DYNAMICAL SYSTEMS ASSOCIATED WITH CROSSED PRODUCTS

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ABSTRACT. In this paper, we consider both algebraic crossed products of commutative complex algebras A with the integers under an automorphism of A, and Banach algebra crossed products of commutative C^*-algebras A with the integers under an automorphism of A. We investigate, in particular, connections between algebraic properties of these crossed products and topological properties of naturally associated dynamical systems. For example, we draw conclusions about the ideal structure of the crossed product by investigating the dynamics of such a system. To begin with, we recall results in this direction in the context of an algebraic crossed product and give simplified proofs of generalizations of some of these results. We also investigate new questions, for example about ideal intersection properties of algebras properly between the coefficient algebra A and its commutant A'. Furthermore, we introduce a Banach algebra crossed product and study the relation between the structure of this algebra and the topological dynamics of a naturally associated system.

1. INTRODUCTION

A lot of work has been done on the connection between certain topological dynamical systems and crossed product C^*-algebras. In [15] and [16], for example, one starts with a homeomorphism σ of a compact Hausdorff space X and constructs the crossed product C^*-algebra $C(X) \rtimes_\alpha \mathbb{Z}$, where $C(X)$ is the algebra of continuous complex valued functions on X and α is the $\mathbb{Z}$-action on $C(X)$ naturally induced by σ. One of many results obtained is equivalence between simplicity of the algebra and minimality of the system, provided that X consists of infinitely many points, see [2], [9], [15], [16] or, for a more general approach in the metrizable case, [17]. In [12], a purely algebraic variant of the crossed product is considered, having more general classes of algebras than merely continuous functions on compact Hausdorff spaces as coefficient algebras. For example, it is proved there that, for such crossed products, the analogue of the equivalence between density of non-periodic points of a dynamical system and maximal commutativity of the coefficient algebra in the associated crossed product C^*-algebra is true for significantly larger classes of coefficient algebras and associated dynamical systems. In [11], further work is done in this setup, mainly for crossed products of complex commutative semi-simple completely regular Banach-algebras A (of which $C(X)$ is an example) with the integers under an automorphism of A. In particular, various properties of the ideal structure in such crossed products are shown to be equivalent to topological properties of the naturally induced topological dynamical system on $\Delta(A)$, the character space of A.

In this paper, we recall some of the most important results from [12] and [11], and in a number of cases provide significantly simplified proofs of generalizations of results occurring in [11], giving a clearer view of the heart of the matter. We also include results of a new type in the algebraic setup, and furthermore start the investigation of the Banach algebra crossed product $\ell_1^\sigma(\mathbb{Z}, A)$ of a commutative C^*-algebra A with the integers under an automorphism σ of A. In the case when A is unital, this algebra is precisely the one whose C^*-envelope is the crossed product C^*-algebra mentioned above.

This paper is organized as follows. In Section 2 we give the most general definition of the kind of crossed product that we will use throughout the first sections of this paper. We also mention the elementary result that the commutant of the coefficient algebra is automatically a maximal commutative subalgebra of the crossed product.

In Section 3 we prove that for any such crossed product $A \rtimes_\Psi \mathbb{Z}$, the commutant A' of the coefficient algebra A has non-zero intersection with any non-zero ideal $I \subseteq A \rtimes_\Psi \mathbb{Z}$. In [11, Theorem 6.1], a more complicated proof of this was given for a restricted class of coefficient algebras A.

In Section 4 we focus on the case when A is a function algebra on a set X with an automorphism $\widetilde{\sigma}$ of A induced by a bijection $\sigma : X \to X$. According to [16, Theorem 5.4], the following three properties are equivalent for a compact Hausdorff space X and a homeomorphism σ of X:

- The non-periodic points of (X, σ) are dense in X;
- Every non-zero closed ideal I of the crossed product C^*-algebra $C(X) \rtimes_\alpha \mathbb{Z}$ is such that $I \cap C(X) \neq \{0\}$;
- $C(X)$ is a maximal abelian C^*-subalgebra of $C(X) \rtimes_\alpha \mathbb{Z}$.

In Theorem 4.5 an analogue of this result is proved for our setup. A reader familiar with the theory of crossed product C^*-algebras will easily recognize that if one chooses $A = C(X)$ for X a compact Hausdorff space in this theorem, then the algebraic crossed product is canonically isomorphic to a norm-dense subalgebra of the crossed product C^*-algebra coming from the considered induced dynamical system.

For a different kind coefficient algebras A than the ones allowed in Theorem 4.5, we prove a similar result in Theorem 4.6. Theorem 4.5 and Theorem 4.6 have no non-trivial situations in common (Remark 4.8).

In Section 5 we show that in many situations we can always find both a subalgebra properly between the coefficient algebra A and its commutant A' (as long as $A \subsetneq A'$, a property we have a precise condition for in Theorem 4.5) and a non-trivial ideal trivially intersecting it, and a subalgebra properly between A and A' intersecting every non-trival ideal non-trivially.

Section 6 is concerned with the algebraic crossed product of a complex commutative semi-simple Banach algebra A with the integers under an automorphism σ of A, naturally inducing a homeomorphism $\widetilde{\sigma}$ of the character space $\Delta(A)$ of A. We extend results from [11].

In Section 7 we introduce the Banach algebra crossed product $\ell_1^\sigma(\mathbb{Z}, A)$ for a commutative C^*-algebra A and an automorphism σ of A. In Theorem 7.4 we give an explicit description of the closed commutator ideal in this algebra in terms of the dynamical system naturally induced on $\Delta(A)$. We determine the characters of

$\ell_1^\sigma(\mathbb{Z}, A)$. The modular ideals which are maximal and contain the commutator ideal are precisely the kernels of the characters.

