\tau_1} \mathcal{K}_\eta^{\tau_0} f. \tag{6.3}$$

Proof. First, consider any $\phi \in X_0^{\tau_0}$ and write $u = E\phi$, then $\dot{u}(\xi) = L(\xi + \tau_0) u_\xi$. Defining $\psi = u_{\tau_1 - \tau_0}$, notice that the function $v = E\psi$ has $v(\xi) = u(\xi + \tau_1 - \tau_0)$ and satisfies $\dot{v}(\xi) = L(\xi + \tau_1) v_\xi$, showing that $\psi \in X_0^{\tau_1}$ as required. Now consider the function y defined by

$$y = T_{\tau_1} \mathcal{K}_\eta^{\tau_0} f - \mathcal{K}_\eta^{\tau_0+\tau_1} T_{\tau_1} f. \tag{6.4}$$

It is easy to compute

$$\dot{y}(\xi) = L(\xi + \tau_0 + \tau_1) y_\xi + f(\xi + \tau_1) - f(\xi + \tau_1) = L(\xi + \tau_0 + \tau_1) y_\xi \tag{6.5}$$

and hence $y \in \mathcal{N}_0^{\tau_0+\tau_1}$. The final statement now follows from $y = E y_0$, together with the computation

$$\begin{array}{rcl} y_0 & = & Q_0^{\tau_0+\tau_1} y_0 = Q_0^{\tau_0+\tau_1} \mathrm{ev}_{\tau_1} \mathcal{K}_\eta^{\tau_0} f - Q_0^{\tau_0+\tau_1} \mathrm{ev}_0 \mathcal{K}_\eta^{\tau_0+\tau_1} T_{\tau_1} f \\ & = & Q_0^{\tau_0+\tau_1} \mathrm{ev}_{\tau_1} \mathcal{K}_\eta^{\tau_0} f, \end{array} \tag{6.6}$$

where (5.18) was used in the last identity. □

An elementary observation that follows from this result and the uniqueness of continuations, is that if $y \in \mathcal{N}_0^\tau$ for any $\tau \in \mathbb{R}$, then

$$\mathrm{ev}_\xi E \mathrm{ev}_{\xi'} y = \mathrm{ev}_{\xi+\xi'} y. \tag{6.7}$$

We will need the ability to relate the different subspaces X_0^τ to one another in a natural fashion. To this end, we recall the matrix $W \in \mathcal{L}(X_0)$ that is related to the monodromy operator $M_{2\pi}$ by $M_{2\pi} = e^{2\pi W}$. For all $\tau \in \mathbb{R}$, we define the bounded linear operators $\Pi_\to^\tau : X_0 \to X_0^\tau$ and $\Pi_\leftarrow^\tau : X_0^\tau \to X_0$, via

$$\begin{array}{rcl} \Pi_\to^\tau \phi & = & \mathrm{ev}_\tau E e^{-\tau W} \phi, \\ \Pi_\leftarrow^\tau \psi & = & e^{\tau W} \mathrm{ev}_{-\tau} E \psi. \end{array} \tag{6.8}$$

In addition, we define a mapping $\Pi : \mathbb{R} \to \mathcal{L}(X, X_0)$ by

$$\Pi(\tau) = \Pi_\leftarrow^\tau Q_0^\tau. \tag{6.9}$$

Using the definition of W and the identity (6.7), it is clear that all three operators defined above are periodic, i.e., $\Pi_\leftarrow^{\tau+2\pi} = \Pi_\leftarrow^\tau$ and similarly for $\Pi_\to$ and Π. Notice also that $\Pi_\to^\tau \Pi_\leftarrow^\tau = I$ and $\Pi_\leftarrow^\tau \Pi_\to^\tau = \Pi(\tau) \Pi_\to^\tau = I$.

In the remainder of this section we will show that the operator Π inherits the C^r-smoothness of the linear operator L. In [26] this was obtained directly, using an equivalence between the Floquet spectrum and the spectrum of an operator Λ'_{per}, that in our setting should be seen as the restriction of Λ to the space $W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \cap C^{2\pi}_{\mathrm{per}}(\mathbb{R}, \mathbb{C}^n)$. In particular, any eigensolution $\Lambda'_{\mathrm{per}} u = -\lambda u$ would lead to a Floquet exponent λ via $x(\xi) = e^{\lambda \xi} u(\xi)$. However, this last observation is only valid in the absence of delayed and advanced arguments in (2.2). This fact forces us to pursue an alternate approach.

Lemma 6.2. *Consider a linear equation of the form (4.3) that satisfies the assumption (HL) and suppose that this equation admits no Floquet exponents on the imaginary axis. Then the function* $\overline{\Lambda^{-1}} : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}, \mathbb{C}^n), W^{1,2}(\mathbb{R}, \mathbb{C}^n))$ *given by* $\tau \mapsto (\Lambda^\tau)^{-1}$ *is* C^1*-smooth, with derivative given by*

$$D\overline{\Lambda^{-1}}(\tau) = \overline{\Lambda^{-1}}(\tau)[DL^\tau]\overline{\Lambda^{-1}}(\tau). \tag{6.10}$$

In addition, for any $\xi \in \mathbb{R}$*, the function* $\overline{G} : \mathbb{R} \to L^2(\mathbb{R}, \mathbb{C}^{n\times n})$ *given by* $\tau \mapsto G^\tau(\xi, \cdot)$ *is* C^1*-smooth, with derivative given by*

$$D\overline{G}(\tau) = \overline{\Lambda^{-1}}(\tau)[DL^\tau]\overline{G}(\tau). \tag{6.11}$$

Proof. Consider for $0 \le j \le N$ the operators $\widetilde{B}^{j,\tau} \in \mathcal{L}(\ell_2, \ell_2)$, given by

$$(B^{j,\tau} w)_n = \sum_{k\in\mathbb{Z}} e^{i(n-k)r_j} e^{ik\tau} B^j_k w_{n-k}. \tag{6.12}$$

We claim that $\tau \mapsto \widetilde{B}^{j,\tau}$ is differentiable at $\tau = 0$ and that the derivative is generated by the operator $DB^j \in \mathcal{L}(X, \mathbb{C}^n)$. Indeed, a similar estimate as in (4.7) yields

$$\begin{array}{rcl} \left\|[\widetilde{B}^{j,\tau} - \widetilde{B}^j - \tau\widetilde{DB}^j]w\right\|_2^2 & = & \sum_{n\in\mathbb{Z}} \left|\sum_{m\in\mathbb{Z}} e^{i(n-m)r_j}[e^{i\tau m} - im\tau - 1]B^j_m w_{n-m}\right|^2 \\ & \le & (\sum_{m\in\mathbb{Z}} \left|[e^{i\tau m} - im\tau - 1]B^j_m\right|)^2 \left\|w\right\|_2^2 . \end{array} \tag{6.13}$$

Now fix $\varepsilon > 0$ and choose $\varepsilon' = \varepsilon[2\sum_{m\in\mathbb{Z}} \left|mB^j_m\right|]^{-1} > 0$. Since the exponential function is differentiable, there exists a $\delta' > 0$ such that

$$|e^z - z - 1| < \varepsilon' |z| \tag{6.14}$$

for all $|z| < \delta'$. Now let $M > 0$ be so large that $(\frac{2}{\delta'} + 1)\sum_{|m|>M} \left|mB^j_m\right| < \frac{\varepsilon}{2}$. Finally, fix $\delta = \frac{\delta'}{M}$. For any $0 < |\tau| < \delta$, write $\Delta = \sum_{m\in\mathbb{Z}} \left|[e^{i\tau m} - im\tau - 1]B^j_m\right|$ and compute

$$\begin{array}{rcl} \Delta & = & \sum_{|m|\le\frac{\delta'}{|\tau|}} \left|[e^{i\tau m} - im\tau - 1]B^j_m\right| + \sum_{|m|>\frac{\delta'}{|\tau|}} \left|[e^{i\tau m} - im\tau - 1]B^j_m\right| \\ & \le & \sum_{|m|\le\frac{\delta'}{|\tau|}} \varepsilon' |\tau| \left|mB^j_m\right| + \sum_{|m|>\frac{\delta'}{|\tau|}} (2 + |m|\,|\tau|) \left|B^j_m\right| \\ & \le & |\tau|\,\varepsilon' \sum_{m\in\mathbb{Z}} \left|mB^j_m\right| + \sum_{|m|>\frac{\delta'}{|\tau|}} (\frac{2}{\delta'} + 1)\,|m|\,|\tau| \left|B^j_m\right| \\ & \le & \frac{\varepsilon}{2}|\tau| + |\tau| \sum_{|m|>M} (\frac{2}{\delta'} + 1) \left|mB^j_m\right| < \varepsilon\,|\tau| . \end{array} \tag{6.15}$$

This proves the differentiability of $\tau \mapsto \widetilde{B}^{j,\tau}$ at $\tau = 0$ and analogously for all $\tau \in \mathbb{R}$. Since Δ_z does not depend on τ, this shows that $\tau \mapsto I - \Delta_z B^\tau_z \in \mathcal{L}(\ell_2)$ and hence the inverse are differentiable in the variable τ, uniformly for $z \in i\mathbb{R}$. We find

$$D\big[\tau \mapsto (I - \Delta_z B^\tau_z)^{-1}\big] = \tau \mapsto [I - \Delta_z B^\tau_z]^{-1}\Delta_z[DB]^\tau_z[I - \Delta_z B^\tau_z]^{-1}. \tag{6.16}$$

An estimate analogous to (4.13) now completes the proof. ∎

The explicit forms (6.10) and (6.11) allow repeated differentiation of $\overline{\Lambda^{-1}}$ and $\overline{G}$, up to the point that the differentiability of L is lost. This observation leads to the following result.

Corollary 6.3. *Consider a linear equation of the form (4.3) that satisfies the assumption (HL) and suppose that this equation admits no Floquet exponents on the imaginary axis. Then the functions* $\overline{\Lambda^{-1}} : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}, \mathbb{C}^n), W^{1,2}(\mathbb{R}, \mathbb{C}^n))$ *and* $\overline{G} : \mathbb{R} \to L^2(\mathbb{R}, \mathbb{C}^{n\times n})$ *are* C^r*-smooth and for any* $1 \le \ell \le r$ *we have*

$$\begin{array}{lcl} D^\ell \overline{\Lambda^{-1}}(\tau) & = & \sum_{(f_1,\ldots,f_q)} c_{(f_1,\ldots,f_q)} \overline{\Lambda^{-1}}(\tau)[D^{f_1} L^\tau]\overline{\Lambda^{-1}}(\tau)\ldots\overline{\Lambda^{-1}}(\tau)[D^{f_q} L^\tau]\overline{\Lambda^{-1}}(\tau), \\ D^\ell \overline{G}(\tau) & = & \sum_{(f_1,\ldots,f_q)} c_{(f_1,\ldots,f_q)} \overline{\Lambda^{-1}}(\tau)[D^{f_1} L^\tau]\overline{\Lambda^{-1}}(\tau)\ldots\overline{\Lambda^{-1}}(\tau)[D^{f_q} L^\tau]\overline{G}(\tau), \end{array} \tag{6.17}$$

in which the sum is taken over tuples $(f_1, \ldots, f_q)$ *with* $f_i \ge 1$ *and* $f_1 + \ldots + f_q = \ell$.

