

LOCAL INVARIANTS ATTACHED TO WEIERSTRASS POINTS

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ABSTRACT. We study certain isomorphisms of line bundles connected to Weierstrass points on semistable fibrations of curves. We give two applications. One is a formula for the order of vanishing of the discriminant of a hyperelliptic curve over a discrete valuation ring in terms of geometrical and combinatorial data determined by its special fiber. The other is a closed explicit formula for the Faltings stable height of a curve over a number field in terms of elementary local data and the Néron-Tate heights of its Weierstrass points. We deduce a lower bound for the height which we compare with a known lower bound for Faltings heights of general abelian varieties.

1. INTRODUCTION

The purpose of this paper is to make a study of Weierstrass points on semistable fibrations of curves, much in the spirit of §2 of E. Viehweg's paper [26]. The main difference with that paper is that we cover more general base schemes, including the possibility of positive characteristics in the fibers. In addition, we want to combine the information that we obtain in this way with Noether's formula, in its form as a canonical isomorphism between certain line bundles [21]. As a result of this combination we find a specific ample line bundle on the moduli space of stable curves which seems to be of independent interest.

We consider two main applications of the theory we arrive at. The first application is to the case of semistable fibrations with a hyperelliptic general fiber. Here it is possible to actually determine a canonical section of our ample line bundle, and a "reduction to the universal case" enables us to identify it, essentially, with the discriminant of a hyperelliptic curve which is a canonical section of a certain power of the determinant of the Hodge bundle [14]. Thus we arrive at an expression for the order of vanishing of the discriminant of a hyperelliptic curve over a discrete valuation ring in terms of data associated to the distribution of Weierstrass points over its special fiber (Corollary 4.10).

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The second application of our theory is to the case of curves over a number field. Here the theory enables us to write down a closed explicit formula for the Faltings stable height [8] of a curve, involving besides the number of singular points in the fibers of a regular minimal model of the curve only elementary data associated with its Weierstrass points (Corollary 5.3). The explicit nature of our formula allows for a certain lower bound on the Faltings stable height which we hope will be sharp enough in order to exclude certain abelian varieties over number fields to be possibly isomorphic (as an abelian variety) to the jacobian of a curve.

Conventions— All schemes will be locally noetherian and separated. If R is a discrete valuation ring we will assume its valuation to be normalized in the sense that its value group is \mathbb{Z} . If D is a Cartier divisor on a scheme X we use the notation $O_X(D)$ for the associated line bundle. However, we will often not distinguish between D and the line bundle $O_X(D)$ it defines. If \mathcal{L}, \mathcal{M} are line bundles the notation $\mathcal{L} = \mathcal{M}$ means that \mathcal{L} and \mathcal{M} are canonically isomorphic or that a specific isomorphism has been given. The natural group operation on line bundles will sometimes be denoted in a multiplicative way, sometimes in an additive way.

Let $\rho: X \rightarrow S$ be a proper flat morphism of relative dimension 1 with geometrically connected fibers. We call $X \rightarrow S$ a *prestable curve* of genus g if the geometric fibers of $X \rightarrow S$ are reduced, nodal curves of arithmetic genus g . We call a prestable curve $X \rightarrow S$ *semistable* (resp. *stable*) if every non-singular rational component of a geometric fiber meets the other components of that fiber in at least 2 (resp. 3) points. Note that the property of being prestable (or semistable resp. stable) is preserved under arbitrary base change.

If $\rho: X \rightarrow S$ is a prestable curve we denote by S^o the open subset of S where the fibers of ρ are smooth. Moreover we put $X^o = \rho^{-1}S^o$. We denote by X^{sm} the locus of x on X such that ρ is smooth at x .

2. WEIERSTRASS POINTS AND NOETHER'S FORMULA

This section is an introduction to the starting point of this paper, Proposition 2.5.

(2.1) Let $\rho: X \rightarrow S$ be a prestable curve of genus $g \geq 1$. According to [7] the morphism ρ has a relative dualizing sheaf $\omega = \omega_{X/S}$. This is an invertible sheaf, whose formation is compatible with base change; the pushforward $\rho_*\omega$ is locally free of rank g and on X^{sm} the sheaf ω can be identified with the sheaf of relative differentials. We use the notation λ for the determinant line bundle $\det \rho_*\omega$ on S .

Assume from now on that ρ is generically smooth, and that S is regular. It follows, since ρ is a locally complete intersection morphism, that X is a Cohen-Macaulay scheme (cf. [16], Corollary 8.2.18). Since in addition $\text{codim}(X - X^{sm}) \geq 2$ we have the following property for line bundles \mathcal{L} on X : every regular section of \mathcal{L} over X^{sm} extends uniquely to a regular section of \mathcal{L} over X .

(2.2) We recall a construction going back to S.Y. Arakelov (cf. [1], p. 1298) generalizing the notion of Weierstrass points on smooth curves over a field.

Proposition 2.1. *Let $\rho: X \rightarrow S$ be a generically smooth prestable curve of genus $g \geq 1$ with S regular. Then the line bundle $\omega^{\otimes g(g+1)/2} \otimes \rho^* \lambda^{-1}$ has a canonical global section. The formation of this section commutes with any base change preserving generic smoothness and regularity of the base.*

Before we give the proof we fix some notation. Let x be a point in X^{sm} , and suppose that x maps to s in S . Choose a parameter t on an open neighbourhood U of x in X^{sm} such that $\widehat{O_{X,x}} = \widehat{O_{S,s}}[[t]]$. Denote by \mathcal{D}_i for $i \geq 0$ the $\widehat{O_{S,s}}$ -linear map $\widehat{O_{X,x}} \rightarrow \widehat{O_{X,x}}$ given by $t^n \mapsto \binom{n}{i} t^{n-i}$. Note that if $\widehat{O_{S,s}}$ is of characteristic zero then \mathcal{D}_i is just the map sending f to $\frac{1}{i!} \frac{d^i f}{dt^i}$. For (f_1, \dots, f_g) in $\widehat{O_{X,x}}^g$ we define

$$[f_1, \dots, f_g] := \det(\mathcal{D}_{i-1} f_j)_{1 \leq i, j \leq g},$$

the *Wronskian* on (f_1, \dots, f_g) .

