

Rate of convergence to equilibrium for interacting particle systems via coupling and concentration

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Abstract

We present a new approach to estimate the relaxation speed to equilibrium of interacting particle systems. It is based on concentration inequalities and coupling. We illustrate our approach in a variety of examples for which we obtain several new results with short and non technical proofs. These examples include the symmetric and asymmetric exclusion process and high-temperature spin-flip dynamics ("Glauber dynamics"). We also give a direct proof of the Poincaré inequality, based on coupling, in the context of one-dimensional Gibbs measures with possibly long-range potentials.

Keywords: L^p estimates, uniform and non-uniform coupling, Poincaré's inequality, Young's inequality, exclusion process, spin-flip dynamics, Glauber dynamics, Gibbs measures.

1 Introduction

In the study of relaxation to equilibrium for interacting particle systems, several approaches have been put forward. In the uniformly ergodic regime (also known under the name " $M < \epsilon$ " regime [14, Chapter I]), relaxation to the unique stationary measure is exponential in the supremum norm, with an estimate in term of the so-called triple norm. In [17] this estimate

(and generalizations of it) is obtained via time discretization and coupling. Exponential relaxation in the L^2 context can be derived from the Poincaré inequality, which is usually obtained via the stronger log-Sobolev inequality, which in turn implies exponential relaxation in L^∞ .

For processes with a conservation law, such as the exclusion process, typically the relaxation is expected to be diffusive, *i.e.*, with a power-law decay. This type of decay has been obtained in the context of Kawasaki dynamics in [1, 2], [5] [18] by the spectral gap method, *i.e.*, by estimating the speed at which the spectral gap of the finite-volume generator vanishes. Alternative methods to obtain power-law decay are Nash inequalities [10], or "attractivity" and "linearity" in [7], [16].

In this paper, we present a new approach based on a combination of concentration inequalities (in the spirit of the Azuma-Hoeffding inequality, see, *e.g.*, [13]), and coupling, thus continuing in the spirit of what we developed in [6], but now in the time-dependent context.

In the realm of concentration inequalities, a crucial quantity is the "vector" of variations of a function. The bounds, *e.g.*, the Gaussian bound or L^p estimates are usually in terms of the ℓ^2 norm of this vector, whereas in the ergodic theory of interacting particle systems mostly the ℓ^1 norm (commonly called triple norm) appears.

The time evolution acts on the vector of variations in a way that can be estimated in terms of a convolution with a time-dependent function ψ_t . This function ψ_t measures how well we can couple at site x if we start with a single discrepancy at the origin. The ℓ^2 norm of this convolution can then be estimated via Young's inequality. Here the advantage of the ℓ^2 (as opposed to ℓ^1) becomes clear, since we have some flexibility in the choice of norms in Young's inequality. Even in the conservative case, where typically the ℓ^1 norm of ψ_t is a constant not depending on time, higher norms can behave better, and can even produce the expected diffusive decay.

For the coupling, we typically have two regimes: a regime where there is a uniform (in the starting configuration) control of the coupling and a regime where there is only a pointwise control, *i.e.*, the coupling behaves badly for a set of exceptional (in the measure theoretical sense) configurations. In the uniform coupling regime we combine coupling with Gaussian bounds, which leads to a time-dependent Gaussian bounds for exponential moments, and via this to L^p relaxation. The non-uniform coupling regime is dealt with via moment-estimates, where the configurations for which the coupling behaves badly are "neutralized" by integration over the stationary measure. This situation is met (unavoidably) in the context of the asymmetric exclusion process, where we can estimate the L^p -relaxation in terms of a quantity related to the equilibrium behavior of a single second class particle.

We illustrate our approach in a variety of examples, for which we obtain several new results with remarkably compact proofs. For the symmetric exclusion process, we obtain sharp Gaussian and L^p bounds in terms of the transition kernel of the underlying random walk, that yield the expected diffusive decay. Similar estimates are obtained in the context of the voter model and the (subcritical) contact process. For high-temperature spin-flip (or Glauber) dynamics, we obtain the usual exponential decay, however with an estimate in terms of the ℓ^2 norm, which allows for a better control of, *e.g.*, spatial averages. In the context of the asymmetric exclusion process, we illustrate our coupling method in the non-uniform situation and obtain L^p bounds in terms of a natural quantity related to the second class particle. Finally we give a direct proof based on coupling of the Poincaré inequality in the context of one-dimensional Gibbs measures with possibly long-range potentials.

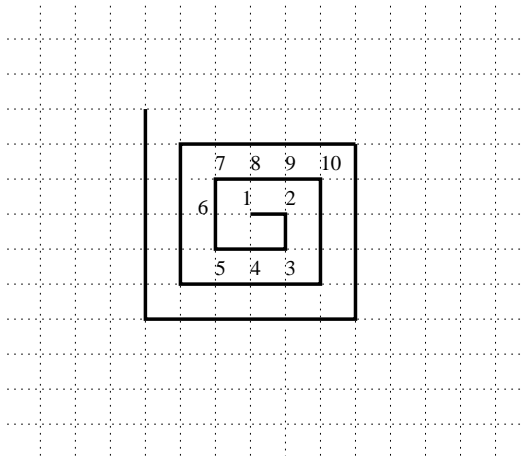
2 Notations, definitions

2.1 Configurations

We work in the context of lattice spin systems, with state space $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, $d \geq 1$ (endowed with product topology). Elements of Ω are denoted σ, η, ξ . We fix a spiraling enumeration of \mathbb{Z}^d

$$\mathbb{Z}^d = \{x_0, x_1, \dots, x_n, \dots\}$$

where $x_0 = 0$, x_1, \dots, x_{2d} are neighbors of 0, etc., see the figure below for $d = 2$. For $x \in \mathbb{Z}^d$ we denote by n_x the index of x in this enumeration. Then



we have an order relation defined via $x \leq y$ iff $n_x \leq n_y$. We further denote

$$(\leq i) := \{x \in \mathbb{Z}^d : n_x \leq i\} \quad (1)$$

and similarly the sets $(< i), (> i), (\geq i), (\neq i)$, where we add the convention $(< 0) := \emptyset$. With a slight abuse of notation, we will use the symbol i both for the index (in the enumeration) of a site $x = x(i)$, in \mathbb{Z}^d as well as for the site itself.

For $\Lambda \subset \mathbb{Z}^d$ we define \mathcal{F}_Λ to be the σ -field generated by $\{\pi_x, x \in \Lambda\}$ where π_x are the natural coordinate maps $\pi_x : \sigma \rightarrow \sigma(x)$. In agreement with the notation (1) we then have the σ -fields $\mathcal{F}_{\leq i}, \mathcal{F}_{< i}$, etc., where $\mathcal{F}_{< 0}$ is defined to be the trivial σ -field $\{\emptyset, \Omega\}$. $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^d}$ is the Borel sigma-field on Ω .

For $\sigma \in \Omega$ we define σ^i to be the configuration obtained from σ by flipping at i , *i.e.*,

$$\sigma^i(j) = \begin{cases} \sigma(j) & \text{if } j \neq i \\ 1 - \sigma(i) & \text{if } j = i \end{cases}.$$

For $\sigma \in \Omega$ and $i, j \in \mathbb{Z}^d$ we define

$$\sigma^{ij}(k) = \begin{cases} \sigma(k) & \text{if } k \notin \{i, j\} \\ \sigma(i) & \text{if } k = j \\ \sigma(j) & \text{if } k = i \end{cases}.$$

For $\sigma^1, \dots, \sigma^n \in \Omega$ and a partition $\Lambda_1, \dots, \Lambda_n$ of \mathbb{Z}^d (*i.e.*, the Λ_i 's are pairwise disjoint and $\cup_{i=1}^n \Lambda_i = \mathbb{Z}^d$), we denote $\sigma_{\Lambda_1}^1 \sigma_{\Lambda_2}^2 \dots \sigma_{\Lambda_n}^n$ to be the configuration that coincides with σ^1 on $\Lambda_1, \dots, \sigma^n$ on Λ_n . For instance we write $\sigma_{< i} \sigma_i \xi_{> i}$, etc.

For $x \in \mathbb{Z}^d, \sigma \in \Omega$, we denote $\tau_x \sigma$ the configuration shifted by x , *i.e.*, $\tau_x \sigma(y) = \sigma(x + y)$.

If A is a finite subset of \mathbb{Z}^d , $|A|$ denotes its cardinality.

2.2 Functions

For a function $f : \Omega \rightarrow \mathbb{R}$ we define the “discrete derivative” in the direction σ_i at the configuration η to be

$$\nabla_i f(\eta) = f(\eta^i) - f(\eta)$$

and the variation in direction σ_i

$$\delta_i f = \sup_{\eta \in \Omega} (f(\eta^i) - f(\eta)).$$

The collection $\{\delta_i f : i \in \mathbb{Z}^d\}$ is denoted by δf .

For all $p \geq 1$, let

$$\|\delta f\|_p := \|\delta f\|_{\ell^p(\mathbb{Z}^d)} = \left(\sum_{i \in \mathbb{Z}^d} (\delta_i f)^p \right)^{\frac{1}{p}}.$$

For $p = 1$ this norm is usually called “triple norm” [14]:

$$\|f\| \equiv \|\delta f\|_1.$$

A function is called local if there exists a finite subset D_f of \mathbb{Z}^d such that $\delta_i f = 0$ for all $i \notin D_f$. For $\Lambda \subset \mathbb{Z}^d$, $\alpha > 0$, and for $f : \Omega \rightarrow \mathbb{R}$, we define its spatial average by

$$\mathcal{A}_{\alpha, \Lambda}(f) = \frac{1}{|\Lambda|^\alpha} \sum_{x \in \Lambda} \tau_x f \quad (2)$$

where $\tau_x f : \sigma \mapsto f(\tau_x \sigma)$.