We will use the representation (5.4) in order to establish the smoothness of Π. We hence need to extend the results above to show the differentiability of $\overline{\Lambda^{-1}}$ acting as an operator into the space of C^{r+1}-smooth functions. To do this, let $K' \subset \mathbb{R}$ be a compact interval and consider the set $C_0(K', \mathbb{C}^n)$ of continuous functions f with support contained in K', i.e., $\mathrm{supp}(f) \subset K'$. Fixing any bounded open interval $\Omega \subset \mathbb{R}$, we now define operators $\overline{\Gamma} = \overline{\Gamma}_{(\eta)} : \mathbb{R} \to \mathcal{L}(C_0(K', \mathbb{C}^n), C^{r+1}(\Omega, \mathbb{C}^n))$ and $\overline{H} = \overline{H}_{(\eta)} : \mathbb{R} \to C^{r+1}(\Omega, \mathbb{C}^{n\times n})$ via

$$\begin{array}{lcl} \overline{\Gamma}(\tau) f & = & \overline{\Lambda^{-1}_{(\eta)}}(\tau) f - \overline{\Lambda^{-1}_{(-\eta)}}(\tau) f, \\ \overline{H}(\tau) & = & \overline{G}_{(\eta)}(\tau) - \overline{G}_{(-\eta)}(\tau). \end{array} \tag{6.18}$$

Notice that indeed $\overline{\Gamma}(\tau) f \in C^{r+1}(\Omega, \mathbb{C}^n)$, since $\Lambda^\tau \overline{\Gamma}(\tau) f = 0$. Throughout the remainder of this paper, we will use the symbol D_ξ to exclusively represent differentiation with respect to a time-like real-valued variable. The details should be clear from the context. We will also write D_τ for the derivative with respect to the variable τ.

For any suitable integer s, a quick calculation shows that $D^s_\xi D^\ell_\tau \overline{\Lambda^{-1}}(\tau)$ can be written as a sum of elements of the form

$$D_{(e_1,\ldots e_p)} \Lambda_{(f_1,\ldots f_q)}, \tag{6.19}$$

for integers $0 \le p \le s$, $e_i \ge 0$ and $f_i \ge 1$ that satisfy $f_1 + \ldots + f_q \le \ell$ and $p + e_1 + \ldots + e_p + f_1 + \ldots + f_q = s + \ell$, in which

$$\begin{array}{lcl} D_{(e_1,\ldots e_p)} & = & [D^{e_1} L^\tau]\ldots[D^{e_p} L^\tau], \\ \Lambda_{(f_1,\ldots f_q)} & = & \overline{\Lambda^{-1}}(\tau)[D^{f_1} L^\tau]\overline{\Lambda^{-1}}(\tau)\ldots\overline{\Lambda^{-1}}(\tau)[D^{f_q} L^\tau]\overline{\Lambda^{-1}}(\tau), \end{array} \tag{6.20}$$

together with elements of the form

$$[D^{e_1} L^\tau]\ldots[D^{e_p} L^\tau] D^f_\xi, \tag{6.21}$$

with $e_i \ge 0$, $f \ge 0$ and $p + e_1 + \ldots + e_p + f + 1 = s + \ell$. Now for any tuples $(e_1, \ldots, e_p)$ and $(f_1, \ldots, f_q)$, define the sets

$$\begin{array}{lcl} (e_1, \ldots, e_p) \oplus 1 & = & \{(e_1 + 1, e_2, \ldots, e_p), (e_1, e_2 + 1, \ldots, e_p), \ldots, (e_1, \ldots, e_p + 1)\}, \\ (f_1, \ldots, f_q) \odot 1 & = & (f_1, \ldots, f_q) \oplus 1 \cup \{(1, f_1, \ldots, f_q), (f_1, 1, \ldots, f_q), \ldots, (f_1, \ldots, f_q, 1)\}. \end{array} \tag{6.22}$$

If $q \geq 1$, an easy calculation shows that

$$\begin{array}{rcl} D_\xi D_{(e_1,\ldots,e_p)}\Lambda_{(f_1,\ldots,f_q)} & = & D_{(e_1,\ldots,e_p)\oplus 1}\Lambda_{(f_1,\ldots,f_q)} + D_{(e_1,\ldots,e_p,0)}\Lambda_{(f_1,\ldots,f_q)} \\ & & +D_{(e_1,\ldots,e_p,f_1)}\Lambda_{(f_2,\ldots,f_q)} \\ D_\tau D_\xi D_{(e_1,\ldots,e_p)}\Lambda_{(f_1,\ldots,f_q)} & = & D_{(e_1,\ldots,e_p)\oplus 1\oplus 1}\Lambda_{(f_1,\ldots,f_q)} + D_{(e_1,\ldots,e_p)\oplus 1}\Lambda_{(f_1,\ldots,f_q)\odot 1} \\ & & +D_{(e_1,\ldots,e_p,0)\oplus 1}\Lambda_{(f_1,\ldots,f_q)} + D_{(e_1,\ldots,e_p,0)}\Lambda_{(f_1,\ldots,f_q)\odot 1} \\ & & +D_{(e_1,\ldots,e_p,f_1)\oplus 1}\Lambda_{(f_2,\ldots,f_q)} + D_{(e_1,\ldots,e_p,f_1)}\Lambda_{(f_2,\ldots,f_q)\odot 1}, \end{array} \tag{6.23}$$

upon understanding that $\Lambda_\emptyset = \Lambda^{-1}(\tau)$ and noting that for any set $\mathcal{E}$, one should read $D_\mathcal{E} = \sum_{e\in\mathcal{E}} D_e$. If $q = 0$, then the same identity holds if one writes $f_1 = 0$, $\Lambda_{(f_2,\ldots,f_q)} = \mathrm{id}$ and $\Lambda_{(f_2,\ldots,f_q)\odot 1} = 0$. The important observation, which can be verified by a simple calculation, is that D_τ and D_ξ commute on elements of the form (6.19), i.e.,

$$D_\tau D_\xi D_{(e_1,\ldots,e_p)}\Lambda_{(f_1,\ldots,f_q)} = D_\xi D_\tau D_{(e_1,\ldots,e_p)}\Lambda_{(f_1,\ldots,f_q)}. \tag{6.24}$$

Lemma 6.4. *Consider a linear equation of the form (4.3) that satisfies the assumption (HL) and suppose that for some $\gamma > 0$ this equation admits no Floquet exponents λ with $0 < |\mathrm{Re}\,\lambda| < \gamma$. Consider an integer $0 \leq \ell \leq r$ and a parameter $\eta \in (0,\gamma)$. Then the maps $\overline{\Gamma}_\ell = \overline{\Gamma}_{(\eta),\ell} : \mathbb{R} \to \mathcal{L}(C_0(K,\mathbb{C}^n), C^{r+1-\ell}(\Omega,\mathbb{C}^n))$ and $\overline{H}_\ell = \overline{H}_{(\eta),\ell} : \mathbb{R} \to C^{r+1-\ell}(\Omega,\mathbb{C}^{n\times n})$ are C^ℓ-smooth.*

Proof. We will only treat the map $\overline{\Gamma}_\ell$, since the differentiability of $\overline{H}_\ell$ follows in a similar fashion. For any $\tau \in \mathbb{R}$, consider the map $\Phi_\ell(\tau) : C_0(K,\mathbb{C}^n) \to C^{r+1-\ell}(\Omega,\mathbb{C}^n)$ given by

$$\Phi_\ell(\tau)f = [D_\tau^\ell\overline{\Lambda^{-1}_{(+\eta)}}](\tau)f - [D_\tau^\ell\overline{\Lambda^{-1}_{(-\eta)}}](\tau)f. \tag{6.25}$$

In order to see that indeed $\Phi_\ell(\tau)f \in C^{r+1-\ell}(\Omega,\mathbb{C}^n)$, notice first that due to the special form of $\Phi_\ell(\tau)$ we can ignore all the terms of the form (6.21) in the expansion of $[D_\xi^{r+1-\ell}D_\tau^\ell\overline{\Lambda^{-1}_{(\pm\eta)}}](\tau)$. We hence need only consider the terms of the form (6.19) with $s = r+1-\ell$. However, since $e_i \leq r$ for all $1 \leq i \leq p$, these terms will yield a continuous function when applied to f, as desired.

For convenience, we will treat each of the $r+1-\ell$ components of $\overline{\Gamma}_\ell f$ separately in order to show that Φ_ℓ is indeed the ℓ-th derivative of $\overline{\Gamma}_\ell$. To this end, define for all $0 \leq s \leq r+1-\ell$, the map $\Gamma^{(s)}(\tau) : C_0(K,\mathbb{C}^n) \to C(\Omega,\mathbb{C}^n)$, given by $\Gamma^{(s)}(\tau)f = D_\xi^s\overline{\Gamma}_\ell(\tau)f$. Observe first that due to the Sobolev embeddings $W^{1,2}_{\pm\eta}(\mathbb{R},\mathbb{C}^n) \subset L^\infty_{\pm\eta}(\mathbb{R},\mathbb{C}^n)$, Corollary 6.3 implies that $D_\tau^\ell\Gamma^{(0)} = \Phi_\ell$ when viewing Φ_ℓ as a function mapping into $C(\Omega,\mathbb{C}^n)$. Now due to the commutation relation (6.24), one may use a similar argument to show that for all $0 \leq s \leq r+1-\ell$, $\Gamma^{(s)}$ is ℓ-fold differentiable, with

$$[D_\tau^\ell\Gamma^{(s)}](\tau)f = [D_\xi^sD_\tau^\ell\overline{\Lambda^{-1}_{(+\eta)}} - D_\xi^sD_\tau^\ell\overline{\Lambda^{-1}_{(-\eta)}}](\tau)f \in C(\Omega,\mathbb{C}^n). \tag{6.26}$$

The continuity of Φ_ℓ follows from the continuity of $\overline{\Lambda^{-1}_{(\pm\eta)}}$ as maps $\mathbb{R} \to \mathcal{L}(L^2_{\pm\eta}(\mathbb{R},\mathbb{C}^n), W^{1,2}_{\pm\eta}(\mathbb{R},\mathbb{C}^n))$. $\square$

