On the irrationality of polynomial Cantor series

Jaroslav Hančl* and Robert Tijdeman

Abstract

In this paper we prove some general results which imply, for example, the irrationality of $\sum_{N=1}^{\infty} \frac{[N^\alpha]}{N!}$ $(\alpha \geq 0)$, $\sum_{N=1}^{\infty} \frac{[N \log^\beta N]}{N!}$ $(\beta \in \mathbb{R})$ and $\sum_{N=1}^{\infty} \frac{[N^\alpha \exp(\log^\beta N)]}{N!}$ $(\alpha \geq 0, 0 < \beta < 1)$. Moreover, we prove that such numbers are linearly independent over the rationals. The method of proof is an extension of the approach used in earlier works of the authors and Pingzhi Yuan.

1 Introduction

Let $(b_n)_{n=1}^{\infty}$ be a sequence of integers. In the present paper we study the irrationality of $R := \sum_{n=1}^{\infty} \frac{b_n}{n!}$ and, more generally, of $R^* := \sum_{N=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N}(an+b)}$ where a and b are given positive integers. In 1766 Lambert [11] proved the irrationality of $e = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}$. In 1873 Hermite [10] established the transcendence of e which implies the irrationality of $\sum_{n=1}^{\infty} \frac{m^n}{n!}$ for any nonzero integer m. In 1869 G. Cantor [2] showed that if $0 \leq b_n < n$, then R is irrational if and only if $b_n > 0$ infinitely often and $b_n < n-1$ infinitely often. On the other hand, if $\frac{b_n}{n-1}$ is constant for n larger than some n_0, then $R \in \mathbb{Q}$. This is an exceptional case in many results. Oppenheim [13] showed that both the condition on $b_n > 0$ and the condition on $b_n < n-1$ can be relaxed. For example, it follows from his results that if $|b_n| < n$ for every n, then R is rational if and only if $\frac{b_n}{n-1}$ is ultimately a fixed integer. Thus if $|b_n| < n-1$ for every n and $R \in \mathbb{Q}$, then b_n is ultimately equal to 0. The results of Oppenheim were extended by the authors [8] who showed that if $n \nmid b_n$ for all n, $b_n = o(n^2)$ and $\liminf_{n\to\infty} \frac{|b_n|}{n} = 0$, then R is irrational. They further proved that R is irrational if $(b_n)_{n=1}^{\infty}$ is a monotonic sequence of positive integers such that $b_n = O(n^2)$ and $\gcd(b_n, n-1) = o(b_n)$. Tijdeman and Yuan [17] extended another result of Oppenheim by showing that R is irrational if $b_n = O(n)$ and the sequence $(\frac{b_n}{n})_{n=1}^{\infty}$ has an irrational limit point. See also Hančl [6] and [7].

*Supported by the grants no. 201/04/0381 and MSM6198898701
Mathematical subject classifitation: 11J72.
Key words and phrases: Irrationality, linear independence, infinite series, factorial series.

Erdős and Straus [5] started a series of results in which the size of the difference $b_{n+1} - b_n$ is a relevant factor. They used such results to establish the irrationality of R in case $(b_n)_{n=1}^{\infty}$ represents a multiplicative or other arithmetic function. It follows from their result that if $b_n > 0$ for all n, $b_{n+1} - b_n = o(n)$ and $\liminf_{n\to\infty} \frac{n}{b_n} = 0$, then R is irrational. The authors [8] showed that the condition $\liminf_{n\to\infty} \frac{n}{b_n} = 0$ can be replaced with the necessary condition that $\frac{b_n}{n-1}$ is not ultimately constant. Tijdeman and Yuan [17] showed that, moreover, the condition $b_n > 0$ for all n can be dropped: if $b_{n+1} - b_n = o(n)$, then $R \in \mathbb{Q}$ if and only if $\frac{b_n}{n-1}$ is ultimately a fixed integer. These results generalize Erdős' result [3] that $\sum_{n=1}^{\infty} \frac{p_n}{n!} \notin \mathbb{Q}$, where $\{p_n\}_{n=1}^{\infty}$ is the sequence of consecutive prime numbers. In fact Erdős claimed the irrationality of $\sum_{n=1}^{\infty} \frac{p_n^k}{n!} \notin \mathbb{Q}$ for $k = 1, 2, \ldots$, but unfortunately he proved only the case $k = 1$. Oppenheim [13] showed that $\sum_{n=1}^{\infty} \frac{\epsilon_n d_n}{n!}, \sum_{n=1}^{\infty} \frac{\epsilon_n \sigma_n}{n!}$ and $\sum_{n=1}^{\infty} \frac{\epsilon_n \phi_n}{n!}$ are irrational for all choices of $\epsilon_n \in \{-1, 1\}$, where $d(n), \sigma(n), \phi(n)$ denote the number of divisors, the sums of divisors, and the Euler function of n, respectively. A special case was treated by Erdős and Kac [4]. Erdős and Straus [5] proved that the numbers $1, \sum_{n=1}^{\infty} \frac{\sigma_n}{n!}, \sum_{n=1}^{\infty} \frac{\phi_n}{n!}$ and $\sum_{n=1}^{\infty} \frac{b_n}{n!}$, where $|b_n| < n^{\frac{1}{2}-\epsilon}$ for all large n and $b_n \neq 0$ infinitely often, are linearly independent over the rationals. Most of the mentioned results were stated in greater generality in the original papers than above.

Tijdeman and Yuan [17] started to compare second order differences (cf. the proof of their Theorem 4.3). In the present paper we pursue this idea by studying K-th order differences. For doing so we have to impose stronger regularity conditions on the numbers b_n. Nevertheless the results are valid for a wide class of sequences $(b_n)_{n=1}^{\infty}$. Corollary 3.1 precisely states for which polynomials $P(x)$ with integer coefficients $\sum_{n=1}^{\infty} \frac{P(n)}{n!}$ is rational. Section 3 further provides a method to establish the irrationality of a large class of numbers $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)}$ where $f(N)$ is an integer-valued function satisfying $f(N) = (aN + b)F(N) + O(1)$ and F is a smooth function which does not grow faster than a polynomial. In particular it yields the irrationality of the following numbers:

$$\sum_{n=1}^{\infty} \frac{[\gamma n^{\alpha}]}{n!} \text{ for } \alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z}, \gamma \in \mathbb{R}_+, \quad \sum_{n=1}^{\infty} \frac{[\log n]}{n!}, \quad \sum_{n=1}^{\infty} \frac{[\exp(\log^{\frac{1}{2}} n)]}{n!}.$$

In Section 4 the linear independence over the rationals of such numbers is treated. For example, linear independence is shown for the numbers

$$1, \ e \text{ and } \sum_{n=1}^{\infty} \frac{[n^{\alpha}]}{n!} \ \text{ for all } \ \alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z}.$$

The results remind us of the result by Loxton and van der Poorten [12] who proved by Mahler's method that $\sum_{n=1}^{\infty} [n\alpha]\beta^n$ is transcendental for α irrational and β algebraic with $0 < |\beta| < 1$.

2 An irrationality criterion for Polynomial Cantor sums

We consider sums of the form

$$S=\sum_{N=0}^{\infty}\frac{P(N)}{\prod_{n=0}^{N}Q(n)}$$

where $P(x)\in\mathbb{Z}[x]$, $Q(x)\in\mathbb{Z}[x]$, $P\not\equiv 0$, the denominators do not vanish, and the sum converges. If Q is a constant then $Q(n)$ is an integer a for all n and $S=\sum_{N=0}^{\infty}\frac{P(N)}{a^{N+1}}\in\mathbb{Q}$ (cf. [16] Section 4.1). In the sequel we assume that Q is non-constant. If $\deg P<\deg Q$, then the irrationality of S is an immediate consequence of results of Oppenheim [13].

Theorem 2.1 *(Oppenheim) Let $P(x)\in\mathbb{Q}[x]$, $Q(x)\in\mathbb{Z}[x]$, $P,Q\not\equiv 0$, $\deg P<\deg Q$ such that $Q(n)\neq 0$ for all positive integer n. Then*

$$S=\sum_{N=0}^{\infty}\frac{P(N)}{\prod_{n=0}^{N}Q(n)}\notin\mathbb{Q}.$$

Proof. Without loss of generality we may assume that $P(x)\in\mathbb{Z}[x]$. If the leading coefficient of Q is positive, the theorem follows immediately from Oppenheim [13], Theorem 4. If the leading coefficient of Q is negative, one can immediatelly apply Theorem 8 of [13]. □

Suppose that $\deg P\geq\deg Q$ We write

$$P(x)=A_0(x)+A_1(x)Q(x)+A_2(x)Q(x)Q(x-1)+\cdots+A_t(x)Q(x)Q(x-1)\cdots Q(x-t) \tag{1}$$

with $A_j(x)\in\mathbb{Q}[x]$, $\deg(A_j)<\deg(Q)$ for $j=0,1,\cdots,t$ and $A_t\not\equiv 0$. Let A be a common denominator of the coefficients of $A_0,A_1,\cdots,A_t$. Hence

$$S=\sum_{j=0}^{t}\sum_{N=0}^{\infty}\frac{A_j(N)Q(N)\cdots Q(N-j)}{\prod_{n=0}^{N}Q(n)}=$$

$$\sum_{j=0}^{t}\sum_{N=0}^{j-1}A_j(N)Q(-1)\cdots Q(N-j)+\sum_{j=0}^{t}\sum_{N=j}^{\infty}\frac{A_j(N)}{\prod_{n=0}^{N-j-1}Q(n)}=$$

$$\sum_{N=0}^{t}\sum_{j=N+1}^{t}A_j(N)Q(-1)\cdots Q(N-j)+\sum_{j=0}^{t}\sum_{N=0}^{\infty}\frac{A_j(N+j)}{\prod_{n=0}^{N-1}Q(n)}.$$

We conclude that

$$S=\sum_{N=0}^{t}\sum_{j=1}^{t-N}A_{N+j}(N)Q(-1)\cdots Q(-j)+\sum_{N=0}^{\infty}\frac{1}{\prod_{n=0}^{N-1}Q(n)}\sum_{j=0}^{t}A_j(N+j). \tag{2}$$

The first term on the right-hand side is a rational number. To the second term on the right-hand side we can apply Theorem 2.1. In view $\deg(\sum_{j=0}^{t} A_j(x+j)) < \deg(Q(x))$, by Theorem 2.1, $S \in \mathbb{Q}$ if and only if the polynomial $\sum_{j=0}^{t} A_j(x+j)$ vanishes for all large integer values of x, i.e. is identically zero. This yields the following general result.

