ALGEBRAIC DEPENDENCE OF COMMUTING ELEMENTS IN ALGEBRAS

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Abstract. The aim of this paper to draw attention to several aspects of the algebraic dependence in algebras. The article starts with discussions of the algebraic dependence problem in commutative algebras. Then the Burchnall-Chaundy construction for proving algebraic dependence and obtaining the corresponding algebraic curves for commuting differential operators in the Heisenberg algebra is reviewed. Next some old and new results on algebraic dependence of commuting q-difference operators and elements in q-deformed Heisenberg algebras are reviewed. The main ideas and essence of two proofs of this are reviewed and compared. One is the algorithmic dimension growth existence proof. The other is the recent proof extending the Burchnall-Chaundy approach from differential operators and the Heisenberg algebra to the q-deformed Heisenberg algebra, showing that the Burchnall-Chaundy eliminant construction indeed provides annihilating curves for commuting elements in the q-deformed Heisenberg algebras for q not a root of unity.

1. Introduction

In 1994, one of the authors of the present paper, S. Silvestrov, based on consideration of the previous literature and a series of trial computations, made the following three part conjecture.

- The first part of the conjecture stated that the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved in greater generality, that is for much more general classes of non-commutative algebras and rings than the Heisenberg algebra and related algebras of differential operators treated by Burchnall and Chaundy and in subsequent literature.
- The second part stated that the Burchnall–Chaundy eliminant construction of annihilating algebraic curves formulated in determinant (resultant) form works well after some appropriate modifications for the most or possibly for all classes of algebras where the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved.
- Finally, the third part of the conjecture stated that the construction and the proof of the vanishing of the corresponding determinant algebraic curves on

2000 *Mathematics Subject Classification.* Primary 16S99; Secondary 81S05, 39A13.

Key words and phrases. q-deformed Heisenberg algebra, commuting elements, algebraic dependence, eliminant.

This work was supported by a visitor's grant of the Netherlands Organisation for Scientific Research (NWO), the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Crafoord Foundation, the Royal Physiographic Society in Lund, and the Royal Swedish Academy of Sciences. We are also grateful to Lars Hellström and Daniel Larsson for helpful comments and discussions.

the commuting elements can be performed for all classes of algebras or rings where this fact is true, using only the internal structure and calculations for the elements in the corresponding algebras or rings and the algebraic combinatorial expansion formulas and methods for the corresponding determinants, that is, without any need of passing to operator representations and use of analytic methods as used in the Burchnall–Chaundy type proofs.

This third part of the conjecture remains widely open with no general such proofs available for any classes of algebras and rings, even in the case of the usual Heisenberg algebra and differential operators, and with only a series of examples calculated for the Heisenberg algebra, q-Heisenberg algebra and some more general algebras, all supporting the conjecture. In the first and the second part of the conjecture progress has been made. In [6], the key Burchnall–Chaundy type theorem on algebraic dependence of commuting elements in q-deformed Heisenberg algebras (and thus as a corollary for q-difference operators as operators representing q-deformed Heisenberg algebras) was obtained. The result and the methods have been extended to more general algebras and rings generalizing q-deformed Heisenberg algebras (generalized Weyl structures and graded rings) in [7]. The proof in [6] is totally different from the Burchnall–Chaundy type proof. It is an existence argument based only on the intrinsic properties of the elements and internal structure of q-deformed Heisenberg algebras, thus supporting the first part of the conjecture. It can be used successfully for an algorithmic implementation for computing the corresponding algebraic curves for given commuting elements. However, it does not give any specific information on the structure or properties of such algebraic curves or any general formulae. It is thus important to have a way of describing such algebraic curves by some explicit formulae, as for example those obtained using the Burchnall–Chaundy eliminant construction for the $q = 1$ case, i.e., for the classical Heisenberg algebra. In [11], a step in that direction was made by offering a number of examples all supporting the claim that the eliminant determinant method should work in the general case. However, no general proof for this was provided. The complete proof following the footsteps of the Burchnall-Chaundy approach in the case of q not a root of unity has been recently obtained [8], by showing that the determinant eliminant construction, properly adjusted for the q-deformed Heisenberg algebras, gives annihilating curves for commuting elements in the q-deformed Heisenberg algebra when q is not a root of unity, thus confirming the second part of the conjecture for these algebras. In our proof we adapt the Burchnall-Chaundy eliminant determinant method of the case $q = 1$ of differential operators to the q-deformed case, after passing to a specific faithful representation of the q-deformed Heisenberg algebra on Laurent series and then performing a detailed analysis of the kernels of arbitrary operators in the image of this representation. While exploring the determinant eliminant construction of the annihilating curves, we also obtain some further information on such curves and some other results on dimensions and bases in the eigenspaces of the q-difference operators in the image of the chosen representation of the q-deformed Heisenberg algebra. In the case of q being a root of unity the algebraic dependence of commuting elements holds only over the center of the q-deformed Heisenberg algebra [6], and it is unknown yet how to modify the eliminant determinant construction to yield annihilating curves for this case.

The present article starts with discussions of the algebraic dependence problem in algebras. Then the Burchnall-Chaundy construction for proving algebraic dependence and obtaining corresponding algebraic curves for commuting differential operators and commuting elements in the Heisenberg algebra is reviewed. Next some old and new results on algebraic dependence of commuting q-difference operators and elements in the q-deformed Heisenberg algebra are discussed. In the final two subsections we review two proofs for algebraic dependence of commuting elements in the q-deformed Heisenberg algebra. The first one is the recent proof from [8] extending the Burchnall-Chaundy approach from differential operators and the Heisenberg algebra to q-difference operators and the q-deformed Heisenberg algebra, showing that the Burchnall-Chaundy eliminant construction indeed provides annihilating curves for q-difference operators and for commuting elements in q-deformed Heisenberg algebras for q not a root of unity. The second one is the algorithmic dimension growth existence proof from [6].