Corollary 6.5. *Consider the setting of Lemma 6.4. The function $\tau \mapsto \Pi(\tau)$ is C^r-smooth as a map from $\mathbb{R}$ into $\mathcal{L}(X, X_0)$.*

Proof. It is sufficient to show that $\tau \mapsto \mathrm{ev}_{-\tau} E Q_0^\tau$ is C^r-smooth as a map from $\mathbb{R} \to \mathcal{L}(X)$. For an appropriate open $\Omega' \subset \mathbb{R}$, notice that the evaluation function $\Omega' \to \mathcal{L}(C^{\ell+1}(\Omega, \mathbb{C}^n), X)$ defined by $\xi \mapsto \mathrm{ev}_\xi$ is C^ℓ-smooth. In view of Lemma 6.4 and the representation (5.4), the C^r-smoothness of Π now follows from Lemma A.3. $\square$

7 The center manifold

We are now ready to construct the center manifold for the nonlinear equation (2.1). As a preparation, we need to modify the nonlinearity R so that it becomes globally Lipschitz continuous. This can be realized by choosing a C^∞-smooth cutoff-function $\chi : [0, \infty) \to \mathbb{R}$ with $\|\chi\|_\infty = 1$, that satisfies $\chi(\xi) = 0$ for $\xi \geq 2$, while $\chi(\xi) = 1$ for $\xi \leq 1$. We subsequently define for any $\delta > 0$ the nonlinearity $R_\delta : \mathbb{R} \times X \to \mathbb{C}^n$, given by

$$R_\delta(\xi, \phi) = \chi(\|\Pi(\xi)\phi\| / \delta')\chi(\left\|(I - Q_0^\xi)\phi\right\| / \delta)R(\xi, \phi), \tag{7.1}$$

in which $\delta' = \delta \sup_{\xi \in \mathbb{R}} \left\|\Pi^\xi_{\to}\right\|$. As in [18], one can show that this map is bounded and globally Lipschitz continuous in the second variable. In particular, the Lipschitz constant L_δ is independent of $\xi \in \mathbb{R}$ and satisfies $L_\delta \to 0$ as $\delta \to 0$, while one has the estimate $|R_\delta(\xi, \phi)| \leq 4\delta L_\delta$ for all $\xi \in \mathbb{R}$ and $\phi \in X$. Associated to R_δ one can define the substitution map $\widetilde{R}_\delta : BC_\eta(\mathbb{R}, \mathbb{C}^n) \to BC_\eta(\mathbb{R}, \mathbb{C}^n)$, given by $[\widetilde{R}_\delta x](\xi) = R_\delta(\xi, x_\xi)$. The Lipschitz constant associated to this substitution map $\widetilde{R}_\delta$ is given by $w^\eta L_\delta$, in which we have introduced the quantity

$$w = \max(e^{-r_{\min}}, e^{r_{\max}}) \geq 1. \tag{7.2}$$

Following these preliminaries, we introduce the operator $\mathcal{G} : BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \times X_0 \times \mathbb{R} \to BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$ that acts as

$$\mathcal{G}(u, \phi, \tau) = E e^{-\tau W} \left[\phi - \Pi(\tau)\mathrm{ev}_\xi \mathcal{K}\widetilde{R}_\delta(u)\right] + \mathcal{K}\widetilde{R}_\delta(u). \tag{7.3}$$

Notice that any fixpoint $u = \mathcal{G}(u, \phi, \tau)$ will satisfy the equation $\dot{u}(\xi) = L(\xi)u_\xi + R_\delta(\xi, u_\xi)$, with $Q_0^\tau u_\tau = \Pi^\tau_{\to}\phi$. For this reason, we set out to show that for any fixed pair $(\phi, \tau) \in X_0 \times \mathbb{R}$, the map $\mathcal{G}(\cdot, \phi, \tau)$ is a contraction on $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, yielding a fixpoint $u = u^*_\eta(\phi, \tau)$.

Theorem 7.1. *Consider the nonlinear equation (2.1) and assume that the conditions (HL), (HF), (HR1) and (HR2) are all satisfied. Pick any $\gamma > 0$ such that there are no Floquet exponents λ with $0 < |\mathrm{Re}\,\lambda| < \gamma$ and consider any interval $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ with $\min(k, r)\eta_{\min} < \eta_{\max}$. Then there exist constants $0 < \varepsilon < \delta$ such that the following properties hold.*

(i) For all $\eta \in [\eta_{\min}, \eta_{\max}]$ and for any pair $(\phi, \tau) \in X_0 \times \mathbb{R}$, the fixpoint equation $u = \mathcal{G}(u, \phi, \tau)$ has a unique solution $u = u^*_\eta(\phi, \tau) \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$.

(ii) For any pair $\xi, \overline{\xi} \in \mathbb{R}$ with $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$, we have

$$u^*\big(\Pi(\xi)\mathrm{ev}_\xi u^*(\phi, \tau), \overline{\xi}\big) = T_{\xi - \overline{\xi}} u^*(\phi, \tau). \tag{7.4}$$

(iii) *For any pair* $\eta_{\min} \leq \eta_1 < \eta_2 \leq \eta_{\max}$*, one has the identity* $u^*_{\eta_2} = \mathcal{J}^1_{\eta_2\eta_1} u^*_{\eta_1}$.

(iv) *For any pair* $(\phi, \tau) \in X_0 \times \mathbb{R}$*, we have the inequality*

$$\left\| (I - Q^{\xi}_0)\mathrm{ev}_{\xi} u^*_{\eta}(\phi, \tau) \right\| < \delta, \tag{7.5}$$

for all $\xi \in \mathbb{R}$*.*

(v) *Consider a pair* $(\phi, \tau) \in X_0 \times \mathbb{R}$ *that has* $\|\phi\| < \varepsilon$*. Then the following inequality holds for all* $r_{\min} \leq \theta \leq r_{\max}$*,*

$$\left\| \Pi(\tau + \theta)\mathrm{ev}_{\tau+\theta} u^*_{\eta}(\phi, \tau) \right\| < \delta. \tag{7.6}$$

(vi) *For all* $\eta \in (\min(k, r)\eta_{\min}, \eta_{\max}]$*, the mapping* $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}} : X_0 \times \mathbb{R} \to BC^1_{\eta}(\mathbb{R}, \mathbb{C}^n)$ *is of class* $C^{\min(k,r)}$*.*

We need a preparatory result to prove this theorem, which allows us to restrict the parameter τ to the interval $[0, 2\pi]$. This in turn will enable us to choose the parameters δ and ε independently of $\tau \in \mathbb{R}$, simplifying the analysis considerably.

Proposition 7.2. *Let* u *satisfy* $u = \mathcal{G}(u, \phi, \tau)$*. Consider any* $\overline{\tau}$ *with* $\tau - \overline{\tau} \in 2\pi\mathbb{Z}$ *and let* $v = T_{\tau-\overline{\tau}}u$*. Then* v *satisfies the fixpoint equation* $v = G(v, \phi, \overline{\tau})$*.*

Proof. First note that Lemma 6.1 implies

$$\mathcal{K}\widetilde{R}_{\delta}(v) = T_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u) - EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u), \tag{7.7}$$

using which we compute

$$\begin{array}{rcl} T_{\overline{\tau}-\tau}G(v, \phi, \overline{\tau}) & = & T_{\overline{\tau}-\tau}Ee^{-\overline{\tau}W}\left[\phi - \Pi(\overline{\tau})\mathrm{ev}_{\overline{\tau}}[T_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u) - EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u)]\right] \\ & & +\mathcal{K}\widetilde{R}_{\delta}(u) - T_{\overline{\tau}-\tau}EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u) \\ & = & Ee^{-\tau W}\left[\phi - \Pi(\overline{\tau})\mathrm{ev}_{\tau}\mathcal{K}\widetilde{R}_{\delta}(u)\right] + \mathcal{K}\widetilde{R}_{\delta}(u) \\ & & +Ee^{-\tau W}\Pi(\overline{\tau})\mathrm{ev}_{\overline{\tau}}EQ_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u) - Ee^{(\overline{\tau}-\tau)W}Q_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u) \\ & = & u + Ee^{-\tau W}\Pi^{\overline{\tau}}_{\leftarrow}Q_0(\overline{\tau})\Pi^{\overline{\tau}}_{\rightarrow}e^{\overline{\tau}W}Q_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u) \\ & & -Ee^{(\overline{\tau}-\tau)W}Q_0\mathrm{ev}_{\tau-\overline{\tau}}\mathcal{K}\widetilde{R}_{\delta}(u) \\ & = & u. \end{array} \tag{7.8}$$

□

We are now ready to prove items (i) through (v) of Theorem 7.1. The remaining item (vi) will be treated in Section 8, where the necessary machinery is developed.

Partial proof of Theorem 7.1. In view of Proposition 7.2, we may assume throughout the proof that $\tau \in [0, 2\pi]$.

(i) Choose $\delta > 0$ in such a way that for all $\eta \in [\eta_{\min}, \eta_{\max}]$ and all $\sigma \in \mathbb{R}$, we have

$$w^{\eta} \|\mathcal{K}_{\eta}\| L_{\delta} \big[1 + \|E\|_{\eta} e^{2\pi|W|} \|\Pi(\sigma)\| w^{\eta} e^{2\pi\eta}\big] < \frac{1}{4}. \tag{7.9}$$

Then for any pair $(\phi, \tau) \in X_0 \times [0, 2\pi]$ and all $\eta \in [\eta_{\min}, \eta_{\max}]$, we have the inequality

$$\|\mathcal{G}(u_1, \phi, \tau) - \mathcal{G}(u_2, \phi, \tau)\|_{BC^1_{\eta}} \leq \frac{1}{4} \|u_1 - u_2\|_{BC^1_{\eta}}. \tag{7.10}$$

In addition, if $\rho \geq 2 \|E\|_{\eta} e^{2\pi|W|} \|\phi\|$, then $\mathcal{G}(\cdot, \phi, \tau)$ maps the ball with radius ρ in $BC^1_{\eta}(\mathbb{R}, \mathbb{C}^n)$ into itself. We can hence use the contraction mapping theorem to define the unique solution $u = u^*_{\eta}(\phi, \tau)$ of the fixpoint equation $u = \mathcal{G}(u, \phi, \tau)$ for $\tau \in [0, 2\pi]$.