Theorem 2.2 *Let $P(x), Q(x) \in \mathbb{Z}[x]$, $P \not\equiv 0$, Q non-constant, such that*

$$S = \sum_{N=0}^{\infty} \frac{P(N)}{\prod_{n=0}^{N} Q(n)}$$

is a well-defined real number. Define t, $A_0(x), \cdots, A_t(x) \in \mathbb{Q}[x]$ by

$$P(x) = A_0(x)+A_1(x)Q(x)+A_2(x)Q(x)Q(x-1)+\cdots+A_t(x)Q(x)Q(x-1)\cdots Q(x-t)$$

with $\deg(A_j) < \deg(Q)$ for $j = 0, 1, \cdots, t$ and $A_t \not\equiv 0$. Then

$$S \in \mathbb{Q} \iff \sum_{j=0}^{t} A_j(x+j) \equiv 0. \qquad (3)$$

Moreover, if Q is monic and $S \in \mathbb{Q}$, then $S \in \mathbb{Z}$.

Remark 2.1 *Observe that if $P(x)$ has rational coefficients, we can multiply P and S by the lowest common multiple of the coefficients of P to get a polynomial with integer coefficients and apply Theorem 2.2.*

Remark 2.2 *If Q is linear, then all the polynomials A_j are rational constants and $S \in \mathbb{Q} \iff \sum_{j=0}^{t} A_j = 0$. It follows from a general theorem of Shidlovski [15] that if $\sum_{j=0}^{t} A_j \neq 0$, then S is transcendental.*

Proof. (of Theorem 2.2). The first statement has already been proved. If Q is monic, then $A_0(x), \cdots, A_t(x) \in \mathbb{Z}[x]$. Hence $A = 1$. Using this we obtain, by (3),

$$S \in \mathbb{Q} \Rightarrow S \in \mathbb{Z} + \sum_{N=0}^{\infty} \frac{1}{\prod_{n=0}^{N-1} Q(n)} \sum_{j=0}^{t} A_j(N+j) = \mathbb{Z}$$

□

Example 2.1 *Let $P(x) = p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0 \in \mathbb{Z}[x]$, $Q(x) = q_2x^2 + q_1x + q_0 \in \mathbb{Z}[x]$ with $q_2 \neq 0$. Then $A_2(x) = \frac{p_4}{q_2^2}$,*

$$A_1(x) = \frac{1}{q_2^3}\{(p_3q_2^2-2p_4q_1q_2+2p_4q_2^2)x+(p_2q_2^2-p_3q_1q_2+p_4q_1^2+p_4q_1q_2-2p_4q_0q_2-p_4q_2^2)\},$$

$$A_0(x) = \frac{1}{q_2^3}(p_1 q_2^3 - p_2 q_1 q_2^2 - p_3 q_0 q_2^2 + p_3 q_1^2 q_2 + 2p_4 q_0 q_1 q_2 - p_4 q_1^3)x +$$
$$\frac{1}{q_2^3}(p_0 q_2^3 - p_2 q_0 q_2^2 + p_3 q_0 q_1 q_2 + p_4 q_0^2 q_2 - p_4 q_0 q_1^2).$$

Hence $S \in \mathbb{Q}$ if and only if

$$p_1 q_2^3 - p_2 q_1 q_2^2 - p_3 q_0 q_2^2 + p_3 q_1^2 q_2 + p_3 q_2^2 + 2p_4 q_0 q_1 q_2 - 2p_4 q_1 q_2 - p_4 q_1^3 + 2p_4 q_2^2 = 0$$

and

$$p_0 q_2^3 - p_2 q_0 q_2^2 + p_2 q_2^2 + p_3 q_0 q_1 q_2 - p_3 q_1 q_2 + p_3 q_2^2 + p_4 q_0^2 q_2 - p_4 q_0 q_1^2$$
$$-2p_4 q_0 q_2 + p_4 q_1^2 - p_4 q_1 q_2 + 2p_4 q_2 + p_4 q_2^2 = 0.$$

Moreover, according to the proof of Theorem 2.2, if $S \in \mathbb{Q}$, then $q_2^3 S \in \mathbb{Z}$.

The special case $p_4 = q_1 = 0$ has been treated in [16]. The even more special case $p_4 = p_2 = p_1 = q_1 = 0$, $p_3 \neq 0$ has been death with in [16]. As mentioned in [9] the condition there should be read as $c = 1$ and $b + ad = 0$. In accordance with the above conditions.

Remark 2.3 *The case of an alternating sum $\sum_{N=0}^{\infty}(-1)^N \frac{P(N)}{\prod_{n=0}^{N} Q(n)}$ can be reduced to the above sum by replacing Q by $-Q$.*

3 Some sufficient conditions for irrationality

This section deals with sums of the type

$$S = \sum_{N=1}^{\infty} \frac{P(n)}{\prod_{n=1}^{N} Q(n)}$$

for $P(x), Q(x) \in \mathbb{Q}[x]$, Q is non-constant such that $P(n), Q(n) \in \mathbb{Z}$ and $Q(n) \neq 0$ for all $n \in \mathbb{N}$. We use the convention that an empty product equals 1. We shall give sufficient conditions for the irrationality of S. The following theorem will be our starting point.

Theorem 3.1 *Let $P(x) \in \mathbb{Q}[x]$, $P \not\equiv 0$, $Q(x) \in \mathbb{Z}[x]$, Q is non-constant and $Q(n) \neq 0$ for all positive integers n. Let $\mathcal{R}$ be a class of polynomials with integral coefficients. Suppose that Q has the property that if $R \in \mathcal{R}$, $R \not\equiv 0$ then there exist $A(x), B(x) \in \mathbb{Q}[x]$ such that*

$$R(x) = A(x)Q(x) + B(x),$$

$\deg(B) < \max(\deg(Q), \deg(R))$ *and $A(x+1) + B(x) \in \mathcal{R}$, $A(x+1) + B(x) \not\equiv 0$. Then $P \in \mathcal{R}$ implies that then $S = \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} \notin \mathbb{Q}$.*

Proof. If $\deg(P) < \deg(Q)$, then we can apply Theorem 2.1. (Here the supposition is irrelevant.) Suppose $\deg(P) \geq \deg(Q)$. By our assumption there exists $A(x), B(x) \in \mathbb{Q}[x]$ such that

$$P(x) = A(x)Q(x) + B(x)$$

with $\deg(B) < \max(\deg(Q), \deg(R))$ and $A(x+1)+B(x) \in \mathcal{R}$, $A(x+1)+B(x) \not\equiv 0$. Futhermore

$$S = \sum_{N=1}^{\infty} \frac{A(N)Q(N) + B(N)}{\prod_{n=1}^{N} Q(n)} = A(1) + \sum_{N=1}^{\infty} \frac{A(N+1) + B(N)}{\prod_{n=1}^{N} Q(n)}.$$

If $\deg(A(x+1) + B(x)) < \deg(Q)$, then we can apply Theorem 2.1 to conclude that $S \notin \mathbb{Q}$ in view of $A(x+1) + B(x) \not\equiv 0$. If $\deg(A(x+1) + B(x)) \geq \deg(Q)$, then we can apply the procedure to $A(x+1) + B(x)$ in place of $P(x)$. After at most $\deg(P)$ iterations we will arrive at a situation where we can apply Theorem 2.1 to conclude that $S \notin \mathbb{Q}$. □

Example 3.1 *Suppose that* $\mathcal{R}$ *is a class of polynomials* $\sum_{i=0}^{T} r_i x^i \in \mathbb{Q}[x]$ *with non-negative coefficients* r_i *and* $Q(x) = \sum_{i=0}^{M} c_i x^i \in \mathbb{Z}[x]$*,* $c_M > 0$*,* $c_i < 0$ *for* $i = 0, 1, \cdots, M-1$*. Then* $R(x) \in \mathcal{R}$*,* $T \geq M$ *implies*

$$R(x) = \frac{r_T}{c_M} x^{T-M} Q(x) + (\sum_{i=0}^{T-1} r_i x^i + \frac{r_T}{c_M}(c_{M-1}x^{T-1} + c_{M-2}x^{T-2} + \cdots + c_0 x^{T-M}))$$

such that the resulting polynomials $A(x)$*,* $B(x)$ *have non-negative coefficients. It follows that* $A(x+1) + B(x) \in \mathcal{R}$ *and* $A(x+1) + B(x) \not\equiv 0$*. If* $T < M$*, then we choose* $A = 0$*,* $B = R$*. Hence by Theorem 3.1 if* $P(x) \in \mathcal{R}$*,* $P \not\equiv 0$*, then* $S \notin \mathbb{Q}$*.*

Example 3.2 *Let* $\mathcal{R}$ *be the class as in Example 3.1 and* $Q(x) = x^2 - x + 1$*. Then* $R(x) \in \mathcal{R}$*,* $T \geq 2$ *implies*

$$R(x) = r_T x^{T-2} Q(x) + \sum_{i=0}^{T-1} r_i x^i + r_T(x^{T-1} - x^{T-2})$$

such that the resulting polynomial

$$A(x+1) + B(x) = r_T(x+1)^{T-2} + \sum_{i=0}^{T-1} r_i x^i + r_T(x^{T-1} - x^{T-2})$$

has non-negative coefficients and is non-trivial. Hence, by Theorem 3.1, if $P(x) \in \mathcal{R}$*,* $P \not\equiv 0$*, then* $S \notin \mathbb{Q}$*.*

Example 3.3 *Suppose that $\mathcal{R}$ is the same class as in previous examples and $Q(x) = x^2 + 1$. Then, if $\deg(R) \geq 4$,*

$$R(x) = r_T(x^{T-2} - x^{T-4})(x^2 + 1) + (\sum_{i=0}^{T-1} r_i x^i + c_T x^{T-4})$$

such that the resulting polynomial $B(x)$ has non-negative coefficients and is non-trivial. Moreover $A(x+1) = r_T((x+1)^{T-2} - (x+1)^{T-4})$ has also non-negative coefficients. It follows that $A(x+1)+B(x) \in \mathcal{R}$ and the coefficient of x is positive. If $2 \leq T < 4$, then, putting $r_3 = 0$ if $T = 2$,

$$R(x) = (r_3 x + r_2)(x^2 + 1) + ((r_1 - r_3)x + (r_0 - r_2))$$

and

$$A(x+1) + B(x) = r_1 x + (r_0 + r_3)$$

which implies that $A(x+1) + B(x) \in \mathcal{R}$, but vanishes if and only if $r_0 = r_1 = r_3 = 0$. Finally if $T < 2$, then we choose again $A = 0$, $B = R$. We conclude that if $P \in \mathcal{R}$, $P \not\equiv 0$, then $S \notin \mathbb{Q}$, unless $T = 2$, $r_1 = r_0 = 0$. In this exceptional case we have

$$S = \sum_{N=1}^{\infty} \frac{N^2}{\prod_{n=1}^{N}(n^2+1)} = \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N-1}(n^2+1)} - \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(n^2+1)} = 1$$

which is rational.