2. DESCRIPTION OF THE PROBLEM: COMMUTING ELEMENTS IN AN ALGEBRA ARE GIVEN, THEN FIND CURVES THEY LIE ON

Any two elements α and β in a field k lie on an algebraic curve of the second degree $F(x,y) = (x-\alpha)(y-\beta) = xy - \alpha y - \beta x + \alpha\beta = 0$. The important feature of this curve is that its coefficients are also elements in the field k. The same holds if the field k is replaced by any commutative k-algebra R, except that then the coefficients in the *annihilating polynomial* F are elements in R, and hence it becomes of interest from the side of building interplay with the algebraic geometry over the field k to determine whether one may find an annihilating polynomial with coefficients from k for any two elements in the commutative algebra R. It is well-known that this is not always possible even in the case of the ordinary commutative polynomial algebras aver a field unless some special conditions are imposed on the considered polynomials. The situation is similar of course for rational functions or many other commutative algebras of functions. The ideals of polynomials annihilating subsets in an algebra are also well-known to be fundamental for algebraic geometry, for Gröbner basis analysis in computational algebra and consequently for various applications in Physics and Engineering.

Another appearance of this type of problems worth mentioning in our context comes from Galois theory and number theory in connection to algebraic and transcendental field extensions. If L is a field extension $K \subset L$ of a field K, then using Zorn's lemma one can show that there always exists a subset of L which is maximal algebraically independent over K. Any such subset is called transcendence basis of the field L over the subfield K (or of the extension $K \subset L$). All transcendence bases have the same cardinality called the transcendence degree of a field extension. If $B = \{b_1, \dots, b_n\}$ is a finite transcendence basis of $K \subset L$, then L is an algebraic extension of the subfield $K(B)$ in L generated by B over the subfield K. This means in particular that for any $l \in L$ the set $\{B, l\}$ is algebraically dependent over K, that is, there exists a polynomial F in $n+1$ indeterminates over K such that $F(b_1, \dots, b_n, a) = 0$. Investigation of algebraic dependence, transcendence basis and transcendence degree is a highly non-trivial important direction in number theory and theory of fields with striking results and many longstanding open problems. For example, if algebraic numbers $a_1, ..., a_n$ are linearly independent over $\mathbb{Q}$, then $e^{a_1}, ..., e^{a_n}$ are algebraically independent over $\mathbb{Q}$ (Lindemann-Weierstrass theorem, 1880's). Whether e and π are algebraically dependent over $\mathbb{Q}$ or not is unknown, and only relatively recently (1996) a long-standing conjecture on algebraic independence of π and e^{π} was confirmed [13]. Comprehensive overviews and references in this direction can be found in [14].

In view of the above, naturally important problems are to describe, analyze and classify:

(C1) commutative algebras over K in which any pair of elements is algebraically dependent over K;

(C2) pairs (A, B) of a commutative algebra A and a subalgebra B over a field K such that any pair of elements in A is algebraically dependent over B.

Problem (C1) is of course a special case of (C2).

In the polynomial algebra $K[x_1, \ldots, x_n]$ generated by n independent commuting indeterminates, for instance the set $\{x_1, \ldots, x_n\}$ as well as any of its non-empty subsets are algebraically independent over K. Thus $K[x_1, \ldots, x_n]$ does not belong to the class of algebras in the problem (C1). The same holds of course for any algebra containing $K[x_1, \ldots, x_n]$. In general algebraic dependence of polynomials happens only under some restrictive conditions on the rank (or vanishing) of their Jacobian and other fine aspects.

This indicates that some principal changes have to be introduced in order to be able to get examples of algebras satisfying (C1) or (C2). It turns out that the range of possibilities expands dramatically within the realm of non-commutative geometry, if the indeterminates (the coordinates) are non-commuting.

In any algebra there are always commutative subalgebras, for instance any subalgebra generated by any single element. If the algebra contains for instance a non-zero nilpotent element ($a^n = 0$ for some $n > 2$), then the subalgebra generated by this element satisfies ($C1$). Indeed, any two elements $f_1 = ap(a) + p_0$ and $f_2 = aq(a) + q_0$ in this commutative subalgebra are annihilated by $F(s, t) = (q_0 s - p_0 t)^n$, since $F(f_1, f_2) = (q_0 f_1 - p_0 f_2)^n = a^n (q_0 p(a) - p_0 q(a))^n = 0$.

For the non-commutative algebras the problems inspired by $C1$ and $C2$ are to describe, analyze and classify:

- (NC1) algebras over a field K such that in all their commutative subalgebras any pair of elements is algebraically dependent over the field;
- (NC2) pairs (A, B) of an algebra A and a subalgebra B over a field K such that any pair of commuting elements in A is algebraically dependent over B.

Problem (NC1) is of course a special case of (NC2).

From the situation in commutative algebras as described above, one might intuitively expect that finding non-commutative algebras satisfying (NC1) is difficult if not impossible task. However, this perception is not quite correct. One of general purposes for the present article is to illuminate this phenomena.