The following lemma shows a contraction property of these spatial averages.

LEMMA 2.1. *For $f : \Omega \rightarrow \mathbb{R}$ bounded measurable we have*

$$\|\delta \mathcal{A}_{\alpha, \Lambda}(f)\|_p \leq |\Lambda|^{-\alpha + \frac{1}{p}} \|\delta f\|_1.$$

PROOF. We use the obvious fact that $\delta_y(\tau_x f) = \delta_{x+y} f$ and Young’s inequality to get

$$\begin{aligned} \|\delta \mathcal{A}_{\alpha, \Lambda}(f)\|_p^p &= \frac{1}{|\Lambda|^{\alpha p}} \sum_i \left(\sum_x \mathbb{1}_\Lambda(x) \delta_{i+x} f \right)^p = \frac{1}{|\Lambda|^{\alpha p}} \|\mathbb{1}_\Lambda * \delta f\|_p^p \\ &\leq \frac{1}{|\Lambda|^{\alpha p}} \|\mathbb{1}_\Lambda\|_p^p \|\delta f\|_1^p = \frac{1}{|\Lambda|^{\alpha p - 1}} \|\delta f\|_1^p \end{aligned}$$

where we denoted by $\mathbb{1}_\Lambda$ the indicator function of the set Λ . \square

2.3 Gibbs measures

In the rest of this paper we will only consider translation-invariant measures, and in many places we will restrict to translation-invariant Gibbs measures μ on (Ω, \mathcal{F}) [8]. We briefly recall a few definitions and facts.

Let \mathcal{S} denote the set of finite subsets of \mathbb{Z}^d .

DEFINITION 2.1. A translation-invariant interaction is a function

$$U : \mathcal{S} \times \Omega \rightarrow \mathbb{R}$$

such that the following conditions are satisfied:

1. $\sigma \mapsto U(A, \sigma)$ is \mathcal{F}_A -measurable for any $A \in \mathcal{S}$.
2. Translation invariance:

$$U(A + x, \tau_{-x}\sigma) = U(A, \sigma) \quad \forall A \in \mathcal{S}, x \in \mathbb{Z}^d, \sigma \in \Omega.$$

3. Uniform summability:

$$\sum_{A \ni 0} \sup_{\sigma \in \Omega} |U(A, \sigma)| < \infty. \quad (3)$$

The set of all such interactions is denoted by \mathcal{U} . An interaction U is called *finite-range* if there exists an $R > 0$ such that $U(A, \sigma) = 0$ for all $A \in \mathcal{S}$ with $\text{diam}(A) > R$. For $U \in \mathcal{U}$, $\zeta \in \Omega$, $\Lambda \in \mathcal{S}$, we define the finite-volume Hamiltonian with boundary condition ζ as

$$H_\Lambda^\zeta(\sigma) = \sum_{A \cap \Lambda \neq \emptyset} U(A, \sigma_\Lambda \zeta_{\Lambda^c}). \quad (4)$$

Corresponding to the Hamiltonian in (4) we have the finite-volume Gibbs measures $\mu_\Lambda^{U, \zeta}$, $\Lambda \in \mathcal{S}$, defined on Ω by

$$\int f(\xi) \mu_\Lambda^{U, \zeta}(d\xi) = \sum_{\sigma_\Lambda \in \Omega_\Lambda} f(\sigma_\Lambda \zeta_{\Lambda^c}) \frac{e^{-H_\Lambda^\zeta(\sigma)}}{Z_\Lambda^\zeta} \quad (5)$$

where f is any continuous function and Z_Λ^ζ denotes the partition function normalizing $\mu_\Lambda^{U, \zeta}$ to a probability measure. Because of the uniform summability condition (3), H_Λ^ζ and $\mu_\Lambda^{U, \zeta}$ are continuous as a function of the boundary condition ζ .

For a probability measure μ on Ω , we denote by μ_Λ^ζ the conditional probability distribution of $\sigma(x), x \in \Lambda$, given $\sigma_{\Lambda^c} = \zeta_{\Lambda^c}$. Of course, this object is only defined on a set of μ -measure one. For Λ, Γ finite subsets of \mathbb{Z}^d , and $\Lambda \subset \Gamma$, we denote by $\mu_\Gamma(\sigma_\Lambda | \zeta)$ the conditional probability to find σ_Λ inside Λ , given that ζ occurs in $\Gamma \setminus \Lambda$.

For $U \in \mathcal{U}$, we call μ a Gibbs measure with interaction U if its conditional probabilities coincide with the ones prescribed in (5), *i.e.*, if

$$\mu_\Lambda^\zeta = \mu_\Lambda^{U,\zeta} \quad \text{for } \mu \text{ almost every } \zeta \in \Omega.$$

We denote by $\mathcal{G}(U)$ the (non-empty) set of all translation-invariant Gibbs measures with interaction U .

For μ a Gibbs measure on Ω , $\Lambda \subset \mathbb{Z}^d$, and $\sigma \in \Omega$ we denote by μ_{σ_Λ} the measure μ conditioned on having the fixed configuration σ_Λ on Λ . For $i \in \mathbb{Z}^d$ we denote by μ^i the image measure of μ under the transformation $\sigma \mapsto \sigma^i$. Since μ is assumed to be a Gibbs measure, the Radon-Nikodym derivatives $\frac{d\mu^i}{d\mu}$ exist and are continuous. Moreover, there is a constant $C > 0$ such that

$$\left\| \frac{d\mu^i}{d\mu} \right\|_\infty \leq C. \quad (6)$$

2.4 Dynamics and semigroups

Associated to a Gibbs measure μ we have natural spin-flip dynamics, usually called Glauber dynamics. These are Markov processes on Ω with generator on local functions defined via

$$L_\mu^G f(\eta) = \sum_i c(i, \eta) (f(\eta^i) - f(\eta))$$

where the rates $0 < \epsilon < c(x, \sigma) < K$ are supposed to be uniformly bounded from below and from above, and satisfy

$$\frac{c(i, \sigma)}{c(i, \sigma^i)} = \frac{d\mu^i}{d\mu}(\sigma)$$

which guarantees that the process with generator L_μ^G started from μ is reversible. We denote by $(S_t)_{t \geq 0}$ the $L^2(\mu)$ -semigroup of this process. Notice that since μ is assumed to be translation-invariant S_t commutes with translations.

In the course of this paper we will also deal with examples of other dynamics such as the exclusion process, the contact process, etc, see below and/or [14] for more details.

Next, we define the quadratic form

$$\mathcal{E}(f, f) = \sum_i \int (\nabla_i f)^2 d\mu. \quad (7)$$

Associated to the generator L_μ^G we have the Dirichlet form

$$\mathcal{E}_G^\mu(f, f) = \frac{1}{2} \sum_i \int c(i, \sigma) (\nabla_i f)^2(\sigma) \mu(d\sigma). \quad (8)$$

Since the rates satisfy $0 < \epsilon < c(x, \sigma) < K$, we have the obvious bounds

$$\frac{\epsilon}{2} \mathcal{E}(f, f) \leq \mathcal{E}_G^\mu(f, f) \leq \frac{K}{2} \mathcal{E}(f, f).$$

Therefore, *e.g.*, in inequalities like the Poincaré inequality (see below) it is equivalent to bound the variance (under μ) by the quadratic form (7) or by the Dirichlet form (8).

2.5 Coupling

For two probability measures ν, μ on Ω , a coupling is a probability measure on $\Omega \times \Omega$ with marginals μ , resp. ν . For an extensive background on coupling, we refer to [21].

We fix the following distance on Ω , though any other distance compatible with the product topology would be suited: $\text{dist}(\eta, \xi) = \sum_i 2^{-i} |\eta(i) - \xi(i)|$. The Vasserstein distance between ν, μ with respect to this distance is then defined by

$$d(\mu, \nu) = \inf \left\{ \int \text{dist}(\eta, \xi) d\mathbb{P}(\eta, \xi) : \mathbb{P} \text{ is a coupling of } \mu \text{ and } \nu \right\}. \quad (9)$$

An optimal coupling is a coupling which achieves the infimum in (9). In our context, by compactness, an optimal coupling always exists.

For two Markov processes $\{\eta_t : t \geq 0\}$, $\{\xi_t : t \geq 0\}$, a coupling is a process $\{(\eta_t^1, \eta_t^2) : t \geq 0\}$ on $\Omega \times \Omega$ with marginals $\{\eta_t : t \geq 0\}$, resp. $\{\xi_t : t \geq 0\}$. For spin-flip processes such as defined in the previous section, there is a natural coupling, called basic coupling, following from the so-called "graphical construction", see [14, Chapter III, Section 1].

For a monotone Markov process [14, Chapter II] there exists a coupling such that if $\eta \leq \xi$ (meaning that for all $x \in \mathbb{Z}^d$ $\eta(x) \leq \xi(x)$), then, in the coupling, the order is preserved in the course of time, *i.e.*, for all $t \geq 0$, $\mathbb{P}_{\eta, \xi}(\eta_t \leq \xi_t) = 1$.

2.6 Inequalities

DEFINITION 2.2. *Let μ be a probability measure μ on Ω .*

a) We say that μ satisfies the **Gaussian exponential-moment bound** with constant $c = c(\mu)$ (abbreviated *GEMB*(c)) if for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable we have

$$\mathbb{E}_\mu(e^{f - \mathbb{E}_\mu f}) \leq e^{c\|f\|_2^2}. \quad (10)$$

b) We say that μ satisfies the **Poincaré inequality** if there exists a constant $c = c(\mu)$ such that for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable

$$\text{Var}_\mu(f) \leq c \mathcal{E}(f, f). \quad (11)$$

If μ is a reversible measure for the Markov process, then the Poincaré inequality for μ implies exponential relaxation in $L^2(\mu)$. More precisely, from (11) and the spectral theorem, we have the estimate (see [14, Theorem 4.16, Chapter IV]),

$$\|S_t f - \mathbb{E}_\mu(f)\|_{L^2(\mu)} \leq e^{-\frac{t}{c}} \|f\|_{L^2(\mu)}^2.$$

3 Gaussian concentration and uniform coupling

3.1 Coupling matrix

We start with a probability measure μ that satisfies *GEMB*(c), and with a Markov process $\{\sigma_t : t \geq 0\}$ with semigroup $(S_t)_{t \geq 0}$.