We shall generalize these examples in the remainder of this section. For $\mathcal{R}$ we shall take the class of polynomials $\sum_{i=0}^{T} r_i x^i \in \mathbb{Q}[x]$ with non-negative coefficients r_i. Observe that by considering $P(x + x_0)$ in place of $P(x)$ for a suitable integer $x_0 \geq 0$ we can always secure that the coefficients are non-negative. Of course we than have to consider

$$S = \sum_{N=1}^{\infty} \frac{P(N + x_0)}{\prod_{n=1}^{N} Q(n + x_0)}$$

so that Q is translated by the same x_0. In the first theorem we allow Q to be the product of polynomials from a certain class.

Theorem 3.2 *Let $P(x), Q(x) \in \mathbb{Q}[x]$, $P(x) = \sum_{i=0}^{T} a_i x^i$ with $a_T > 0$ and $a_i \geq 0$ for every $i = 0, 1, \ldots, T-1$, $Q(x) = \prod_{m=1}^{M} C_m(x)$ with $C_m(x) = \sum_{k=0}^{K_m} c_{m,k} x^k \in \mathbb{R}[x]$ for $m = 1, 2, \ldots, M$ such that $c_{m,M_m} > 0$, but $c_{m,k} \leq 0$ for $k = 0, 1, \ldots, K_m - 1$. Assume that $Q(x)$ is non-constant and $Q(n) \in \mathbb{Z}$, $Q(n) \neq 0$ for every positive integer n. Then*

$$S = \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} \notin \mathbb{Q}.$$

Proof. If $\deg P < \deg Q$, then from Theorem 2.1 we conclude that $S \notin \mathbb{Q}$ (even without using that Q is a special product).

Assume that $\deg P \geq \deg Q$. Then we obtain, by applying division with reminder,

$$\begin{aligned}
P(x) &= A_1(x)C_1(x) + B_1(x), && \deg B_1 < \deg A_1 \\
A_1(x) &= A_2(x)C_2(x) + B_2(x), && \deg B_2 < \deg A_2 \\
&\vdots \\
A_{M-1}(x) &= A_M(x)C_M(x) + B_M(x), && \deg B_M < \deg A_M.
\end{aligned}$$

Hence $P(x) = A(x)Q(x) + B(x)$ with $A(x) = A_1(x) \cdots A_M(x)$ and

$$B(x) = B_1(x) + B_2(x)C_1(x) + B_3(x)C_1(x)C_2(x) + \cdots + B_M(x)C_1(x) \ldots C_{M-1}(x).$$

So $\deg(B) < K_1 + \cdots + K_M = \deg(Q)$. Since $P(x) \in \mathcal{R}$, we find by induction on m that $A_m(x)$ and $B_m(x)$ have non-negative coefficients in view of

$$x^j = \frac{1}{c_{i,M_j}} x^{i-M_i} C_i(x) + \sum_{k=0}^{M_i-1} \frac{-c_{i,k}}{c_{i,M_j}} x^{i+k-M_i} \quad (i = 1, \cdots M_j, j = 1, 2, \cdots)$$

which is repeatedly used and yields non-negative coefficients, and therefore $A(x+1) + B(x)$ too. Thus $A(x+1) + B(x) \in \mathcal{R}$. Moreover, since $A(x)$ is nontrivial, $A(x+1) + B(x)$ is non-trivial too. As before

$$S = \sum_{N=1}^{\infty} \frac{A(N+1) + B(N)}{\prod_{n=1}^{N} Q(n)}.$$

We now repeat the above argument with $P(x)$ replaced by $A(x+1)+B(x)$ which has lower degree than $P(x)$. If $\deg(A(x+1) + B(x)) < \deg(Q)$, then we apply Theorem 2.1 to conclude $S \notin \mathbb{Q}$, otherwise we apply division with remainder. After at most $\deg(P)$ iterations we apply Oppenheim's Theorem 2.1 to conclude $S \notin \mathbb{Q}$. □

Example 3.4 *For every positive integer k the number*

$$\sum_{N=2}^{\infty} \frac{(N^2 + N + 1)^k}{\prod_{n=2}^{N} (n^3 - 2n^2 + 1)}$$

is irrational. Indeed we have $(x^3 - 2x^2 + 1) = (x-1)(x^2 - x - 1)$.

Corollary 3.1 *Let $P(x), Q(x) \in \mathbb{Q}[x]$ and $P(x) = \sum_{i=0}^{T} a_i x^i$. Suppose that $a_T > 0$ and $a_i \geq 0$ for every $i = 0, 1, \ldots, T-1$. Let all roots of the polynomial $Q(x)$ are real, noninteger and non-negative. Assume that $Q(x)$ is not constant and that $Q(n) \in \mathbb{Z}$ and $Q(n) \neq 0$ for all $n \in \mathbb{N}$. Then $S = \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} \notin \mathbb{Q}$.*

Example 3.5 *For every positive integer k the number*

$$\sum_{N=1}^{\infty} \frac{N^k}{\prod_{n=1}^{N}(n^2-3n+1)}$$

is irrational.

Remark 3.1 *The condition that the roots of Q are non-integral is only needed to guarantee that the terms of the series are well-defined.*

Theorem 3.3 *Let d, k and r be positive integers with $rd \leq k$ and let $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{Z}[x]$. Assume that $a_T > 0$ and $a_i \geq 0$ for $i = 0, 1, \ldots, T-1$. If*

$$S = \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(n^k + n^{k-d} + \ldots + n^{k-rd})} \in \mathbb{Q}$$

then $k = rd$. If $k = rd$ and $d > 1$ then $S \in \mathbb{Q}$ if and only if $P(x) = c(x^k + x^{k-d} + \ldots + x^d)$ where c is a fixed positive integer.

Proof. Step 1. The case $T \geq k + d$. We write $Q(x) = x^k + x^{k-d} + \cdots + x^{k-rd}$ and

$$P(N) = \sum_{i=0}^{k+d-1} a_i N^i + \sum_{i=k+d}^{T} a_i N^i =$$

$$\sum_{i=0}^{k+d-1} a_i N^i + \sum_{i=k+d}^{T} a_i \{(N^{i-k} - N^{i-k-d})(N^k + N^{k-d} + \ldots + N^{k-rd}) + N^{i-(r+1)d}\} =$$

$$D(N-1)(N^k + N^{k-d} + \ldots + N^{k-rd}) + E(N)$$

where

$$D(x-1) = \sum_{i=k+d}^{T} a_i x^{i-k-d}(x^d - 1)$$

and

$$E(x) = \sum_{i=0}^{k+d-1} a_i x^i + \sum_{i=k+d}^{T} x^{i-(r+1)d}.$$

Hence

$$D(x) = \sum_{i=k+d}^{T} a_i (x+1)^{i-k-d} \sum_{j=1}^{d} \binom{d}{j} x^j.$$

Since $a_T > 0$, $a_i \geq 0$ for $i \geq 0$, the coefficient of x of $D(x)$ is positive. Futhermore

$$S = \sum_{N=0}^{\infty} \frac{P(N)}{\prod_{n=0}^{N} Q(n)} = \sum_{N=0}^{\infty} \left(\frac{D(N-1)}{\prod_{n=0}^{N-1} Q(n)} + \frac{E(N)}{\prod_{n=0}^{N} Q(n)}\right) = \sum_{N=0}^{\infty} \frac{D(N) + E(N)}{\prod_{n=0}^{N} Q(n)}.$$

The degree of $D(x)+E(x)=P^*(x)$ is at most $T^* \le \max(T-k, k+d-1, T-(r+1)d$, its coefficients are non-negative integers and the coefficient of x of $P^*(x)$ is positive. We may iterate this procedure with $P*$ in place of P and continue until $\deg(P^*) < k+d$. From now we assume that the conditions of the Theorem 3.3 are satisfied with $T < k+d$ and remember that $a_1 > 0$ if we have applied Step 1.

Step 2: The case $T < k+d$. If $T < k$, then it follows immediatelly from Oppenheim's theorem that $S \in \mathbb{Q}$ in view of $a_T > 0$ if we have not applied Step 1 and otherwise $a_1 > 0$. Therefore we suppose $k \le T < k+d$. We write

$$P(N) = \sum_{i=0}^{k-1} a_i N^i + \sum_{i=k}^{T} a_i N^i =$$

$$\sum_{i=0}^{k-1} a_i N^i + \sum_{i=k}^{T} a_i N^{i-k}(N^k + N^{k-d} + \ldots + N^{k-rd}) - \sum_{i=k}^{T} a_i (N^{i-d} + \cdots + N^{i-rd}) =$$

$$D(N-1)(N^k + N^{k-d} + \ldots + N^{k-rd}) + E(N)$$

where

$$D(x-1) = \sum_{i=k}^{T} a_i x^{i-k}$$

and

$$E(x) = \sum_{i=0}^{k-1} a_i x^i - \sum_{i=k}^{T} a_i (x^{i-d} + \cdots + x^{i-rd}).$$

Hence

$$D(x) = \sum_{i=k}^{T} a_i (x+1)^{i-k}$$

has positive coefficients in view of $a_T > 0$, $a_i \ge 0$ for $i = k, \cdots, T$, and

$$S = \sum_{N=0}^{\infty} \frac{D(N) + E(N)}{\prod_{n=0}^{N} Q(n)}$$

where $P^*(x) = D(x) + E(x)$ has integral coefficients and degree $< k$. We distinguish between two cases.

Case 2A. Let $k > rd$. Then we have $P^*(0) = \sum_{i=k}^{T} a_i i^{i-k} + a_0 \ge a_T > 0$ so that P^* is non-trivial. By Oppenheim's theorem [13] we obtain that $S \notin \mathbb{Q}$.