3. Burchnall-Chaundy construction for differential operators

Among the longest known, constantly studied and used all over Mathematics non-commutative algebras is the so called Heisenberg algebra, also called Weyl algebra or Heisenberg-Weyl algebra. It is defined as an algebra over a field K with two generators A and B and defining relations $AB - BA = I$, or equivalently as $\mathcal{H}_1 = K \langle A, B \rangle / \langle AB - BA - I \rangle$, the quotient algebra of a free algebra on two generators A and B by the two-sided ideal generated by $AB - BA - I$ where I is the unit element.

It's ubiquity is due to the fact that the basic operators of differential calculus $A = D = \frac{d}{dx}$ and $M_x : f(x) \mapsto xf(x)$ satisfy the Heisenberg algebra defining relation $DM_x - M_xD = I$ on polynomials, formal power series, differentiable functions or any linear spaces of functions invariant under these operators due to the Leibniz rule. On any such space, the representation (D, M_x) of the commutation relation $AB - BA = I$ defines a representation of the algebra $\mathcal{H}_1$, called sometimes, especially in Physics, the (Heisenberg) canonical representation. We will use sometimes this term as well, for shortage of presentation.

If K is a field of characteristic zero, then the algebra

$$\mathcal{H}_1 = K\langle A, B\rangle / \langle AB - BA - I\rangle$$

is simple, meaning that it does not contain any two-sided ideals different from zero ideal and the whole algebra. The kernel $\mathrm{Ker}(\pi) = \{a \in A \mid \pi(a) = 0\}$ of any representation π of an algebra is a two-sided ideal in the algebra. Thus any non-zero representation of $\mathcal{H}_1$ is faithful, which is in particular holds also for a canonical representation. Thus the algebra $\mathcal{H}_1$ is actually isomorphic to the ring of differential operators with polynomial coefficients acting for instance on the linear space of all polynomials or on the space of formal power series in a single variable.

In the literature on algebraic dependence of commuting elements in the Heisenberg algebra and its generalizations – a result which is fundamental for the algebro-geometric method of constructing and solving certain important non-linear partial differential equations – one can find several different proofs of this fact, each with its own advantages and disadvantages. The first proof of such result utilizes analytical and operator theoretical methods. It was first discovered by Burchnall and Chaundy [2] in the 1920's. Their articles [2, 3, 4] contain also pioneering results in the direction of in-depth connections to algebraic geometry. These fundamental papers were largely forgotten for almost fifty years when the main results and the method of the proofs of Burchnall and Chaundy were rediscovered in the context of integrable systems and non-linear differential equations [9, 10, 12]. Since the 1970's, deep connections between algebraic geometry and solutions of non-linear differential equations have been revealed, indicating an enormous richness largely yet to be explored, in the intersection where non-linear differential equations and algebraic geometry meet. This connection is of interest both for its own theoretical beauty and because non-linear differential equations appear naturally in a large variety of applications, thus providing further external motivation and a source of inspiration for further research into commutative subalgebras and their interplay with algebraic geometry.

A second, more algebraic approach to proving the algebraic dependence of commuting differential operators was obtained in a different context by Amitsur [1] in the 1950's. Amitsur's approach is more in the direction of the classical connections with field extensions we have already mentioned. Recently in the 1990's a more algorithmic combinatorial method of proof based on dimension growth considerations has been found in [5, 6]. The main motivating problem for these developments was to describe, as detailed as possible, algebras of commuting differential operators and their properties. The solution of this problem is where the interplay with algebraic geometry enters the scene. The Burchnall-Chaundy result is responsible for this connection as it states that commuting differential operators satisfy equations for certain algebraic curves, which can be explicitly calculated for each pair of commuting operators by the so called eliminant method. The formulas for these curves, obtained from this method by using the corresponding determinants, are important for their further analysis and for applications and further development of the general method and interplay with algebraic geometry.

In the rest of this section we will briefly review the basic steps of the Burchnall–Chaundy construction. For simplicity of exposition until the end of this section we assume that the field of scalars is $K = \mathbb{C}$.

Commutativity of a pair of differential operators

$$P = \sum_{i=0}^{m} p_i(t)\partial^i, \qquad Q = \sum_{i=0}^{n} q_i(t)\partial^i$$

where p_i, q_i are analytic functions in t and $\partial := \frac{\mathrm{d}}{\mathrm{d}t}$, puts severe restrictive conditions on the functions p_i and q_i. In its original formulation the result of Burchnall and Chaundy can be stated as follows.

Theorem 3.1 (Burchnall–Chaundy, 1922)**.** *For any two commuting differential operators P and Q, there is a polynomial $F(x, y) \in \mathbb{C}[x, y]$ such that $F(P, Q) = 0$.*

The polynomial appearing in this theorem is often referred to as the *Burchnall–Chaundy polynomial*.

It is worth mentioning that in their papers Burchnall and Chaundy have neither specified any conditions on what kind of functions the coefficients in the differential operators are, nor the spaces on which these operators act. Thus more precise formulations of the result should contain such a specification. Any space of functions where the construction is valid would be fine (e.g., polynomials or analytic functions in the complex domain). Algebraic steps in the construction are generic. However in order to reach the main conclusions on the existence and annihilating property of the curves, the Burchnall-Chaundy considerations use existence of solutions of an eigenvalue problem for ordinary differential operators and the property that the dimension of the solution space of a homogeneous differential equation does not exceed the order of the operator. To ensure these properties, coefficients of differential operators must be required to belong to not very restrictive but nevertheless specific classes of functions.

We will now sketch Burchnall-Chaundy construction for the convenience of the reader and in connection with further considerations in the present article. In spite of the fact that the Burchnall-Chaundy arguments actually do not constitute a complete proof due to some serious gaps, they provide important insight for building annihilating curves which can be developed into a complete proof and a well functioning construction after appropriate adjustments and restructuring.