We apply *GEMB*(c) to the function $S_t f$. Therefore, we have to estimate $\delta(S_t f)$:

$$\begin{aligned} \delta_i(S_t f) &= \sup_\sigma |S_t f(\sigma^i) - S_t f(\sigma)| \\ &\leq \sum_k D_t(i, k) \delta_{k, f} \end{aligned} \quad (12)$$

where we introduced the matrix

$$D_t(i, k) = \sup_\sigma D_t^\sigma(i, k) = \sup_\sigma \inf_{\mathbb{P}} \mathbb{P}_{\sigma^i, \sigma}(\sigma_t^1(k) \neq \sigma_t^2(k)). \quad (13)$$

Here the infimum runs over all couplings of $\{\sigma_t : t \geq 0\}$ starting at σ^i in the first copy and σ in the second copy.

In the translation-invariant case we have

$$D_t(i, k) =: \psi_t(k - i). \quad (14)$$

Since we consider translation-invariant measures, ψ_t will be the relevant quantity.

In the case of *monotone* dynamics, the coupling can be chosen such that the order between configurations is preserved, which implies that

$$\begin{aligned} \mathbb{P}_{\sigma^i, \sigma}(\sigma_t^1(k) \neq \sigma_t^2(k)) &= \mathbb{E}_{\sigma_{\neq i}^1 1_i, \sigma_{\neq i}^2 0_i}(\sigma_t^1(k) - \sigma_t^2(k)) \\ &= \mathbb{E}_{\sigma_{\neq i}^1 1_i}(\sigma_t(k)) - \mathbb{E}_{\sigma_{\neq i}^2 0_i}(\sigma_t(k)). \end{aligned} \quad (15)$$

Therefore, in this case, the matrix $D_t^\sigma(i, k)$ is completely controlled by expectations of $\sigma_t(k)$.

3.2 Time-dependent deviation bounds

THEOREM 3.1. *If μ satisfies GEMB(c) (10), then for any pair $u, v \geq 1$ such that $\frac{1}{u} + \frac{1}{v} = \frac{3}{2}$, and for all $t \geq 0$, one has*

$$\mathbb{E}_\mu (e^{S_t f - \mathbb{E}_\mu(S_t f)}) \leq e^{c \|\psi_t\|_u^2 \|\delta f\|_v^2}. \quad (16)$$

PROOF. By combining (12),(13),(14), we obtain

$$\delta(S_t f) \leq \psi_t * \delta f.$$

Therefore, Young's inequality yields

$$\|\delta(S_t f)\|_2^2 \leq \|\psi_t\|_u^2 \|\delta f\|_v^2$$

for any $u, v > 1$ such that $\frac{1}{u} + \frac{1}{v} = \frac{3}{2}$. The theorem is proved. \square

COROLLARY 3.1. *Under the conditions of Theorem 3.1, for all $t \geq 0$, and for all $a \geq 0$, one has the deviation bounds*

$$\mu(S_t f - \mathbb{E}_\mu(S_t f) \geq a) \leq \exp\left(-\frac{a^2}{4c \|\psi_t\|_u^2 \|\delta f\|_v^2}\right) \quad (17)$$

and

$$\mu(|S_t f - \mathbb{E}_\mu(S_t f)| \geq a) \leq 2 \exp\left(-\frac{a^2}{4c \|\psi_t\|_u^2 \|\delta f\|_v^2}\right). \quad (18)$$

Moreover, one has the following estimate for the variance

$$\mathrm{Var}_\mu(S_t f) \leq c \|\psi_t\|_u^2 \|\delta f\|_v^2 \quad (19)$$

and, more generally, for all $p \geq 1$,

$$\|S_t f - \mathbb{E}_\mu(S_t f)\|_{L^p(\mu)} \leq 2\sqrt{c} \left(p\Gamma\left(\frac{p}{2}\right)\right)^{\frac{1}{p}} \|\psi_t\|_u \|\delta f\|_v. \quad (20)$$

PROOF. The deviation bound (17) follows easily from (16) and a standard application of the (exponential) Chebychev inequality. The deviation bound (18) follows at once from (17) applied to f and $-f$.

In order to obtain the L^p -bounds, we start from the deviation bound (18) and use the following elementary lemma.

LEMMA 3.1. *Suppose that X is a random variable such that for all $a \geq 0$*

$$\mathbb{P}(|X| \geq a) \leq 2e^{-\frac{a^2}{\kappa}}$$

for some $\kappa > 0$. Then

$$\mathbb{E}(|X|^p) \leq p\Gamma\left(\frac{p}{2}\right) \kappa^{\frac{p}{2}}$$

for all $p \geq 1$ (where Γ is Euler's Gamma function).

PROOF.

$$\mathbb{E}(|X|^p) = \int_0^\infty p t^{p-1} \mathbb{P}(|X| \geq t) dt \leq 2 \int_0^\infty p t^{p-1} e^{-\frac{t^2}{\kappa}} dt = p\Gamma\left(\frac{p}{2}\right) \kappa^{\frac{p}{2}}.$$

□

The proof of Corollary 3.1 is now complete. □

As we will see in the examples below, these bounds are sharp as far as the t -dependence is concerned, *e.g.*, in the case of the symmetric exclusion process with μ a Bernoulli measure, they give the correct decay behavior.

The next corollary is about spatial averages defined in (2). It exploits the fact that in (18) and (20) we have the $\|\cdot\|_v$ -norm of δf (with $v > 1$), and combines with the contraction property of Lemma 2.1.

COROLLARY 3.2. *Suppose that μ satisfies GEMB(c). Then, for all $t \geq 0$, for all $a \geq 0$, for all $\Lambda \subset \mathbb{Z}^d$ and for all $\alpha \geq 1/2$, for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable, for all $u, v > 1$ such that $\frac{1}{u} + \frac{1}{v} = \frac{3}{2}$, we have the estimates*

$$\mu\left(|S_t(\mathcal{A}_{\alpha,\Lambda}(f)) - |\Lambda|^{1-\alpha} \mathbb{E}_\mu(S_t f)| \geq a\right) \leq 2 \exp\left(-\frac{|\Lambda|^{2\alpha - \frac{2}{v}} a^2}{4c \|\psi_t\|_u^2 \|\delta f\|_1^2}\right)$$

and for all $p \geq 1$:

$$\|S_t(\mathcal{A}_{\alpha,\Lambda}(f)) - |\Lambda|^{1-\alpha}\mathbb{E}_\mu(S_t f)\|_{L^p(\mu)} \leq 2\sqrt{c} \left(p\Gamma\left(\frac{p}{2}\right)\right)^{\frac{1}{p}} |\Lambda|^{-\alpha+\frac{1}{v}} \|\psi_t\|_u \|\delta f\|_1.$$

REMARK 3.1. A possible generalization of the Gaussian exponential-moment bound with constant c is the following. Suppose G is a positive convolution operator on $\ell^2(\mathbb{Z}^d)$, i.e.,

$$(G\varphi)_i = \sum_k G(i-k)\varphi(k)$$

with $G : \mathbb{Z}^d \rightarrow \mathbb{R}$ a non-negative function. A typical example of G we have in mind here is the lattice Green's function. Associated to G , we have the quadratic form on the domain of $G^{1/2}$ defined as usual by

$$V_G(\varphi) = \langle \varphi, G\varphi \rangle.$$

We then say that a measure satisfies the Gaussian exponential moment inequality with covariance kernel G if for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable we have the inequality

$$\mathbb{E}_\mu(e^{f - \mathbb{E}_\mu f}) \leq e^{cV_G(\delta f)}.$$

The analogue of the time-dependent estimate in Theorem 3.1 then becomes

$$\mathbb{E}_\mu(e^{S_t f - \mathbb{E}_\mu(S_t f)}) \leq e^{cV_G(\psi_t * \delta g)}$$

and, by an application of Young's inequality, we have, e.g., as a possible estimate

$$V_G(\psi_t * \delta g) \leq \|\psi_t\|_2 \|G\|_2 \|\delta g\|_1^2.$$

3.3 Examples

3.3.1 Symmetric exclusion process

The symmetric exclusion process (SEP) is the process defined by the generator acting on local functions

$$Lf(\eta) = \sum_{x,y} p(x,y)(f(\eta^{xy}) - f(\eta)),$$

where η^{xy} is obtained from η by exchanging occupations in x and y in the configuration η , and where $p(x,y) = p(0,y-x)$ is supposed to be an irreducible, symmetric and translation-invariant random walk transition probability with finite second moment. In that case the ergodic stationary measures are Bernoulli, i.e., $\mu = \nu_\rho$ (see [14, Chapter VIII]).