Case 2B. Let $k = rd$ and $d > 1$. Then we have $P^*(0) = \sum_{i=k}^{T} a_i i^{i-k} + a_0 - a_k = \sum_{i=k+1}^{T} a_i i^{i-k} + a_0$ and this is positive if $T > k$. Thus $S \notin \mathbb{Q}$ if $T > k$ by Oppenheim's theorem. Suppose $T = k$. Then the coefficient of x of $P^*(x)$ equals a_1 because of $T = k = rd$ and $d > 1$. If $a_1 > 0$, then we know that $S \notin \mathbb{Q}$ by Oppenheims theorem. This is certainly the case if we have applied Step 1. We conclude that $a_1 = 0$ is only possible if the original T equals k. Then $D(x) = a_k$ and $E(x) = \sum_{i=0}^{k-1} a_i x^i - a_k(x^{k-d} + x^{k-2d} + \cdots + 1)$. According to Oppenheim's theorem $S \in \mathbb{Q}$ if and only if $D(x) + E(x) \equiv 0$. Hence $S \in \mathbb{Q}$ if and only if

$$P(x) = \sum_{i=0}^{k} a_i x^i = a_k x^k + a_k(x^{k-d} + x^{k-2d} + \cdots + 1) - a_k$$

$$= a_k(x^k + x^{k-d} + \cdots + x^d).$$

This completes the proof of the theorem.

□

Example 3.6 *Let $T, k \in \mathbb{N}$. Then*

$$\sum_{N=1}^{\infty} \frac{N^T}{\prod_{n=1}^{N}(n^k + n^{k-1} + \ldots + n)} \notin \mathbb{Q}.$$

Remark 3.2 *The following example shows that we cannot omit the condition $d > 1$ if $k = rd$ in Theorem 3.3.*

$$\sum_{N=1}^{\infty} \frac{N^5 + 2N^3 + N^2}{\prod_{n=1}^{N}(n^3 + n^2 + n + 1)} =$$

$$\sum_{N=1}^{\infty} \frac{(N^2 - N + 2)(N^3 + N^2 + N + 1) - N^2 - N - 2}{\prod_{n=1}^{N}(n^3 + n^2 + n + 1)} =$$

$$\sum_{N=1}^{\infty} \left(\frac{N^2 - N + 2}{\prod_{n=1}^{N-1}(N^3 + N^2 + N + 1)} - \frac{N^2 + N + 2}{\prod_{n=1}^{N}(N^3 + N^2 + N + 1)}\right) = 1 \in \mathbb{Q}.$$

Theorem 3.4 *Let $Q(x) \in \mathbb{Q}[x]$ and $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{R}[x]$. Suppose that Q is not constant, that $Q(n) \in \mathbb{Z}$ and $Q(n) \neq 0$ for all $n \in \mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{Z}$ be a sequence such that $f(N) = Q(N)P(N) + O(\frac{Q(N)}{N})$ as $N \to \infty$. Suppose $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)} \in \mathbb{Q}$. Then $a_T, a_{T-1}, \ldots, a_1, a_0 \in \mathbb{Q}$.*

Proof. Suppose $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)} = \frac{p}{q}$ where p and $q > 0$ are coprime integers. Let U be the largest index i with $a_i \notin \mathbb{Q}$. Let d be a common denominator of $a_{U+1}, \ldots, a_T$. Put $P_1(x) = \sum_{i=0}^{U} a_i x^i$ and $P_2(x) = Q(x) \sum_{i=U+1}^{T} a_i x^i$. Then we have, by Lemma 6.5,

$$dp = dq \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)} =$$

$$q \sum_{N=1}^{\infty} \frac{d(f(N) - P_2(N) + P_2(N))}{\prod_{n=1}^{N} Q(n)} =$$

$$qdQ_1 + q\sum_{N=1}^{\infty}\frac{d(f(N)-P_2(N))+dQ_2Q_3(N)}{\prod_{n=1}^{N}Q(n)}$$

where $Q_1, Q_2 \in \mathbb{Q}$, $Q_3(x) \in \mathbb{Q}[x]$ correspond to P_2 according to Lemma 6.5. ¿From this and Lemma 6.1 we obtain that for every positive integer N the number

$$R_N^* := qd\sum_{s=0}^{\infty}\frac{f(N+s)-P_2(N+s)+Q_2Q_3(N+s)}{\prod_{n=N}^{N+s}Q(n)}$$

is an integer. From the definition of P_1, P_2 and the assumption on f it follows that

$$R_N^* = qd\sum_{s=0}^{\infty}\frac{P_1(N+s)}{\prod_{n=N}^{N+s-1}Q(n)} + O(\frac{1}{N}).$$

This combined with Lemma 6.4 applied to $H(X) = qdP_1(X+j)$ implies that the number

$$R_U^*(N) := \sum_{j=0}^{U}(-1)^j\binom{U}{j}R_{N+j}^* = (-1)^U qdP_1^{(U)}(N) + O(\frac{1}{N}) =$$

$$(-1)^U qU!da_U + O(\frac{1}{N})$$

is an integer. This is a contradiction for a sufficiently large number N. □

Theorem 3.5 *Let d, k and r be positive integers with $rd \le k$ and let $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{Z}[x]$. Assume that $a_T > 0$ and $a_i \ge 0$ for $i = 0, 1, \ldots, T-1$. If*

$$S = \sum_{N=2}^{\infty}\frac{P(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})} \in \mathbb{Q}$$

then $k = rd$. If $k = rd$ and $d > 1$ then $S \in \mathbb{Q}$ if and only if $P(x) = c(x^k - x^{k-d} + \ldots + x^d)$ where c is a fixed positive integer.

Proof. Let $\deg(P) < k$. Then we apply Oppenheim's Theorem 2.1 to conclude $S \notin \mathbb{Q}$. Assume that $\deg(P) \ge k$. Now the proof falls into two cases.

1. Suppose that r is odd. If $i \ge k+d$ then we can write

$$N^i = N^{i-k-d}(N^{k+d} - N^{k-rd}) + N^{i-(r+1)d} =$$

$$N^{i-k-d}(N^d+1)(N^k - N^{k-d} + \cdots - N^{k-rd}) + N^{i-(r+1)d}.$$

Thus

$$P(x) = A(x)(x^k - x^{k-d} + \ldots + (-1)^r x^{k-rd}) + B(x)$$

where $A(x), B(x) \in Z[x]$, $\deg(B) < k + d$, $\deg(A) \leq \deg(P) - k$, $A(x)B(x) \not\equiv 0$ and the polynomials $A(x)$, $B(x)$ have non-negative coefficients. Hence

$$S = \sum_{N=2}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})} =$$

$$\sum_{N=2}^{\infty} \frac{A(N)(N^k - N^{k-d} + \ldots + (-1)^r N^{k-rd}) + B(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})} =$$

$$q + \sum_{N=2}^{\infty} \frac{A(N+1) + B(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})}$$

where q is the suitable rational number. By induction we repeat this procedure. Therefore we obtain

$$S = q^* + \sum_{N=2}^{\infty} \frac{P^*(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})}$$

where $P(x) \in Z[x]$, $P^* \not\equiv 0$, $\deg(P) < k + d$ and all coefficient of P are non-negative. If $k \leq i < k + d$ then we can write

$$N^i = N^{i-k}(N^k - N^{k-d} + N^{k-2d} - \cdots - N^{k-rd}) + N^{i-d} - N^{i-2d} + N^{i-3d} + \cdots + N^{i-kd}.$$

Thus

$$P^*(x) = A^*(x)(x^k - x^{k-d} + \ldots + (-1)^r x^{k-rd}) + B^*(x)$$

where $A^*(x), B^*(x) \in Z[x]$, $\deg(B^*) < k$, $\deg(A^*) \leq d$, $A^*(x)B^*(x) \not\equiv 0$, the polynomial $A^*(x)$ has non-negative coefficients and the coefficient of the lowest pover of polynomial $B^*(x)$ is positive. This implies that $A^*(x+1) + B^*(x) \not\equiv 0$. Now we can write

$$S = q^* + \sum_{N=2}^{\infty} \frac{P^*(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})} =$$

$$\sum_{N=2}^{\infty} \frac{A^*(N)(N^k - N^{k-d} + \ldots + (-1)^r N^{k-rd}) + B^*(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})} =$$

$$q^{**} + \sum_{N=2}^{\infty} \frac{A^*(N+1) + B^*(N)}{\prod_{n=1}^{N}(n^k - n^{k-d} + \ldots + (-1)^r n^{k-rd})}$$

where q^{**} is the suitable rational number. Note that $A^*(N+1) + B^*(N) \not\equiv 0$ and $\deg(A^*(N+1) + B^*(N)) < k$. Then we apply Oppenheim's Theorem 2.1 to conclude $S \notin \mathbb{Q}$.

1. Suppose that r is even.

□

4 Main criterion for the irrationality

Theorem 4.1 *Let $K \geq 0$ be an integer, $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant and $Q(n) \neq 0$, $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that*

$$F(N+j) = \sum_{r=0}^{K} \frac{F^{(r)}(N)}{r!} j^r + O\left(\frac{F(N)}{N^{K+1}}\right) \quad for\ j = 0, 1, \ldots, K\ as\ N \to \infty, \tag{4}$$

$$F^{(r)}(N) = O\left(\frac{F(N)}{N^r}\right) \quad for\ r = 0, 1, \ldots, K\ as\ N \to \infty, \tag{5}$$

$$\lim_{N\to\infty} F^{(K)}(N) = 0,\ \lim_{N\to\infty} \frac{N^{K+1}|F^{(K)}(N)|}{F(N)} = \infty \tag{6}$$

and

$$\limsup_{N\to\infty} N|F^{(K)}(N)| = \infty. \tag{7}$$

Let $f : \mathbb{N} \to \mathbb{Z}$ be a sequence such that $R^* := \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)}$ is absolutely convergent and $f(N) = Q(N)F(N) + O(\frac{Q(N)}{N})$ as $N \to \infty$. Then R^* is irrational.

Proof. Let N be sufficiently large. Suppose $R^* = \frac{p}{q}$ where p and $q > 0$ are coprime integers. Put

$$R_K^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=0}^{\infty} \frac{f(N+j+s)}{\prod_{n=N+j}^{N+j+s} Q(n)}.$$

By Lemma 6.1, $qR_K^*(N)$ is an integer for every positive integer N. We have, by Lemma 6.4,

$$R_K^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=0}^{\infty} \frac{F(N+j+s)}{\prod_{n=N+j}^{N+j+s-1} Q(n)} + O(\frac{1}{N}) =$$

$$(-1)^K F^{(K)}(N) + O\left(\frac{F(N)}{N^{K+1}} + \frac{1}{N}\right) =$$

$$(-1)^K F^{(K)}(N) \left(1 + O\left(\frac{F(N)}{N^{K+1}|F^{(K)}(N)|}\right)\right) + O(\frac{1}{N}).$$

By (6) the right-hand side tends to 0 as $N \to \infty$. Since $qR_K^*(N)$ is an integer, we infer that $R_K^*(N) = 0$ for $N \geq N_0$. It follows that $0 = (1+o(1))NF^{(K)}(N) + O(1)$ for $N \geq N_0$ which contradicts (7). □

Corollary 4.1 *Let $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant, $Q(n) \neq 0$ and $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Let $\alpha \in \mathbb{R}_{\geq 0} \setminus \mathbb{Z}, \gamma \in \mathbb{R}_{\geq 0}$. If $\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha]}{\prod_{n=1}^{N} Q(n)} \in \mathbb{Q}$, then $\alpha = 0$ and $[\gamma] = 0$.*

Proof. Apply Theorem 4.1 with $K = [\alpha], f(N) = [\gamma N^{\alpha}], F(N) = \gamma N^{\alpha-1}$. □

Remark It is remarkable that Corollary 4.1 holds for all $\alpha \geq 0$ and $\gamma > 0$ so that $\sum_{N=1}^{\infty} \frac{[\gamma N^{\alpha}]}{N!}$ is a strictly monotonic function of α and γ missing all rational values.