2. Definition and a basic result

Let A be an associative commutative complex algebra and let $\Psi : A \to A$ be an algebra automorphism. Consider the set

$$A \rtimes_\Psi \mathbb{Z} = \{f : \mathbb{Z} \to A \mid f(n) = 0 \text{ except for a finite number of } n\}.$$

We endow it with the structure of an associative complex algebra by defining scalar multiplication and addition as the usual pointwise operations. Multiplication is defined by *twisted convolution*, $*$, as follows;

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k) \cdot \Psi^k(g(n-k)),$$

where Ψ^k denotes the k-fold composition of Ψ with itself. It is trivially verified that $A \rtimes_\Psi \mathbb{Z}$ *is* an associative $\mathbb{C}$-algebra under these operations. We call it the *crossed product of A and $\mathbb{Z}$ under Ψ.*

A useful way of working with $A \rtimes_\Psi \mathbb{Z}$ is to write elements $f, g \in A \rtimes_\Psi \mathbb{Z}$ in the form $f = \sum_{n \in \mathbb{Z}} f_n \delta^n, g = \sum_{m \in \mathbb{Z}} g_m \delta^n$, where $f_n = f(n), g_m = g(m)$, addition and scalar multiplication are canonically defined, and multiplication is determined by $(f_n \delta^n) * (g_m \delta^m) = f_n \cdot \Psi^n(g_m) \delta^{n+m}$, where $n, m \in \mathbb{Z}$ and $f_n, g_m \in A$ are arbitrary.

Clearly one may canonically view A as an abelian subalgebra of $A \rtimes_\Psi \mathbb{Z}$, namely as $\{f_0 \delta^0 \mid f_0 \in A\}$. The following elementary result is proved in [12, Proposition 2.1].

Proposition 2.1. *The commutant A' of A is abelian, and thus it is the unique maximal abelian subalgebra containing A.*

3. Every non-zero ideal has non-zero intersection with A'

Throughout the whole paper, when speaking of an ideal we shall always mean a *two-sided* ideal. We shall now show that any non-zero ideal in $A \rtimes_\Psi \mathbb{Z}$ has non-zero intersection with A'. This result, Theorem 3.1, should be compared with Theorem 4.5, which says that a non-zero ideal may well intersect A solely in 0. An analogue of Theorem 3.1 in the context of crossed product C^*-algebras is found in [13, Theorem 4.3]. Note that in [11] a proof of Theorem 3.1 was given for the case when A was completely regular semi-simple Banach algebra, and that this proof heavily relied upon A having these properties. The present proof is elementary and valid for arbitrary commutative algebras.

Theorem 3.1. *Let A be an associative commutative complex algebra, and let $\Psi : A \to A$ be an automorphism. Then every non-zero ideal of $A \rtimes_\Psi \mathbb{Z}$ has non-zero intersection with the commutant A' of A.*

Proof. Let I be a non-zero ideal, and let $f = \sum_n f_n \delta^n \in I$ be non-zero. Suppose that $f \notin A'$. Then there must be an f_{n_i} and $a \in A$ such that $f_{n_i} \cdot a \neq 0$. Hence $f' := (\sum_n f_n \delta^n) * \Psi^{-n_i}(a)\delta^{-n_i}$ is a non-zero element of I, having $f_{n_i} \cdot a$ as coefficient of δ^0 and having at most as many non-zero coefficients as f. If $f' \in A'$ we are done, so assume $f' \notin A'$. Then there exists $b \in A$ such that $F := b * f' - f' * b \neq 0$. Clearly $F \in I$ and it is easy to see that F has strictly less non-zero coefficients than f' (the coefficient of δ^0 in F is zero), hence strictly less than f. Now if $F \in A'$, we

are done. If not, we repeat the above procedure. Ultimately, if we do not happen to obtain a non-zero element of $I \cap A'$ along the way, we will be left with a non-zero monomial $G := g_m \delta^m \in I$. If this does not lie in A', there is an $a \in A$ such that $g_m \cdot a \neq 0$. Hence $G * \Psi^{-m}(a)\delta^{-m} = g_m \cdot a \in I \cap A \subseteq I \cap A'$. $\square$

Note that the fact that all elements in $A \rtimes_\Psi \mathbb{Z}$ are *finite* sums of the form $\sum_n f_n \delta^n$ is crucial for the argument used in the proof.

4. Automorphisms induced by bijections

Fix a non-empty set X, a bijection $\sigma : X \to X$, and an algebra of functions $A \subseteq \mathbb{C}^X$ that is invariant under σ and σ^{-1}, i.e., such that if $h \in A$, then $h \circ \sigma \in A$ and $h \circ \sigma^{-1} \in A$. Then (X, σ) is a discrete dynamical system (the action of $n \in \mathbb{Z}$ on $x \in X$ is given by $n : x \mapsto \sigma^n(x)$) and σ induces an automorphism $\widetilde{\sigma} : A \to A$ defined by $\widetilde{\sigma}(f) = f \circ \sigma^{-1}$ by which $\mathbb{Z}$ acts on A via iterations.

In this section we will consider the crossed product $A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$ for the above setup, and explicitly describe the commutant A' of A. Furthermore, we will investigate equivalences between properties of non-periodic points of the system (X, σ), and properties of A'. First we make a few definitions.

Definition 4.1. For any nonzero $n \in \mathbb{Z}$ we set

$$\mathrm{Sep}_A^n(X) = \{x \in X | \exists h \in A : h(x) \neq \widetilde{\sigma}^n(h)(x)\},$$
$$\mathrm{Per}_A^n(X) = \{x \in X | \forall h \in A : h(x) = \widetilde{\sigma}^n(h)(x)\},$$
$$\mathrm{Sep}^n(X) = \{x \in X | x \neq \sigma^n(x))\},$$
$$\mathrm{Per}^n(X) = \{x \in X | x = \sigma^n(x)\}.$$

Furthermore, let

$$\mathrm{Per}_A^\infty(X) = \bigcap_{n \in \mathbb{Z}\setminus\{0\}} \mathrm{Sep}_A^n(X),$$
$$\mathrm{Per}^\infty(X) = \bigcap_{n \in \mathbb{Z}\setminus\{0\}} \mathrm{Sep}^n(X).$$

Finally, for $f \in A$, put

$$\mathrm{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

It is easy to check that all these sets, except for $\mathrm{supp}(f)$, are $\mathbb{Z}$-invariant and that if A separates the points of X, then $\mathrm{Sep}_A^n(X) = \mathrm{Sep}^n(X)$ and $\mathrm{Per}_A^n(X) = \mathrm{Per}^n(X)$. Note also that $X \setminus \mathrm{Per}_A^n(X) = \mathrm{Sep}_A^n(X)$, and $X \setminus \mathrm{Per}^n(X) = \mathrm{Sep}^n(X)$. Furthermore $\mathrm{Sep}_A^n(X) = \mathrm{Sep}_A^{-n}(X)$ with similar equalities for n and $-n$ ($n \in \mathbb{Z}$) holding for $\mathrm{Per}_A^n(X)$, $\mathrm{Sep}^n(X)$ and $\mathrm{Per}^n(X)$ as well.