Proof of Proposition 2.1. By what we have said in (2.1) it suffices to construct a section of $\omega^{\otimes g(g+1)/2} \otimes \rho^* \lambda^{-1}$ restricted to X^{sm} . Let x be a point in X^{sm} , and suppose that x maps to s in S . Let (η_1, \dots, η_g) be a basis of $\rho_* \omega$ around s , and write $\eta_i = f_i dt$ with f_i in $\widehat{O_{X,x}}$ and with t a local parameter on an open neighbourhood of x as above. If we put

$$[\eta_1, \dots, \eta_g] = [f_1, \dots, f_g] \cdot (dt)^{\otimes g(g+1)/2}$$

it can be checked that $[\eta_1, \dots, \eta_g]$ is independent of the chosen t and defines a section of $\omega^{\otimes g(g+1)/2}$ locally around x . If $(\eta'_1, \dots, \eta'_g)$ is another basis of $\rho_* \omega$ around s connected to (η_1, \dots, η_g) via

$$\eta'_i = \sum_{j=1}^g a_{ij} \cdot \eta_j \quad \text{for } i = 1, \dots, g$$

then one easily verifies that $[\eta'_1, \dots, \eta'_g] = \det(a_{ij})[\eta_1, \dots, \eta_g]$. This implies that if we take

$$[\eta_1, \dots, \eta_g] \cdot (\eta_1 \wedge \dots \wedge \eta_g)^{-1}$$

we obtain a section that is independent of the chosen basis as well. Note that this element can be interpreted as a section of the line bundle $\omega^{\otimes g(g+1)/2} \otimes \rho^* \lambda^{-1}$. It is clear that the formation of this section is compatible with any base change preserving generic smoothness and regularity of the base. \square

We call the canonical global section of $\omega^{\otimes g(g+1)/2} \otimes \rho^* \lambda^{-1}$ constructed in the above proof the *Wronskian differential* of $X \rightarrow S$, notation Wr . If $S = \text{Spec } k$ with k a field and Wr is not identically zero we call $\text{div } Wr$ the *divisor of Weierstrass points* of X . It is possible that Wr is identically zero.

Proposition 2.2. *Let $X \rightarrow S$ be a smooth proper curve of genus $g \geq 1$ with $S = \text{Spec } k$ for a field k . If Wr is identically zero then k is of positive characteristic p with $p < 2g - 1$.*

Proof. We may assume k algebraically closed. Let x be a closed point of X , and let $\Gamma_x = \{a_1 < a_2 < \dots\}$ be the set of natural numbers a such that there is an η in $H^0(X, \omega)$ with a zero of exact order $a - 1$ at x . It is clear that Γ_x gives rise to a filtration with simple quotients of the g -dimensional vector space $H^0(X, \omega)$, hence Γ_x consists of g elements. Since ω has no base points we have $a_1 = 1$. On the other hand we have $a_g \leq 2g - 1$. Moreover we can find a local parameter t around x and a basis $(f_1 dt, \dots, f_g dt)$ of $H^0(X, \omega)$ such that $f_i = (\text{unit}) \cdot t^{a_i - 1}$ for $i = 1, \dots, g$. This implies that there is an A in the image of \mathbb{Z} in $O_{X,x}$ such that

$$\begin{aligned} A \cdot [f_1, \dots, f_g] &= \begin{vmatrix} t^{a_1-1} & (a_1-1)t^{a_1-2} & \dots & (a_1-1) \cdots (a_1-g+1)t^{a_1-g} \\ t^{a_2-1} & (a_2-1)t^{a_2-2} & \dots & (a_2-1) \cdots (a_2-g+1)t^{a_2-g} \\ \vdots & \vdots & & \vdots \\ t^{a_g-1} & (a_g-1)t^{a_g-2} & \dots & (a_g-1) \cdots (a_g-g+1)t^{a_g-g} \end{vmatrix} \cdot (\text{unit}) \\ &= \begin{vmatrix} 1 & (a_1-1) & \dots & (a_1-1) \cdots (a_1-g+1) \\ 1 & (a_2-1) & \dots & (a_2-1) \cdots (a_2-g+1) \\ \vdots & \vdots & & \vdots \\ 1 & (a_g-1) & \dots & (a_g-1) \cdots (a_g-g+1) \end{vmatrix} \cdot (\text{unit}) \cdot t^w \\ &= \prod_{1 \leq i < j \leq g} (a_i - a_j) \cdot (\text{unit}) \cdot t^w \end{aligned}$$

where $w = \sum_{i=1}^g (a_i - i)$. Thus if $[f_1 dt, \dots, f_g dt]$ is zero, so is $\prod_{1 \leq i < j \leq g} (a_i - a_j)$, whence k of positive characteristic p with $p < 2g - 1$. \square

Let again $X \rightarrow S$ be a generically smooth prestable curve of genus $g \geq 1$ with S an arbitrary regular scheme. Assume that Wr is not identically zero. Then we call the

closure in X of the Weierstrass points of the general fibers of $X \rightarrow S$ the *Weierstrass divisor* of $X \rightarrow S$, notation W . Note that the generic degree of W is $g^3 - g$. We can write $\text{div } Wr = W + E$ for some effective divisor E and we call E the *residual divisor* of $X \rightarrow S$. Note that W and E themselves are in general only Weil divisors.

The following proposition is clear from the above.

Proposition 2.3. *Let $X \rightarrow S$ be a generically smooth prestable curve of genus $g \geq 1$ with S regular. If Wr is not identically zero there exists a canonical isomorphism*

$$\omega^{\otimes g(g+1)/2} \otimes \rho^* \lambda^{-1} \xrightarrow{\sim} O_X(W + E)$$

with W the Weierstrass divisor of $X \rightarrow S$ and with E the residual divisor of $X \rightarrow S$. E has support in the union of the degenerate fibers of $X \rightarrow S$ and the fibers of $X \rightarrow S$ of positive characteristic p with $p < 2g - 1$. The formation of the isomorphism commutes with any base change preserving generic smoothness, regularity of the base and non-vanishing of Wr .