THEOREM 3.2. *Let (S_t) be the semigroup of the symmetric exclusion process. Then, for any probability measure μ on Ω satisfying GEMB(c) (10), for all $t \geq 0$, for all $p \geq 1$, and for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable, we have the estimates*

$$\|S_t f - \mathbb{E}_\mu(S_t f)\|_{L^p(\mu)} \leq 2\sqrt{c} \left(p\Gamma\left(\frac{p}{2}\right) \right)^{\frac{1}{p}} \sqrt{p_{2t}(0,0)} \|\delta f\|_1 \quad (21)$$

and

$$\mu(|S_t f - \mathbb{E}_\mu(S_t f)| \geq a) \leq 2 \exp\left(-\frac{a^2}{4c p_{2t}(0,0) \|\delta f\|_1^2}\right). \quad (22)$$

In particular, if ν_ρ denotes the Bernoulli measure with density ρ , then we have GEMB(c) with $c = 1/8$, see [13], and hence

$$\|S_t f - \mathbb{E}_{\nu_\rho}(f)\|_{L^p(\nu_\rho)} \leq \frac{C(p) \|\delta f\|_1}{\sqrt{p_{2t}(0,0)}} \quad (23)$$

where

$$C(p) = 2^{-\frac{1}{2}} \left(p\Gamma\left(\frac{p}{2}\right) \right)^{\frac{1}{p}}.$$

PROOF. Since the SEP is monotone, we can apply (15), which gives

$$\begin{aligned} \mathbb{P}_{\sigma^i, \sigma}(\sigma_t^1(k) \neq \sigma_t^2(k)) &= \mathbb{E}_{\sigma_{\neq i} 1_i, \sigma_{\neq i} 0_i}(\sigma_t^1(k) - \sigma_t^2(k)) \\ &= \mathbb{E}_{\sigma_{\neq i} 1_i}(\sigma_t(k)) - \mathbb{E}_{\sigma_{\neq i} 0_i}(\sigma_t(k)). \end{aligned} \quad (24)$$

Moreover the SEP is self-dual, [14, Chapter VIII, Section 1]. Therefore, for all $\eta \in \Omega$, we have

$$\mathbb{E}_\eta(\eta_t(k)) = \hat{\mathbb{E}}_k(\eta(X_t)) \quad (25)$$

where X_t is the position of simple symmetric random walk jumping at rate one according to $p(x, y)$, and $\hat{\mathbb{E}}_k$ denotes expectation in this random walk, starting at k . Combining (24) and (25), we obtain

$$\mathbb{E}_{\sigma_{\neq i} 1_i}(\sigma_t(k)) - \mathbb{E}_{\sigma_{\neq i} 0_i}(\sigma_t(k)) = p_t(i, k) \quad (26)$$

and hence

$$\psi_t(k) = p_t(0, k)$$

which gives

$$\|\psi_t\|_2^2 = \sum_k p_t(0, k)^2 = p_{2t}(0, 0).$$

To finish the proof apply Corollary 3.1 with the choice $u = 2$, $v = 1$. \square

REMARK 3.2. *The L^p -estimates of Theorem 3.2 have the correct asymptotic behavior in $t^{-d/4}$, since by the local limit theorem [19],*

$$p_t(0, 0) \sim 2 \left(\frac{d}{2\pi\vartheta} \right)^{\frac{d}{2}} t^{-\frac{d}{2}} \quad (27)$$

for large t , where

$$\vartheta = \sum_x x^2 p(0, x)$$

is the variance of the underlying random walk.

REMARK 3.3. *In [1], similar L^2 -estimates in terms of the $\|\delta f\|_1$ -norm are obtained via generalized Nash inequalities combined with the spectral gap approach. Besides we have the explicit exponential estimate (22), and the L^p -estimates (23) hold for all $p \geq 1$.*

Combining the estimates of Corollary 3.2 with (27), we obtain the following estimates for "macroscopic averages" evolved over a "macroscopic" period of time.

COROLLARY 3.3. *Let $g : \Omega \rightarrow \mathbb{R}$ be a bounded measurable function and assume that $\mathbb{E}_{\nu_\rho}(g) = 0$. For $\alpha \geq 1/2$, $\kappa > 0$, define*

$$Y(\Lambda, t, g, \alpha, \kappa) = S_{t|\Lambda|^\kappa}(\mathcal{A}_{\alpha, \Lambda}(g)).$$

Then, for all $p \geq 1$, for all $t > 0$ such that $t|\Lambda|^\kappa$ is large enough, and for all $0 < \epsilon < 1$, we have the estimates

$$\|Y(\Lambda, t, g, \alpha, \kappa)\|_p \leq C'(p) t^{-\frac{d\epsilon}{2+2\epsilon}} |\Lambda|^{\frac{\epsilon}{1+\epsilon}(1-\frac{\kappa d}{2})} \|\delta g\|_1 \quad (28)$$

where $C'(p)$ is some positive constant proportional to $C(p)$.

PROOF. Apply Corollary 3.2 with

$$u = 1 + \epsilon, \quad v = \left(\frac{3}{2} - \frac{1}{u} \right)^{-1} = \frac{2 + 2\epsilon}{1 + 3\epsilon}$$

and use the inequality $p_t(0, k) \leq p_t(0, 0)$, which gives

$$\|\psi_t\|_p^p \leq (p_t(0, 0))^{p-1}.$$

Then use (27) to finish the proof. \square

REMARK 3.4. *Remark that the usual central limit scaling associated to the fluctuation fields corresponds to the choice $\kappa = 2/d$, which is the critical case (as far as the volume dependence is concerned) in (28).*

3.3.2 Monotone dynamics with duality: contact process and voter model

To deal with more general monotone systems with *duality* [14], let us come back to (15). Duality means that there exists a Markov process $\{A_t : t \geq 0\}$, the so-called dual process, on the set of finite subsets of \mathbb{Z}^d such that we have the “duality relation”

$$\mathbb{E}_\eta H(A, \eta_t) = \hat{\mathbb{E}}_A H(A_t, \eta)$$

where $H(A, \eta) = \prod_{x \in A} \eta_x$ and $\hat{\mathbb{E}}$ denotes the expectation in the dual process starting from the finite subset A . Then we have the analogue of (26) with $A = \{k\}$:

$$\mathbb{E}_{\sigma_{\neq i} 1_i}(\sigma_t(k)) - \mathbb{E}_{\sigma_{\neq i} 0_i}(\sigma_t(k)) = \hat{\mathbb{P}}_{\{k\}}(A_t \ni i) = \sum_{A \ni i} p_t(k, A).$$

Hence, in the translation-invariant case we obtain

$$\psi_t(m) = \hat{\mathbb{P}}_{\{0\}}(m \in A_t).$$

For $\|\psi_t\|_2^2$ we have a natural probabilistic interpretation:

$$\begin{aligned} \|\psi_t\|_2^2 &= \sum_k \sum_{A \ni k} p_t(0, A) p_t(0, A) = \sum_A |A| p_t(0, A) p_t(0, A) \\ &= \hat{\mathbb{E}}_0 \times \hat{\mathbb{E}}_0(|A_t| \mathbb{1}_{\{A_t=B_t\}}) \end{aligned}$$

where in the last equality by $\hat{\mathbb{E}}_0 \times \hat{\mathbb{E}}_0$ we denote expectation in two independent copies of the dual process starting at $A_0 = \{0\}$.

If $\{\eta_t : t \geq 0\}$ is the *voter model* [14, Chapter V], *i.e.*, the spin system with rates

$$c(x, \eta) = \begin{cases} \sum_y p(x, y) \eta(y) & \text{if } \eta(x) = 0 \\ \sum_y p(x, y) (1 - \eta(y)) & \text{if } \eta(x) = 1 \end{cases}$$

where $p(x, y) = p(0, y - x) \geq 0$ and $\sum_y p(x, y) = 1$, $\sum_y (y - x)^2 p(x, y) < \infty$. The dual process then consists of coalescent random walkers with kernel $p(x, y)$, and our quantity of interest is

$$\|\psi_t\|_2^2 = \sum p_t(0, x) p_t(0, x) = \mathbb{P}_{x,y}(X_t - Y_t = 0) = \tilde{\mathbb{P}}_{x-y}(Z_t = 0)$$

where $\mathbb{P}_{x,y}$ denotes expectation for two independent random walkers starting at x , resp. y , and jumping at rate one according to $p(x, y)$, and $\tilde{\mathbb{P}}_{x-y}$ denotes translation-invariant continuous-time random walk jumping from 0 to a at

rate $p(a) + p(-a)$. The latter random walk is symmetric and hence we recover estimates (21) in that case. Of course, since we do not know neither expect that the stationary measures of the voter model satisfy GEMB(c), these estimates only serve in the transient regime. In fact, the heavy correlation structure of the non-trivial stationary measures of the voter model (see Theorem 2.8 and formula (2.7) p. 242 in [14]) suggests rather a GEMB with operator G (see Remark (3.1)), where G is the Green's function associated to the random walk Z_t .

Let $\{\eta_t : t \geq 0\}$ be the *subcritical contact process* [14, Chapter VI], *i.e.*, the spin system with rates

$$c(x, \eta) = \begin{cases} \lambda \sum_y \eta(y) & \text{if } \eta(x) = 0 \\ 1 & \text{if } \eta(x) = 1 \end{cases}$$

and $\lambda < \lambda_c$. The contact process is self-dual, and hence in the subcritical case we get from [14, Theorem 3.4, p. 290],

$$\|\psi_t\|_2^2 = \sum_A |A| p_t(0, A) p_t(0, A) \leq \sup_{A \neq \emptyset} p_t(0, A) \sup_{t \geq 0} \mathbb{E}_0(|A_t|) \leq e^{-\epsilon t}$$

for some $\epsilon > 0$, which gives the corresponding Gaussian and L^p -estimates of Theorem 3.1 if we start from a measure μ satisfying the GEMB(c).

$$\|S_t f - \mathbb{E}_\mu(S_t f)\|_{L^p(\mu)} \leq 2\sqrt{c} \left(p\Gamma\left(\frac{p}{2}\right) \right)^{\frac{1}{p}} \|\delta f\|_1 e^{-\frac{\epsilon}{2}t}.$$

Combining this with the estimate

$$\mathbb{E}_\mu(S_t f) \leq \|\delta f\|_1 e^{-\epsilon t}$$

for all f with $f(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the all-zero configuration, we obtain that for all $p \geq 1$

$$\|S_t f\|_p \leq C_p \|\delta f\|_1 e^{-\frac{\epsilon}{2}t}.$$

For $\lambda < 1/(2d)$ this follows immediately from the uniform estimates in the “ $M < \epsilon$ ” regime [14, p. 33]. But for $\lambda_c \in (1/(2d), \lambda_c)$, as far as we know these estimates for general f are new.