Corollary 4.2 *Let $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant, $Q(n) \neq 0$ and $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Let $\alpha \in \mathbb{R}_{\geq 0} \setminus \mathbb{Z}, \gamma \in \mathbb{R}_{+}$. Then $\sum_{N=1}^{\infty} \frac{[\gamma N^{\alpha} \log\ N]}{\prod_{n=1}^{N} Q(n)} \notin \mathbb{Q}$.*

Proof. Apply Theorem 4.1 with $K = [\alpha], f(N) = [\gamma N^{\alpha} \log N], F(N) = \gamma N^{\alpha-1} \log N$. □

Theorem 4.2 *Let $K \geq 0$ be an integer, $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant, $Q(n) \neq 0$ and $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Let $F : \mathbb{R}_{+} \to \mathbb{R}_{+}$ be a function such that*

$$F(N+x) = \sum_{r=0}^{\infty} \frac{F^{(r)}(N)}{r!} x^r \quad \textit{for } x = o(N) \text{ as } N \to \infty, \tag{8}$$

$$F^{(r)}(N) = O\left(r! \frac{F(N)}{N^r} \right) \quad \textit{uniformly for } r = 0, 1, \ldots \text{ as } N \to \infty, \tag{9}$$

$$\lim_{x \to \infty} F^{(K)}(x) = 0,\ \lim_{x \to \infty} \frac{x^{K+1} |F^{(K)}(x)|}{F(x)} = \infty \tag{10}$$

and

$$\lim_{x \to \infty} x^2 |F^{(K)}(x)| = \infty. \tag{11}$$

Let $f : \mathbb{N} \to \mathbb{Z}$ be a sequence such that $R^* := \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)}$ is absolutely convergent and $f(N) = Q(N)F(N) + O(\frac{Q(N)}{N})$ as $N \to \infty$. Then R^* is irrational.

Proof. Let N be sufficiently large. Suppose $R^* = \frac{p}{q}$ where p and $q > 0$ are coprime integers. Conditions (8) and (9) imply conditions (4) and (5). Hence, by Theorem 4.1, we may assume without loss of generality that

$$N F^{(K)}(N) = O(1). \tag{12}$$

Observe that by (9) and (8), for $x = o(N)$,

$$F(N+x) \leq \sum_{r=0}^{\infty} |F^{(r)}(N)| \frac{x^r}{r!} = O\left(F(N) \sum_{r=0}^{\infty} (\frac{x}{N})^r \right) = O(F(N)). \tag{13}$$

Put

$$R_{K-1}^*(N) = \sum_{j=0}^{K-1} (-1)^j \binom{K-1}{j} \sum_{s=0}^{\infty} \frac{f(N+j+s)}{\prod_{n=N+j}^{N+j+s} Q(n)}.$$

By Lemma 6.1, $qR^*_{K-1}(N)$ is an integer for every positive integer N. We have, by Lemma 6.4,

$$R^*_{K-1}(N) = \sum_{j=0}^{K-1}(-1)^j\binom{K-1}{j}\sum_{s=0}^{\infty}\frac{F(N+j+s)}{\prod_{n=N+j}^{N+j+s-1}Q(n)} + O(\frac{1}{N}) =$$

$$(-1)^{K-1}F^{(K-1)}(N) + O\left(\frac{F(N)}{N^K}+\frac{1}{N}\right). \tag{14}$$

Put

$$t = 1 + \left[\frac{N}{\min_{x\in[N,2N]}(x^2|F^{(K)}(x)|)^{\frac{1}{2}}} + \frac{N}{\min_{x\in[N,2N]}(\frac{x^{K+1}|F^{(K)}(x)|}{F(x)})^{\frac{1}{2}}}\right]. \tag{15}$$

It follows from (11) and (10) that $t = o(N)$ as $N \to \infty$. Hence, by (13), similar to (14),

$$qR^*_{K-1}(N+t) = (-1)^{K-1}qF^{(K-1)}(N+t) + O\left(\frac{F(N)}{N^K}+\frac{1}{N}\right) \tag{16}$$

is an integer. We apply the Mean Value Theorem to (16) and (14). Hence there exists a real number τ with $0 < \tau < t$ such that

$$M(N) := qR^*_{K-1}(N+t) - qR^*_{K-1}(N) = (-1)^{K-1}qtF^{(K)}(N+\tau) + O\left(\frac{F(N)}{N^K}+\frac{1}{N}\right)$$

is an integer. It follows from (15), (13) and (10) that, for some positive constant c,

$$\frac{tN^K|F^{(K)}(N+\tau)|}{F(N)} \ge \frac{N^{K+1}|F^{(K)}(N+\tau)|}{F(N)\min_{x\in[N,2N]}(\frac{x^{K+1}|F^{(K)}(x)|}{F(x)})^{\frac{1}{2}}}$$

$$\ge c\left(\frac{(N+\tau)^{K+1}|F^{(K)}(N+\tau)|}{F(N+\tau)}\right)^{\frac{1}{2}} \to \infty \text{ as } N \to \infty.$$

Hence, by (12) and $t = o(N)$,

$$M(N) = (-1)^{K-1}(1+o(1))qtF^{(K)}(N+\tau) + O(\frac{1}{N}) \to 0. \tag{17}$$

Thus, since $M(N)$ represents an integer, $M(N) = 0$ for $N \ge N_1$. It follows from (17), (15) and (13) that

$$0 = Nt|F^{(K)}(N+\tau)| + O(1) \ge$$

$$\frac{N^2|F^{(K)}(N+\tau)|}{\min_{x\in[N,2N]}(x^2|F^{(K)}(x)|)^{\frac{1}{2}}} + O(1) \ge \left((N+\tau)^2|F^{(K)}(N+\tau)|\right)^{\frac{1}{2}} + O(1)$$

which contradicts (11). □

Corollary 4.3 *Let $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant, $Q(n) \neq 0$ and $Q(n) \in \mathbb{N}$ for every $n \in \mathbb{N}$. Let $\alpha \in \mathbb{R}_{\geq 0}, \beta \in \mathbb{R}, \beta \neq 0, \gamma \in \mathbb{R}_+$. Suppose $\beta > 0$ whenever $\alpha = 0$. Then $\sum_{N=1}^{\infty} \frac{[\gamma N^{\alpha} \log^{\beta} N]}{\prod_{n=1}^{N} Q(n)} \notin \mathbb{Q}$.*

Proof. Apply Theorem 4.2 with $K = [\alpha]$ if $\alpha \notin \mathbb{Z}$ or $\beta > 0$, and $K = \alpha - 1$ otherwise, $f(N) = [\gamma N^{\alpha} \log^{\beta} N], F(N) = \gamma N^{\alpha-1} \log^{\beta} N$. □

Corollary 4.4 *Let $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant, $Q(n) \neq 0$ and $Q(n) \in \mathbb{N}$ for every $n \in \mathbb{N}$. Let $\alpha \in \mathbb{R}_{\geq 0}, 0 < \beta < 1, \gamma \in \mathbb{R}_+$. Then $\sum_{N=1}^{\infty} \frac{[\gamma N^{\alpha} \exp(\log^{\beta} N)]}{\prod_{n=1}^{N} Q(n)} \notin \mathbb{Q}$.*

Proof. Apply Theorem 4.2 with $K = [\alpha], f(N) = [\gamma N^{\alpha} \exp(\log^{\beta} N)]$,
$F(N) = \gamma N^{\alpha-1} \exp(\log^{\beta} N)$. □

5 Linear independence

Theorem 5.1 *Let $P(x) \in \mathbb{Z}[x]$, $P(x) = \sum_{i=0}^{T} a_i x^i$ and let $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant such that $Q(n) \neq 0$ and $Q(n) \in \mathbb{N}$ for every $n \in \mathbb{N}$. Suppose that and that $\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)}$ is irrational. Let W be a set of functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the following properties.*

(i)
$$F(N+x) = \sum_{r=0}^{\infty} \frac{F^{(r)}(N)}{r!} x^r \quad for\ x = o(N) \ \ as \ \ N \to \infty, \tag{18}$$

(ii)
$$F^{(r)}(N) = O(\frac{r! F(N)}{N^r}) \ \ uniformly\ for\ r = 0, 1, \ldots \ as\ n \to \infty, \tag{19}$$

(iii) *either there exists a positive integer K such that*

$$F^{(K)}(x) = o(1), \ \frac{F(x)}{x^{K+1}} = o(|F^{(K)}(x)|), \ \lim_{x \to \infty} x^2 |F^{(K)}(x)| = \infty \tag{20}$$

or

$$K = 0, \ \lim_{x \to \infty} F(x) = 0, \ \lim_{x \to \infty} xF(x) = \infty, \tag{21}$$

(iv) *for every pair of functions $F, G \in W$ with corresponding integers $K > L$ one has $\lim_{x \to \infty} \frac{G^{(k)}(x)}{F^{(k)}(x)} = 0$ for $k = 0, 1, \ldots, K$; for every pair of functions $F, G \in W$ with $F \neq G$ and corresponding integers $K = L$ one has either $\lim_{x \to \infty} \frac{G^{(k)}(x)}{F^{(k)}(x)} = 0$ for $k = 0, 1, \ldots, K$ or $\lim_{x \to \infty} \frac{F^{(k)}(x)}{G^{(k)}(x)} = 0$ for $k = 0, 1, \ldots, K$.*
Suppose that for every function $F \in W$ there exists a function $f : \mathbb{N} \to \mathbb{Z}$ such that $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)}$ is absolutely convergent and $f(N) = Q(N)F(N) + O(\frac{Q(N)}{N})$ as $N \to \infty$. Then the numbers $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)} (F \in W), \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)}$ and 1 are linearly independent over the rationals.