Definition 4.2. We say that a non-empty subset of X is a *domain of uniqueness for A* if every function in A that vanishes on it, vanishes on the whole of X.

For example, using results from elementary topology one easily shows that for a completely regular topological space X, a subset of X is a domain of uniqueness for $C(X)$ if and only if it is dense in X. In the following theorem we recall some elementary results from [12].

Theorem 4.3. *The unique maximal abelian subalgebra of $A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$ that contains A is precisely the set of elements*

$$A' = \{\sum_{n \in \mathbb{Z}} f_n \delta^n \mid f_n\!\restriction_{\mathrm{Sep}_A^n(X)} \equiv 0 \text{ for all } n \in \mathbb{Z}\}.$$

So if A separates the points of X, then

$$A' = \{\sum_{n \in \mathbb{Z}} f_n \delta^n \mid \mathrm{supp}(f_n) \subseteq \mathrm{Per}^n(X) \text{ for all } n \in \mathbb{Z}\}.$$

Furthermore, the subalgebra A is maximal abelian in $A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$ if and only if, for every $n \in \mathbb{Z} \setminus \{0\}$, $\mathrm{Sep}_A^n(X)$ is a domain of uniqueness for A.

We now focus solely on topological contexts. In order to prove one of the main theorems of this section, we need the following topological lemma.

Lemma 4.4. *Let X be a Baire space which is also Hausdorff, and let $\sigma : X \to X$ be a homeomorphism. Then the non-periodic points of (X, σ) are dense if and only if $\mathrm{Per}^n(X)$ has empty interior for all positive integers n.*

Proof. Clearly, if there is a positive integer n such that $\mathrm{Per}^n(X)$ has non-empty interior, the non-periodic points are not dense. For the converse we note that we may write

$$X \setminus \mathrm{Per}^\infty(X) = \bigcup_{n > 0} \mathrm{Per}^n(X).$$

If the set of non-periodic points is not dense, its complement has non-empty interior, and as the sets $\mathrm{Per}^n(\Delta(A))$ are clearly all closed since X is Hausdorff, there must exist an integer $n_0 > 0$ such that $\mathrm{Per}^{n_0}(X)$ has non-empty interior since X is a Baire space. $\qquad\square$

We are now ready to prove the following theorem.

Theorem 4.5. *Let X be a Baire space which is also Hausdorff, and let $\sigma : X \to X$ be a homeomorphism inducing, as usual, an automorphism $\widetilde{\sigma}$ of $C(X)$. Suppose A is a subalgebra of $C(X)$ that is invariant under $\widetilde{\sigma}$ and its inverse, separates the points of X and is such that for every non-empty open set $U \subseteq X$ there is a non-zero $f \in A$ that vanishes on the complement of U. Then the following three statements are equivalent.*

- *A is a maximal abelian subalgebra of $A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$;*
- *$\mathrm{Per}^\infty(X)$ is dense in X;*
- *Every non-zero ideal $I \subseteq A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$ is such that $I \cap A \neq \{0\}$.*

Proof. Equivalence of the first two statements is precisely the result in [12, Theorem 3.7]. The first property implies the third by Proposition 2.1 and Theorem 3.1. Finally, to show that the third statement implies the second, assume that $\mathrm{Per}^\infty(X)$ is not dense. It follows from Lemma 4.4 that there exists an integer $n > 0$ such that $\mathrm{Per}^n(X)$ has non-empty interior. By the assumptions on A there exists a non-zero $f \in A$ such that $\mathrm{supp}(f) \subseteq \mathrm{Per}^n(X)$. Consider now the non-zero ideal I generated by $f + f\delta^n$. It is spanned by elements of the form $a_i \delta^i * (f + f\delta^n) * a_j \delta^j$, $(f + f\delta^n) * a_j \delta^j$, $a_i \delta^i * (f + f\delta^n)$ and $f + f\delta^n$. Using that f vanishes outside $\mathrm{Per}^n(X)$, so that $f\delta^n * a_j \delta^j = a_j f\delta^{n+j}$, we may for example rewrite

$$a_i \delta^i * (f + f\delta^n) * a_j \delta^j = [a_i \cdot (a_j \circ \widetilde{\sigma}^{-i})\delta^i] * [f\delta^j + f\delta^{n+j}]$$

$$= [a_i \cdot (a_j \circ \widetilde{\sigma}^{-i}) \cdot (f \circ \widetilde{\sigma}^{-i})]\delta^{i+j} + [a_i \cdot (a_j \circ \widetilde{\sigma}^{-i}) \cdot (f \circ \widetilde{\sigma}^{-i})]\delta^{i+j+n}.$$

A similar calculation for the other three kinds of elements that span I now makes it clear that any element in I may be written in the form $\sum_i (b_i \delta^i + b_i \delta^{n+i})$. As i runs only through a finite subset of $\mathbb{Z}$, this is not a non-zero monomial. In particular, it is not a non-zero element in A. Hence I intersects A trivially. $\square$

We also have the following result for a different kind of subalgebras of $C(X)$.