3.3.3 High-temperature Glauber dynamics

In this case, we consider the process with generator acting on local functions given by

$$Lf(\sigma) = \sum_i c(i, \sigma)(f(\sigma^i) - f(\sigma)).$$

The rates are chosen to be strictly positive, bounded and such that the detailed balance condition

$$\frac{c(i, \sigma)}{c(i, \sigma^i)} = \frac{d\mu^i}{d\mu}(\sigma) \quad (29)$$

holds. Here μ^i denotes the image measure of μ under the spin-flip transformation $\sigma \mapsto \sigma^i$. The detailed balance condition (29) ensures that μ is a reversible measure for the dynamics.

The reversible measure μ is now supposed to be a Gibbs measure in the *Dobrushin uniqueness regime*, i.e., such that the Dobrushin matrix

$$C_{ij} = \sup_{x \in \{0,1\}, \sigma \in \Omega} \left| \mu(\sigma_i = x | (\sigma^j)_{\mathbb{Z}^d \setminus \{i\}}) - \mu(\sigma_i = x | \sigma_{\mathbb{Z}^d \setminus \{i\}}) \right|$$

satisfies

$$\sup_i \sum_j C_{ij} < 1$$

which implies in particular that $(I - C)$ is an invertible and positive operator in $\ell^2(\mathbb{Z}^d)$.

THEOREM 3.3. *For all $t \geq 0$, for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable*

$$\mathbb{E}_\mu \left(e^{S_t f - \mathbb{E}_\mu(f)} \right) \leq e^{c e^{-\alpha t} \|\delta f\|_2^2}. \quad (30)$$

PROOF. By [22, Proposition 2.5.], we have the estimate

$$\delta_i(S_t f) \leq \sum_j (e^{-t(I-C)})_{ji} \delta_j f$$

which gives

$$\|\delta(S_t f)\|_2^2 \leq \|e^{-t(I-C)}\|_2^2 \|\delta f\|_2^2 \leq \|\delta f\|_2^2 e^{-\alpha t}$$

where the second inequality follows with some $\alpha > 0$, from $\|C\|_2 < 1$, which implies that $I - C$ is a strictly positive operator. The fact $\|C\|_2 < 1$ also follows from [22, Proposition 2.5] and $\|C\|_2^2 \leq \|C\|_\infty \|C\|_1$. To finish the proof, we apply Theorem 3.1: it was proved in [12, Theorem 1] that a Gibbs measure in the Dobrushin uniqueness regime satisfies GEMB(c) with c explicitly given in terms of the Dobrushin matrix. \square

The estimate (30) in turn leads to exponential relaxation in $L^p(\mu)$ via Corollary 3.1, which is the content of the next proposition.

COROLLARY 3.4. *For all $p > 1$, for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable,*

$$\|S_t f - \mathbb{E}_\mu(f)\|_{L^p(\mu)} \leq C(p) \|\delta f\|_2^2 e^{-\alpha t}.$$

Compared with the bounds coming from the “ $M < \epsilon$ ” criterion [14, Chapter I] we have the $\|\cdot\|_2$ norm (instead of the triple norm), which can be an advantage, especially in view of taking spatial averages, as in Corollary 3.2.

4 Moment bounds and non-uniform coupling

In the previous section, we obtained useful estimates only in the case $\psi_t \rightarrow 0$ as $t \rightarrow \infty$. There are natural situations, such as the asymmetric exclusion process, where taking the supremum over σ in (13) spoils the decay of the matrix elements $D_t^\sigma(i, k)$ (as $k - i$ becomes large). The configurations which are responsible for this absence of decay can however still be exceptional in the sense of the measure μ , so that for “typical” configurations σ , the decay of $D_t^\sigma(i, k)$ can still be controled. First, we illustrate this in the context of the estimation of the variance of $S_t f$.

We start by the martingale decomposition (telescoping) of the quantity

$$S_t f - \mathbb{E}_\mu(S_t f) = \sum_i V_i$$

where

$$V_i = \mathbb{E}_\mu(S_t f | \mathcal{F}_{\leq i}) - \mathbb{E}_\mu(S_t f | \mathcal{F}_{< i}).$$

We recall the notation $\mu_{\sigma_{\leq i}}$ for the measure μ conditioned on having $\sigma_{\leq i}$ on the set $(\leq i)$, and similarly $\mu_{\sigma_{< i} 1_i}$, $\mu_{\sigma_{< i} 0_i}$. By $\mu_{\sigma_{< i} 1_i, \sigma_{< i} 0_i}$ we denote a coupling of $\mu_{\sigma_{< i} 1_i}$ with $\mu_{\sigma_{< i} 0_i}$, and by $\mathbb{P}_{\sigma, \eta}$ we denote a coupling of the processes with semigroup S_t starting from σ in the first copy, η in the second copy. Later on we optimize over the choice of the coupling.

Using this notation we can estimate $|V_i|$:

$$\begin{aligned} |V_i(\sigma)| &= \left| \int S_t f(\eta) \mu_{\sigma_{\leq i}}(d\eta) - \int S_t f(\eta) \mu_{\sigma_{< i}}(d\eta) \right| \\ &\leq \left| \int S_t f(\eta) \mu_{\sigma_{< i} 1_i}(d\eta) - \int S_t f(\eta) \mu_{\sigma_{< i} 0_i}(d\eta) \right| \\ &\leq \sum_k \left(\int \mu_{\sigma_{< i} 1_i, \sigma_{< i} 0_i}(d\eta^1 d\eta^2) \mathbb{P}_{\eta^1, \eta^2}(\eta_t^1(k) \neq \eta_t^2(k)) \right) \delta_k f \\ &= \sum_k D_t^\sigma(i, k) \delta_k f \end{aligned} \tag{31}$$

where we introduced the matrix D_t^σ with elements $D_t^\sigma(i, k)$ given by

$$D_t^\sigma(i, k) = \int \mu_{\sigma_{<i}1_i, \sigma_{<i}0_i}(d\eta^1 d\eta^2) \mathbb{P}_{\eta_1, \eta_2}(\eta_t^1(k) \neq \eta_t^2(k)).$$

We then have the pointwise estimate

$$|V_i(\sigma)| = |V_i(\sigma_{\leq i})| \leq (D_t^\sigma \delta f)_i$$

and hence

$$\text{Var}_\mu(S_t f) = \sum_i \mathbb{E}_\mu(V_i^2) \leq \int \|D_t^\sigma \delta f\|_2^2 \mu(d\sigma). \quad (32)$$

The advantage of this expression is that it contains integration over σ so that “exceptional σ ” for which $D_t^\sigma(i, k)$ does not decay properly (as $k - i$ gets large) are integrated out.

Higher moment bounds are obtained via the Burkholder-Gundy inequality, exactly as in [6, Theorems 3 and 6]. If we define

$$\mathcal{D}_t^p(i, j) = (\mathbb{E}_\mu(D_t^\sigma(i, j))^p)^{1/p}$$

we have the following result.

THEOREM 4.1. *For all $t \geq 0$, for all $p \in \mathbb{N}$, for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable we have the estimate*

$$\|S_t f - \mathbb{E}_\mu(S_t f)\|_{L^{2p}(\mu)} \leq 20p \|D_t^{2p}\|_2 \|\delta f\|_2.$$

4.1 Example: the asymmetric exclusion process

The asymmetric exclusion process is defined via the generator on local functions

$$Lf(\eta) = \sum_{x, y} p(x, y) \eta(x)(1 - \eta(y))(f(\eta^{xy}) - f(\eta))$$

where $p(x, y)$ is a translation-invariant nearest-neighbor random walk kernel with non-zero mean.

For the asymmetric exclusion process, when we start in the basic coupling from $(\sigma_{\neq i}1_i, \sigma_{\neq i}0_i)$, then at later times there is exactly one lattice site $k = X_t$ where $\sigma_t^1(k) \neq \sigma_t^2(k)$. X_t is the position of the so-called second class particle [15], starting initially at lattice site i and with the other particles distributed according to the configuration $\sigma_{\neq i}$. So in this case, we can write,

$$D_t^\sigma(i, k) = \mathbb{P}_{\sigma^i, \sigma}(\sigma_t^1(k) \neq \sigma_t^2(k)) = \mathbb{P}_{\sigma^i, \sigma}(X_t = k). \quad (33)$$

First we remark that taking the supremum over σ in (33) spoils the decay of the matrix elements. To see this, first consider the totally asymmetric nearest neighbor case in dimension one. The configuration σ is then chosen to be

$$\begin{cases} \sigma^*(x) = 0 & \text{for } x < 0 \\ \sigma^*(x) = 1 & \text{for } x \geq 0 \end{cases}$$

and $i = 0$. In this case, the second class particle is stuck at 0, *i.e.*,

$$D_t^{\sigma^*}(0, k) = \delta_{0, k}$$

which does not decay as a function of t .

Similarly, in the (not totally asymmetric) case starting from σ^* , the distribution of the second class particle is tight [3], *i.e.*,

$$\liminf_{t \rightarrow \infty} D_t^{\sigma^*}(0, k) > 0.$$

Therefore, we cannot apply Theorem 3.1 to obtain (useful) L^p estimates. Instead of applying Theorem 4.1, we obtain in the next theorem a variance estimate in terms of a quantity of special interest in the case of the asymmetric exclusion process, as we comment on hereafter.