Remark 2.1. The divisor W is in general not finite over S , nor is the splitting of $\text{div } Wr$ as $W + E$ in general well-behaved under base change (cf. [26], Remark 2.11). The situation is better if $\dim S = 1$ or if $X \rightarrow S$ is a fibration with hyperelliptic general fibers. In this paper we will be primarily interested in these two cases.

Example 2.2. If $g = 1$ and X is regular or ρ is stable then one has a canonical isomorphism $\omega = \rho^* \lambda$ and both W and E are empty. For $X \rightarrow S$ a fibration with hyperelliptic general fibers and $\dim S = 1$ we have an explicit formula for E , see Proposition 4.2 below. More examples of computations of E can be found in [26], Section 4.

(2.3) Let $\rho: X \rightarrow S$ be a semistable curve. We recall from [6] [19] the notion of Deligne pairing for $X \rightarrow S$. This pairing is a rule that associates to each pair $(\mathcal{L}, \mathcal{M})$ of line bundles on X a line bundle $\langle \mathcal{L}, \mathcal{M} \rangle$ on S , satisfying, among others, the following properties:

(i) for given line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}_1, \mathcal{M}_2$ on X we have canonical isomorphisms

$$\langle \mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{M} \rangle = \langle \mathcal{L}_1, \mathcal{M} \rangle \otimes \langle \mathcal{L}_2, \mathcal{M} \rangle, \quad \langle \mathcal{L}, \mathcal{M}_1 \otimes \mathcal{M}_2 \rangle = \langle \mathcal{L}, \mathcal{M}_1 \rangle \otimes \langle \mathcal{L}, \mathcal{M}_2 \rangle.$$

(ii) for given line bundles \mathcal{L}, \mathcal{M} we have a canonical isomorphism $\langle \mathcal{L}, \mathcal{M} \rangle = \langle \mathcal{M}, \mathcal{L} \rangle$.

(iii) the Deligne pairing commutes with base change.

(iv) if $\det R\rho_* \mathcal{L}$ denotes the determinant of cohomology [6] of \mathcal{L} one has a canonical isomorphism

$$(\det R\rho_* \mathcal{L})^{\otimes 2} = \langle \mathcal{L}, \mathcal{L} \otimes \omega^{-1} \rangle \otimes \lambda^{\otimes 2}.$$

(v) if $P \in X(S)$ is a section of ρ such that the image of P can be seen as a Cartier divisor on X , then one has a canonical isomorphism $P^*\mathcal{L} = \langle O_X(P), \mathcal{L} \rangle$.

(vi) if D is an effective Cartier divisor on X then $\langle D, D \otimes \omega \rangle$ has a canonical rational section Ad given by “adjunction”. In the case that $D = O_X(P)$ for a section $P \in X(S)$ of $X \rightarrow S$ one has in addition a canonical isomorphism $\langle D, D \otimes \omega \rangle = O_S$ with Ad on the left hand side corresponding to 1 on the right hand side.

(vii) if \mathcal{M} is a line bundle of generic degree d in the fibers and \mathcal{L} is a line bundle on S then one has a canonical isomorphism $\langle \rho^*\mathcal{L}, \mathcal{M} \rangle = \mathcal{L}^{\otimes d}$.

(viii) assume $S = \text{Spec } R$ with R a discrete valuation ring, assume $X \rightarrow S$ is the minimal regular model of its generic fiber, and let $X' \rightarrow S$ be the corresponding stable model. Then one has a canonical S -morphism $\pi: X \rightarrow X'$ and for any two line bundles $\mathcal{L}', \mathcal{M}'$ on X' one has a canonical isomorphism $\langle \pi^*\mathcal{L}', \pi^*\mathcal{M}' \rangle = \langle \mathcal{L}', \mathcal{M}' \rangle$.

(2.4) Let $\rho: X \rightarrow S$ for the moment be a smooth curve, with S any scheme. According to [6], Théorème 9.9 there exists an up to sign canonical isomorphism

$$\mu: \det R\rho_*(\omega^{\otimes 2}) \xrightarrow{\sim} \lambda^{\otimes 13}$$

of line bundles on S , compatible with base change. Letting $X \rightarrow S$ again be an arbitrary generically smooth semistable curve one has μ on the open subset S^o of S and it extends canonically to a non-zero morphism

$$\det R\rho_*(\omega^{\otimes 2}) \longrightarrow \lambda^{\otimes 13}$$

on S . We denote the zero divisor of this morphism by Δ , and put $\delta = O_S(\Delta)$. Thus we have, up to sign, a canonical isomorphism

$$\lambda^{\otimes 13} = \det R\rho_*(\omega^{\otimes 2}) \otimes \delta,$$

compatible with base change. By property (iv) of the Deligne pairing we obtain

Proposition 2.4. *(Cf. [21], Theorem 5.10 or [20], Théorème 2.1) Let $X \rightarrow S$ be a generically smooth semistable curve. Then there is, up to sign, a canonical isomorphism of line bundles*

$$\lambda^{\otimes 12} = \langle \omega, \omega \rangle \otimes \delta$$

on S , commuting with any base change preserving generic smoothness.

We remark that if $S = \text{Spec } R$ with R a discrete valuation ring, then if X is regular, the valuation $\text{ord}_s \Delta$ at the closed point s of S is equal to the number of singular points

in the geometric fiber at s .

(2.5) A suitable combination of Propositions 2.3 and 2.4 yields

Proposition 2.5. *Let $X \rightarrow S$ be a generically smooth semistable curve of genus $g \geq 1$ with S regular. Assume that Wr is not identically zero. Then we have, up to a sign, a canonical isomorphism of line bundles*

$$(3g - 1)(8g + 4)\lambda = \langle 4W - (g + 1)\omega + 4E, \omega \rangle \otimes (2g - 1)(g + 1)\delta$$

on S . The formation of this isomorphism commutes with any base change preserving generic smoothness, regularity of the base, and non-vanishing of Wr .