Theorem 4.6. *Let X be a topological space, $\sigma : X \to X$ a homeomorphism, and A a non-zero subalgebra of $C(X)$, invariant both under the usual induced automorphism $\widetilde{\sigma} : C(X) \to C(X)$ and under its inverse. Assume that A separates the points of X and is such that every non-empty open set $U \subseteq X$ is a domain of uniqueness for A. Then the following three statements are equivalent.*

- *A is maximal abelian in $A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$;*
- *σ is not of finite order;*
- *Every non-zero ideal $I \subseteq A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$ is such that $I \cap A \neq \{0\}$.*

Proof. Equivalence of the first two statements is precisely the result in [12, Theorem 3.11]. That the first statement implies the third follows immediately from Proposition 2.1 and Theorem 3.1. Finally, to show that the third statement implies the second, assume that there exists an n, which we may clearly choose to be non-negative, such that $\sigma^n = \mathrm{id}_X$. Now take any non-zero $f \in A$ and consider the non-zero ideal $I = (f + f\delta^n)$. Using an argument similar to the one in the proof of Theorem 4.5 one concludes that $I \cap A = \{0\}$. $\square$

Corollary 4.7. *Let M be a connected complex manifold and suppose the function $\sigma : M \to M$ is biholomorphic. If $A \subseteq H(M)$ is a subalgebra of the algebra of holomorphic functions that separates the points of M and which is invariant under the induced automorphism $\widetilde{\sigma}$ of $H(M)$ and its inverse, then the following three statements are equivalent.*

- *A is maximal abelian in $A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$;*
- *σ is not of finite order;*
- *Every non-zero ideal $I \subseteq A \rtimes_{\widetilde{\sigma}} \mathbb{Z}$ is such that $I \cap A \neq \{0\}$.*

Proof. On connected complex manifolds, open sets are domains of uniqueness for $H(M)$. See for example [5]. $\square$

Remark 4.8. It is worth mentioning that the required conditions in Theorem 4.5 and Theorem 4.6 can only be simultaneously satisfied in case X consists of a single point and $A = \mathbb{C}$. This is explained in [12, Remark 3.13]

5. Algebras properly between the coefficient algebra and its commutant

From Theorem 4.5 it is clear that for spaces X which are Baire and Hausdorff and subalgebras $A \subseteq C(X)$ with sufficient separation properties, A is equal to its own commutant in the associated crossed product precisely when the aperiodic points, $\mathrm{Per}^\infty(X)$, constitute a dense subset of X. This theorem also tells us that whenever $\mathrm{Per}^\infty(X)$ is not dense there exists a non-zero ideal I having zero intersection with A, while the general Theorem 3.1 tells us that every non-zero ideal has non-zero intersection with A', regardless of the system (X, σ).

Definition 5.1. We say that a subalgebra has the *intersection property* if it intersects every non-zero ideal non-trivially.

A subalgebra B such that $A \subsetneq B \subsetneq A'$ is said to be properly between A and A'. Two natural questions comes to mind in case $\mathrm{Per}^{\infty}(X)$ is not dense:

(i) Do there exist subalgebras properly between A and A' having the intersection property?

(ii) Do there exist subalgebras properly between A and A' *not* having the intersection property?

We shall show that for a significant class of systems the answer to both these questions is positive.

Proposition 5.2. *Let X be a Hausdorff space, and let $\sigma : X \to X$ be a homeomorphism inducing, as usual, an automorphism $\widetilde{\sigma}$ of $C(X)$. Suppose A is a subalgebra of $C(X)$ that is invariant under $\widetilde{\sigma}$ and its inverse, separates the points of X and is such that for every non-empty open set $U \subseteq X$ there is a non-zero $f \in A$ that vanishes on the complement of U. Suppose furthermore that there exists an integer $n > 0$ such that the interior of $\mathrm{Per}^n(X)$ contains at least two orbits. Then there exists a subalgebra B such that $A \subsetneq B \subsetneq A'$ which does not have the intersection property.*

Proof. Using the Hausdorff property of X and the fact that $\mathrm{Per}^n(X)$ contains two orbits we can find two non-empty disjoint invariant open subsets U_1 and U_2 contained in $\mathrm{Per}^n(X)$. Consider

$$B = \{f_0 + \sum_{k \neq 0} f_k \delta^k : f_0 \in A, \mathrm{supp}(f_k) \subseteq U_1 \cap \mathrm{Per}^k(X) \text{ for } k \neq 0\}.$$

Then B is a subalgebra and $B \subseteq A'$. The assumptions on A and the definitions of U_1 and U_2 now make it clear that $A \subsetneq B \subsetneq A'$ since there exist, for example, non-zero functions $F_1, F_2 \in A$ such that $\mathrm{supp}(F_1) \subseteq U_1$ and $\mathrm{supp}(F_2) \subseteq U_2$, and thus $F_1 \delta^n \in B \setminus A$ and $F_2 \delta^n \in A' \setminus B$. Consider the non-zero ideal I generated by $F_2 + F_2 \delta^n$. Using an argument similar to the one used in the proof of Theorem 4.5 we see that $I \cap A = \{0\}$. It is also easy to see that $I \subseteq \{\sum_k f_k \delta^k : \mathrm{supp}(f_k) \subseteq U_2\}$ since U_2 is invariant. As $U_1 \cap U_2 = \emptyset$, we see from the description of B that $I \cap B \subseteq A$, so that $I \cap B \subseteq I \cap A = \{0\}$. $\square$

We now exhibit algebras properly between A and A' that do have the intersection property.