THEOREM 4.2. *Let ν_ρ be the Bernoulli measure with density ρ , *i.e.*, with $\nu_\rho(\eta_x = 1) = \rho$. Define*

$$\Psi_t(k) = \left(\int (\mathbb{P}_{\eta_{\neq 0} 1_0, \eta_{\neq 0} 0_0}(X_t = k))^2 d\nu_\rho(\eta) \right)^{1/2}. \quad (34)$$

Then, for all $f : \Omega \rightarrow \mathbb{R}$ bounded measurable, and for all $t \geq 0$, we have the variance estimate

$$\text{Var}_{\nu_\rho}(S_t f) \leq \|\Psi_t\|_2^2 \|\delta f\|_1^2.$$

PROOF. The conditional distribution $\mu^{\sigma \leq i}$ appearing in (31) is now of course simply the Bernoulli measure on the configuration outside the region ($\leq i$), where we have conditioned, *i.e.*, on $\{0, 1\}^{(>i)}$. Therefore, using (33), the estimate for the variance (32) becomes

$$\begin{aligned} \text{Var}_\mu(S_t f) &\leq \sum_{i, k, l} \delta_k f \delta_l f \int d\nu_\rho(\sigma) d\nu_\rho(\xi) d\nu_\rho(\xi') \times \\ &\quad \mathbb{P}_{\sigma < i 1_i \xi_{> i}, \sigma < i 0_i \xi_{> i}}(X_t = k) \mathbb{P}_{\sigma < i 1_i \xi'_{> i}, \sigma < i 0_i \xi'_{> i}}(X_t = l) \end{aligned} \quad (35)$$

where we use the basic coupling [14, Chapter III, Section 1]. Then, by using the Cauchy-Schwarz inequality and translation invariance in (35), we then obtain

$$\text{Var}_{\nu_\rho}(S_t f) \leq \|\Psi_t * \delta f\|_2^2$$

where Ψ_t is defined in (34). Applying Young's inequality then yields the result of the theorem. \square

In order to get a feeling about the decay of the quantity $\|\Psi_t\|_2^2$, introduce, for $q \in \mathbb{T}_d = (-\pi, \pi]^d$,

$$S(q, t, \eta) = \mathbb{E}_{\eta \neq 0, \eta \neq 0_0} (e^{iq \cdot X_t}).$$

Then, by Parseval's identity,

$$\|\Psi_t\|_2^2 = \frac{1}{(2\pi)^d} \int \int_{\mathbb{T}_d} |S(q, t, \eta)|^2 dq \nu_\rho(d\eta).$$

If we *first* average over η , *i.e.*, introduce

$$S(k, t) = \int \mathbb{P}_{\eta \neq 0, \eta \neq 0_0}(X_t = k) \nu_\rho(d\eta) = \frac{1}{\rho(1-\rho)} \mathbb{E}_{\nu_\rho} ((\eta_t(k) - \rho)(\eta_0(0) - \rho)).$$

For this quantity we have the conjectured diffusive behavior in $d \geq 3$

$$S(k, t) \sim t^{-\frac{d}{2}} \mathcal{N} \left(\frac{k - a(\rho)t}{D(\rho)t^{1/2}} \right)$$

where \mathcal{N} is the standard normal density, and $a(\rho) = (1 - 2\rho)b$, with b the first moment of the underlying random walk, whereas for $d = 1$ the conjectured behavior is superdiffusive, more precisely

$$S(k, t) \sim t^{-\frac{2}{3}} \Phi \left((k - (1 - 2\rho)bt)t^{-\frac{2}{3}} \right)$$

with Φ an unknown scaling function, see [20].

This means that for the Fourier transform

$$S(q, t) = \int S(q, t, \eta) \nu_\rho(d\eta)$$

we have the conjectured diffusive behavior

$$S(q, t) \sim \exp \left(ia(\rho, t) \cdot q - \frac{1}{2} q^2 D(\rho) t \right)$$

in dimension $d \geq 3$, with $a(\rho, t) = (1 - 2\rho)b$, where b is the first moment of the underlying random walk, whereas in $d = 1$,

$$S(q, t) \sim e^{ia(\rho)qt} \hat{\Phi}(qt^{\frac{2}{3}})$$

where

$$\hat{\Phi}(q) = \int_{\mathbb{R}^d} e^{iq \cdot x} \Phi(x) dx.$$

For t large, it is reasonable to expect that $S(q, t, \eta)$ behaves (in leading order in t) as its average over η , for ν_ρ typical η . If we insert this, we find the following corresponding large t behavior of $\|\Psi_t\|_2^2$:

$$\|\Psi_t\|_2^2 \sim \begin{cases} Ct^{-\frac{d}{2}} & \text{for } d \geq 3 \\ Ct^{-\frac{2}{3}} & \text{for } d = 1. \end{cases}$$

which gives the corresponding variance estimates

$$\text{Var}_{\nu_\rho}(S_t f) \leq \begin{cases} C\|\delta f\|_1^2 t^{-\frac{d}{2}} & \text{for } d \geq 3 \\ C\|\delta f\|_1^2 t^{-\frac{2}{3}} & \text{for } d = 1. \end{cases}$$

5 The Poincaré inequality for one-dimensional Gibbs measures

In this section we give a direct (*i.e.*, without renormalization) proof the Poincaré inequality via coupling in the context of one-dimensional Gibbs measures for a large class of potentials. For finite-range potentials the log-Sobolev inequality is obtained in [11], which is stronger than the Poincaré inequality, and implies exponential relaxation in L^∞ .

The idea to derive the Poincaré inequality is to estimate the V_i appearing in the telescoping identity for $f - \mathbb{E}_\mu(f)$ by introducing the coupling matrix as before, but also taking into account the integration over the coupling of the conditional distributions $\mu_{\sigma_{<i}\bar{\sigma}_i}$ and $\mu_{\sigma_{<i}\sigma_i}$, instead of replacing it by the supremum of the integrand. In the sequel, we use the notation $\bar{\sigma}_i = 1 - \sigma_i$.

Let $\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}$ be a coupling of the conditional probabilities $\mu_{\sigma_{<i}\bar{\sigma}_i}$ and $\mu_{\sigma_{<i}\sigma_i}$. We measure its "quality" by the quantity

$$\Theta(j) := \sup_{\sigma} \sup_i \left(\int \mathbb{1}_{\{\xi_{i+j}^1 \neq \xi_{i+j}^2\}} \mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}(d\xi_{>i}^1 d\xi_{>i}^2) \right), \quad \forall j \in \mathbb{N}. \quad (36)$$

Typically, for one-dimensional Gibbs measures, we expect $\Theta(j)$ to be small for j large. Indeed, if we are far from the boundary, the boundary condition is not felt and we can couple successfully for different boundary conditions.

We state our result in terms of a summability condition for Θ and hereafter show that this condition is satisfied for the long-range Ising model. In the following theorem, by "interaction" we mean an interaction in the sense

of Definition 2.1. In particular, it is translation-invariant and uniformly summable.

Moreover, we need to assume the following condition on the interaction:

$$\sum_{A \ni 0} \text{diam}(A) \|U(A, \cdot)\|_\infty < \infty. \quad (37)$$

Notice that this implies that there is a unique Gibbs measure for U . This condition is a bit stronger than the classical condition found in [8, Chapter 8, Section 8.3].

In order to assure the existence of a coupling that leads to the Poincaré inequality, we will also need the following stronger condition. There exists $\alpha > 3$ and $C > 0$ such that for all m :

$$\sum_{A \ni 0, \text{diam}(A) > m} \|U(A, \cdot)\|_\infty \leq \frac{C}{m^\alpha}. \quad (38)$$

THEOREM 5.1. *Let U be an interaction on \mathbb{Z} satisfying condition (37). If there exists a coupling $\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}$ of the conditional probabilities $\mu_{\sigma_{<i}\bar{\sigma}_i}$ and $\mu_{\sigma_{<i}\sigma_i}$ such that, if*

$$\sum_{j \geq 1} (\Theta(j))^{1/q} < \infty. \quad (39)$$

for some $q > 2$, then the Gibbs measure associated to U satisfies the Poincaré inequality (11). If the interaction U of the Gibbs measure μ satisfies (38), then such a coupling exists.

PROOF. We will prove the Poincaré inequality under the condition (39). The existence of a coupling satisfying this condition under (38) is proved in the appendix.

One starts with the telescoping identity

$$f(\sigma) - \mathbb{E}_\mu(f) = \sum_i V_i(\sigma)$$

where

$$V_i = \mathbb{E}_\mu(f | \mathcal{F}_{\leq i}) - \mathbb{E}_\mu(f | \mathcal{F}_{< i}).$$

Then, estimate

$$|V_i(\sigma)| = |V_i(\sigma_{\leq i})| \leq \left| \int f(\sigma_{<i}\sigma_i\xi_{>i}) \mu_{\sigma_{<i}\sigma_i}(d\xi_{>i}) - \int f(\sigma_{<i}\sigma_i\xi_{>i}) \mu_{\sigma_{<i}\bar{\sigma}_i}(d\xi_{>i}) \right|$$

and telescope further to obtain

$$|V_i(\sigma)| \leq \int \mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}(d\xi_{>i}^1 d\xi_{>i}^2) \times \left(|\nabla_i f(\sigma_{<i}\sigma_i \xi_{>i}^2)| + \sum_{j \geq i+1} \mathbb{1}_{\{\xi_j^2 \neq \xi_j^1\}} \left| \nabla_j f(\sigma_{<i}\bar{\sigma}_i \xi_{(i+1,j)}^1 \xi_j^2 \xi_{(j+1,\infty)}^2) \right| \right).$$

To alleviate notations we set, for $j \geq i+1$,

$$(\sigma\xi)_{i,j}^{1,2} := \sigma_{<i}\bar{\sigma}_i \xi_{(i+1,j)}^1 \xi_j^2 \xi_{(j+1,\infty)}^2$$

then we can rewrite

$$\begin{aligned} V_i(\sigma) &\leq \int |\nabla_i f(\sigma_{\leq i} \xi_{>i})| \mu_{\sigma_{<i}\sigma_i}(d\xi_{>i}) \\ &+ \int \sum_{j \geq 1} \mathbb{1}_{\{\xi_{i+j}^1 \neq \xi_{i+j}^2\}} |\nabla_{i+j} f((\sigma\xi)_{i,j}^{1,2})| \mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}(d\xi_{>i}^1 d\xi_{>i}^2). \end{aligned} \quad (40)$$

Apply Hölder's inequality with $1 < p < 2$ in the second term of (40) to estimate

$$\begin{aligned} V_i(\sigma) &\leq \int |\nabla_i f(\sigma_{\leq i} \xi_{>i})| \mu_{\sigma_{<i}\sigma_i}(d\xi_{>i}) \\ &+ \sum_{j \geq 1} \Theta^{1/q}(j) \left(\int |\nabla_{i+j} f((\sigma\xi)_{i,j}^{1,2})|^p \mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}(d\xi_{>i}^1 d\xi_{>i}^2) \right)^{1/p} \end{aligned} \quad (41)$$

where Θ is defined in (36).