Proof. By Proposition 2.3 we have

$$\begin{aligned} 4W - (g + 1)\omega + 4E &= 2g(g + 1)\omega - 4\rho^*\lambda - (g + 1)\omega \\ &= (2g - 1)(g + 1)\omega - 4\rho^*\lambda. \end{aligned}$$

Pairing with ω yields a canonical isomorphism

$$\langle 4W - (g + 1)\omega + 4E, \omega \rangle = (2g - 1)(g + 1)\langle \omega, \omega \rangle - (8g - 8)\lambda$$

on S ; here we used property (vii) of the Deligne pairing. Applying Proposition 2.4 one finds

$$\langle 4W - (g + 1)\omega + 4E, \omega \rangle = (2g - 1)(g + 1)(12\lambda - \delta) - (8g - 8)\lambda$$

and hence by rewriting

$$\langle 4W - (g + 1)\omega + 4E, \omega \rangle = (3g - 1)(8g + 4)\lambda - (2g - 1)(g + 1)\delta,$$

canonically up to sign. □

We remark that the line bundle $(8g + 4)\lambda$ seems to have various natural meanings. For example in [5] its restriction to the hyperelliptic locus is used to obtain positivity properties of certain linear combinations of λ and δ . In [11] it appears (up to non-canonical isomorphism) as the “biextension line bundle” associated to a certain algebraic 1-cycle on the jacobian of a curve.

The aim of the rest of this paper is to make a further study of the isomorphism from Proposition 2.5 in some special cases.

3. CURVES OVER FUNCTION FIELDS

We start with the case that the base S is a smooth proper connected complex curve. Let $X \rightarrow S$ be a generically smooth semistable curve of genus $g \geq 1$. It follows from Proposition 2.2 that Wr is not identically zero. Assume that X is regular. Then X is a smooth proper surface over \mathbb{C} and hence both W and E can be viewed as Cartier divisors on X . Moreover one has intersection theory for line bundles on X and one property of the Deligne pairing is that $(\mathcal{L}, \mathcal{M}) = \deg \langle \mathcal{L}, \mathcal{M} \rangle$ for any two line bundles \mathcal{L}, \mathcal{M} on X . Since S is a Dedekind scheme, if the Weierstrass points on the generic fiber of $X \rightarrow S$ are rational W can be written as a sum of images of sections, multiplicities allowed. In any case there exists a finite cover $T \rightarrow S$ with T regular such that $X \times_S T$ has its Weierstrass points rational on the generic fiber.

Proposition 3.1. *Let $X \rightarrow S$ be a generically smooth semistable curve of genus $g \geq 1$ with S a smooth proper connected complex curve and X regular. Then the equality*

$$(3g - 1)(8g + 4) \deg \lambda = (4W - (g + 1)\omega + 4E, \omega) + (2g - 1)(g + 1) \deg \delta$$

holds.

Proof. Immediate from Proposition 2.5. □

We derive some positivity results for the intersection number $(4W - (g + 1)\omega + 4E, \omega)$. Recall from [5] that if the generic fiber of $X \rightarrow S$ is hyperelliptic, each degenerate fiber belongs to at least one of a collection of types Δ_i with $i = 1, \dots, [g/2]$ or Ξ_i with $i = 0, \dots, [(g - 1)/2]$.

Proposition 3.2. *Take the assumptions of Proposition 3.1, and assume furthermore $g \geq 2$ and $X \rightarrow S$ not isotrivial. Then we have the lower bound*

$$(4W - (g + 1)\omega + 4E, \omega) \geq (g - 1)^2 \deg \delta$$

with equality if and only if the generic fiber of $X \rightarrow S$ is hyperelliptic and the degenerate fibers do not belong to Δ_i or Ξ_i for $i \geq 1$. In particular, the intersection number $(4W - (g + 1)\omega + 4E, \omega)$ is positive.

Proof. According to the Cornalba-Harris-Xiao inequality (cf. [5], Theorem 4.12) we have $(8g + 4) \deg \lambda \geq g \deg \delta$ with equality if and only if the generic fiber of $X \rightarrow S$ is hyperelliptic and the degenerate fibers do not belong to Δ_i or Ξ_i for $i \geq 1$. Using this to eliminate $(8g + 4) \deg \lambda$ from Proposition 3.1 we find the first statement. The second statement follows from the first except that we have to consider the case of a smooth

fibration with hyperelliptic fibers. But in this case we have $(8g + 4)\lambda$ trivial on S , hence such a fibration is isotrivial by [1], Corollary 1. \square

Remark 3.1. In fact the line bundle

$$\langle 4W - (g + 1)\omega + 4E, \omega \rangle$$

is ample on the moduli space of complex stable genus g curves. Indeed, by Proposition 2.5 its “slope” is

$$\frac{(3g - 1)(8g + 4)}{(2g - 1)(g + 1)} > 11$$

and therefore another result of Cornalba-Harris applies (cf. [5], Theorem 1.3).

Proposition 3.3. *Take the assumptions of Proposition 3.1, and assume furthermore that $X \rightarrow S$ is not isotrivial. Let $P \in X(S)$ be a section of $X \rightarrow S$. Then the inequality*

$$4g(g - 1)(P, \omega) \geq (\omega, \omega)$$

holds.

Proof. This follows from the Hodge index theorem on X . See [23], Proposition 2. \square

We call a section $P \in X(S)$ *special* if equality holds in the above proposition.