Proposition 5.3. *Let X be a Hausdorff space, and let $\sigma : X \to X$ be a homeomorphism inducing, as usual, an automorphism $\widetilde{\sigma}$ of $C(X)$. Suppose A is a subalgebra of $C(X)$ that is invariant under $\widetilde{\sigma}$ and its inverse, separates the points of X and is such that for every non-empty open set $U \subseteq X$ there is a non-zero $f \in A$ that vanishes on the complement of U. Suppose furthermore that there exist an integer $n > 0$ such that the interior of $\mathrm{Per}^n(X)$ contains a point x_0 which is not isolated, and an $f \in A$ with $\mathrm{supp}(f) \subseteq \mathrm{Per}^n(X)$ and $f(x_0) \neq 0$. Then there exists a subalgebra B such that $A \subsetneq B \subsetneq A'$ which has the intersection property.*

Proof. Define

$$B = \{\sum_{k \in \mathbb{Z}} f_k \delta^k \in A' \,|\, f_k(x_0) = 0 \text{ for all } k \neq 0\},$$

where x_0 is as in the statement of the theorem. Clearly B is a subalgebra and $A \subseteq B$. Since x_0 is not isolated, we can use the assumptions on A and the fact that X is Hausdorff to first find a point different from x_0 in the interior of $\mathrm{Per}^n(X)$

and subsequently a non-zero function $g \in A$ such that $\mathrm{supp}(g) \subseteq \mathrm{Per}^n(X)$ and $g(x_0) = 0$. Then $g\delta^n \in B \setminus A$. Also, by the assumptions on A there is a non-zero $f \in A$ with $\mathrm{supp}(f) \subseteq \mathrm{Per}^n(X)$ such that $f(x_0) \neq 0$, whence $f\delta^n \in A' \setminus B$. This shows that B is a subalgebra properly between A and A'. To see that it has the intersection property, let I be an arbitrary non-zero ideal in the crossed product and note that by Theorem 3.1 there is a non-zero $F = \sum_{k \in \mathbb{Z}} f_k \delta^k$ in $I \cap A'$. Now if for all $k \neq 0$ we have that $f_k(x_0) = 0$, we are done. So suppose there is some $k \neq 0$ such that $f_k(x_0) \neq 0$. Since f_k is continuous and x_0 is not isolated, we may use the Hausdorff property of X to conclude that there exists a non-empty open set V contained in the interior of $\mathrm{Per}^n(X)$ such that $x_0 \notin V$ and $f_k(x) \neq 0$ for all $x \in V$. The assumptions on A now imply that there is an $h \in A$ such that $h(x_0) = 0$ and $h(x_1) \neq 0$ for some $x_1 \in V \subseteq \mathrm{supp}(f_n)$. Clearly $0 \neq h * F \in I \cap B$. $\square$

Theorem 5.4. *Let X be a Baire space which is Hausdorff and connected. Let $\sigma : X \to X$ be a homeomorphism inducing an automorphism $\tilde{\sigma}$ of $C(X)$ in the usual way. Suppose A is a subalgebra of $C(X)$ that is invariant under $\tilde{\sigma}$ and its inverse, such that for every open set $U \subseteq X$ and $x \in U$ there is an $f \in A$ such that $f(x) \neq 0$ and $\mathrm{supp}(f) \subseteq U$. Then precisely one of the following situations occurs:*

 (i) *$A = A'$, which happens precisely when $\mathrm{Per}^\infty(X)$ is dense;*
 (ii) *$A \subsetneq A'$ and there exist both subalgebras properly between A and A' which have the intersection property, and subalgebras which do not. This happens precisely when $\mathrm{Per}^\infty(X)$ is not dense and X is infinite;*
 (iii) *$A \subsetneq A'$ and every subalgebra properly between A and A' has the intersection property. This happens precisely when X consists of one point.*

Proof. By Theorem 4.5, (i) is clear and we may assume that $\mathrm{Per}^\infty(X)$ is not dense. Suppose first that X is infinite and note that by Lemma 4.4 there exists $n_0 > 0$ such that $\mathrm{Per}^{n_0}(X)$ has non-empty interior. If this interior consists of one single orbit then as X is Hausdorff every point in the interior is both closed and open, so that X consists of one point by connectedness, which is a contradiction. Hence there are at least two orbits in the interior of $\mathrm{Per}^{n_0}(X)$. Furthermore, no point of X can be isolated. Thus by Proposition 5.2 and Proposition 5.3 there are subalgebras properly between A and A' which have the intersection property, and subalgebras which do not. Suppose next that X is finite, so that $X = \{x\}$ by connectedness. Then σ is the identity map, and $A = \mathbb{C}$. In this case, $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ may be canonically identified with $\mathbb{C}[t, t^{-1}]$. Let B be a subalgebra such that $\mathbb{C} \subsetneq B \subsetneq \mathbb{C}[t, t^{-1}]$, and let I be a non-zero ideal of $\mathbb{C}[t, t^{-1}]$. We will show that $I \cap B \neq \{0\}$ and hence may assume that $I \neq \mathbb{C}[t, t^{-1}]$. Since $\mathbb{C}[t, t^{-1}]$ is the ring of fractions of $\mathbb{C}[t]$ with respect to the multiplicatively closed subset $\{t^n \mid n \text{ is a non-negative integer}\}$ and $\mathbb{C}[t]$ is a principal ideal domain, it follows from [1, Proposition 3.11, (i)] that I is of the form $(t - \alpha_1) \cdots (t - \alpha_n)\mathbb{C}[t, t^{-1}]$ for some $n > 0$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. There exists a non-constant f in B, and then the element $(f - f(\alpha_1)) \cdots (f - f(\alpha_n))$ is a non-zero element of B. It is clearly also in I since it vanishes at $\alpha_1, \ldots, \alpha_n$ and hence has $(t - \alpha_1) \cdots (t - \alpha_n)$ as a factor. Hence $I \cap B \neq \{0\}$ and the proof is completed. $\square$

It is interesting to mention that arguments similar to the ones used in Propositions 5.2 and 5.3 work in the context of the crossed product C^*-algebra $C(X) \rtimes_\alpha \mathbb{Z}$ where X is a compact Hausdorff space and α the automorphism induced by a homeomorphism of X. See [13, Section 5] for details.