(ii) We first write $\psi = \Pi(\xi)\mathrm{ev}_{\xi} u^*(\phi, \tau)$ and compute

$$\begin{array}{rcl} \psi & = & \Pi(\xi)\mathrm{ev}_{\xi} E e^{-\tau W}\phi - \Pi(\xi)\mathrm{ev}_{\xi} E e^{-\tau W}\Pi(\tau)\mathrm{ev}_{\tau}\mathcal{K}\widetilde{R}_{\delta}(u^*(\phi,\tau)) + \Pi(\xi)\mathrm{ev}_{\xi}\mathcal{K}\widetilde{R}_{\delta}(u^*(\phi,\tau)) \\ & = & e^{(\xi-\tau)W}\phi - e^{(\xi-\tau)W}\Pi(\tau)\mathrm{ev}_{\tau}\mathcal{K}\widetilde{R}_{\delta}(u^*(\phi,\tau)) + \Pi(\xi)\mathrm{ev}_{\xi}\mathcal{K}\widetilde{R}_{\delta}(u^*(\phi,\tau)). \end{array} \tag{7.11}$$

Now writing $u = u^*(\phi, \tau)$ and $v = T_{\xi - \overline{\xi}} u$, it suffices to show that $u = T_{\overline{\xi} - \xi}\mathcal{G}(v, \psi, \overline{\xi})$. We can closely follow the computation (7.8) in Proposition 7.2 and substitute (7.11) to obtain

$$\begin{array}{rcl} T_{\overline{\xi}-\xi}\mathcal{G}(v, \psi, \overline{\xi}) & = & E e^{-\xi W}\big[\psi - \Pi(\overline{\xi})\mathrm{ev}_{\xi}\mathcal{K}\widetilde{R}_{\delta}(u)\big] + \mathcal{K}\widetilde{R}_{\delta}(u) \\ & = & E e^{-\tau W}[\phi - \Pi(\tau)\mathrm{ev}_{\tau}\mathcal{K}\widetilde{R}_{\delta}(u)] + \mathcal{K}\widetilde{R}_{\delta}(u) = u. \end{array} \tag{7.12}$$

(iii) This follows immediately using the fact that $\mathcal{K}_{\eta_1}$ and $\mathcal{K}_{\eta_2}$ agree on $BC_0(\mathbb{R}, \mathbb{C}^n)$, together with the estimate $|R_{\delta}(\xi, \mathrm{ev}_{\xi} u^*(\phi, \tau))| \leq 4\delta L_{\delta}$.

(iv) If $\delta > 0$ is chosen sufficiently small to ensure that for some $0 < \eta_0 < \gamma$ and all $\sigma \in \mathbb{R}$ we have

$$w^{\eta_0} L_{\delta} < (4 \left\|\mathcal{K}^{\sigma}_{\eta_0}\right\|)^{-1}, \tag{7.13}$$

then we may use Lemma 6.1 to compute

$$\begin{array}{rcl} (I - Q^{\xi}_0)\mathrm{ev}_{\xi} u^*(\phi, \tau) & = & (I - Q^{\xi}_0)\mathrm{ev}_{\xi} E e^{-\xi W}\big[\phi - \Pi(\xi)\mathrm{ev}_{\xi}\mathcal{K}_{\eta_0}\widetilde{R}_{\delta}(u^*(\phi,\tau))\big] \\ & & +(I - Q^{\xi}_0)\mathrm{ev}_{\xi}\mathcal{K}_{\eta_0}\widetilde{R}_{\delta}(u^*(\phi,\tau)) \\ & = & (I - Q^{\xi}_0)\mathrm{ev}_{\xi}\mathcal{K}_{\eta_0}\widetilde{R}_{\delta}(u^*(\phi,\tau)) = \mathrm{ev}_0\mathcal{K}^{\xi}_{\eta_0} T_{\xi}\widetilde{R}_{\delta}(u^*(\phi,\tau)) \end{array} \tag{7.14}$$

and hence

$$\left\|(I - Q^{\xi}_0)\mathrm{ev}_{\xi} u^*(\phi, \tau)\right\| \leq w^{\eta_0} \left\|\mathcal{K}^{\xi}_{\eta_0}\right\| \left\|T_{\xi}\widetilde{R}_{\delta}(u^*(\phi,\tau))\right\|_{\eta_0} \leq w^{\eta_0} \left\|\mathcal{K}^{\xi}_{\eta_0}\right\| 4\delta L_{\delta} < \delta. \tag{7.15}$$

(v) Choose $\delta > 0$ and $\varepsilon > 0$ sufficiently small to ensure that for some $0 < \eta_0 < \gamma$ and all $\tau, \tau' \in \mathbb{R}$,

$$\begin{array}{lcl} w^{2\eta_0} e^{2\pi\eta_0} \|E\|_{\eta_0} e^{2\pi|W|} \|\Pi^{\tau}_{\leftarrow}\| \varepsilon & < & \frac{1}{2}\delta, \\ 4L_{\delta} \|\mathcal{K}_{\eta_0}\| w^{2\eta_0} e^{2\pi\eta_0} \big[\, \|\Pi(\tau)\| + e^{2\pi|W|} \|\Pi(\tau')\| e^{2\pi\eta_0} w^{\eta_0} \|E\|_{\eta_0} \|\Pi^{\tau}_{\leftarrow}\| \,\big] & < & \frac{1}{2}\delta. \end{array} \tag{7.16}$$

Recalling that $\tau \in [0, 2\pi]$ and writing $\Delta = \|\Pi(\tau+\theta)\mathrm{ev}_{\tau+\theta} u^*(\phi,\tau)\|$, we compute

$$\begin{array}{rcl} \Delta & = & \left\| \Pi_{\leftarrow}^{\tau+\theta} \mathrm{ev}_{\tau+\theta} E e^{-\tau W} \big[\phi - \Pi(\tau) \mathrm{ev}_{\tau} \mathcal{K}_{\eta_0} \widetilde{R}_\delta(u) \big] + \Pi(\tau+\theta) \mathrm{ev}_{\tau+\theta} \mathcal{K}_{\eta_0} \widetilde{R}_\delta(u) \right\| \\ & \leq & \left\| \Pi_{\leftarrow}^{\tau+\theta} \right\| w^{\eta_0} w^{\eta_0} e^{2\pi\eta_0} \left\| E \right\|_{\eta_0} e^{2\pi|W|} \big[\varepsilon + \|\Pi(\tau)\| e^{2\pi\eta_0} w^{\eta_0} \left\| \mathcal{K}_{\eta_0} \right\| 4\delta L_\delta \big] \\ & & + \left\| \Pi(\tau+\theta) \right\| w^{\eta_0} w^{\eta_0} e^{2\pi\eta_0} \left\| \mathcal{K}_{\eta_0} \right\| 4\delta L_\delta \\ & < & \frac{\delta}{2} + \frac{\delta}{2}. \end{array} \tag{7.17}$$

□

In the remainder of this section we will derive an ODE that is satisfied on the finite dimensional center manifold. To this end, we consider an arbitrary pair $(\phi, \tau) \in X_0 \times \mathbb{R}$, and introduce the function $\Phi : \mathbb{R} \to X_0$, given by

$$\Phi(\xi) = \Pi(\xi)\mathrm{ev}_\xi u^*(\phi,\tau). \tag{7.18}$$

Notice that we can apply the identity (7.4) to invert this and express $u^*(\phi,\tau)$ in terms of $\Phi(\xi)$. In particular, for any $\overline{\xi}$ for which $\xi - \overline{\xi} \in 2\pi\mathbb{Z}$, we find

$$u^*(\phi,\tau) = T_{\overline{\xi}-\xi} u^*\big(\Phi(\xi), \overline{\xi}\big). \tag{7.19}$$

Setting out to obtain an ODE for Φ, we introduce the shorthand $u = u^*(\tau,\phi)$ and differentiate (7.18) to find

$$\begin{array}{rcl} \dot{\Phi}(\xi) & = & [D\Pi(\xi)]\mathrm{ev}_\xi u + \Pi(\xi) D[\mathrm{ev}_\xi u] \\ & = & [D\Pi(\xi)]\mathrm{ev}_\xi u + \Pi(\xi)\mathrm{ev}_\xi D u \\ & = & [D\Pi(\xi)]\mathrm{ev}_\xi u + \Pi(\xi)\mathrm{ev}_\xi L u + \Pi(\xi)\mathrm{ev}_\xi \widetilde{R}_\delta(u) \\ & = & [D\Pi(\xi)]\mathrm{ev}_\xi u^*(\Phi(\xi),\xi) + \Pi(\xi)\mathrm{ev}_\xi L u^*(\Phi(\xi),\xi) + \Pi(\xi)\mathrm{ev}_\xi \widetilde{R}(u^*(\Phi(\xi),\xi)) \\ & = & [D\Pi(\xi)]\mathrm{ev}_\xi E e^{-\xi W}\psi + \Pi(\xi)\mathrm{ev}_\xi L E e^{-\xi W}\psi + f(\xi, \Phi(\xi)). \end{array} \tag{7.20}$$

Here the nonlinearity $f(\xi,\psi)$ is of order $O(\|\psi\|^2)$ as $\psi \to 0$ and is explicitly given by

$$\begin{array}{rcl} f(\xi,\psi) & = & [D\Pi(\xi)]\mathrm{ev}_\xi[u^*(\psi,\xi) - E e^{-\xi W}\psi] \\ & & +\Pi(\xi)\mathrm{ev}_\xi L[u^*(\psi,\xi) - E e^{-\xi W}\psi] \\ & & +\Pi(\xi)\mathrm{ev}_\xi \widetilde{R}_\delta(u^*(\psi,\xi)). \end{array} \tag{7.21}$$

Using Proposition 7.2 one easily sees that f is 2π-periodic in the first variable, i.e., $f(\xi+2\pi,\psi) = f(\xi,\psi)$ for all $\xi \in \mathbb{R}$ and $\psi \in X_0$. In addition, the C^r-smoothness of Π and the $C^{\min(r,k)}$-smoothness of u^* imply that $f \in C^{\min(r-1,k)}(\mathbb{R} \times X_0, X_0)$.

It remains to treat the linear part of (7.20). Defining $y = E e^{-\xi W}\psi \in \mathcal{N}_0$, notice that

$$\begin{array}{rcl} [D\Pi(\xi)]\mathrm{ev}_\xi y + \Pi(\xi)\mathrm{ev}_\xi L y & = & [D\Pi(\xi)]\mathrm{ev}_\xi y + \Pi(\xi)\mathrm{ev}_\xi D y \\ & = & [D\Pi(\xi)]\mathrm{ev}_\xi y + \Pi(\xi) D \mathrm{ev}_\xi y = D[\Pi(\xi)\mathrm{ev}_\xi y] \\ & = & D[e^{\xi W}\mathrm{ev}_{-\xi} E \mathrm{ev}_\xi y] = D[e^{\xi W} ev_0 y] = W e^{\xi W} ev_0 y = W\psi. \end{array} \tag{7.22}$$

We have hence established the following result.