If differential operators P and Q of orders m and n, respectively, commute then $P - h$ and Q commute for any constant $h \in \mathbb{C}$. Thus $Q(\mathrm{Ker}(P - h)) \subseteq \mathrm{Ker}(P - h)$. Consequently, if $y_1, \dots, y_m$ is a basis of $\mathrm{Ker}(P - h)$, the fundamental set of solutions of the eigenvalue problem for the differential equation $P(y) - hy = 0$ (note that the existence and the dimension properties of the solution space are assumed to hold here), then $Q(y_1), \dots, Q(y_m)$ are also elements of $\mathrm{Ker}(P - h)$ and hence $Q(\vec{y}) = A\vec{y}$ for some matrix $A = (a_{i,j})_{i,j=1}^{m}$ with entries from $\mathbb{C}$. Let $k \in \mathbb{C}$ be another arbitrary constant. A common nonzero solution, $Y = \vec{c}^T\vec{y}, \vec{c} \in \mathbb{C}^n$, of eigenvalue problems

$PY = hY$, $QY = hY$, or equivalently a nonzero $Y \in \mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k)$, exists if and only if $(Q-k)Y = \vec{c}^{\,T}(A-k)\vec{y} = 0$ has a nonzero solution $\vec{c}$, which happens only if $\det(A-k) = 0$. This is a polynomial in k of order m and hence corresponding to each h there exists only m values of the constant k (not necessarily all distinct) such that there exists nonzero $Y \in \mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k)$ Note that here it was used that the scalar field is algebraically closed. Similarly, corresponding to each k there exists n values of h such that $\mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k) \neq \{0\}$. From this Burchnall and Chaundy conclude that if $\mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k) \neq \{0\}$, then h and k satisfy some polynomial equation $F(h,k) = 0$, where F is a polynomial of degree n in h and m in k with coefficients from $\mathbb{C}$. There is however a problem with this key conclusion. Surely, the equation $h = 2k + \sin(k)$ gives a bijection between the h's and the k's, but the resulting curve is not algebraic. Thus as we already pointed out, substantial adjustments are necessary in order to rearrange these arguments into a functioning construction and a complete proof. Suppose, however, that the proper adjustments have been made, and that a polynomial F with the above annihilating properties has been found. Then, any $Y \in \mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k)$ is also a solution of the differential equation $F(P,Q)Y = F(h,k)Y = 0$ which is of order mn unless it happens that $F(P,Q) = 0$. Thus there can be at most mn linearly independent $Y \in \mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k)$. Note that here again a specific property of the dimension of the solution space of a differential equation is assumed to hold. For each h there exists k such that $\mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k) \neq \{0\}$. Since the field $\mathbb{C}$ is infinite (note that it is another special property of the field), one can chose infinitely many pairwise distinct numbers h with corresponding k and nonzero functions $Y_{h,k} \in \mathrm{Ker}(P-h) \cap \mathrm{Ker}(Q-k)$. But any nonempty set of eigenfunctions with pairwise distinct eigenvalues for a linear operator is always linearly independent. Thus the dimension of the solution space $\mathrm{Ker}(F(P,Q))$ is infinite. But this contradicts to the already proved $\mathrm{Ker}(F(P,Q)) \leq mn$ unless $F(P,Q) = 0$. Therefore, indeed $F(P,Q) = 0$ which is exactly what was claimed.

A beautiful feature of the Burchnall-Chaundy arguments in the differential operator case, however, is that they are almost constructive in the sense that they actually tell us, after taking a closer look, how to compute such annihilating curves, given the commuting operators. This is done by constructing the *resultant* (or *eliminant*) of operators P and Q. We sketch this construction, as it is important to have in mind for this article. To this end, for complex variables h and k, one writes:

$$\partial^r(P - h\mathbf{1}) = \sum_{i=0}^{m+r} \theta_{i,r}\partial^i - h\partial^r, \qquad r = 0, 1, \ldots, n-1 \tag{1}$$

$$\partial^r(Q - k\mathbf{1}) = \sum_{i=0}^{n+r} \omega_{i,r}\partial^i - k\partial^r, \qquad r = 0, 1, \ldots, m-1 \tag{2}$$

where $\theta_{i,r}$ and $\omega_{i,r}$ are certain functions built from the coefficients of P and Q respectively, whose exact form is calculated by moving ∂^r through to the right of the coefficients, using the Leibniz rule. The coefficients of the powers of ∂ on the right hand side in (1) and (2) build up the rows of a matrix exactly as written. That is, as the first row we take the coefficients in $\sum_{i=0}^{m} \theta_{i,0}\partial^i - h\partial^0$, and as the second row the coefficients in $\sum_{i=0}^{m+1} \theta_{i,1}\partial^i - h\partial$, continuing this until $k = n-1$. As the n^{th} row we take the coefficients in $\sum_{i=0}^{n} \omega_{i,0}\partial^i - k\partial^0$, and as the $(n+1)^{\text{th}}$ row we take the coefficients in $\sum_{i=0}^{n+1} \omega_{i,1}\partial^i - k\partial$ and so on. In this manner we get a

$(n+m)\times(n+m)$-matrix using (1) and (2). The determinant of this matrix yields a trivariate polynomial $F(x,h,k)$ over $\mathbb{C}$. When written as $F(x,h,k)=\sum_i \delta_i(h,k)x^i$, it can be proved, using existence and uniqueness results for ordinary differential equations, that $\delta_i(P,Q)=0$ for all i. It is not difficult to see that the δ_i are not all zero.