6. Semi-simple Banach algebras

In what follows, we shall focus on cases where A is a commutative complex Banach algebra, and freely make use of the basic theory for such A, see e.g. [6]. As conventions tend to differ slightly in the literature, however, we mention that we call a commutative Banach algebra A *completely regular* (the term *regular* is also frequently used in the literature) if, for every subset $F \subseteq \Delta(A)$ (where $\Delta(A)$ denotes the character space of A) that is closed in the Gelfand topology and for every $\phi_0 \in \Delta(A) \setminus F$, there exists an $a \in A$ such that $\widehat{a}(\phi) = 0$ for all $\phi \in F$ and $\widehat{a}(\phi_0) \neq 0$. All topological considerations of $\Delta(A)$ will be done with respect to its Gelfand topology.

Now let A be a complex commutative semi-simple completely regular Banach algebra, and let $\sigma : A \to A$ be an algebra automorphism. As in [12], σ induces a map $\widetilde{\sigma} : \Delta(A) \to \Delta(A)$ defined by $\widetilde{\sigma}(\mu) = \mu \circ \sigma^{-1}, \mu \in \Delta(A)$, which is automatically a homeomorphism when $\Delta(A)$ is endowed with the Gelfand topology. Hence we obtain a topological dynamical system $(\Delta(A), \widetilde{\sigma})$. In turn, $\widetilde{\sigma}$ induces an automorphism $\widehat{\sigma} : \widehat{A} \to \widehat{A}$ (where $\widehat{A}$ denotes the algebra of Gelfand transforms of all elements of A) defined by $\widehat{\sigma}(\widehat{a}) = \widehat{a} \circ \widetilde{\sigma}^{-1} = \widehat{\sigma(a)}$. Therefore we can form the crossed product $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$.

In what follows, we shall make frequent use of the following fact. Its proof consists of a trivial direct verification.

Theorem 6.1. *Let A be a commutative semi-simple Banach algebra and σ an automorphism, inducing an automorphism $\widehat{\sigma} : \widehat{A} \to \widehat{A}$ as above. Then the map $\Phi : A \rtimes_\sigma \mathbb{Z} \to \widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ defined by $\sum_{n \in \mathbb{Z}} a_n \delta^n \mapsto \sum_{n \in \mathbb{Z}} \widehat{a_n} \delta^n$ is an isomorphism of algebras mapping A onto $\widehat{A}$.*

We shall now conclude that, for certain A, two different algebraic properties of $A \rtimes_\sigma \mathbb{Z}$ are equivalent to density of the non-periodic points of the naturally associated dynamical system on the character space $\Delta(A)$. The analogue of this result in the context of crossed product C^*-algebras is [16, Theorem 5.4]. We shall also combine this with a theorem from [12] to conclude a stronger result for the Banach algebra $L_1(G)$, where G is a locally compact abelian group with connected dual group.

Theorem 6.2. *Let A be a complex commutative semi-simple completely regular Banach algebra, $\sigma : A \to A$ an automorphism and $\widetilde{\sigma}$ the homeomorphism of $\Delta(A)$ in the Gelfand topology induced by σ as described above. Then the following three properties are equivalent:*

- *The non-periodic points $\mathrm{Per}^\infty(\Delta(A))$ of $(\Delta(A), \widetilde{\sigma})$ are dense in $\Delta(A)$;*
- *Every non-zero ideal $I \subseteq A \rtimes_\sigma \mathbb{Z}$ is such that $I \cap A \neq \{0\}$;*
- *A is a maximal abelian subalgebra of $A \rtimes_\sigma \mathbb{Z}$.*

Proof. As A is completely regular, and $\Delta(A)$ is Baire since it is locally compact and Hausdorff, it is immediate from Theorem 4.5 that the following three statements are equivalent.

- The non-periodic points $\mathrm{Per}^\infty(\Delta(A))$ of $(\Delta(A), \widetilde{\sigma})$ are dense in $\Delta(A)$;
- Every non-zero ideal $I \subseteq \widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ is such that $I \cap \widehat{A} \neq \{0\}$;
- $\widehat{A}$ is a maximal abelian subalgebra of $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$.

Now applying Theorem 6.1 we can pull everything back to $A \rtimes_\sigma \mathbb{Z}$ and the result follows. □

The following result for a more specific class of Banach algebras is an immediate consequence of Theorem 6.2 together with [12, Theorem 4.16].

Theorem 6.3. *Let G be a locally compact abelian group with connected dual group and let $\sigma : L_1(G) \to L_1(G)$ be an automorphism. Then the following three statements are equivalent.*

- *σ is not of finite order;*
- *Every non-zero ideal $I \subseteq L_1(G) \rtimes_\sigma \mathbb{Z}$ is such that $I \cap L_1(G) \neq \{0\}$;*
- *$L_1(G)$ is a maximal abelian subalgebra of $L_1(G) \rtimes_\sigma \mathbb{Z}$.*

To give a more complete picture, we also include the results [11, Theorem 5.1] and [11, Theorem 7.6].

Theorem 6.4. *Let A be a complex commutative semi-simple completely regular unital Banach algebra such that $\Delta(A)$ consists of infinitely many points, and let σ be an automorphism of A. Then*

- *$A \rtimes_\sigma \mathbb{Z}$ is simple if and only if the associated system $(\Delta(A), \widetilde{\sigma})$ on the character space is minimal.*
- *$A \rtimes_\sigma \mathbb{Z}$ is prime if and only if $(\Delta(A), \widetilde{\sigma})$ is topologically transitive.*

7. The Banach algebra crossed product $\ell_1^\sigma(\mathbb{Z}, A)$ for a commutative C^*-algebra A

Let A be a commutative C^*-algebra with spectrum $\Delta(A)$ and $\sigma : A \to A$ an automorphism. We identify the set $\ell^1(\mathbb{Z}, A)$ with the set $\{\sum_{n \in \mathbb{Z}} f_n \delta^n | f_n \in A, \sum_{n \in \mathbb{Z}} \|f_n\| < \infty\}$ and endow it with the same operations as for the finite sums in Section 2. Using that σ is isometric one easily checks that the operations are well defined, and that the usual norm on this set is an algebra norm with respect to the convolution product.