Proposition 7.3. *Consider the setting of Theorem 7.1. For any $(\phi, \tau) \in X_0 \times \mathbb{R}$, define the function $\Phi : \mathbb{R} \to X_0$ given by $\Phi(\xi) = \Pi(\xi)\mathrm{ev}_\xi u^*(\phi, \tau)$. Then Φ is $C^{\min(r,k+1)}$-smooth and satisfies the ordinary differential equation*

$$\dot{\Phi}(\xi) = W\Phi(\xi) + f(\xi, \Phi(\xi)). \tag{7.23}$$

Here the function $f : \mathbb{R} \times X_0 \to X_0$, which is explicitly given by (7.21), is $C^{\min(r-1,k)}$-smooth and satisfies $f(\xi + 2\pi, \psi) = f(\xi, \psi)$ for all $(\xi, \psi) \in \mathbb{R} \times X_0$. Finally, we have $f(\xi, 0) = 0$ and $Df(\xi, 0) = 0$ for all $\xi \in \mathbb{R}$.

In a standard fashion [15, 18], one may now use the ODE derived above in conjunction with the properties of u^* established in Theorem 7.1 to prove our main results in Theorem 2.2. As a final remark, we observe that in the constant coefficient situation where $L(\xi) = L$, we have $T_\xi u^*(\cdot, \xi) = u^*(\cdot, 0)$ and $\Pi(\xi) = Q_0$ for all $\xi \in \mathbb{R}$, which shows that the definition of f reduces correctly to the form derived in [18].

8 Smoothness of the center manifold

In this section we address the smoothness of the center manifold established above. In particular, we set out to prove item (vi) of Theorem 7.1. Throughout this section we consider a fixed system (2.1) that satisfies the conditions (HL), (HF), (HR1) and (HR2) and recall the corresponding integers r and k. In addition, we fix an interval $[\eta_{\min}, \eta_{\max}] \subset (0, \gamma)$ as in the setting of Theorem 7.1. In order to ease notation we will assume that $r \geq k$, but we remark that upon interchanging k and r all our arguments here remain valid when in fact $r < k$. Our arguments here are based on the strategy developed in [9, Section IX.7] and will extend the proof given in [18] for autonomous versions of (2.1).

Due to the presence of the cutoff function on the infinite dimensional complement of X_0, the nonlinearity R_δ loses the C^k-smoothness on X and becomes merely Lipschitz continuous. To correct for this situation, we introduce for any $\eta > 0$ the Banach space

$$V^1_\eta(\mathbb{R}, \mathbb{C}^n) = \big\{u \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \mid \|u\|_{V^1_\eta} := \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|} \|\Pi(\xi)u_\xi\| + \sup_{\xi\in\mathbb{R}} \Big\|(I - Q^\xi_0)u_\xi\Big\| + \|\dot{u}\|_\eta < \infty\big\}, \tag{8.1}$$

which is continuously embedded in $BC^1_\eta(\mathbb{R}, \mathbb{C}^n)$, together with the open set

$$V^{1,\delta}_\eta(\mathbb{R}, \mathbb{C}^n) = \big\{u \in BC^1_\eta(\mathbb{R}, \mathbb{C}^n) \mid \sup_{\xi\in\mathbb{R}} \Big\|(I - Q^\xi_0)u_\xi\Big\| < \delta\big\} \subset V^1_\eta(\mathbb{R}, \mathbb{C}^n). \tag{8.2}$$

We start by establishing conditions under which the substitution maps $\widetilde{R}_\delta : V^{1,\delta}_\sigma(\mathbb{R}, \mathbb{C}^n) \to BC^1_\zeta(\mathbb{R}, \mathbb{C}^n)$ are smooth. Notice that R_δ is of class C^k on the set B^h_δ, in which

$$B^h_\delta = \big\{(\xi, \phi) \in \mathbb{R} \times X \mid \Big\|(I - Q^\xi_0)\phi\Big\| < \delta\big\}. \tag{8.3}$$

Considering any pair of integers $p \geq 0$, $q \geq 0$ with $p+q \leq k$, observe that the norms $\|D_1^p D_2^q R_\delta(\xi, \phi)\|$ are uniformly bounded on B_δ^h. Thus, for any $u \in C(\mathbb{R}, \mathbb{C}^n)$ for which $\sup_{\xi\in\mathbb{R}} \left\|(I - Q_0^\xi)u_\xi\right\| < \delta$ and for any $0 \leq p \leq k$, we can define a map $\widetilde{R}_\delta^{(p,q)}(u) \in \mathcal{L}^{(q)}(C(\mathbb{R}, \mathbb{C}^n), C(\mathbb{R}, \mathbb{C}^n))$ by

$$\widetilde{R}_\delta^{(p,q)}(u)(v_1, \ldots, v_q)(\xi) = D_1^p D_2^q R_\delta(\xi, u_\xi)\big((v_1)_\xi, \ldots, (v_q)_\xi\big). \tag{8.4}$$

Here the symbol $\mathcal{L}^{(q)}(Y, Z)$ denotes the space of q-linear mappings from $Y \times \ldots \times Y$ into Z. Note that the map $\widetilde{R}_\delta^{(p,q)}(u)$ defined above is well-defined, since $D_1^p D_2^q R_\delta$ is a continuous map from $B_\delta^h \times X^q$ into $\mathbb{C}^n$, as is the map $i_x : \mathbb{R} \to X$ which sends $\xi \mapsto x_\xi$, for any $x \in C(\mathbb{R}, \mathbb{C}^n)$. Throughout the remainder of this section we will adopt the shorthand $BC_\zeta^1 = BC_\zeta^1(\mathbb{R}, \mathbb{C}^n)$, together with analogous ones for the other function spaces. The following two results are stated without proof, as they are very similar to their counterparts in [18].

Proposition 8.1. *Let $p \geq 0$ and $q \geq 0$ be positive integers with $p + q \leq k$. Pick $\eta \geq q\zeta > 0$. Then for any $u \in C(\mathbb{R}, \mathbb{C}^n)$ such that $\sup_{\xi\in\mathbb{R}} \left\|(I - Q_0^\xi)u_\xi\right\| < \delta$, we have*

$$\widetilde{R}_\delta^{(p,q)}(u) \in \mathcal{L}^{(q)}(BC_\zeta^1, BC_\eta) \cap \mathcal{L}^{(q)}(V_\zeta^1, BC_\eta), \tag{8.5}$$

where the norm is bounded by

$$\left\|\widetilde{R}_\delta^{(p,q)}\right\|_{\mathcal{L}^{(q)}} \leq w^\zeta \sup_{\xi\in\mathbb{R}} e^{-(\eta - q\zeta)|\xi|} \|D_1^p D_2^q R_\delta(\xi, u_\xi)\| < \infty. \tag{8.6}$$

Furthermore, consider any $0 \leq \ell \leq k - (p + q)$ and any $\sigma > 0$. If $\eta > q\zeta + \ell\sigma$, then in addition the map $u \mapsto \widetilde{R}_\delta^{(p,q)}(u)$ from $V_\sigma^{1,\delta}$ into $\mathcal{L}^{(q)}(BC_\zeta^1, BC_\eta)$ is C^ℓ-smooth, with $D^\ell \widetilde{R}_\delta^{(p,q)} = \widetilde{R}_\delta^{(p,q+l)}$. The same holds when considering $u \mapsto \widetilde{R}_\delta^{(p,q)}(u)$ as a map from $V_\sigma^{1,\delta}$ into $\mathcal{L}^{(q)}(V_\zeta^1, BC_\eta)$.

Finally, if $p + q < k$, consider any $u \in V_\sigma^{1,\delta}$. Then for any q-tuple of functions $v_1, \ldots, v_q \in BC_\zeta^1$, we have $\widetilde{R}_\delta^{(p,q)}(u)(v_1, \ldots, v_q) \in C^1(\mathbb{R}, \mathbb{C}^n)$, with

$$\begin{array}{rcl} D_\xi \widetilde{R}_\delta^{(p,q)}(u)(v_1, \ldots, v_q) & = & \widetilde{R}_\delta^{(p+1,q)}(u)(v_1, \ldots, v_q) + \widetilde{R}_\delta^{(p,q+1)}(u)(\dot{u}, v_1, \ldots, v_q) \\ & & + \widetilde{R}_\delta^{(p,q)}(u)(\dot{v}_1, v_2, \ldots, v_q) + \ldots + \widetilde{R}_\delta^{(p,q)}(u)(v_1, v_2, \ldots, \dot{v}_q). \end{array} \tag{8.7}$$

Proposition 8.2. *Consider integers $p \geq 0$ and $q \geq 0$ with $p + q < k$. Let $\eta > q\zeta + \sigma$ for some $\zeta > 0$ and $\sigma > 0$. Let Φ be a mapping of class C^1 from $X_0 \times \mathbb{R}$ into $V_\sigma^{1,\delta}$. Then the mapping $\widetilde{R}_\delta^{(p,q)} \circ \Phi$ from $X_0 \times \mathbb{R}$ into $\mathcal{L}^{(q)}(BC_\zeta^1, BC_\eta)$ is of class C^1 with*

$$D(\widetilde{R}_\delta^{(p,q)} \circ \Phi)(\phi, \tau)(v_1, \ldots, v_q, (\psi, \xi)) = \widetilde{R}^{(p,q+1)}(\Phi(\phi, \tau))(v_1, \ldots, v_q, D\Phi(\phi, \tau)(\psi, \xi)). \tag{8.8}$$

For convenience, we introduce for any $\eta \in [\eta_{\min}, \eta_{\max}]$ the function $\mathcal{E} : V_\eta^{1,\delta} \times (X_0 \times [0, 2\pi]) \to BC_\eta^1$ via

$$\mathcal{E}\big(u, (\phi, \tau)\big) = Ee^{-\tau W}\big[\phi - \Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u)\big]. \tag{8.9}$$

One may compute the partial derivatives

$$\begin{array}{rcl} D_1\mathcal{E}\big(u, (\phi, \tau)\big) & = & -Ee^{-\tau W}\Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta^{(0,1)}(u), \\ D_2\mathcal{E}\big(u, (\phi, \tau)\big) & = & Ee^{-\tau W} \times \Big(- EWe^{-\tau W}[\phi - \Pi(\tau)\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u)] \\ & & -Ee^{-\tau W}[D\Pi(\tau)]\mathrm{ev}_\tau \mathcal{K}\widetilde{R}_\delta(u) + \Pi(\tau)\mathrm{ev}_\tau[L\mathcal{K}\widetilde{R}_\delta(u) + \widetilde{R}_\delta(u)]\Big) \end{array} \tag{8.10}$$

and easily conclude that these are both continuous functions. This means that $\mathcal{E}$ is at least C^1-smooth and in addition enables us to define the continuous auxiliary functions $\mathcal{F}_1 : X_0 \times [0, 2\pi] \to \mathcal{L}(V^1_\eta, BC^1_\eta) \cap \mathcal{L}(BC^1_\eta, BC^1_\eta)$ and $\mathcal{F}_2 : X_0 \times [0, 2\pi] \to \mathcal{L}(X_0 \times \mathbb{R}, BC^1_\eta)$ by

$$\begin{array}{rcl} \mathcal{F}_1(\phi, \tau) & = & D_1\mathcal{E}\big(u^*(\phi, \tau), (\phi, \tau)\big), \\ \mathcal{F}_2(\phi, \tau) & = & D_2\mathcal{E}\big(u^*(\phi, \tau), (\phi, \tau)\big). \end{array} \tag{8.11}$$

Notice that Proposition 8.1 implies that $\mathcal{F}_1$ is indeed well-defined as an element in $\mathcal{L}(BC^1_\eta, BC^1_\eta)$.