We denote by $\mu_{\sigma}^{ij}(d\xi_{>i})$ the distribution of $((\sigma\xi)_{i,j}^{1,2})_{>i}$ under the measure $\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}(d\xi_{>i}^1 d\xi_{>i}^2)$ (where the dependence on σ is in fact only on $\sigma_{\leq i}$). Further we denote

$$R_{\sigma}^{ij}(\eta_{>i}) = \frac{d\mu_{\sigma}^{ij}}{d\mu_{\sigma_{<i}\bar{\sigma}_i}}(\eta_{>i}).$$

With this notation, we rewrite (41)

$$\begin{aligned} V_i(\sigma) &\leq \int |\nabla_i f(\sigma_{\leq i} \xi_{>i})| \mu_{\sigma_{<i}\sigma_i}(d\xi_{>i}) \\ &+ \sum_{j \geq 1} \Theta^{1/q}(j) \left(\int |\nabla_{i+j} f(\sigma_{<i}\bar{\sigma}_i \eta_{>i})|^p R_{\sigma}^{ij}(\eta_{>i}) \mu_{\sigma_{<i}\bar{\sigma}_i}(d\eta_{>i}) \right)^{1/p}. \end{aligned} \quad (42)$$

The following lemma tells us that we can find a coupling $\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}$ such that we have a uniform control on $R_{\sigma}^{ij}(\eta_{>i})$.

LEMMA 5.1. *Under the assumption (37), there exists a coupling $\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}$ such that*

$$R_{\sigma}^{ij}(\eta_{>i}) \leq C \quad (43)$$

for some constant $C > 0$ only depending on U .

The proof of this lemma is given in the appendix. It uses the classical so-called ‘‘house of cards coupling’’, which under the stronger condition (38) will also satisfy (39).

Using lemma 5.1 we proceed to rewrite (42)

$$\begin{aligned} V_i(\sigma) &\leq \int |\nabla_i f(\sigma_{\leq i} \xi_{>i})| \mu_{\sigma_{<i}\sigma_i}(d\xi_{>i}) \\ &+ C^{1/p} \sum_{j \geq 1} \Theta^{1/q}(\sigma_{\leq i}) \left(\int |\nabla_{i+j} f(\sigma_{<i}\bar{\sigma}_i \eta_{>i})|^p \mu_{\sigma_{<i}\bar{\sigma}_i}(d\eta_{>i}) \right)^{1/p}. \end{aligned}$$

Introduce

$$\Xi_{\sigma}(i, k) = \left(\int |\nabla_k f(\sigma_{<i}\bar{\sigma}_i \eta_{>i})|^p \mu_{\sigma_{<i}\bar{\sigma}_i}(d\eta_{>i}) \right)^{1/p}$$

then, using Cauchy-Schwarz’s inequality, we have

$$\begin{aligned} V_i^2(\sigma) &\leq 2 \int (\nabla_i f)^2(\sigma_{<i}\sigma_i \xi_{>i}) \mu_{\sigma_{<i}\sigma_i}(d\xi_{>i}) \\ &+ 2C^{2/p} \left(\sum_{j \geq 1} \Xi_{\sigma}(i, i+j) \Theta^{1/q}(j) \right)^2 \\ &\leq 2 \int (\nabla_i f)^2(\sigma_{<i}\sigma_i \xi_{>i}) \mu_{\sigma_{<i}\sigma_i}(d\xi_{>i}) \\ &+ 2C^{2/p} \left(\sum_{j \geq 1} (\Xi_{\sigma}(i, i+j))^2 \Theta^{1/q}(j) \right) \sum_{j \geq 1} \Theta^{1/q}(j). \end{aligned}$$

Now use Jensen’s inequality, remembering that $2/p > 1$, to estimate

$$\begin{aligned} \Xi_{\sigma}(i, k)^2 &= \left(\int |\nabla_k f(\sigma_{<i}\bar{\sigma}_i \eta_{>i})|^p \mu_{\sigma_{<i}\bar{\sigma}_i}(d\eta_{>i}) \right)^{2/p} \\ &\leq \int (\nabla_k f(\sigma_{<i}\bar{\sigma}_i \eta_{>i}))^2 \mu_{\sigma_{<i}\bar{\sigma}_i}(d\eta_{>i}). \end{aligned} \quad (44)$$

Integrating w.r.t. μ then gives, using (6),

$$\begin{aligned} \int \Xi_{\sigma}(i, k)^2 d\mu &\leq \int (\nabla_k f(\sigma_{<i}\bar{\sigma}_i \eta_{>i}))^2 \mu_{\sigma_{<i}\bar{\sigma}_i}(d\eta_{>i}) \mu(d\sigma) \\ &\leq C' \int (\nabla_k f(\sigma_{<i}\bar{\sigma}_i \sigma_{>i}))^2 \mu(d\sigma). \end{aligned} \quad (45)$$

Combining now (45) with (44) we arrive at the estimate

$$\begin{aligned}
\text{Var}_\mu(f) &= \sum_i \int V_i^2 d\mu \\
&\leq 2 \sum_i \int (\nabla_i f)^2 d\mu \\
&\quad + 2C'' \|\Theta^{1/q}\|_1 \sum_i \sum_{j \geq 1} \Theta^{1/q}(j) \int (\nabla_{i+j} f)^2 d\mu \quad (46)
\end{aligned}$$

where $C''' = C^{2/p} C''$. Putting

$$\Upsilon(k) = \int (\nabla_k f)^2 d\mu \quad \text{and} \quad \Theta'_q(j) = \Theta^{1/q}(j) \mathbb{1}_{\{j \geq 1\}}$$

we can rewrite and estimate the double sum in (46), using Young's inequality,

$$\sum_i \sum_{j \geq 1} \Theta^{1/q}(j) \int (\nabla_{i+j} f)^2 d\mu = \|\Upsilon * \Theta'_q\|_1 \leq \|\Upsilon\|_1 \|\Theta^{1/q}\|_1$$

which finally yields

$$\text{Var}_\mu(f) \leq 2(1 + C'' \|\Theta^{1/q}\|_1^2) \mathcal{E}(f, f).$$

The proof of the theorem is complete. \square

REMARK 5.1. *It is clear that there could be many couplings satisfying the conclusion of Lemma 5.1. The trivial example is the independent (product) coupling. But then we want a coupling having the property (39).*

REMARK 5.2. *In the case of product measures, the second term in (40) is absent since in that case we can perfectly couple the conditional distributions, i.e., for all $j > i$*

$$\int \mathbb{1}_{\{\xi_j^1 \neq \xi_j^2\}} \mu_{\sigma_{<i} \bar{\sigma}_i, \sigma_{<i} \sigma_i} (d\xi_{>i}^1 d\xi_{>i}^2) = 0.$$

So we obtain the estimate

$$V_i(\sigma) \leq \int \nabla_i f(\xi) \mu_{\sigma_{<i} \sigma_i} (d\xi)$$

and using Cauchy-Schwarz's inequality one gets

$$\int V_i(\sigma)^2 \mu(d\sigma) \leq \int (\nabla_i f(\sigma_{\leq i} \xi_{>i}))^2 \mu_{\sigma_{\leq i}} (d\xi_{>i}) \mu(d\sigma) = \int (\nabla_i f)^2 d\mu,$$

which gives the Poincaré inequality for product measures:

$$\text{Var}_\mu(f) = \int \sum_i V_i^2 d\mu \leq \sum_i \int (\nabla_i f)^2 d\mu.$$

6 Appendix: the house of cards coupling

In this appendix we first show that the “house of cards coupling”, which is an explicit coupling of the conditional probabilities $\mu_{\sigma_{<i}\bar{\sigma}_i}$ and $\mu_{\sigma_{<i}\sigma_i}$, satisfies the estimate (43), under the uniqueness condition (37). Next, we show that under the condition (38), the coupling also satisfies (39).

6.1 Estimate of Lemma 5.1

The house of cards coupling of the conditional distributions $\mu_{\sigma_{<i}\bar{\sigma}_i}$ and $\mu_{\sigma_{<i}\sigma_i}$ runs as follows. We start by generating the symbols $(\sigma_{i+1}^1, \sigma_{i+1}^2)$ as the optimal coupling of the conditional distribution $\mu_{\sigma_{<i}\bar{\sigma}_i}(\cdot_{i+1})$ with $\mu_{\sigma_{<i}\sigma_i}(\cdot_{i+1})$. The symbols $(\sigma_{i+1}^1, \sigma_{i+1}^2)$ being generated, we generate $(\sigma_{i+2}^1, \sigma_{i+2}^2)$ as the optimal coupling of the conditional distributions $\mu_{\sigma_{<i}\bar{\sigma}_i\sigma_{i+1}^1}(\cdot_{i+2})$ with $\mu_{\sigma_{<i}\sigma_i\sigma_{i+1}^2}(\cdot_{i+2})$, etc.