We denote this algebra by $\ell_1^\sigma(\mathbb{Z}, A)$, and note that it is a Banach algebra. By basic theory of C^*-algebras, we have the isometric automorphism $A \cong \widehat{A} = C_0(\Delta(A))$. As in Section 6, σ induces a homeomorphism, $\widetilde{\sigma} : \Delta(A) \to \Delta(A)$ and an automorphism $\widehat{\sigma} : C_0(\Delta(A)) \to C_0(\Delta(A))$ and we have a canonical isometric isomorphism of $\ell_1^\sigma(\mathbb{Z}, A)$ onto $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$ as in Theorem 6.1.

We will work in the concrete crossed product $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$. We shall describe the closed commutator ideal $\mathscr{C}$ in terms of $(\Delta(A), \sigma)$. In analogy with the notation used in [14], we make the following definitions.

Definition 7.1. Given a subset $S \subseteq \Delta(A)$, we set

$$\ker(S) = \{f \in C_0(\Delta(A)) \,|\, f(x) = 0 \text{ for all } x \in S\},$$

$$\mathrm{Ker}(S) = \{\sum_{n \in \mathbb{Z}} f_n \in \ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A))) \,|\, f_n(x) = 0 \text{ for all } x \in S, n \in \mathbb{Z}\}.$$

Clearly $\mathrm{Ker}(S)$ is always a closed subspace, and in case S is invariant, it is a closed ideal.

We shall also need the following version of the Stone-Weierstrass theorem.

Theorem 7.2. *Let X be a locally compact Hausdorff space and let C be a closed subset of X. Let B be a self-adjoint subalgebra of $C_0(X)$ vanishing on C. Suppose that for any pair of points $x, y \in X$, with $x \neq y$, such that at least one of them is not in C, there exists $f \in B$ such that $f(x) \neq f(y)$. Then $\overline{B} = \{f \in C_0(X) : f(x) = 0 \text{ for all } x \in C\}$.*

Proof. This follows from the more general result [3, Theorem 11.1.8], as it is well known that the pure states of $C_0(\Delta(A))$ are precisely the point evaluations on the locally compact Hausdorff space $\Delta(A)$, and that a pure state of a sub-C^*-algebra always has a pure state extension to the whole C^*-algebra. By passing to the one-point compactification of $\Delta(A)$, one may also easily derive the result from the more elementary [4, Theorem 2.47]. □

Definition 7.3. Let A be a normed algebra. An approximate unit of A is a net $\{E_\lambda\}_{\lambda \in \Lambda}$ such that for every $a \in A$ we have $\lim_\lambda \|E_\lambda a - a\| = \lim_\lambda \|aE_\lambda - a\| = 0$.

Recall that any C^*-algebra has an approximate unit such that $\|E_\lambda\| \leq 1$ for all $\lambda \in \Lambda$. In general, however, an approximate identity need not be bounded. We are now ready to prove the following result, which is the analogue of the first part of [14, Proposition 4.9].

Theorem 7.4. $\mathscr{C} = \mathrm{Ker}(\mathrm{Per}^1(\Delta(A)))$.

Proof. It it easily seen that $\mathscr{C} \subseteq \mathrm{Ker}(\mathrm{Per}^1(\Delta(A)))$. For the converse inclusion we choose an approximate identity $\{E_\lambda\}_{\lambda \in \Lambda}$ for $C_0(\Delta(A))$ and note first of all that for any $f \in C_0(\Delta(A))$ we have $f * (E_\lambda \delta) - (E_\lambda \delta) * f = E_\lambda(f - f \circ \widetilde{\sigma}^{-1})\delta \in \mathscr{C}$. Hence as $\mathscr{C}$ is closed, $(f - f \circ \widetilde{\sigma}^{-1})\delta \in \mathscr{C}$ for all $f \in C_0(\Delta)$. Clearly the set $J = \{g \in C_0(\Delta(A)) \mid g\delta \in \mathscr{C}\}$ is a closed subalgebra (and even an ideal) of $C_0(\Delta(A))$. Denote by I the (self-adjoint) ideal of $C_0(\Delta(A))$ generated by the set of elements of the form $f - f \circ \widetilde{\sigma}^{-1}$. Note that I vanishes on $\mathrm{Per}^1(\Delta(A))$ and that it is contained in J. Using complete regularity of $C_0(\Delta(A))$, it is straightforward to check that for any pair of distinct points $x, y \in \Delta(A)$, at least one of which is not in $\mathrm{Per}^1(\Delta(A))$, there exists a function $f \in I$ such that $f(x) \neq f(y)$. Hence by Theorem 7.2 I is dense in $\mathrm{ker}(\mathrm{Per}^1(\Delta(A)))$, and thus $\{f\delta \mid f \in \mathrm{ker}(\mathrm{Per}^1(\Delta(A)))\} \subseteq \mathscr{C}$ since J is closed. So for any $n \in \mathbb{Z}$ and $f \in \mathrm{ker}(\mathrm{Per}^1(\Delta(A)))$ we have $(f\delta) * (E_\lambda \circ \widetilde{\sigma})\delta^{n-1} = (fE_\lambda)\delta^n \in \mathscr{C}$. This converges to $f\delta^n$, and hence $\mathscr{C} \supseteq \mathrm{Ker}(\mathrm{Per}^1(\Delta(A)))$. □

Denote the set of non-zero multiplicative linear functionals of $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$ by Ξ. We shall now determine a bijection between Ξ and $\mathrm{Per}^1(\Delta(A)) \times \mathbb{T}$. It is a standard result from Banach algebra theory that any $\mu \in \Xi$ is bounded and of norm at most one. Since one may choose an approximate identity $\{E_\lambda\}_{\lambda \in \Lambda}$ for $C_0(\Delta(A))$ such that $\|E_\lambda\| \leq 1$ for all $\lambda \in \Lambda$ it is also easy to see that $\|\mu\| = 1$. Namely, given $\mu \in \Xi$ we may choose an $f \in C_0(\Delta(A))$ such that $\mu(f) \neq 0$. Then by continuity of μ we have $\mu(f) = \lim_\lambda \mu(fE_\lambda) = \mu(f)\lim_\lambda(E_\lambda)$ and hence $\lim_\lambda \mu(E_\lambda) = 1$.