For clarity, we include the following example.

Example 3.2. Let P and Q be as above, with $m=3$ and $n=2$. We then have $F(x,h,k)=$

$$\begin{vmatrix} p_{0,0}(x)-h & p_{0,1}(x) & p_{0,2}(x) & p_{0,3}(x) & 0 \\ p_{1,0}(x) & p_{1,1}(x)-h & p_{1,2}(x) & p_{1,3}(x) & p_{1,4}(x) \\ q_{0,0}(x)-k & q_{0,1}(x) & q_{0,2}(x) & 0 & 0 \\ q_{1,0}(x) & q_{1,1}(x)-k & q_{1,2}(x) & q_{1,3}(x) & 0 \\ q_{2,0}(x) & q_{2,1}(x) & q_{2,2}(x)-k & q_{2,3}(x) & q_{2,4}(x) \end{vmatrix}.$$

Remark 3.3. Thus the pairs of eigenvalues $(h,k)\in\mathbb{C}^2$ corresponding to the same joint eigenfunction, i.e., corresponding to the same non-zero y such that

$$Py=hy \qquad \text{and} \qquad Qy=ky,$$

lie on curves Z_i defined by the δ_i. There is a "dictionary" between certain geometric and analytic data [12] allowing in particular construction of common eigenfunctions for commuting operators using geometric data associated to the compactification of the annihilating algebraic curve.

Burchnall–Chaundy result for operators with polynomial coefficients can be reformulated in more general terms for the abstract Heisenberg algebra $\mathcal{H}_1$ rather than in terms of its specific canonical representation by differential operators. This specialization of coefficients to polynomials does not influence the Burchnall–Chaundy construction of the annihilating algebraic curves. Thus restricting to this context does not effect the main ingredients needed for building the interplay with algebraic geometry. At the same time, when moreover reformulated entirely in the general terms of the algebra $\mathcal{H}_1$, this specialization becomes a generalization in another way, because establishing algebraic dependence of commuting elements directly in the Heisenberg algebra, without passing to the canonical representation, makes the result valid not just for differential operators, but also for any other representations of $\mathcal{H}_1$ by other kinds of operators.

Theorem 3.4 ("Burchnall–Chaundy", algebraic version)**.** *Let P and Q be two commuting elements in $\mathcal{H}_1$, the Heisenberg algebra. Then there is a bivariate polynomial $F(x,y)\in\mathbb{C}[x,y]$ such that $F(P,Q)=0$.*

4. Burchnall–Chaundy theory for the q-deformed Heisenberg algebra

Let K be a field. If $q\in K$ then $\mathcal{H}_q=\mathcal{H}_K(q)$, the q-deformed Heisenberg algebra over K, is the unital associative K-algebra which is generated by two elements A and B, subject to the commutation relation $AB-qBA=I$. Though this algebra sometimes is also called the q-deformed Weyl or q-deformed Heisenberg-Weyl algebra in various contexts, we will follow systematically the terminology in [6]. We will indicate how one can prove that - under a condition on q - for any commuting $P,Q\in\mathcal{H}_q$ of order at least one (where "order" will be defined below), there exist finitely many explicitly calculable polynomials $p_i \in K[X,Y]$ such that $p_i(P,Q) = 0$ for all i, and at least one of the p_i is non-zero. The number of polynomials depends on the coefficients of P and Q, as well as on their orders. The polynomials p_i can be obtained by the mentioned before so-called eliminant construction which was introduced for differential operators (the case of $q = 1$) by Burchnall and Chaundy in [2, 3, 4] in 1920's, and which we will employ and extend to the context of general q-deformed Heisenberg algebras showing that analogous determinant (resultant) construction of the annihilating algebraic curves works for q-deformed Heisenberg algebras well.

We assume $q \neq 0$ throughout (our method breaks down when $q = 0$) and define the q-integer $\{n\}_q$, for $n \in \mathbb{Z}$, by

$$\{n\}_q = \begin{cases} \frac{q^n-1}{q-1} & q \neq 1; \\ n & q = 1. \end{cases}$$

Following [6, Definition 5.2] we will say that $q \in K^* = K \setminus \{0,1\}$ is of *torsion type* if there is a positive integer solution p to $q^p = 1$, and of *free type* if the only integer solution to $q^p = 1$ is $p = 0$. In the torsion type case the least such positive integer p is called the order of q and in free type case the order of q is said to be zero. If $q = 1$ then it is said to be of a torsion type if K is a field of non-zero characteristic, and of free type if the field is of characteristic zero.

Remark 4.1. The following are equivalent for $q \neq 0$:

(1) for $n \in \mathbb{Z}$, $\{n\}_q = 0$ if and only if $n = 0$;
(2) for $n_1, n_2 \in \mathbb{Z}$, $\{n_1\}_q = \{n_2\}_q$ if and only if $n_1 = n_2$;
(3) $\begin{cases} q \text{ is not a root of unity other than } 1, & \text{if } \operatorname{char} k = 0; \\ q \text{ is not a root of unity}, & \text{if } \operatorname{char} k \neq 0. \end{cases}$

Part (2) of this remark is essential when one considers the dimension of eigenspaces for q-difference operators which is important for our extension of the Burchnall-Chaundy construction.

Note also that under our assumptions K is infinite.