Remark that at each stage where we generate new symbols, we simply couple optimally two probability measures on $\{0, 1\}$. More explicitly, if P_p gives mass p to $\{1\}$ and mass $1 - p$ to $\{0\}$, and Q gives mass q to $\{1\}$ and mass $1 - q$ to $\{0\}$, then the optimal coupling is gives mass $p \wedge q$ to $\{(1, 1)\}$, $p - p \wedge q$ to $\{(1, 0)\}$, $q - p \wedge q$ to $\{(0, 1)\}$ and $1 - p - q + p \wedge q$ to $\{(0, 0)\}$.

Abbreviate $\Lambda_j = (i, i + j]$, $\Lambda_{j,N} = (i + j + 1, N]$, and the coupling $\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i} =: \mu_{\mathbf{i}}^\sigma$. We now have to estimate the ratio

$$\frac{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}, \sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}})}{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \eta_{\Lambda_j}, \sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}})} \quad (47)$$

uniformly in σ, η, j, N .

$$\begin{aligned} & \frac{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}, \sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}})}{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \eta_{\Lambda_j}, \sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}})} \\ &= \frac{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}} | \sigma_{\Lambda_j}^1 = \eta_{\Lambda_j})}{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}} | \sigma_{\Lambda_j}^2 = \eta_{\Lambda_j})} \\ &= \frac{\sum_{\zeta_{\Lambda_j}} \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \zeta_{\Lambda_j} | \sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}} | \sigma_{\Lambda_j}^1 = \eta_{\Lambda_j} \cap \sigma_{\Lambda_j}^2 = \zeta_{\Lambda_j})}{\sum_{\zeta_{\Lambda_j}} \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^1 = \zeta_{\Lambda_j} | \sigma_{\Lambda_j}^2 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}} | \sigma_{\Lambda_j}^2 = \eta_{\Lambda_j} \cap \sigma_{\Lambda_j}^1 = \zeta_{\Lambda_j})}. \end{aligned}$$

From the construction of the coupling, we have the following “consistency” property

$$\begin{aligned} \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}} | \sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}, \sigma_{\Lambda_j}^2 = \zeta_{\Lambda_j}) &= \mu_{\sigma_{<i}\bar{\sigma}_i\eta_{\Lambda_{j,N}}, \sigma_{<i}\sigma_i\zeta_{\Lambda_j}}(\sigma_{\Lambda_{j,N}}^2 = \eta_{\Lambda_{j,N}}) \\ &= \mu_{\sigma_{<i}\sigma_i\zeta_{\Lambda_j}}(\eta_{\Lambda_{j,N}}) \end{aligned} \quad (48)$$

where the last line follows simply because $\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i, \eta_{\Lambda_j, N}, \sigma_{<i}\sigma_i, \zeta_{\Lambda_j}}$ is a coupling of $\mu_{\sigma_{<i}\bar{\sigma}_i, \eta_{\Lambda_j, N}}$ and $\mu_{\sigma_{<i}\sigma_i, \zeta_{\Lambda_j}}$. Now we use that under the uniqueness condition (37) on the potential U of the one-dimensional Gibbs measure μ , we have the uniform estimate (see e.g. [8])

$$\frac{\mu_{\sigma_{<i}\sigma_i, \zeta_{\Lambda_j}}(\eta_{\Lambda_j, N})}{\mu_{\sigma_{<i}\sigma_i, \eta_{\Lambda_j}}(\eta_{\Lambda_j, N})} \leq C. \quad (49)$$

So we obtain, combining the previous estimates with (49), that

$$\begin{aligned} & \frac{\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}(\sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}, \sigma_{\Lambda_j, N}^2 = \eta_{\Lambda_j, N})}{\mu_{\sigma_{<i}\bar{\sigma}_i, \sigma_{<i}\sigma_i}(\sigma_{\Lambda_j}^2 = \eta_{\Lambda_j}, \sigma_{\Lambda_j, N}^2 = \eta_{\Lambda_j, N})} \\ & \leq C \frac{\sum_{\zeta_{\Lambda_j}} \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^1 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \zeta_{\Lambda_j} | \sigma_{\Lambda_j}^1 = \eta_{\Lambda_j})}{\sum_{\zeta_{\Lambda_j}} \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \eta_{\Lambda_j}) \mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \zeta_{\Lambda_j} | \sigma_{\Lambda_j}^2 = \eta_{\Lambda_j})} \\ & = C \frac{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^1 = \eta_{\Lambda_j})}{\mu_{\mathbf{i}}^\sigma(\sigma_{\Lambda_j}^2 = \eta_{\Lambda_j})} \\ & = C \frac{\mu_{\sigma_{<i}\bar{\sigma}_i}(\eta_{\Lambda_j})}{\mu_{\sigma_{<i}\sigma_i}(\eta_{\Lambda_j})} \leq C^2. \end{aligned} \quad (50)$$

6.2 The behavior of Θ for the house of cards process

We now specify the relation between the decay of $\Theta(j)$ and the decay of the potential of the one-dimensional Gibbs measure. The coupling of $\mu_{\sigma_{<i}\bar{\sigma}_i}$ and $\mu_{\sigma_{<i}\sigma_i}$ is as in the previous subsection, via sequentially generating the symbols $\sigma_{>i}^1, \sigma_{>i}^2$ by iteratively using the optimal coupling of the conditional distributions of the next symbol given the symbols already generated.

The crucial quantity appearing in [4] which is used to compare with a house of cards process (*i.e.*, a Markov chain with state space $\mathbb{N} \cup \{0\}$ which can go up by one unit or go down to zero in a single time step) is

$$\inf_{a, \sigma, \eta: \sigma_{\{-m, \dots, m\}} = \eta_{\{-m, \dots, m\}}} \frac{\mu(\sigma_0 = a | \sigma_{\mathbb{Z} \setminus \{0\}})}{\mu(\sigma_0 = a | \eta_{\mathbb{Z} \setminus \{0\}})} \geq 1 - \gamma_m.$$

The house of cards process is then the Markov chain $\{Z_n : n \in \mathbb{N}\}$ on \mathbb{N} with transition probabilities

$$\mathbb{P}(Z_{n+1} = m + 1 | Z_n = m) = 1 - \gamma_m = 1 - \mathbb{P}(Z_{n+1} = 0 | Z_n = m).$$

The chain Z_n dominates the process counting the number of matches in the optimal coupling of $\mu_{\sigma_{<i}\bar{\sigma}_i}$ and $\mu_{\sigma_{<i}\sigma_i}$. The transience of Z_n is thus sufficient

to have a successful coupling. More precisely, we have the following relation between Θ and the return probabilities of the house of cards process:

$$\Theta(k) \leq \sum_{l=k}^{\infty} \mathbb{P}(Z_l = 0). \quad (51)$$

If we have $\gamma_m \leq m^{-\alpha}$, then the corresponding return probabilities satisfy $\mathbb{P}(Z_m = 0) \leq Cm^{-\alpha}$, and if $\gamma_m \leq e^{-\alpha m}$, then also $\mathbb{P}(Z_m = 0) \leq Ce^{-\alpha m}$.

To estimate γ_m in terms of the potential U of the Gibbs measure $\mu \in \mathcal{G}(U)$, we proceed as follows. Let $\sigma, \sigma' \in \Omega$ be such that $\sigma_{\{-m, \dots, m\}} = \sigma'_{\{-m, \dots, m\}}$, then

$$\begin{aligned} \log \frac{\mu(\sigma_0 = a | \sigma_{\mathbb{Z} \setminus \{0\}})}{\mu(\sigma_0 = a | \sigma'_{\mathbb{Z} \setminus \{0\}})} &\leq \sup_a |H_{\{0\}}^{\sigma}(a) - H_{\{0\}}^{\sigma'}(a)| \\ &\leq \sum_{A \ni 0, \text{diam}(A) \geq m} \|U(A, \cdot)\|_{\infty} \end{aligned}$$

which gives an upper bound for γ_m

$$\gamma_m \leq \exp \left(\sum_{A \ni 0, \text{diam}(A) \geq m} \|U(A, \cdot)\|_{\infty} \right) - 1. \quad (52)$$

To satisfy condition (39) it is sufficient, according to (51), to have

$$\sum_{k=1}^{\infty} \left(\sum_{m \geq k} \mathbb{P}(Z_m = 0) \right)^{\frac{1}{q}} < \infty \quad (53)$$

for some $q > 2$.

Therefore if there exists $\alpha > 3$ such that for all m large enough

$$\sum_{A \ni 0, \text{diam}(A) \geq m} \|U(A, \cdot)\|_{\infty} \leq \frac{C}{m^{\alpha}} \quad (54)$$

then there exists $q > 2$ such that (53) is satisfied.

As an example, take the *long-range Ising model* with interaction

$$U(\{i, j\}, \sigma) = \frac{\beta(2\sigma_i - 1)(2\sigma_j - 1)}{|i - j|^{\kappa}}$$

for $i \neq j \in \mathbb{Z}$, and $U(A, \sigma) = 0$ for all other subsets $A \subset \mathbb{Z}$, and where $\beta \in \mathbb{R}$. It is immediate to check that this interaction satisfies (37) for all $\kappa > 2$ and for all β . Using (52), we can choose

$$\gamma_m = C \sum_{k \geq m} \frac{1}{k^\kappa} \sim C' \frac{1}{m^{\kappa-1}}.$$

Therefore, combining (53), we conclude that (39) holds for all $\kappa > 4$.

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