Proposition 3.4. *Let S be a smooth proper connected complex curve, let $X \rightarrow S$ be a smooth non-isotrivial relative curve, and assume all Weierstrass points of the generic fiber are rational. Then at least one of the sections of $W \rightarrow S$ is not special.*

Proof. Let g be the genus of the fibers of $X \rightarrow S$. We may assume that $g \geq 3$. By the assumption that $X \rightarrow S$ is smooth we have E empty and hence $4(W, \omega) > (g + 1)(\omega, \omega)$ from Proposition 3.2. As W has generic degree $g^3 - g$ the assumption $4g(g - 1)(P, \omega) = (\omega, \omega)$ for each section P of $W \rightarrow S$ leads to a contradiction. \square

4. HYPERELLIPTIC CURVES

The aim of this section is to study the isomorphism of Proposition 2.5 for a fibration with hyperelliptic general fibers. Our main result is an expression for the order of vanishing of the discriminant of a hyperelliptic curve over a discrete valuation ring in terms of geometrical data related to the distribution of Weierstrass points over the special fiber (Corollary 4.10).

(4.1) We start with some definitions. Let X be a (semi)stable curve of genus $g \geq 2$ over an algebraically closed field k equipped with an automorphism $\sigma \in \text{Aut}_k X$. We call the pair (X, σ) a *hyperelliptic (semi)stable curve* if σ is of order 2 and $X/\langle\sigma\rangle$ is a prestable curve of genus 0 over k . Next, let a (semi)stable curve $X \rightarrow S$ be given with an element $\sigma \in \text{Aut}_S X$ of order 2. We call the pair $(X \rightarrow S, \sigma)$ a *hyperelliptic (semi)stable curve* if each geometric fiber of $X \rightarrow S$ is hyperelliptic for the automorphism induced by σ . According to [27], Appendix one has a Deligne-Mumford moduli stack $\overline{\mathcal{H}}_g$ proper over $\text{Spec } \mathbb{Z}$ parametrizing stable hyperelliptic curves of genus g . It has an open dense substack \mathcal{H}_g corresponding to smooth curves as well as an open dense substack $\overline{\mathcal{H}}_g^o$ corresponding to stable curves whose fibers in characteristic 2 are smooth. According to [27], Remark 1.4 the stack $\overline{\mathcal{H}}_g^o$ is smooth over $\text{Spec } \mathbb{Z}$.

Proposition 4.1. *Let $(X \rightarrow S, \sigma)$ be a generically smooth hyperelliptic semistable curve with S regular. Let Wr be the Wronskian differential of $X \rightarrow S$. Then Wr is not identically zero and the residual divisor E is supported in the degenerate fibers.*

Note that we put no restrictions on the characteristics of the fibers of $X \rightarrow S$.

Proof. We are done if we prove that Wr is not identically zero in the case $S = \text{Spec } k$ where k is an algebraically closed field. Let $y^2 + ay = b$ with a, b in $k[x]$ be an equation for $X \rightarrow S$. Then σ is given by $y \mapsto -y + a$. As $X/\langle\sigma\rangle$ is of genus 0 the quotient map $X \rightarrow X/\langle\sigma\rangle$ is separable. This implies $2y + a$ not identically zero, and a basis of $H^0(X, \omega)$ is given by $x^i dx / (2y + a)$ for $i = 0, \dots, g - 1$ where g is the genus of X . A computation yields

$$\left[\frac{dx}{2y + a}, \dots, \frac{x^{g-1} dx}{2y + a} \right] = (2y + a)^{g(g-1)/2} \left(\frac{dx}{2y + a} \right)^{\otimes g(g+1)/2},$$

and the result follows. \square

(4.2) We give an explicit formula for the residual divisor in the case that the base scheme is the spectrum of a discrete valuation ring with residue field not of characteristic 2. We use a number of methods and techniques from I. Kausz's article [14].

Let $(X \rightarrow S, \sigma)$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 2$. Assume $S = \text{Spec } R$ with R a discrete valuation ring of residue characteristic $\neq 2$, and assume furthermore that X is regular. Denote by \mathfrak{m} the maximal ideal of R and by v the normalized valuation of R . Possibly after making a finite extension of the base we may assume that the following holds (cf. [14], Lemma 4.1): the generic fiber of $X \rightarrow S$ is given by an equation $y^2 = A \cdot f(x)$ with $A \in R^*$ and $f(x) = \prod_{i=1}^{2g+2} (x - a_i)$ for certain

$a_i \in R$. We have $a_i \neq a_j$ and $v(a_i - a_j)$ even for $i \neq j$, and $\#\{\overline{a_i} \in R/\mathfrak{m} : a_i \in R\} \geq 3$. Under these assumptions we have an explicit description of the special fiber of $X \rightarrow S$.

One starts by constructing a finite tree $T = (\mathcal{V}, \mathcal{E})$ from the above data. For non-negative integer n denote by $r_n: \{a_1, \dots, a_{2g+2}\} \rightarrow R/\mathfrak{m}^n$ the natural map sending a_i to its residue class modulo \mathfrak{m}^n . The vertices of T are then the elements of the set $\mathcal{V} = \sqcup_{n \geq 0} \mathcal{V}_n$ where $\mathcal{V}_n = \{V \in R/\mathfrak{m}^n : \#r_n^{-1}(V) \geq 2\}$. The set \mathcal{E} of edges of T consists of the pairs (V, V') where $V \in \mathcal{V}_n$ and $V' \in \mathcal{V}_{n+1}$ for some $n \geq 0$ and $V' \mapsto V$ under the canonical map $\mathcal{V}_{n+1} \rightarrow \mathcal{V}_n$. Note that \mathcal{V} has a canonical partial ordering and that there is an absolute minimum V_0 with respect to this ordering. It is clear that T is canonically isomorphic to the dual graph of the special fiber of the prestable curve $Y' \rightarrow S$ of genus 0 that is obtained by taking the smooth curve \mathbb{P}_S^1 and then successively blowing up closed points of the special fiber where the sections P_i given by the a_i meet, until the strict transform of $\sum_i P_i$ in Y' becomes regular. In particular, the vertices of T can be viewed as the irreducible components of the special fiber of Y' .