Proof. Suppose that there exist functions $f_0 := P, f_1, f_2, \dots, f_M \in W$ such that the numbers $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)}$ $(i = 0, 1, \dots, M)$ and the number 1 are linearly dependent over the rationals. Then there exist integers $A_0, A_1, A_2, \dots, A_M, p$ and $q > 0$, not all zero, such that

$$\frac{p}{q} = \sum_{m=0}^{M} A_m \sum_{N=1}^{\infty} \frac{f_m(N)}{\prod_{n=1}^{N} Q(n)}. \tag{22}$$

It is excluded that $A_1 = \dots = A_M = 0$. Without loss of generality we may assume that $A_1, \dots, A_M$ are all nonzero and $M > 0$. Lemma 6.5 and equation (22) imply

$$\frac{p}{q} = A_0 \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} + \sum_{m=1}^{M} A_m \sum_{N=1}^{\infty} \frac{f_m(N)}{\prod_{n=1}^{N} Q(n)} =$$

$$= \frac{p_1}{q_1} + \frac{p_2}{q_2} \sum_{N=1}^{\infty} \frac{Q_1(N)}{\prod_{n=1}^{N} Q(n)} + \sum_{N=1}^{\infty} \frac{\sum_{m=1}^{M} A_m f_m(N)}{\prod_{n=1}^{N} Q(n)}$$

where $p_1, p_2, q_1 > 0$ and $q_2 > 0$ are suitable integers which do not depend on N, $Q_1(x) \in \mathbb{Z}[x]$, $Q_1(x) \neq 0$, $\deg(Q_1) < \deg(Q)$ and the coefficients of $Q_1(x)$ also do not depend on N. Thus

$$A := q_1 q_2 q (\frac{p}{q} - \frac{p_1}{q_1}) = q_1 q \sum_{N=1}^{\infty} \frac{Q_1(N) + q_2 \sum_{m=1}^{M} A_m f_m(N)}{\prod_{n=1}^{N} Q(n)}$$

is an integer. By Lemma 6.1 we obtain that for every positive integer N the number

$$S_N := q_1 q \sum_{s=0}^{\infty} \frac{Q_1(N+s) + q_2 \sum_{m=1}^{M} A_m f_m(N+s)}{\prod_{n=N}^{N+s} Q(n)}$$

is an integer. Hence

$$S_N = q_1 q_2 q \sum_{s=0}^{\infty} \frac{\sum_{m=1}^{M} A_m F_m(N+s)}{\prod_{n=N}^{N+s-1} Q(n)} + O(\frac{1}{N}). \tag{23}$$

Without loss of generality we may assume by (iv) and (iii) that if K is the integer corresponding to F_M, then

$$\lim_{x \to \infty} \frac{F_m^{(k)}(x)}{F_M^{(k)}(x)} = 0 \ \text{ for } k = 0, 1, \dots, K \text{ and } m = 0, 1, \dots, M-1. \tag{24}$$

Let L be a nonnegative integer with $L \leq K$. We obtain from (23) that

$$R_L^*(N) := \sum_{j=0}^{L} (-1)^j \binom{L}{j} S_{N+j} =$$

$$q_1q_2q\sum_{j=0}^{L}(-1)^j\binom{L}{j}\sum_{s=0}^{\infty}\left(\frac{\sum_{m=1}^{M}A_mF_m(N+s+j)}{\prod_{n=N}^{N+s+j-1}Q(n)}\right)+O(\frac{1}{N})$$

is an integer too. From Lemma 6.4, (18), and (19) we deduce that

$$R_L^*(N)=(-1)^Lq_1q_2q\sum_{m=1}^{M}A_mF_m^{(L)}(N)+\sum_{m=1}^{M}O(F_m(N)N^{-L-1})+O(\frac{1}{N}).$$

Hence, by (24),

$$R_L^*(N)=(-1)^Lq_1q_2q\sum_{m=1}^{M}A_mF_m^{(L)}(N)+O(F_M(N)N^{-L-1})+O(\frac{1}{N}). \tag{25}$$

We distinguish between two cases.
(a) Suppose $\limsup_{N\to\infty}N|F_M^{(K)}(N)|=\infty$. Then we put $L=K$ and find by (24) that

$$R_K^*(N)=(-1)^Kq_1q_2qA_MF_M^{(K)}(N)(1+o(1))+O(F_M(N)N^{-K-1})+O(\frac{1}{N}).$$

Hence we derive a contradiction in the same way as we did in the last lines of the proof of Theorem 4.1.
(b) Suppose $\limsup_{N\to\infty}N|F_M^{(K)}(N)|<\infty$, i.e. $F_M^{(K)}(N)=O(\frac{1}{N})$ as $N\to\infty$. Note that $K\geq 1$ in view of (21). By (13) and (24) we have, for $m=1,\dots,M$ and $x=o(N)$, that $F_m(N+x)=O(F_m(N))=O(F_M(N))$. Put $L=K-1$ in (25). Hence

$$R_{K-1}^*(N)=(-1)^{K-1}q_1q_2q\sum_{m=1}^{M}A_mF_m^{(K-1)}(N)+O(F_M(N)N^{-K})+O(\frac{1}{N}).$$

Define t as in (15). We apply the Mean Value Theorem to the integer $M(N):=R_{K-1}^*(N+t)-R_K^*(N)$. We obtain that

$$M(N)=(-1)^{K-1}q_1q_2qt\sum_{m=1}^{M}A_mF_m^{(K)}(N+\tau)+O(F_M(N)N^{-K})+O(\frac{1}{N})$$

for some τ with $0<\tau<t$. By (24) we have

$$M(N)=(-1)^{K-1}q_1q_2qt(1+o(1))A_MF_M^{(K)}(N+\tau)+O\left(\frac{F_M(N)}{N^K}+\frac{1}{N}\right)$$

The further proof proceeds as the proof of Theorem 4.2 from the introduction of $M(N)$ on. □

Remark. It follows from a repeated use of l'Hôpital's rule that condition (iv) can be relaxed. If $\lim_{x\to\infty} \frac{F^{(K)}(x)}{G^{(K)}(x)} = 0$ and $\lim_{x\to\infty} G^{(K-1)}(x) = \infty$, then $\lim_{x\to\infty} \frac{F^{(k)}(x)}{G^{(k)}(x)} = 0$ for $k = 0, 1, \ldots, K$. (See [14], Theorem 5.13 with $A = 0$ and $a = \infty$.)

Corollary 5.1 *Let $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant, $Q(n) \neq 0$ and $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Then the numbers 1, e and $\sum_{n=1}^{\infty} \frac{[\gamma n^{\alpha}]}{\prod_{i=1}^{n} Q(i)}$ $(\alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z}, \gamma \in \mathbb{R}_{>0})$ are linearly independent over the rationals.*

Remark Conditions (i), (ii) and (iii) of Theorem 5.1 are satisfied by a large class of functions comprising

$$\gamma x^{\alpha} \quad (\alpha > -1, \alpha \notin \mathbb{Z}, \gamma \in \mathbb{R}_+) \quad \text{with } K = [\alpha] + 1$$

$$\gamma e^{\beta(\log x)^{\alpha}} \quad (0 < \alpha < 1, \beta \in \mathbb{R}_+, \gamma \in \mathbb{R}_+) \quad \text{with } K = 1$$

$$\gamma(\log x)^{\alpha} \quad (\alpha \neq 0, \gamma \in \mathbb{R}_+) \quad \text{with } K = 1 \text{ if } \alpha > 0, K = 0 \text{ if } \alpha < 0$$

$$\gamma(\log\log x)^{\alpha} \quad (\alpha \neq 0, \gamma \in \mathbb{R}_+) \quad \text{with } K = 1 \text{ if } \alpha > 0, K = 0 \text{ if } \alpha < 0.$$

It is therefore possible to apply Theorem 5.1 to sums and products of such functions and polynomials provided that condition (iv) is satisfied.

Example 5.1 *Let $Q(x) \in \mathbb{Q}[x]$, $Q(x)$ is non-constant, $Q(n) \neq 0$ and $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Then the numbers 1, $\sum_{n=1}^{\infty} \frac{[(\log n)^{\frac{1}{2}}]}{\prod_{i=1}^{n} Q(i)}$ and $\sum_{n=1}^{\infty} \frac{[e^{(\log n)^{\frac{1}{2}}}]}{\prod_{i=1}^{n} Q(i)}$ are linearly independent over the rationals.*

6 General case

Theorem 6.1 (Oppenheim [13], Theorem 8). *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of integers such that $a_n \geq 2$, $\mid b_n \mid < a_n$ and $\liminf_{n\to\infty} \frac{|b_n|}{a_n} = 0$. Then there is an n_0 such that*

$$S^* = \sum_{n=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N} a_n} \in \mathbb{Q}$$

if and only if $b_n = 0$ for all $n > n_0$.

Suppose there is k such that

$$b_n = c_{0,n} + c_{1,n}a_n + c_{2,n}a_n a_{n-1} + \ldots + c_{k,n}a_n a_{n-1} \ldots a_{n-k}$$

for every n. Then there exist integers a, b with $a \neq 0$ such that

$$\sum_{n=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N} a_n} = \sum_{n=1}^{\infty} (\frac{c_{0,n}}{a_1 \ldots a_n} + \frac{c_{1,n}}{a_1 \ldots a_{n-1}} + \ldots + \frac{c_{k,n}}{a_1 \ldots a_{n-k}}) =$$

$$\frac{b}{a} + \sum_{n=1}^{\infty} \frac{c_{0,n} + c_{1,n+1} + \ldots + c_{k,n+k}}{\prod_{n=1}^{N} a_n}.$$

¿From Theorem 6.1 we obtain that if

$$\lim_{n\to\infty} \frac{c_{0,n} + c_{1,n+1} + \ldots + c_{k,n+k}}{a_n} = 0,$$

and $\mid c_{0,n} + c_{1,n+1} + \ldots + c_{k,n+k} \mid < a_n$ for all large n then there is an n_0 such that $S^* \in \mathbb{Q}$ if and only if $c_{0,n} + c_{1,n+1} + \ldots + c_{k,n+k} = 0$ for every $n > n_0$.