We will employ an induction approach towards establishing the smoothness of u^*. The next result serves as a starting point by obtaining the C^1-smoothness.

Proposition 8.3. *For all $\eta \in (\eta_{\min}, \eta_{\max}]$, the function $\mathcal{J}^1_{\eta\eta_{\min}} u^*_{\eta_{\min}} : X_0 \times [0, 2\pi] \to BC^1_\eta$ is C^1-smooth. In addition, for each $1 \le p \le k$ and all $\eta \in (p\eta_{\min}, \eta_{\max}]$, the function*

$$(\phi, \tau) \mapsto \mathcal{J}^1_{\eta\ p\eta_{\min}} D^p_\xi u^*_{\eta_{\min}}(\phi, \tau), \tag{8.12}$$

which maps $X_0 \times [0, 2\pi]$ into BC^1_η, is continuous.

Proof. Consider any $\eta \in (\eta_{\min}, \eta_{\max}]$. We will apply Lemma A.2 in the setting $Y_0 = V^1_{\eta_{\min}}$, $Y = BC^1_{\eta_{\min}}$ and $Y_1 = BC^1_\eta$, together with their natural inclusions. Furthermore, we choose $\Omega_0 = V^{1,\delta}_{\eta_{\min}} \subset V^1_{\eta_{\min}}$ and let $\Lambda = X_0 \times \mathbb{R}$ with $\Lambda_0 = X_0 \times [0, 2\pi]$. For any $(\phi, \tau) \in X \times [0, 2\pi]$, the operators featuring in Appendix A are defined by

$$\begin{array}{rcll} F(u, \phi, \tau) & = & \mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_{\eta_{\min}} \widetilde{R}_\delta(u), & u \in BC^1_{\eta_{\min}}, \\ F^{(1)}(u, \phi, \tau) & = & D_1\mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(0,1)}_\delta(u) \in \mathcal{L}(BC^1_{\eta_{\min}}), & u \in V^{1,\delta}_{\eta_{\min}}, \\ F^{(1)}_1(u, \phi, \tau) & = & D_1\mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_\eta \circ \widetilde{R}^{(0,1)}_\delta(u) \in \mathcal{L}(BC^1_\eta), & u \in V^{1,\delta}_{\eta_{\min}}. \end{array} \tag{8.13}$$

In the context of Lemma A.2 this means that $G : V^{1,\delta}_{\eta_{\min}} \times X_0 \times [0, 2\pi] \to BC^1_\eta$ is defined by

$$\begin{array}{rcl} G(u, \phi, \tau) & = & \mathcal{E}(u, (\phi, \tau)) + \mathcal{J}^1_{\eta\eta_{\min}} \mathcal{K}_{\eta_{\min}} \widetilde{R}_\delta(u) \\ & = & \mathcal{E}(u, (\phi, \tau)) + \mathcal{K}_\eta \widetilde{R}_\delta(u), \end{array} \tag{8.14}$$

in which the final equality follows from the fact that $\mathcal{K}_{\eta_{\min}}$ and $\mathcal{K}_\eta$ agree on BC_0.

Conditions (HC1), (HC3) and (HC4) are satisfied due to the C^1-smoothness of $\mathcal{E}$, together with Proposition 8.1. The inequality (7.9) implies (HC2) and (HC5), while (HC6) follows from (7.13). We conclude that $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^1 and that $D(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}})(\phi, \xi) = \mathcal{J}^1_{\eta\eta_{\min}} \circ u^{*(1)}_{\eta_{\min}}(\phi, \xi) \in \mathcal{L}(X_0 \times \mathbb{R}, BC^1_\eta)$, where $u^{*(1)}_{\eta_{\min}}(\phi, \xi)$ is the unique solution of the equation

$$u^{(1)} \quad = \quad [\mathcal{F}_1(\phi, \tau) + \mathcal{K}_{\eta_{\min}} \circ \widetilde{R}^{(0,1)}(u^*_{\eta_{\min}}(\phi, \tau))]u^{(1)} + \mathcal{F}_2(\phi, \tau) \tag{8.15}$$

in the space $\mathcal{L}(X_0 \times \mathbb{R}, BC^1_{\eta_{\min}})$. We compute

$$\begin{array}{rcl} D_\xi u^*_{\eta_{\min}}(\phi, \tau) & = & L u^*_{\eta_{\min}}(\phi, \tau) + \widetilde{R}_\delta(u^*_{\eta_{\min}}(\phi, \tau)) \\ D^2_\xi u^*_{\eta_{\min}}(\phi, \tau) & = & [DL] u^*_{\eta_{\min}}(\phi, \tau) + L[D u^*_{\eta_{\min}}(\phi, \tau)] + \\ & & \widetilde{R}^{(1,0)}(u^*_{\eta_{\min}}(\phi, \xi)) + \widetilde{R}^{(0,1)}(u^*_{\eta_{\min}}(\phi, \tau)) D u^*_{\eta_{\min}}(\phi, \tau) \end{array} \tag{8.16}$$

and hence $(\phi,\tau) \mapsto \mathcal{J}^1_{\eta\eta_{\min}} D_\xi u^*_{\eta_{\min}}(\phi,\tau)$ is continuous. It is easy to see that (8.16) can be differentiated another $k-1$ times, showing that in general $(\phi,\tau) \mapsto \mathcal{J}^1_{\eta\ \ell\eta_{\min}} D^\ell_\xi u^*_{\eta_{\min}}(\phi,\tau)$ is continuous for $1 \leq \ell \leq k$. □

In the interest of clarity, we specify in some detail the induction hypothesis that we use prior to performing the induction step, To this end, consider any integer ℓ that satisfies $1 \leq \ell < k$ and suppose that for all $1 \leq q \leq \ell$, there exist mappings

$$u^{*(q)}_{\eta_{\min}} : X_0 \times [0,2\pi] \to \mathcal{L}^{(q)}(X_0 \times \mathbb{R}, BC^1_{q\eta_{\min}}), \tag{8.17}$$

such that the following properties are satisfied.

(IH1) For all $1 \leq q \leq \ell$ and for all $\eta \in (q\eta_{\min}, \eta_{\max}]$, the mapping $\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}$ is of class C^q with

$$D^q(\mathcal{J}^1_{\eta\eta_{\min}} \circ u^*_{\eta_{\min}}) = \mathcal{J}^1_{\eta\ q\eta_{\min}} \circ u^{*(q)}_{\eta_{\min}}. \tag{8.18}$$

(IH2) For all integer pairs (p,q) with $0 \leq q \leq \ell$ and $1 \leq p \leq k-q$ and all $\eta \in ((p+q)\eta_{\min}, \eta_{\max}]$, the function $X_0 \times [0,2\pi] \to \mathcal{L}^{(q)}(X_0 \times \mathbb{R}, BC^1_\eta)$, defined by

$$(\phi,\tau) \mapsto \mathcal{J}^1_{\eta\ (p+q)\eta_{\min}} D^p_\xi u^{*(q)}(\phi,\tau), \tag{8.19}$$

is continuous.

(IH3) For any pair $(\phi,\tau) \in X_0 \times [0,2\pi]$, the map $u^{*(\ell)}_{\eta_{\min}}(\phi,\tau)$ is the unique solution at $\overline{\eta} = \eta_{\min}$ of an equation of the form

$$u^{(\ell)} = F^{(\ell)}_{\overline{\eta}}(u^{(\ell)},\phi,\tau) \tag{8.20}$$

in the space $\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\overline{\eta}})$, with

$$F^{(\ell)}_{\overline{\eta}}(u^{(\ell)},\phi,\tau) = \left[\mathcal{F}_1(\phi,\tau) + \mathcal{K}_{\ell\overline{\eta}} \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))\right] u^{(\ell)} + D^{\ell-1}\mathcal{F}_2(\phi,\xi) + H^{(\ell)}_{\overline{\eta}}(\phi,\tau). \tag{8.21}$$

Here we have $H^{(1)}(\phi,\tau) = 0$ and for $\ell \geq 2$ we can write $H^{(\ell)}_{\overline{\eta}}(\phi,\tau)$ as a finite sum of terms of two different forms, the first of which is given by

$$\mathcal{K}_{q\overline{\eta}} \circ \widetilde{R}^{(0,q)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))(u^{*(e_1)}_{\eta_{\min}}(\phi,\tau),\ldots,u^{*(e_q)}_{\eta_{\min}}(\phi,\tau)), \tag{8.22}$$

with $2 \leq q \leq \ell$ and integers $e_i \geq 1$ such that $e_1 + \ldots + e_q = \ell$. The second form can be written as

$$D^{f_1}\mathcal{F}_1(\phi,\tau) u^{*(f_2)}_{\eta_{\min}}(\phi,\tau), \tag{8.23}$$

with integers $f_1 \geq 1$ and $f_2 \geq 1$ that satisfy $f_1 + f_2 = \ell$.

Using Proposition 8.3 it is easily verified that the assumptions above are satisfied for $\ell = 1$. Before proceeding with the remaining cases, we need to study the smoothness of the operators $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proposition 8.4. *Suppose that for some integer $1 \le \ell < k$ the induction assumptions (IH1) through (IH3) all hold. Then for any $\eta \in [\eta_{\min}, \eta_{\max}]$, the functions $\mathcal{F}_1 : X_0 \times [0, 2\pi] \to \mathcal{L}(BC^{\ell}_{\eta}, BC^1_{\eta})$ and $\mathcal{F}_2 : X_0 \times [0, 2\pi] \to \mathcal{L}(X_0 \times \mathbb{R}, BC^1_{\eta})$ are C^{ℓ}-smooth.*

Proof. Upon defining $\mathcal{E}'(u, (\phi, \tau)) = \mathcal{E}(u, (\phi, \tau)) - Ee^{-\tau W}\phi$, we remark that it is sufficient to establish the claim for the operators $\mathcal{F}_1'$ and $\mathcal{F}_2'$ associated to $\mathcal{E}'$. Observe first that for $i = 1, 2$ we can write $D^{\ell}\mathcal{F}_i'(\phi, \tau)$ as a sum of terms of the form

$$D_1^{\alpha_1} D_2^{\alpha_2} \mathcal{E}'(u^*(\phi, \tau), (\phi, \tau)) u^{*(\beta_1)}(\phi, \tau) \dots u^{*\beta_{n_\beta}}(\phi, \tau), \tag{8.24}$$

in which $\beta_j \ge 1$ for $1 \le j \le n_\beta$. If $i = 1$, then we have in addition that $\alpha_1 \ge 1$, $n_\beta = \alpha_1 - 1$ and $\alpha_2 + \beta_1 + \ldots + \beta_{n_\beta} = \ell$. If however $i = 2$, then we have $\alpha_2 \ge 1$, $n_\beta = \alpha_1$ and $\alpha_2 + \beta_1 + \ldots + \beta_{n_\beta} = \ell + 1$.