We can construct X from Y' : first, for every $V \in \mathcal{V}$ put $n(V) = n$ if $V \in \mathcal{V}_n$, and put $\varphi(V) = \#r_n^{-1}(V)$. If v_V is the normalized discrete valuation of the function field of Y' corresponding to the irreducible component V of its special fiber, then

$$v_V(x - a_i) = \min(n(V), v(a - a_i))$$

for all i if a is a representative of V . By a counting argument one finds

$$\sum_{i=1}^{2g+2} \min(n(V), v(a - a_i)) = \sum_{i=1}^{n(V)} \varphi(V_i)$$

if $V_0, V_1, \dots, V_n = V$ are the vertices of the unique linear subgraph of T that connects V and V_0 .

For a vertex V in \mathcal{V} define $C(V)$ to be 1 if both n and $\varphi(V)$ are odd, and 0 otherwise. This gives rise to an effective divisor $C := \sum_{i=1}^{2g+2} P_i + \sum_{V \in \mathcal{V}} C(V) \cdot V$ on Y' which has the properties that C is regular and that the class of C is divisible by 2 in the Picard group of Y' . Thus we have an S -scheme X' and a finite flat morphism $\pi': X' \rightarrow Y'$ of degree 2 such that X' is regular and π' is branched exactly along C . It can be seen that $X' \rightarrow S$ is prestable with generic fiber isomorphic to the generic fiber of X . For V an irreducible component of the special fiber of Y' , set $V^* = \pi'^*V = X' \times_{Y'} V$. By construction, if $C(V) = 1$ then V^* is a double line $V^* = 2L$ with L exceptional, and if $C(V) = 0$ then V^* is reduced and $V^* \rightarrow V$ is finite of degree 2, ramified over precisely (C, V) points of V . If we blow down all exceptional fibers of X' we find $X \rightarrow S$, up

to S -isomorphism. Likewise, one can contract all V on Y' with $C(V) = 1$ and obtain a model $Y \rightarrow S$ of \mathbb{P}^1 over the function field of R . We have a canonical map $X \rightarrow Y$, denoted π .

Define (cf. [14], p. 56)

$$e := \frac{1}{2} \sum_{V > V_0: \varphi(V) \text{ even}} \frac{\varphi(V)}{2} \left(\frac{\varphi(V)}{2} - 1 \right) + \frac{1}{2} \sum_{V > V_0: \varphi(V) \text{ odd}} \left(\frac{\varphi(V) - 1}{2} \right)^2,$$

the sums running over V in the set of vertices \mathcal{V} of the tree T .

Proposition 4.2. *Let $(X \rightarrow S = \text{Spec } R, \sigma)$ be a generically smooth hyperelliptic semistable curve of genus $g \geq 2$ over a discrete valuation ring with residue field not of characteristic 2. Assume that X is regular and that the generic fiber of $X \rightarrow S$ is given by an equation $y^2 = A \cdot f(x)$ with $A \in R^*$ and $f(x) = \prod_{i=1}^{2g+2} (x - a_i)$ for certain $a_i \in R$ with $a_i \neq a_j$ and $v(a_i - a_j)$ even for $i \neq j$, and with $\#\{\bar{a}_i \in R/\mathfrak{m} : a_i \in R\} \geq 3$. Let $T = (\mathcal{V}, \mathcal{E})$ be the tree associated to $\{a_1, \dots, a_{2g+2}\}$. Then for the residual divisor E of $X \rightarrow S$ the formula*

$$E = \sum_{\substack{V \in \mathcal{V} \\ C(V)=0}} \left(e - \frac{g}{2} \sum_{i=1}^{n(V)} \varphi(V_i) + \frac{g(g+1)}{2} n(V) \right) \cdot V^*$$

holds, where for $V \in \mathcal{V}$ we denote by $V_0, V_1, \dots, V_n = V$ the vertices of the unique linear subgraph of T that connects V and V_0 .

Proof. The divisor E is the vertical part of the Wronskian $[\omega_0, \dots, \omega_{g-1}]$ on an R -basis $(\omega_0, \dots, \omega_{g-1})$ of $H^0(X, \omega)$. According to [14], Proposition 5.5 there are $e_i \in \mathbb{Z}$ with $\sum_{i=0}^{g-1} e_i = e$ and $b_j \in \{a_1, \dots, a_{2g+2}\}$ such that

$$\omega_i = t^{e_i} \left(\prod_{j=1}^i (x - b_j) \right) \frac{dx}{y}$$

can be chosen. Here t is a generator of the maximal ideal \mathfrak{m} of R . In general, if x is a point on X and f_1, \dots, f_g, u are in $O_{X,x}$ with u a unit, then $(\mathcal{D}_{i-1}(uf_j))$ is equal to a lower-triangular matrix with u 's on the diagonal times $(\mathcal{D}_{i-1}(f_j))$ hence $[uf_1, \dots, uf_g] =$

$u^g[f_1, \dots, f_g]$. In our case we take $u = y^{-1}$ and find, putting $h_i(x) := \prod_{j=1}^i (x - b_j)$,

$$\begin{aligned} [\omega_0, \dots, \omega_{g-1}] &= y^{-g} t^e [h_0, \dots, h_{g-1}] (dx)^{\otimes g(g+1)/2} \\ &= y^{g(g-1)/2} t^e [h_0, \dots, h_{g-1}] \left(\frac{dx}{y} \right)^{\otimes g(g+1)/2} \\ &= y^{g(g-1)/2} t^e \left(\frac{dx}{y} \right)^{\otimes g(g+1)/2}. \end{aligned}$$

Let $(y)_{\text{vert}}$ be the vertical part of the divisor of y on X , and let P_∞ be the section of $Y \rightarrow S$ corresponding to the point at infinity of the generic fiber of Y . According to Lemma 5.2 of [14] we have

$$\operatorname{div} \left(\frac{dx}{y} \right) = (g-1)\pi^* P_\infty - (y)_{\text{vert}} + \sum_{\substack{V \in \mathcal{V} \\ C(V)=0}} n(V) \cdot V^*.$$