Let $\mathcal{L}$ be the K-vector space of all formal Laurent series in a single variable t with coefficients in K. Define

$$M(\sum_{n=-\infty}^{\infty} a_n t^n) = \sum_{n=-\infty}^{\infty} a_n t^{n+1} = \sum_{n=-\infty}^{\infty} a_{n-1} t^n,$$

$$D_q(\sum_{n=-\infty}^{\infty} a_n t^n) = \sum_{n=-\infty}^{\infty} a_n \{n\}_q t^{n-1} = \sum_{n=-\infty}^{\infty} a_{n+1} \{n+1\}_q t^n.$$

Alternatively, one could introduce $\mathcal{L}$ as the vector space of all functions from $\mathbb{Z}$ to K and let M act as the right shift and D_q as a weighted left shift, but the Laurent series model is more appealing.

The algebra $\mathcal{H}_q$ has $\{I, A, A^2, \dots\}$ as a free basis in its natural structure as a left $K[X]$-module with X mapped to B.

So an arbitrary non-zero element P of $\mathcal{H}_q$ can be written as

$$P = \sum_{j=0}^{m} p_j(B) A^j, p_m \neq 0,$$

for uniquely determined $p_j \in K[X]$ and $m \geq 0$. The integer m is called the *order* of P (or degree with respect to A) [6].

By sending A to D_q and B to M, $\mathcal{L}$ becomes an $\mathcal{H}_q$-module. If we make the additional assumption that $\{n\}_q \neq 0$ or equivalently $q^n \neq 1$ for all non-zero $n \in \mathbb{Z}$, then this representation is faithful [6, Theorem 8.3].

We will assume that $q \neq 0$ and $\{n\}_q \neq 0$ for $n \neq 0$ throughout this paper and identify $\mathcal{H}_q$ with its image in $\mathrm{End}_K(\mathcal{L})$ under the previously defined representation without further notice. In the image of $\mathcal{H}_q$ under the representation, $\{1, D_q, D_q^2, \ldots\}$ is a free basis in its natural structure as a left $K[X]$-module with X being mapped to M, and so any P can be written as

$$P = \sum_{j=0}^{m} p_j(M) D_q^j, \; p_m \neq 0,$$

for uniquely determined $p_j \in K[X]$ and $m \geq 0$. The integer m is called the *order* of P.

The important result in the context of this article is an extension of the Burchnall–Chaundy theorem in algebraic version for the q-deformed Heisenberg algebra $\mathcal{H}_q$, due to Hellström and Silvestrov [6, Therem 7.5].

Theorem 4.2 (Hellström–Silvestrov, [6]). *Let $q \in K^* = K \setminus \{0\}$, and let P and Q be two commuting elements in $\mathcal{H}_q$. Then there is a bivariate polynomial $F(x, y) \in \mathrm{Z}(\mathcal{H}_q)[x, y]$, with coefficients in the center $\mathrm{Z}(\mathcal{H}_q)$ of $\mathcal{H}_q$, such that $F(P, Q) = 0$.*

If q is not a root of unity ($\{n\}_q$ are different for different n), the center of $\mathcal{H}_q$ is trivial, i.e., consists of scalar multiples of the identity $\mathrm{Z}(\mathcal{H}_q) = KI$. Thus in this case there exists a "genuine" algebraic curve over the scalar field K as Theorem 4.2 takes the following form [6, Theorem 7.4].

Theorem 4.3 (Hellström–Silvestrov, [6]). *Let $q \in K^* = K \setminus \{0\}$ be of free type, and let P and Q be two commuting elements in $\mathcal{H}_q$. Then there is a bivariate polynomial $F(x, y) \in K[x, y]$, with coefficients in K, such that*

$$F(P, Q) = 0.$$

Note however, that when q is of torsion type and order d (i.e., d is the smallest positive integer such that $q^d = 1$), then the center is

$$\mathrm{Z}(\mathcal{H}_q) = K[A^d, B^d],$$

the subalgebra spanned by $\{A^d, B^d\}$, where d is the order of q, the minimal positive integer such that $q^d = 1$ ([6, Corollary 6.12]). The conclusion of Theorem 4.3, that is the algebraic dependence of commuting elements over the field K, does not hold for q of torsion type, since if p is the order of q, then $\alpha = A^p$ and $\beta = B^p$ commute, but do not satisfy any commutative polynomial relation. Thus in this case Theorem 4.2 has indeed to be invoked.

The proof of Theorem 4.2 as given in [6] is purely existential. However, while it says essentially nothing theoretically on the form or properties of the annihilating curves, the construction used in the proof actually provides an explicit computationally implementable algorithm for producing annihilating polynomials.

A specialization of a part of the general conjecture made by S. Silvestrov in 1994 is that the determinant scheme devised by Burchnall and Chaundy could be used to calculate the polynomial even in the case of $\mathcal{H}_q$. The Burchnall-Chaundy eliminant construction adaptation to $\mathcal{H}_q$ and a series of examples, indicating that the construction indeed yields annihilating algebraic curves for commuting elements in $\mathcal{H}_q$, was first presented in [11]. But no full proof of the conjecture that this adaption is possible for all commuting elements of $\mathcal{H}_q$ was given. Even though a direct generalization of the classical arguments of Burchnall and Chaundy was attempted some important technical steps were missing. The main reason why an analogous proof for q-difference operators (i.e., elements in $\mathcal{H}_q$) is problematic is that the solution space for q-difference equations may not in general be as well-behaved as for ordinary differential operators, and the proof of the classical Burchnall–Chaundy theorem relies heavily on properties of the solution spaces to the eigenvalue-problems for the differential operators P and Q.