Now notice that the only nonzero component of $D_1^{\alpha_1} D_2^{\alpha_2} \mathcal{E}'(u, (\phi, \tau))$ can be written as a sum of terms of the form

$$EW^{\gamma_0} e^{-\tau W} (D^{\gamma_1}\Pi)(\tau) \mathrm{ev}_\tau D_\xi^{\gamma_2} \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u), \tag{8.25}$$

in which $\gamma_j \ge 0$ for $0 \le j \le 2$ with $\gamma_0 + \gamma_1 + \gamma_2 = \alpha_2$. Setting out to compute the derivatives with respect to ξ appearing in (8.25), notice first that

$$D_\xi \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u) = L\mathcal{K}\widetilde{R}_\delta^{(0,\alpha_1)}(u) + \widetilde{R}_\delta^{(0,\alpha_1)}(u). \tag{8.26}$$

Generalizing, we obtain that $D_\xi^{\gamma_2} \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u)$ can be written as a sum of terms of two different forms, the first of which is given by

$$[D^{e_1}L] \dots [D^{e_{n_e}}L] \widetilde{R}^{(p, \alpha_1 + q)}(u)(D^{f_1}u, \dots, D^{f_q}u)(D^{g_1}, \dots D^{g_{\alpha_1}}), \tag{8.27}$$

in which we have $p \ge 0$, $q \ge 0$ and $n_e \ge 0$, together with $e_j \ge 0$ for all $1 \le j \le n_e$, $f_j \ge 1$ for all $1 \le j \le q$ and $g_j \ge 0$ for all $1 \le j \le \alpha_1$. In addition, we must have

$$1 + (1 + e_1) + \ldots + (1 + e_{n_e}) + p + f_1 + \ldots + f_q + g_1 + \ldots + g_{\alpha_1} = \gamma_2. \tag{8.28}$$

The second form is given by

$$[D^{e_1}L] \dots [D^{e_{n_e}}L] \mathcal{K} \widetilde{R}_\delta^{(0,\alpha_1)}(u), \tag{8.29}$$

in which $n_e \ge 0$, $e_j \ge 0$ for all $1 \le j \le n_e$ and

$$(1 + e_1) + \ldots + (1 + e_{n_e}) = \gamma_2. \tag{8.30}$$

Indeed, this can be verified directly for $\gamma_2 = 1$ and differentiation of the terms in (8.27) and (8.29) again gives terms of these forms.

It remains to show that the terms (8.27) and (8.29) are continuous after substituting $u = u^*(\phi, \tau)$. In view of Proposition (IH2), it suffices to check that we have $\alpha_1 + p + q \le k$, $e_j \le r$ for $1 \le j \le n_e$, $f_j \le \ell$ for $1 \le j \le q$ and $g_j + \beta_j \le \ell$ for $1 \le j \le n_\beta$. If in fact we have $i = 1$, i.e., we are considering $D^{\ell}\mathcal{F}_1'$, then we in addition need $g_{\alpha_1} \le \ell - 1$ to ensure that BC^{ℓ}_{η} is mapped into BC^1_{η} under the operator $D_\xi^{g_{\alpha_1}}$. All these inequalities can easily be verified by using the conditions (8.28) and (8.30). □

Proof of item (vi) of Theorem 7.1. Assume that for some $1 \le \ell < k$, the induction assumptions (IH1) through (IH3) are satisfied. Notice that these conditions ensure that $F^{(\ell)}_{\overline{\eta}} : \mathcal{L}^{(\ell)}(X_0, BC^1_{p\overline{\eta}}) \times X_0 \to \mathcal{L}^{(\ell)}(X_0, BC^1_{\ell\overline{\eta}})$ is well-defined for all $\overline{\eta} \in [\eta_{\min}, \frac{1}{\ell}\eta_{\max}]$ and, in addition, is a uniform contraction for these values of $\overline{\eta}$. We now fix $\eta \in ((\ell+1)\eta_{\min}, \eta_{\max}]$ and choose σ and ζ such that $\eta_{\min} < \sigma < (\ell+1)\sigma < \zeta < \eta$. We wish to apply Lemma A.2 in the setting $\Omega_0 = Y_0 = \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\sigma})$, $Y = \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\zeta})$, $Y_1 = \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\eta})$ with the corresponding natural inclusions, with the parameter space given by $\Lambda_0 = X_0 \times [0, 2\pi]$ with $\Lambda_0 \subset \Lambda = X_0 \times \mathbb{R}$. For any $(\phi, \tau) \in X_0 \times [0, 2\pi]$, we define the functions

$$
\begin{array}{rcl}
F(u^{(\ell)}, \phi, \tau) & = & [\mathcal{F}_1(\phi,\tau) + \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))]u^{(\ell)} + D^{\ell-1}\mathcal{F}_2(\phi,\tau) + H^{(\ell)}_{\zeta/\ell}(\phi), \\
F^{(1)}(u^{(\ell)}, \phi, \tau) & = & \mathcal{F}_1(\phi,\tau) + \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau)) \in \mathcal{L}(\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\zeta)), \\
F^{(1)}_1(u^{(\ell)}, \phi, \tau) & = & \mathcal{F}_1(\phi,\tau) + \mathcal{K}_\eta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau)) \in \mathcal{L}(\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\eta)),
\end{array}
\tag{8.31}
$$

in which we take $u^{(\ell)} \in \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\zeta)$ in the definition of F and $u^{(\ell)} \in \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\sigma})$ for $F^{(1)}$ and $F^{(1)}_1$. To check (HC1), we need to show that the map $G : \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\sigma}) \times X_0 \times [0, 2\pi] \to \mathcal{L}^{(\ell)}(X_0, BC^1_\eta)$ given by

$$
G(u^{(\ell)}, \phi, \tau) = \big[\mathcal{F}_1(\phi,\tau) + \mathcal{J}^1_{\eta\zeta} \circ \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))\big]u^{(\ell)} + D^{\ell-1}\mathcal{F}_2(\phi,\tau) + \mathcal{J}^1_{\eta\zeta} H^{(\ell)}_{\zeta/\ell}(\phi,\tau) \tag{8.32}
$$

is of class C^1. In view of the linearity of this map with respect to $u^{(\ell)}$, together with the smoothness of $\mathcal{F}_1$ and $\mathcal{F}_2$ as established in Proposition 8.4, it is sufficient to show that $(\phi,\tau) \mapsto \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))$ is of class C^1 as a map from $X_0 \times \mathbb{R}$ into $\mathcal{L}(BC^1_{\ell\sigma}, BC^1_\zeta)$ and, in addition, that $(\phi,\tau) \mapsto H^{(\ell)}_{\zeta/\ell}(\phi,\tau)$ is of class C^1 as a map from $X_0 \times \mathbb{R}$ into $\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_\zeta)$. The first fact follows from Proposition 8.2 using $\zeta > (\ell+1)\sigma$ and the C^1-smoothness of the map $(\phi,\tau) \mapsto \mathcal{J}^1_{\sigma\eta_{\min}} u^*_{\eta_{\min}}(\phi,\tau)$. To verify the second fact, we again use Proposition 8.2 to differentiate the components of $H^{(\ell)}$ given in (8.22) and (8.23). The first component yields

$$
\begin{array}{l}
D\mathcal{K}_\zeta \circ \widetilde{R}^{(0,q)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))(u^{*(e_1)}_{\eta_{\min}}(\phi,\tau), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi,\tau)) \\
\quad = \mathcal{K}_\zeta \circ \widetilde{R}^{(0,q+1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))(u^{*(e_1)}_{\eta_{\min}}(\phi,\tau), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi,\tau), u^{*(1)}_{\eta_{\min}}(\phi,\tau)) \\
\quad + \sum_{j=1}^q \mathcal{K}_\zeta \circ \widetilde{R}^{(0,q)}(u^*_{\eta_{\min}}(\phi,\tau))(u^{*(e_1)}_{\eta_{\min}}(\phi,\tau), \ldots, u^{*(e_j+1)}_{\eta_{\min}}(\phi,\tau), \ldots, u^{*(e_q)}_{\eta_{\min}}(\phi,\tau)),
\end{array}
\tag{8.33}
$$

in which each occurrence of $u^{*(j)}_{\eta_{\min}}$ is understood to map into $BC^1_{j\sigma}$. An application of Proposition 8.1 with $\zeta > (\ell+1)\sigma$, shows that the above map is indeed continuous from $X_0 \times \mathbb{R}$ into $\mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_\zeta)$. The second component can be treated using similar arguments in conjunction with Proposition 8.4. These arguments immediately show that also (HC4) is satisfied. Conditions (HC2), (HC3) and (HC5) can be verified much as before. (HC6) follows from the fact that $\mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\eta_{\min}}) \subset \mathcal{L}^{(\ell)}(X_0 \times \mathbb{R}, BC^1_{\ell\sigma})$.