Example 6.1 *By Oppenheim's theorem we obtain*

$$\sum_{n=2}^{\infty} \frac{n^n + 1}{\prod_{j=2}^{n}(j^j - 1)} = \sum_{n=2}^{\infty} \frac{1}{\prod_{j=2}^{n-1}(j^j - 1)} + \frac{2}{\prod_{j=2}^{n}(j^j - 1)} =$$

$$-\frac{1}{3} + \sum_{n=2}^{\infty} \frac{2}{\prod_{j=2}^{n}(j^j - 1)} \notin \mathbb{Q}.$$

Let $Q(x) \in \mathbb{Q}[x]$. Assume that $Q(n) \neq 0$ and $Q(n) \in \mathbb{Z}$ for every positive integer n. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers. We investigate under which conditions

$$R^* = \sum_{N=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N} Q(n)} \tag{26}$$

is irrational.

The following lemma dealing with the sum

$$R_N^* := \sum_{m=N}^{\infty} \frac{b_m}{\prod_{n=N}^{m} Q(n)} \tag{27}$$

is crucial.

Lemma 6.1 *If $R^* = \frac{p}{q}$ for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$, then $qR_N^* \in \mathbb{Z}$ for all N.*

Proof. We have

$$p \prod_{n=1}^{N-1} Q(n) = q \sum_{m=1}^{N-1} b_m \prod_{n=m+1}^{N-1} Q(n) + q \sum_{m=N}^{\infty} \frac{b_m}{\prod_{n=N}^{m} Q(n)}$$

and the first two terms are integers. □

Remark. If q divides $\prod_{n=1}^{N-1} Q(n)$, then we need not multiply by q to obtain integers and can conclude that R_N^* itself is an integer.

Theorem 6.2 *Let $P_n(x), Q_n(x) \in \mathbb{Z}[x]$ and $P_n(x)$ is non-constant for every $n \in \mathbb{N}$. Assume that there exists a V such that P_n can be written as*

$$P_n(x) = \sum_{i=1}^{V} B_{n,i}(x-i) \prod_{j=1}^{i} Q_{n-j+1}(x+1-j)$$

with $\deg(B_{n,i}) < \deg(Q_{n-i})$ for every i. Then

$$S = \sum_{N=1}^{\infty} \frac{P_N(N)}{\prod_{n=1}^{N} Q_n(n)} \in \mathbb{Q}$$

if and only if

$$\sum_{i=0}^{V} B_{n,i} = 0$$

for all large n.

The following consequence of a theorem of Oppenheim implies that R^* is irrational if $b_n = o(Q(n))$, but not ultimately constant 0.

Lemma 6.2 (Oppenheim [13], Theorem 8). *If $|b_n| < Q(n)$ for all $n > n_0$ and $\liminf_{n\to\infty} \frac{|b_n|}{Q(n)} = 0$, then R^* is rational if and only if $b_n = 0$ for all $n > n_0$.*

The next lemma displays some well known properties of Stirling numbers of the second kind.

Lemma 6.3 *Let K be a nonnegative integer. Put*

$$S(r,K) = \frac{1}{K!} \sum_{j=0}^{K} (-1)^{K-j} \binom{K}{j} j^r.$$

Then $S(r,K) = 0$ if $r < K$, $S(r,K) = 1$ if $r = K$ and $S(r,K) \in \mathbb{N}$ if $r > K > 0$.

For a proof see [1] Section III.2.

The last lemma of the section will be used for all theorems except for Theorem 6.4.

Lemma 6.4 *Let $K \geq 0$ be given integer and $Q(x) \in \mathbb{Q}[x]$ such that $Q(n) \neq 0$ and $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Let $H : \mathbb{R} \to \mathbb{R}_+$ be a K times continuously differentiable function such that $H(x) \neq 0$ for $x > x_0$. Suppose we have*

$$H(N+j) = \sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} j^r + O\left(\frac{H(N)}{N^{K+1}}\right) \quad \text{for } j = 0, 1, \ldots, K \text{ as } N \to \infty \qquad (28)$$

and

$$H^{(r)}(N) = O(\frac{H(N)}{N^r}) \quad for\ r = 0, 1, \ldots, K\ as\ N \to \infty. \tag{29}$$

Put

$$R_K^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=0}^{\infty} \frac{H(N+j+s)}{\prod_{n=N+j}^{N+j+s-1} Q(n)}.$$

Then

$$R_K^*(N) = (-1)^K H^{(K)}(N) + O(H(N)N^{-K-1})\ as\ N \to \infty. \tag{30}$$

Proof. Put

$$Q_s(N) = \frac{1}{\prod_{i=0}^{s-1} Q(N+i)}.$$

Let N be sufficiently large. Note that for $j = 0, 1, \ldots, K$, by (28) and (29),

$$H(N+j) = O\left(H(N) \sum_{r=0}^{K+1} \frac{1}{r!} (\frac{j}{N})^r \right) = O(H(N)e^{j/N}) = O(H(N)) \text{ as } N \to \infty. \tag{31}$$

Write

$$R_K^*(N) = R_{K,1}^*(N) + R_{K,2}^*(N) + R_{K,3}^*(N) \tag{32}$$

where

$$R_{K,1}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} H(N+j),$$

$$R_{K,2}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=1}^{K} \frac{H(N+j+s)}{\prod_{n=N+j}^{N+j+s-1} Q(n)},$$

$$R_{K,3}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=K+1}^{\infty} \frac{H(N+j+s)}{\prod_{n=N+j}^{N+j+s-1} Q(n)}.$$

By Lemma 6.3 and (28) we have

$$R_{K,1}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} j^r + O(H(N)N^{-K-1}) =$$

$$\sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} \sum_{j=0}^{K} (-1)^j \binom{K}{j} j^r + O(H(N)N^{-K-1}) =$$

$$(-1)^K H^{(K)}(N) + O(H(N)N^{-K-1}) \text{ as } N \to \infty. \tag{33}$$

We now turn to $R_{K,2}^*(N)$. ¿From the facts $0 \leq j \leq K$, $1 \leq s \leq K$, equations (28), (29) and the definition of $Q_s(N)$ we obtain that $H(N+s+j) = O(H(N))$,

$Q_s(N+j) = \sum_{m=0}^{K} \frac{Q_s^{(m)}(N)j^m}{m!} + O(N^{-(K+1)})$ and $Q_s(N+j) = O(N^{-s})$. This, (28) and (29) imply

$$R^*_{K,2}(N) = \sum_{j=0}^{K}(-1)^j \binom{K}{j} \sum_{s=1}^{K} H(N+s+j)Q_s(N+j) =$$

$$\sum_{j=0}^{K}(-1)^j \binom{K}{j} \sum_{s=1}^{K}\sum_{r=0}^{K} \frac{H^{(r)}(N+s)j^r}{r!} \sum_{m=0}^{K} \frac{Q_s^{(m)}(N)j^m}{m!} + O\left(\frac{H(N)}{N^{K+1}}\right) =$$

$$\sum_{s=1}^{K}\sum_{r=0}^{K} \frac{H^{(r)}(N+s)}{r!} \sum_{m=0}^{K} \frac{Q_s^{(m)}(N)}{m!} \sum_{j=0}^{K}(-1)^j j^{r+m} \binom{K}{j} + O\left(\frac{H(N)}{N^{K+1}}\right).$$

¿From this and Lemma 6.3 we obtain that

$$R^*_{K,2}(N) = \sum_{s=1}^{K}\sum_{r=0}^{K} \frac{H^{(r)}(N+s)Q_s^{(K-r)}(N)}{r!(K-r)!}(-1)^K K! + O\left(\frac{H(N)}{N^{K+1}}\right)$$

This and the facts that $H^{(r)}(N+s) = O(H(N)N^{-r})$ and $Q_s^{(K-r)}(N) = O(N^{-(K-r+s)})$ hold for all $r = 0, 1, \ldots, K$ and $s = 1, 2, \ldots, K$ imply that

$$R^*_{K,2}(N) = O\left(\frac{H(N)}{N^{K+1}}\right). \tag{34}$$

Finally we estimate $R^*_{K,3}(N)$. By (31) there exists a constant $c > 1$ such that $H(n+1) < cH(n)$ for all sufficiently large n. Hence $H(N+s) < c^s H(N)$ for every positive integer s. It follows that

$$R^*_{K,3}(N) = O\left(\sum_{s=K+1}^{\infty} \frac{H(N+s)}{\prod_{n=N}^{N+s-1} Q(n)}\right) = O\left(\sum_{s=K+1}^{\infty} \frac{H(N)c^s}{N^s}\right).$$

Hence

$$R^*_{K,3}(N) = O\left(\frac{H(N)}{N^{K+1}}\right) \quad \text{as } N \to \infty. \tag{35}$$

The combination of (32), (33), (34) and (35) yields (30). □

Remark In applications of Lemma 6.4 the integer K is usually chosen as the smallest nonnegative integer such that $H^{(K)}(N) \to 0$ as $N \to \infty$.

By the previous result we obtain

Consequence 6.1 *If* $\sum_{N=1}^{\infty} \frac{Q(N+a)}{\prod_{n=1}^{N} Q(n)} \in \mathbb{Q}$ *for some integer* a*, then there exists an integer* s *such that* $Q(x) = x + s$, $a = -1$ *or* $Q(x) = x - s$, $a = 1$.

Proof. We apply the above result with $P(x) = Q(x+a)$. Hence $c = 1$ and $P^*(N) = -1$ for all N. Since $Q(x+a) - Q(x) = P^*(x)$ is a non-zero constant we obtain that $Q(x)$ is a linear polynomial $rx + s$. Since $Q(x)$ attains integer values at integer points, we have $r, s \in \mathbb{Z}$. From $-1 = Q(x+a) - Q(x) = ra$, we conclude that $(a, r) = (1, -1)$ or $(a, r) = (-1, 1)$. Thus $a \in \{-1, 1\}$ and $Q(x) = x + s$ if $a = -1$ and $Q(x) = -x + s$ if $a = 1$. □

Lemma 6.5 *Let $P(x), Q(x) \in \mathbb{Q}[x]$. Suppose that Q is not constant. Assume that $Q(n) \in \mathbb{Z}$ and $Q(n) \neq 0$ for all $n \in \mathbb{N}$. Then there exist $Q_1, Q_2 \in \mathbb{Q}$ and $Q_3(x) \in \mathbb{Z}[x]$ with $\deg(Q_3) < \deg(Q)$ such that*

$$\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} = Q_1 + Q_2 \sum_{N=1}^{\infty} \frac{Q_3(N)}{\prod_{n=1}^{N} Q(n)}. \tag{36}$$