Lemma 7.5. *The limit $\xi := \lim_\lambda \mu(E_\lambda \delta)$ exists for all $\mu \in \Xi$, and is independent of the approximate unit $\{E_\lambda\}_{\lambda \in \Lambda}$. Furthermore, $\xi \in \mathbb{T}$ and $\lim_\lambda \mu(E_\lambda \delta^n) = \xi^n$ for all integers n.*

Proof. By continuity and multiplicativity of μ we have that $\lim_\lambda \mu(f)\mu(E_\lambda \delta) = \mu(f\delta)$ for all $f \in C_0(X)$. So for any f such that $\mu(f) \neq 0$ we have that $\lim_\lambda \mu(E_\lambda \delta) = \frac{\mu(f\delta)}{\mu(f)}$. This shows that the limit ξ exists and is the same for any approximate unit,

and using a similar argument one easily sees that $\lim_\lambda \mu(E_\lambda \delta^n)$ also exists and is independent of $\{E_\lambda\}_{\lambda \in \Lambda}$. For the rest of the proof, we fix an approximate unit $\{E_\lambda\}_{\lambda \in \Lambda}$ such that $\|E_\lambda\| \le 1$ for all $\lambda \in \Lambda$. As we know that $\|\mu\| = 1$, we see that $|\xi| \le 1$. Now suppose $|\xi| < 1$. It is easy to see that $\lim_\lambda \mu(E_\lambda) = 1 = \xi^0$. Hence also $1 = \lim_\lambda \mu(E_\lambda)^2 = \lim_\lambda \mu(E_\lambda^2) = \lim_\lambda \mu((E_\lambda \delta) * ((E_\lambda \circ \widetilde{\sigma})\delta^{-1})) = \lim_\lambda \mu((E_\lambda \delta)) \cdot \lim_\lambda \mu((E_\lambda \circ \widetilde{\sigma})\delta^{-1})$. Now as we assumed $|\xi| < 1$, this forces $|\lim_\lambda \mu([(E_\lambda \circ \widetilde{\sigma})\delta^{-1}])| > 1$, which is clearly a contradiction since $\|\mu\| = 1$. To prove the last statement we note that for any n, $\{E_\lambda \circ \widetilde{\sigma}^{-n}\}_{\lambda \in \Lambda}$ is an approximate unit for $C_0(X)$, and that if $\{F_\lambda\}_{\lambda \in \Lambda}$ is another approximate unit for $C_0(X)$ indexed by the same set Λ, we have that $\{E_\lambda F_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit as well. Now note that $\mu(E_\lambda \delta) \cdot \mu(E_\lambda \delta) = \mu((E_\lambda \delta) * (E_\lambda \delta)) = \mu(E_\lambda(E_\lambda \circ \widetilde{\sigma}^{-1})\delta^2)$. Using what we concluded above about independence of approximate units, this shows that $\xi^2 = \lim_\lambda \mu(E_\lambda \delta)^2 = \lim_\lambda \mu(E_\lambda(E_\lambda \circ \widetilde{\sigma}^{-1})\delta^2) = \lim_\lambda \mu(E_\lambda \delta^2)$. Inductively, we see that $\lim_\lambda \mu(E_\lambda \delta^n) = \xi^n$ for non-negative n. As $\mu((E_\lambda \delta^{-1}) * (E_\lambda \delta)) = \mu(E_\lambda(E_\lambda \circ \widetilde{\sigma}))$, we conclude that $\lim_\lambda \mu(E_\lambda \delta^{-1}) = \xi^{-1}$, and an argument similar to the one above allows us to draw the desired conclusion for all negative n. $\qquad\square$

We may use this to see that $\Xi = \emptyset$ if $(\Delta(A), \widetilde{\sigma})$ lacks fixed points. This is because the restriction of a map $\mu \in \Xi$ to $C_0(\Delta(A))$ must be a point evaluation, μ_x say, by basic Banach algebra theory. If $x \ne \sigma(x)$ there exists an $h \in C_0(\Delta(A))$ such that $h(x) = 1$ and $(h \circ \sigma)(x) = 0$. By Lemma 7.5 we see that $\mu(h\delta) = \lim_\lambda \mu(hE_\lambda \delta) = \lim_\lambda \mu(h)\mu(E_\lambda \delta)) = h(x)\xi = \xi$ and likewise $\mu(h\delta^{-1}) = \xi^{-1}$. But then $1 = \xi^{-1}\xi = \mu((h\delta^{-1}) * (h\delta)) = \mu(h \cdot (h \circ \sigma)) = h(x) \cdot (h \circ \sigma)(x) = 0$, which is a contradiction.

Now for any $x \in \mathrm{Per}^1(\Delta(A))$ and $\xi \in \mathbb{T}$ there is a unique element $\mu \in \Xi$ such that $\mu(f_n \delta^n) = f_n(x)\xi^n$ for all n and by the above every element of Ξ must be of this form for a unique x and ξ. Thus we have a bijection between Ξ and $\mathrm{Per}^1(\Delta(A)) \times \mathbb{T}$. Denote by $I(x, \xi)$ the kernel of such μ. This is clearly a modular ideal of $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$ which is maximal and contains $\mathscr{C}$ by multiplicativity and continuity of elements in Ξ.

Theorem 7.6. *The modular ideals of $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$ which are maximal and contain the commutator ideal $\mathscr{C}$ are precisely the ideals $I(x, \xi)$, where $x \in \mathrm{Per}^1(\Delta(A))$ and $\xi \in \mathbb{T}$.*

Proof. One inclusion is clear from the discussion above. For the converse, let M be such an ideal and note that it is easy to show that a maximal ideal containing $\mathscr{C}$ is not properly contained in any proper left or right ideal. Thus as $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$ is a spectral algebra, [7, Theorem 2.4.13] implies that $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))/M$ is isomorphic to the complex field. This clearly implies that M is the kernel of a non-zero element of Ξ. $\qquad\square$

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