Nevertheless, in the case of q of free type we have succeeded now to extend the original Burchnall-Chaundy eliminant method to the q-difference operators and hence to $\mathcal{H}_q$. Roughly speaking, the key technical idea is to choose an appropriate representation space for the canonical representation of $\mathcal{H}_q$ generated by M_x, D_q, so that the necessary key ingredients about dimensions of eigenspaces of q-difference operators in the Burchnall-Chaundy type proof are still available. The module $\mathcal{L}$ introduced above has all these properties. It is thus that, for q of free type, we again have a determinant based construction providing annihilating curves. The complete proofs will be presented in [8]; they are considerably more involved than the original work by Burchnall and Chaundy.

An extension of the result and the Burchnall-Chaundy type construction to the case of q of torsion type, i.e., when $\mathcal{H}_q$ has non-trivial center, is still not available. This is just one of many reasons why another proof of the possibility of adapting to $\mathcal{H}_q$ the determinant construction of the annihilating curves, relying on purely algebraic methods, i.e., without passing to a specific representation of $\mathcal{H}_q$, is desirable. The existence of such proof is a specialization to $\mathcal{H}_q$ of the third part of the aforementioned general conjecture by S. Silvestrov.

4.1. **Eliminant determinant construction for q-deformed Heisenberg algebra.** In this section we will briefly outline our extension of the Burchnall-Chaundy result and constructions to the q-difference operators, or to the abstract algebra context of the q-deformed Heisenberg algebra. The complete details of the proofs will be presented in [8].

Let $P, Q \in \mathcal{H}_q$ be of order $m \geq 1$ and $n \geq 1$ respectively. Write for $k = 0, \dots, n-1$,

$$D_q^k P = \sum_{j=0}^{m+k} p_{k,j}(M) D_q^j, \text{ with } p_{k,j} \in K[X],$$

and, for $l = 0, \dots, m-1$, write

$$D_q^l Q = \sum_{j=0}^{n+l} q_{l,j}(M) D_q^j, \text{ with } q_{l,j} \in K[X].$$

By analogy with the Burchnall-Chaundy method for differential operators, we build up an $(m+n) \times (m+n)$-matrix as follows. For $k = 1, \dots, n$, the k-th row is given by the coefficients of powers of D_q in the expression $D_q^{k-1}P - \lambda D_q^{k-1} = \sum_{j=0}^{m+k-1} p_{k-1,j}(M) D_q^j - \lambda D_q^{k-1}$, where λ is a formal variable. For

$k \in \{n+1,\ldots,m+n\}$, the k-th row is given by the coefficients of D_q in $D_q^{k-n-1}Q - \mu D_q^{k-n-1} = \sum_{j=0}^{k-1} p_{k-n-1,j}(M)D_q^j - \mu D_q^{k-n-1}$, where μ is a formal variable different from λ. The determinant of this matrix is called the *eliminant* of P and Q. We denote it $\Delta_{(P,Q)}(M,\lambda,\mu)$. It is a polynomial with coefficients in K.

Theorem 4.4. ([8]) *Let K be a field and $0 \neq q \in K$ be such that $\{n\}_q = 0$ if and only if $n = 0$. Suppose*

$$P = \sum_{j=0}^{m} p_j(M)D_q^j \quad (m \geq 1, p_m \neq 0)$$

and

$$Q = \sum_{j=0}^{n} q_j(M)D_q^j \quad (n \geq 1, p_n \neq 0)$$

are commuting elements of $\mathcal{H}_q$, and let $\Delta_{P,Q}(M,\lambda,\mu)$ be the eliminant constructed as above. Then $\Delta_{P,Q} \neq 0$. In fact, if $q_n(M) = \sum_i a_i M^i$ ($a_i \in K$) then $\Delta_{P,Q}$ has degree n as a polynomial in λ, and its non-zero coefficient of λ^n is equal to $(-1)^n \prod_{k=0}^{m-1}(\sum_i a_i q^{k\cdot i} M^i)$. Likewise, if $p_m(M) = \sum_i b_i M^i$, then $\Delta_{P,Q}$ has degree m as a polynomial in μ, and its non-zero coefficient of μ^m is equal to $(-1)^m \prod_{k=0}^{n-1}(\sum_i b_i q^{k\cdot i} M^i)$.

Let

$$s = n \cdot \max_j \deg(p_j) + m \cdot \max_j \deg(q_j),$$

$$t = \frac{1}{2} \cdot n \cdot (n-1) \cdot \max_j \deg(p_j) + \frac{1}{2} \cdot m \cdot (m-1) \cdot \max_j \deg(q_j),$$

and define the polynomials δ_i ($i = 1,\ldots,s$) by $\Delta_{P,Q}(M,\lambda,\mu) = \sum_{i=0}^{s} \delta_i(\lambda,\mu)M^i$.

Then

1. *each of the coefficients of the δ_i can be expressed as $\sum_{l=0}^{t} r_l q^l$ with the r_l in the subring of K which is generated by the coefficients of all the p_j and the q_j,*
2. *at least one of the δ_i is non-zero,*
3. *$\delta_i(P,Q) = 0$ for all $i = 0,\ldots,s$.*

The reader will easily convince himself of all statements in the theorem other than (3). We will now sketch the main idea of the proof of (3) as given in [8].