It follows that

$$\begin{aligned} \operatorname{div} [\omega_0, \dots, \omega_{g-1}] &= eF + \frac{g(g-1)}{2} \operatorname{div} y + \frac{g(g-1)(g+1)}{2} \pi^* P_\infty \\ &\quad - \frac{g(g+1)}{2} (y)_{\text{vert}} + \frac{g(g+1)}{2} \sum_{\substack{V \in \mathcal{V} \\ C(V)=0}} n(V) \cdot V^* \end{aligned}$$

with F the special fiber of X . Noting that

$$\frac{g(g-1)}{2} \operatorname{div} y = \frac{g(g-1)}{2} (y)_{\text{vert}} + W - \frac{g(g-1)(g+1)}{2} \pi^* P_\infty$$

we derive

$$\operatorname{div} [\omega_0, \dots, \omega_{g-1}] = eF + W - g(y)_{\text{vert}} + \frac{g(g+1)}{2} \sum_{\substack{V \in \mathcal{V} \\ C(V)=0}} n(V) \cdot V^*.$$

Since $y^2 = \prod_{i=1}^{2g+2} (x - a_i)$ we have for V with $C(V) = 0$

$$v_{V^*}(y) = \frac{1}{2} \sum_{i=1}^{2g+2} v_{V^*}(x - a_i) = \frac{1}{2} \sum_{i=1}^{2g+2} \min(n(V), v(a - a_i)) = \frac{1}{2} \sum_{i=1}^{n(V)} \varphi(V_i)$$

where $V_0, V_1, \dots, V_n = V$ are the vertices of the unique linear subgraph of T that connects V and V_0 . We arrive at

$$\operatorname{div} [\omega_0, \dots, \omega_{g-1}] = W + \sum_{\substack{V \in \mathcal{V} \\ C(V)=0}} \left(e - \frac{g}{2} \sum_{i=1}^{n(V)} \varphi(V_i) + \frac{g(g+1)}{2} n(V) \right) \cdot V^*$$

and the claim follows. \square

Apparently, the number e can be interpreted as the multiplicity in E of the irreducible component of the special fiber of $X \rightarrow S$ that maps to V_0 . We note however that this component may depend on the particular equation for $X \rightarrow S$ chosen at the beginning. The reader who wishes to verify that the divisor on the right hand side in Proposition 4.2 is effective will have no difficulty to see that it suffices to have

$$\varphi \leq g + 1 + \frac{1}{g} \left(\frac{\varphi^2}{4} - \frac{\varphi}{2} \right)$$

for all $\varphi = \varphi(V)$ with V running through the vertices of the tree T . This inequality is satisfied for each positive integer φ with $\varphi \leq 2g$. Note that we have $\varphi(V) \leq 2g$ for each V since we assumed that the set $\{\bar{a}_i \in R/\mathfrak{m} : a_i \in R\}$ consists of at least 3 elements.

Example 4.1. We calculate the residual divisor E in an explicit example. Let k be an algebraically closed field of characteristic 0, and let X'_k be a stable curve of genus 2 over k consisting of an elliptic curve A with a one-noded rational curve B attached to it. Let $X_k \rightarrow X'_k$ be the modification of X'_k obtained by partially normalizing X'_k at the node ν of B , and attaching a projective line D at the two points in the preimage of ν . The semi-stable curve obtained in this way has type $I_2 - I_0 - 1$ in the classification of Namikawa-Ueno [22]. Its intersection matrix is

	A	B	D
A	-1	1	0
B	1	-3	2
D	0	2	-2

and for the arithmetic genera resp. the intersections with ω we have

C	$p_a(C)$	(C, ω)
A	1	1
B	0	1
D	0	0

Assume that X_k is the special fiber of a generically smooth curve $X \rightarrow S$ with S the spectrum of a discrete valuation ring R with residue field k and with X regular. Assume that the Weierstrass points P_1, \dots, P_6 of the generic fiber are rational, and that their closures in X are distributed over the special fiber as follows: P_1, P_2, P_3 intersect A ,

P_4 intersects B , and P_5, P_6 intersect D . The curve $X \rightarrow S$ is the minimal model of a genus 2 curve given by an equation $y^2 = (x - a_1) \cdots (x - a_6)$ with $a_1, \dots, a_6 \in R$ giving rise to a linear tree $V_0 - V_1 - V_2 - V_3$ with V_0 represented by a_1, \dots, a_6 , V_1, V_2 represented by a_4, a_5, a_6 and V_3 represented by a_5, a_6 . Thus $\varphi(V_1) = \varphi(V_2) = 3$ and $\varphi(V_3) = 2$. The correspondence with the P_i is via $P_i \leftrightarrow a_i$, the component A corresponds to V_0 , the component B corresponds to V_2 and the component D corresponds to V_3 . The vertex V_1 has $C(V)$ equal to 1. We compute $e = 1$ and Proposition 4.2 gives $E = A + B + 2D$.

(4.3) Let $(X \rightarrow S, \sigma)$ again be an arbitrary generically smooth semistable hyperelliptic curve of genus $g \geq 2$. We prove that under certain weak assumptions both left and right hand side of the isomorphism in Proposition 2.5 have canonical rational sections, and that these correspond, up to a sign, under that isomorphism.

We start with some definitions. Denote by Γ^σ the fixed point subscheme of σ on $X^\sigma \rightarrow S^\sigma$, i.e., the subscheme of X^σ representing the functor $Sch_{S^\sigma} \rightarrow Sets$ given by sending $T \rightarrow S^\sigma$ to $(X^\sigma(T))^\sigma$. The scheme Γ^σ is a closed subscheme of X^σ . We denote by Γ its closure in X . We can view Γ as a Weil divisor on X .