Proof. Put $D_0(x) = P(x)$ and $d = [\frac{\deg(P)}{\deg(Q)}]$. Then for each $i = 0, 1, \ldots, d-1$ there exist unique polynomials $B_i(x), D_i(x) \in \mathbb{Q}[x]$ with $\deg(B_i) < \deg(Q)$ and such that

$$D_i(x-i) = B_i(x-i) + D_{i+1}(x-i-1)Q(x-i).$$

Put

$$B_d(x-d) = D_d(x-d).$$

Then we have

$$P(x) = \sum_{i=0}^{d} B_i(x-i) \prod_{j=1}^{i} Q(x+1-j).$$

Thus

$$\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} =$$

$$\sum_{N=1}^{d} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} + \sum_{N=d+1}^{\infty} \frac{\sum_{i=0}^{d} B_i(N-i) \prod_{j=1}^{i} Q(N+1-j)}{\prod_{n=1}^{N} Q(n)} =$$

$$\sum_{N=1}^{d} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} + \sum_{N=d+1}^{\infty} \sum_{i=0}^{d} B_i(N-i) \frac{1}{\prod_{n=1}^{N-i} Q(n)} =$$

$$\sum_{N=1}^{d} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} + \sum_{N=1}^{\infty} \frac{\sum_{i=0}^{d} B_i(N)}{\prod_{n=1}^{N} Q(n)} - \sum_{N=1}^{d} \frac{\sum_{i=0}^{d-N} B_i(N)}{\prod_{n=1}^{N} Q(n)}.$$

The latter expression is of the form $Q_1 + Q_2 \sum_{N=1}^{\infty} \frac{Q_3(N)}{\prod_{n=1}^{N} Q(n)}$ with Q_1 the rational number which is the sum of the first and third term, Q_2 the inverse of the denominator of $\sum_{i=0}^{d} B_i(N)$ and $Q_3(x) \in \mathbb{Z}[x]$. □

Theorem 6.3 *Let $V \in \mathbb{R}$. Let $P_n(x), Q_n(x) \in \mathbb{Z}[x]$, $P_n(x) = \sum_{i=0}^{T_n} a_{n,i}x^i$, $\deg P_n < V$, $Q_n(x) = \prod_{m=1}^{M_n} C_{n,m}(x)$ and $C_{n,m}(x) = \sum_{k=1}^{M_{n,m}} c_{n,m,k}x^k \in \mathbb{R}[x]$ for all $n \in \mathbb{N}$. Suppose that $a_{T_n} > 0$ and $a_{n,i} \geq 0$ for every $i = 0, 1, \ldots, T_n - 1$. Let $c_{n,m,k} \leq 0$ and $V > c_{n,m,M_{n,m}} > 0$ for all $m = 1, 2, \ldots, M_n$, $k = 0, 1, \ldots, M_{n,m-1}$. Assume that $Q_n(x)$ is not constant, $Q_n(n) \neq 0$ and*

$$(\max_{\sum_{s=0}^{S} \deg Q_{N-s} \leq \deg P_N, m=1,2,\ldots,M_{N-s}, k=0,1,\ldots,M_{n,m} S \in \mathbb{N}} c_{N-s,m,k})^{\deg P_N} \max_{j=0,1,\ldots,P_N} a_{j,N} = o(N) \tag{37}$$

for all $n \in \mathbb{N}$. Then $S = \sum_{N=1}^{\infty} \frac{P_N(N)}{\prod_{n=1}^{N} Q_n(n)} \notin \mathbb{Q}$.

Proof. Follow the proof of Theorem 3.2. Let N be sufficiently large such that $P_N \neq 0$. Then we have

$$P_N(x) = \sum_{i=0}^{V} B_{N,i}(x - i) \prod_{j=1}^{i} Q_{N-j+1}(x + 1 - j).$$

Note that every polynomial $B_{N,i}(x - i) = B_{N,i}(y)$ $(N = 1, 2, \ldots, i = 0, 1, \ldots, T_N$ has nonnegative coefficient of the term with the highest power and every his coefficient consists of the sum of elements. The number of the elements in such a sum is bounded (in fact less then $(\deg P_N)^{\deg P_N} < V^V$). Each such an element has in the numerator product which cosists of one coefficient of the polynomial P_N and no more than $\deg P_N$ coefficients $c_{N-s,m,k}$ with $\sum_{s=0}^{S} \deg Q_{N-s} \leq \deg P_N$, $m = 1, 2, \ldots, M_{N-s}$, $k = 0, 1, \ldots, M_{N,m}$, $S \in \mathbb{N}$ and has in denominator product no more than $\deg P_N$ of the coefficients $c_{N,m,M_{N,m}}$ with $m = 1, 2, \ldots, M_N$, $k = 0, 1, \ldots, M_{N,m-1}$. This fact and (37) imply that each an element in the sum has bounded denominator and numerator is $o(N)$. Hence every polynomial $B_{N,i}(x - i) = B_{N,i}(y)$ $(N = 1, 2, \ldots, i = 0, 1, \ldots, T_N)$ has all coefficients with numerator of $o(N)$ and with the bounded denominator. ¿From the fact that $P_N \neq 0$ we obtain that there exists $I \in \{0, 1, \ldots, T_N\}$ such that $B_{N,I} \neq 0$. Without loss of generality put $I = 0$. Now following the proof of Lemma 6.5 we obtain that

$$\sum_{n=1}^{\infty} \frac{P_n(n)}{\prod_{j=1}^{n} Q_j(j)} = Q_1^* + Q_2^* \sum_{n=1}^{\infty} \frac{Q_{3,n}^*(n)}{\prod_{j=1}^{n} Q_j(j)}$$

where $Q_1^*, Q_2^* \in \mathbb{Q}$, $\frac{1}{Q_2^*} \in \mathbb{N}$ and $Q_{3,n}^*(x) \in \mathbb{Z}[x]$, $\deg Q_{3,n}^* < \deg Q_n$, all the coefficients of $Q_{3,n}^*(x)$ are $o(n)$ and the coefficient of the highest power of the polynomial $Q_{3,n}^*(x)$ is non-negative for all $n \in \mathbb{N}$. ¿From this, Lemma 6.2 and the fact that $Q_{3,N}^*(x) \neq 0$ since $B_{N,0} \neq 0$ we obtain that $S \notin \mathbb{Q}$. □

Example 6.2 *Let $k \in \mathbb{N}$ and $d(n)$ be the number of divisors of n. Then*

$$\sum_{N=2}^{\infty} \frac{d(N)N^k}{\prod_{n=2}^{N}(n - d(n))} \notin \mathbb{Q}.$$

Theorem 6.4 *Let $P(x), Q(x) \in \mathbb{Q}[x]$ and $P(x) = \sum_{i=0}^{T} a_i x^i$. Suppose that Q is not constant. Assume that $Q(n) \in \mathbb{Z}$ and $Q(n) \neq 0$ for all $n \in \mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{Z}$ be a sequence such that $f(N) = P(N) + o(Q(N))$ as $N \to \infty$. Suppose $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)} \in \mathbb{Q}$. Then there exists N_0 such that*

$$f(N) = P(N) - Q_2 Q_3(N) \qquad for\ N > N_0$$

where Q_2 and Q_3 are given by (36).

Proof. We have, by Lemma 6.5,

$$\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} Q(n)} = \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)} + \sum_{N=1}^{\infty} \frac{f(N) - P(N)}{\prod_{n=1}^{N} Q(n)}$$

$$= Q_1 + Q_2 \sum_{N=1}^{\infty} \frac{Q_3(N)}{\prod_{n=1}^{N} Q(n)} + \sum_{N=1}^{\infty} \frac{f(N) - P(N)}{\prod_{n=1}^{N} Q(n)}$$

$$= Q_1 + \sum_{N=1}^{\infty} \frac{Q_2 Q_3(N) + f(N) - P(N)}{\prod_{n=1}^{N} Q(n)}.$$

The numerator of the fraction is a rational number which is $o(Q(N))$ as $N \to \infty$ and has a denominator which is independent of N. Hence, by Lemma 6.2, $Q_2 Q_3(N) + f(N) - P(N) = 0 \quad for\ N > N_0$. □

Corollary 6.1 *Under the conditions of Theorem 6.4 we have that for all N, $P(N) \equiv Q_2 Q_3(N) \bmod 1$ and therefore P has denominator $\frac{1}{Q_2}$. Hence the value of Q_2 is determined by P.*

References

[1] Aigner M. : Combinatorial Theory, Classics in Mathematics, Springer 1997.

[2] Cantor, G.: Über die einfachen Zahlensysteme, Zeit. für Math. 14 (1869), 121-128. Coll. Papers, Berlin, 1932, pp. 35-53.

[3] Erdős P.: Sur certaines series à valeur irrationnelle, Enseignement Math. 4 (1958), 93–100.

[4] Erdős P. and Kac M.: Problem 4518, Amer. Math. Monthly 61 (1954), 264.

[5] Erdős P. and Straus E. G.: On the irrationality of certain series, Pacific J. Math. 55 (1974), 85-92.

[6] Hančl J.: A note to the rationality of infinite series I, Acta Math. Inf. Univ. Ostr. 5 (1997), 5–11.

[7] Hančl J.: A note to a paper of Oppenheim and Šalát concerning the series of Cantor type, Acta Math. Inf. Univ. Ostr. 10 (2002), 35-41.

[8] Hančl J. and Tijdeman, R.: On the irrationality of Cantor series, J. reine angew. Math. 571 (2004), 145-158.

[9] Hančl J., Tijdeman R. : On the rationality of Cantor and Ahmes series, Publicat. Math. 65, no. 3-4, (2004), 371–380.

[10] Hermite C.: Sur la fonction exponentielle, C.R. Acad. Sci. Paris 77 (1873), 18-24, 74-79, 226-233, 285-293; Oeuvres III, 150-181.

[11] Lambert, J.: Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques, Histoire Acad. Berlin (1761), 265-322; Opera Math. II, 112-159.

[12] Loxton, J.H. and van der Poorten, A.J.: Arithmetic properties of certain functions in several variables III, Bull. Austral. Math. Soc. 16 (1977), 15-47.

[13] Oppenheim A.: Criteria for irrationality of certain classes of numbers, Amer. Math. Monthly 61 (1954), 235–241.

[14] Rudin, W.: Principles of Mathematical Analysis, Mc Graw-Hill, New York etc., 1964.

[15] Shidlovski, :Transcendental Numbers, Moscow.

[16] Tijdeman R.: Mambu

[17] Tijdeman, R. and Yuan, P.Z.: On the rationality of Cantor and Ahmes series, Indag. Math. N.S. 13 (2002), 407-418.

Department of Mathematics and Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic,

Mathematisch Instituut, Universiteit Leiden, Niels Bohrweg 1, Postbus 9512, 2300 RA Leiden, The Netherlands, e-mail: tijdeman@math.leidenuniv.nl