The idea is as follows. Suppose $\lambda_0, \mu_0 \in K$ and $0 \neq v_{(\lambda_0,\mu_0)} \in \mathcal{L}$ is a common eigenvector of P and Q:

$$P v_{(\lambda_0,\mu_0)} = \lambda_0 v_{(\lambda_0,\mu_0)},$$

$$Q v_{(\lambda_0,\mu_0)} = \mu_0 v_{(\lambda_0,\mu_0)}.$$

Then the specialization $\lambda = \lambda_0, \mu = \mu_0$ of the matrix of endomorphisms of $\mathcal{L}$ that defines the eliminant has the column vector $(v_{(\lambda_0,\mu_0)},\ldots,D_q^{m+n-1}v_{(\lambda_0,\mu_0)})$ in its kernel. Hence $\Delta_{P,Q}(M,\lambda_0,\mu_0)\, v_{(\lambda_0,\mu_0)} = 0$. Now it does not follow automatically from this that $\Delta_{P,Q}(M,\lambda_0,\mu_0) = 0$ in $\mathcal{H}_q$ since a polynomial in M might have non-trivial kernel, as the example $(M-1)\sum_n t^n = 0$ shows. However, embedding K in an algebraically closed field if necessary, we are able to show that there exist infinitely many such (λ_0,μ_0) where we *can* conclude that $\Delta_{P,Q}(M,\lambda_0,\mu_0) = 0$ in $\mathcal{H}_q$. Thus the operators $\delta_i(P,Q)$ have an infinite-dimensional kernel, and it is possible to show that this implies that $\delta_i(P,Q) = 0$ in $\mathcal{H}_q$.

It is more complicated to show that there exist infinitely many (λ_0, μ_0) with the required property. The idea is to exploit the fact that $v_{(\lambda_0,\mu_0)}$ is both in the kernel of the operator $P - \lambda_0$ of order $m \geq 1$ and of the operator $\Delta_{P,Q}(M, \lambda_0, \mu_0)$ which, if it is not zero, is not constant. We can describe the kernel of a non-constant polynomial element $p(M)$ of $\mathcal{H}_q$ and the action of $P - \lambda_0$ on it explicitly enough to show that any such $v_{(\lambda_0,\mu_0)}$ is in a subspace of *finite* dimension which (for fixed q) depends only on the leading coefficient of P and the degree of P, but not on λ_0 or μ_0. Hence for the infinity of pairs (λ_0, μ_0) that can be shown to exist, it can, by linear independence, only for finitely many pairs be the case that $\Delta_{P,Q}(M, \lambda_0, \mu_0)$ is not zero. For the remaining pairs, the specialized eliminant must be zero.

4.2. **Hellström-Silvestrov proof of algebraic dependence in $\mathcal{H}_q$.** In this section we review some ideas of the proof of Theorems 4.2 and 4.3 obtained by Hellström and Silvestrov in [6].

Let us first q be of free type. Then $\mathrm{Z}(\mathcal{H}_q)$ is isomorphic to K, and hence our aim is to prove Theorem 4.3. There are three cases to consider. In the simplest case, when α is of the form cI for some $c \in K$, the polynomial $P(x, y) = x - c$ satisfies is annihilating for (α, β) since $\alpha - cI = 0$.

In the second case assume that α, β are linear combinations of monomials with equal degrees in A and B (denoted by $\alpha, \beta \sqsubseteq K_0$ as in [6]), and that there is no $c \in K$ such that $\alpha = cI$. Let $a = \deg \alpha > 0$ and $b = \deg \beta$. A general expression for $P(\alpha, \beta)$, where P has at most degree b in the first variable and at most degree a in the second is

$$\sum_{\substack{0 \leqslant i \leqslant b \\ 0 \leqslant j \leqslant a}} p_{ij} \alpha^i \beta^j. \tag{3}$$

This sum is a linear combination of the $(a+1)(b+1)$ vectors $\{\alpha^i \beta^j\}_{i=0;j=0}^{b;a}$, and all these vectors belong to the vector space $\mathrm{Cen}\big(2ab, 0, \alpha\big) = \{\beta \sqsubseteq K_0 \mid [\alpha, \beta] = 0 \text{ and } \deg \beta \leq 2ab\}$. Now the point is that it can be shown that the dimension of this space is strictly smaller than $(a+1)(b+1)$. Hence there exist numbers $\{p_{ij}\}_{i,j}$, not all zero, which make (3) zero. Thus there exists a P as required.

The similar algorithmic dimension growth type proof in the case when $\alpha \not\sqsubseteq K_0$ or $\beta \not\sqsubseteq K_0$ is the most technical part. It is based on application of [6, Theorem 7.3] and [6, Corollary 6.9] which are proved using other concepts and results in the book. Thus we refer the reader to [6] for details.

Let us now assume that q is of torsion type and that p is its order. Observe that $\mathcal{H}_q$, seen as a $\mathrm{Z}(\mathcal{H}_q)$-module, contains a spanning set of p^2 elements, namely $\{B^i A^j\}_{0 \leqslant i,j < p}$. Also observe that $\mathrm{Z}(\mathcal{H}_q)$, by [6, Theorem 4.9] and [6, Theorem 5.7], is an integral domain. Hence any subset of $\mathcal{H}_q$ that contains more elements than a spanning set will be linearly dependent. Consider the polynomial

$$P(x, y) = \sum_{i=0}^{p} \sum_{j=0}^{p} c_{ij} x^i y^j$$

where $c_{ij} \in \mathrm{Z}(\mathcal{H}_q)$. Clearly $P(\alpha, \beta)$ will be a linear combination of $(p+1)^2 > p^2$ elements in $\mathcal{H}_q$ and these elements are linearly dependent. Thus there are coefficients $\{c_{ij}\}_{0 \leqslant i,j \leqslant p} \subset \mathrm{Z}(\mathcal{H}_q)$, not all zero, such that $P(\alpha, \beta) = 0$. This is exactly what the theorem claims.

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