We thus conclude from Lemma A.2 that $\mathcal{J}^1_{\eta\,\ell\eta_{\min}} \circ u^{*(\ell)}_{\eta_{\min}}$ is of class C^1 with $D(\mathcal{J}^1_{\eta\,\ell\eta_{\min}} \circ u^{*(\ell)}_{\eta_{\min}})(\phi,\tau) = \mathcal{J}^1_{\eta\zeta} \circ u^{*(\ell+1)}(\phi,\tau)$, in which $u^{*(\ell+1)}(\phi,\tau)$ is the unique solution of the equation

$$
u^{(\ell+1)} = \big[\mathcal{F}_1(\phi,\tau) + \mathcal{K}_\zeta \circ \widetilde{R}^{(0,1)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))\big]u^{(\ell+1)} + D^\ell\mathcal{F}_2(\phi,\tau) + H^{(\ell+1)}_{\zeta/(\ell+1)}(\phi,\tau) \tag{8.34}
$$

in $\mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_\zeta)$, with

$$H^{(\ell+1)}_{\zeta/(\ell+1)}(\phi,\tau) = \mathcal{K}_\zeta \circ \widetilde{R}^{(0,2)}_\delta(u^*_{\eta_{\min}}(\phi,\tau))(u^{*(\ell)}(\phi,\tau), u^{*(1)}(\phi,\tau)) + DH^{(\ell)}_{\zeta/\ell}(\phi,\tau). \tag{8.35}$$

However, the definition (8.35) remains valid upon writing $\zeta = (\ell+1)\eta_{\min}$. This allows one to define $H^{(\ell+1)}_{\eta_{\min}} \in \mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_{(\ell+1)\eta_{\min}})$ in a natural fashion, with $H^{(\ell+1)}_{\zeta/(\ell+1)} = \mathcal{J}^1_{\zeta\ (\ell+1)\eta_{\min}} H^{(\ell+1)}_{\eta_{\min}}$. We hence conclude that the fixed point $u^{*(\ell+1)}(\phi,\tau)$ of (8.34) is also contained in $\mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_{(\ell+1)\eta_{\min}})$. We can hence define $u^{*(\ell+1)}_{\eta_{\min}} = u^{*(\ell+1)}(\phi,\tau) \in \mathcal{L}^{(\ell+1)}(X_0 \times \mathbb{R}, BC^1_{(\ell+1)\eta_{\min}})$. In order to complete the proof, it remains only to consider the statements in (IH2) that involve the D_ξ derivatives. However, these follow from inspection, repeatedly using $D_\xi \mathcal{K} f = L\mathcal{K} + f$ together with (8.7). □

A Embedded Contractions

In this appendix we outline a version of the embedded contraction theorem that we used to prove that the center manifold is C^k-smooth. The presentation given here contains slight adaptations of results given in [32], therefore the proofs are omitted.

Let Y_0, Y, Y_1 and Λ be Banach spaces with norms denoted respectively by

$$\|\cdot\|_0\,, \quad \|\cdot\|\,, \quad \|\cdot\|_1 \quad \text{and} \quad |\cdot|\,, \tag{A.1}$$

and suppose that we have continuous embeddings $J_0 : Y_0 \hookrightarrow Y$ and $J : Y \hookrightarrow Y_1$. Let $\Omega_0 \subset Y_0$ and $\Lambda_0 \subset \Lambda$ be two open convex open subsets of Y_0 respectively Λ. We consider the fixed-point equation

$$y = F(y,\lambda) \tag{A.2}$$

for some $F : Y \times \Lambda \to Y$. Associated to F we define a function $F_0 : \Omega_0 \times \Lambda_0 \to Y$ via

$$F_0(y_0,\lambda_0) = F(J_0 y_0, \lambda_0) \tag{A.3}$$

and also a function $G : \Omega_0 \times \Lambda_0 \to Y_1$ by $G = J \circ F_0$. We shall need the following assumptions on F and G.

(HC1) The function G is of class C^1. Fix any $\omega_0 \in \Omega_0$ and $\lambda_0 \in \Lambda_0$ and consider the partial derivative $D_1 G(\omega_0,\lambda_0) \in \mathcal{L}(Y_0, Y_1)$. Then there exist $F^{(1)}(\omega_0,\lambda_0) \in \mathcal{L}(Y)$ and $F^{(1)}_1(\omega_0,\lambda_0) \in \mathcal{L}(Y_1)$ such that for any $v_0 \in Y_0$ we have

$$\begin{aligned} D_1 G(\omega_0,\lambda_0) v_0 &= J F^{(1)}(\omega_0,\lambda_0) J_0 v_0, \\ J F^{(1)}(\omega_0,\lambda_0) y &= F^{(1)}_1(\omega_0,\lambda_0) J y. \end{aligned} \tag{A.4}$$

(HC2) There exists some $\kappa_1 \in [0,1)$ such that for all $\omega_0 \in \Omega_0$ and $\lambda_0 \in \Lambda_0$ we have

$$\left\|F^{(1)}(\omega_0,\lambda_0)\right\|_{\mathcal{L}(Y)} \le \kappa_1 \quad \text{and} \quad \left\|F^{(1)}_1(\omega_0,\lambda_0)\right\|_{\mathcal{L}(Y_1)} \le \kappa_1. \tag{A.5}$$

(HC3) The mapping $(\omega_0, \lambda_0) \to J \circ F^{(1)}(\omega_0, \lambda_0)$ is continuous as a map from $\Omega_0 \times \Lambda_0$ into $\mathcal{L}(Y, Y_1)$.

(HC4) The function F_0 has a continuous partial derivative

$$D_2 F_0 : \Omega_0 \times \Lambda_0 \to \mathcal{L}(\Lambda, Y). \tag{A.6}$$

(HC5) There exists some $\kappa_2 \in [0, 1)$ such that for all $y, \overline{y} \in Y$ and all $\lambda_0 \in \Lambda_0$ we have

$$\|F(y, \lambda_0) - F(\overline{y}, \lambda_0)\| \leq \kappa_2 \|y - \overline{y}\|. \tag{A.7}$$

It follows from (HC5) that (A.2) has for each $\lambda_0 \in \Lambda_0$ a unique solution $\Psi = \Psi(\lambda)$. We assume that

(HC6) For some continuous $\Phi : \Lambda_0 \to \Omega_0$ we have $\Psi = J_0 \circ \Phi$.

We define $\kappa = \max(\kappa_1, \kappa_2)$.

Lemma A.1. *Assume that assumptions (HC1) through (HC6) hold, except possibly (HC3). Then Ψ is locally Lipschitz continuous.*

Assuming that (HC1) through (HC6) hold, we can consider the following equation for $A \in \mathcal{L}(\Lambda, Y)$,

$$A = F^{(1)}(\Phi(\lambda_0), \lambda_0)A + D_2 F_0(\Phi(\lambda_0), \lambda_0). \tag{A.8}$$

Since $\left\|F^{(1)}\right\|_{\mathcal{L}(Y)} \leq \kappa < 1$ by (HC2), we see that $I - F^{(1)}(\Phi(\lambda_0), \lambda_0)$ is invertible in $\mathcal{L}(Y)$ and hence for each $\lambda_0 \in \Lambda_0$ (A.8) has a unique solution $A = \mathcal{A}(\lambda_0) \in \mathcal{L}(\Lambda, Y)$.

Lemma A.2. *Assume that (HC1) through (HC6) hold. Then the mapping $J \circ \Psi$ is of class C^1 and $D(J \circ \Psi)(\lambda_0) = J \circ \mathcal{A}(\lambda_0)$ for all $\lambda_0 \in \Lambda_0$.*

We conclude with the following result on differentiation in nested spaces.

Lemma A.3. *Consider an integer $\ell > 1$ and a sequence of Banach spaces $Y_0 \subset Y_1 \subset \ldots \subset Y_\ell$, in which each inclusion $\mathcal{J}_{ji} : Y_i \to Y_j$ with $j > i$ is continuous. Let Z_0 and Z_1 be Banach spaces and $[a, b] \subset \mathbb{R}$ be an interval. Consider functions $M : [a, b] \to \mathcal{L}(Z_0, Y_0)$ and $L : [a, b] \to \mathcal{L}(Y_\ell, Z_1)$ with the following properties.*

(i) For each $0 \leq j \leq \ell$, we have that the map $[a, b] \to \mathcal{L}(Z_0, Y_j)$ given by $\xi \mapsto \mathcal{J}_{j0} M(\xi)$ is C^j-smooth.

(ii) For every $0 \leq q \leq j$, we have that

$$D^q \mathcal{J}_{j0} M(\xi) = \mathcal{J}_{jq} D^q \mathcal{J}_{q0} M(\xi). \tag{A.9}$$

(iii) For each $0 \leq j \leq \ell$, we have that the restriction map $[a, b] \to \mathcal{L}(Y_j, Z_1)$ given by $\xi \mapsto (L(\xi))_{|Y_j}$ is $C^{\ell - j}$-smooth.

Then the map $[a, b] \to \mathcal{L}(Z_0, Z_1)$ given by $\xi \mapsto L(\xi)\mathcal{J}_{\ell 0} M(\xi)$ is C^ℓ-smooth.

Proof. For convenience, define the maps $L_j = L_{|Y_j}$ and $M_j = \mathcal{J}_{j0}M$. Notice first that for any $p \geq 0$ and $q \geq 0$ with $p + q \leq \ell$, we have that the function $W^{p,q} : [a,b] \to \mathcal{L}(Z_0, Z_1)$ defined by

$$W^{p,q}(\xi) = D^q L_p(\xi) D^p M_p(\xi) \tag{A.10}$$

is well-defined and continuous. Associated to a given C^j-smooth operator $S : [a,b] \to \Omega$, we define the usual remainder functions $R_S^{(j)} : [a,b] \times [a,b] \to \Omega$ by

$$R_S^{(j)}(\xi, \xi') = S(\xi') - \sum_{k=0}^{j} D^k S(\xi) \frac{(\xi' - \xi)^k}{k!} \tag{A.11}$$

and observe that $\left\| R_S^{(j)}(\xi, \xi') \right\| = o(|\xi - \xi'|^j)$.

Now notice that

$$\begin{array}{rcl} L(\xi')M_\ell(\xi') & = & L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi') + \sum_{k=0}^{\ell} L(\xi')D^k M_\ell(\xi)\frac{(\xi'-\xi)^k}{k!} \\ & = & L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi') + \sum_{k=0}^{\ell} L(\xi')\mathcal{J}_{\ell k} D^k M_k(\xi)\frac{(\xi'-\xi)^k}{k!} \\ & = & L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi') + \sum_{k=0}^{\ell} L_k(\xi')D^k M_k(\xi)\frac{(\xi'-\xi)^k}{k!}. \end{array} \tag{A.12}$$

Recalling that

$$L_k(\xi') = R_{L_k}^{(\ell-k)}(\xi,\xi') + \textstyle\sum_{m=0}^{\ell-k} D^m L_k(\xi)\frac{(\xi'-\xi)^m}{m!}, \tag{A.13}$$

one can write

$$\begin{array}{rcl} L(\xi')M_\ell(\xi') - L(\xi)M_\ell(\xi) & = & \sum_{(p\geq 0, q\geq 0)|1\leq p+q\leq \ell} c_{p,q} W^{p,q}(\xi'-\xi)^{p+q} \\ & & + \sum_{k=0}^{\ell} R_{L_k}^{(\ell-k)}(\xi,\xi')D^k M_k(\xi)(\xi'-\xi)^k \frac{1}{k!} \\ & & + L(\xi')R_{M_\ell}^{(\ell)}(\xi,\xi'), \end{array} \tag{A.14}$$

for appropriate constants $c_{p,q}$, which shows that indeed $D^\ell[L\mathcal{J}_{\ell 0}M]$ can be properly defined in a continuous fashion. □

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