Proposition 4.3. *Let $(X \rightarrow S, \sigma)$ be a smooth hyperelliptic curve of genus $g \geq 2$. Then Γ is finite and flat of degree $2g + 2$ over S . The formation of Γ commutes with base change. The scheme Γ is étale over those s in S with $\kappa(s)$ not of characteristic 2. If S is regular we have $W = g(g - 1)/2 \cdot \Gamma$.*

Proof. See [15], Section 6 for all statements except the last one. The last statement follows from the proof of Proposition 4.1. \square

If $(X \rightarrow S, \sigma)$ is a generically smooth semistable hyperelliptic curve we say that $\Gamma \rightarrow S$ is *generically étale* if $\Gamma \rightarrow S$ is étale over a dense open subset of S . Note that by the previous proposition this is equivalent to saying that no connected component of S is a scheme of characteristic 2.

Proposition 4.4. *Let $(\rho: X \rightarrow S, \sigma)$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 2$ with S regular and with $\Gamma \rightarrow S$ generically étale. Then the square of the line bundle*

$$\langle 4W - (g + 1)\omega + 4E, \omega \rangle \otimes (2g - 1)(g + 1)\delta$$

has a canonical rational section. This rational section is not identically zero and has support on $S - S^\sigma$. Moreover its formation commutes with any base change preserving generic smoothness of $X \rightarrow S$, generic étaleness of $\Gamma \rightarrow S$, and regularity of the base.

We need the following lemma.

Lemma 4.5. *Let $(\rho: X \rightarrow S, \sigma)$ be a smooth hyperelliptic curve of genus $g \geq 2$. Let $P \in X(S)^\sigma$ be a σ -invariant section of $X \rightarrow S$. Then there exists a unique isomorphism*

$$\omega \xrightarrow{\sim} (2g - 2)O_X(P) \otimes \rho^* \langle P, P \rangle^{\otimes -(2g-1)}$$

that induces, by pulling back along P , the adjunction isomorphism $\langle P, \omega \rangle = \langle P, P \rangle^{\otimes -1}$. The formation of this isomorphism commutes with base change.

Proof. See Lemma 6.2 of [13]. □

Note that in the above lemma we do not assume that the base S be regular.

Proof of Proposition 4.4. It suffices to give a canonical construction of a global trivializing section of the square of the given line bundle restricted to S° , and to prove that the formation of this section commutes with any base change preserving generic étaleness of the closure of the fixed point scheme of σ on the smooth fibers and regularity of the base. Restricting to S° we find E to be empty and W itself to be a Cartier divisor. The line bundle we need to consider thus simplifies to $\langle 4W - (g + 1)\omega, \omega \rangle$. We start by constructing a canonical rational section of the square of this line bundle by considering its restriction to the open and dense subset $S^{\circ\circ}$ of S° of points where the residue characteristic is not 2. Note that $\Gamma_{S^{\circ\circ}} \rightarrow S^{\circ\circ}$ is étale, hence a suitable finite étale base change $T \rightarrow S^{\circ\circ}$ gives a covering $\Gamma_{S^{\circ\circ}} \times_{S^{\circ\circ}} T \rightarrow T$ that can be written as the disjoint union of $2g + 2$ sections. As the formation of $\langle 4W - (g + 1)\omega, \omega \rangle$ commutes with base change, by faithfully flat descent we may assume that already $\Gamma_{S^{\circ\circ}} \rightarrow S^{\circ\circ}$ itself is trivial. Let P be a section of $\Gamma_{S^{\circ\circ}} \rightarrow S^{\circ\circ}$. According to Lemma 4.5 one has on $X_{S^{\circ\circ}}$ a unique isomorphism

$$\omega = (2g - 2)O_{X_{S^{\circ\circ}}}(P) \otimes \rho^* \langle P, P \rangle^{\otimes -(2g-1)}$$

that induces, by pulling back along P , the adjunction isomorphism $\langle P, \omega \rangle = \langle P, P \rangle^{\otimes -1}$. Pairing with ω one obtains an isomorphism $\langle \omega, \omega \rangle = 4g(g - 1)\langle P, \omega \rangle$ over $S^{\circ\circ}$. Summing over all P in $\Gamma(S^{\circ\circ})$ one finds a canonical isomorphism $(2g + 2)\langle \omega, \omega \rangle = 8\langle W, \omega \rangle$ and hence a canonical global trivializing section of $2\langle 4W - (g + 1)\omega, \omega \rangle$. It remains to prove that this global trivializing section over $S^{\circ\circ}$ extends as a global trivializing section over S° . This we do as follows: for every point s on S° of codimension 1 we use the discrete valuation ring $O_{S^\circ, s}$ as a test curve in S° . If the pullback to $\text{Spec } O_{S^\circ, s}$ of the canonical rational section of $2\langle 4W - (g + 1)\omega, \omega \rangle$ on S° that follows from the above construction is trivializing for each s of codimension 1 in S° , then that rational section is in fact

trivializing on all of S^o . But note that $\text{Spec } O_{S^o, s}$ is a Dedekind scheme; this implies that the pullback of $X^o \rightarrow S^o$ to $\text{Spec } O_{S^o, s}$ has the property that after a faithfully flat base change, the Weierstrass divisor can be written as a sum of $2g + 2$ Weierstrass sections. An application once more of base change, faithfully flat descent and Lemma 4.5 yields then that also on $\text{Spec } O_{S^o, s}$ the bundle $2\langle 4W - (g + 1)\omega, \omega \rangle$ has a canonical global trivializing section. \square

(4.4) We denote the canonical rational section constructed in the above proof by ξ . We have a formula for the order of vanishing of ξ along $S - S^o$ in the case that $\dim S = 1$. We begin with a lemma.

Lemma 4.6. *Let $X \rightarrow S = \text{Spec } R$ be a semistable curve over a discrete valuation ring with X regular. If D, D' are Cartier divisors on X then $\langle D, D' \rangle$ has a canonical rational section $\alpha(D, D')$. If one of D, D' is vertical and s is the closed point of S then $\text{ord}_s \alpha(D, D') = (D, D')$, the local intersection multiplicity of D, D' above s . If D is vertical and effective then $\langle D, \omega \rangle$ has a canonical rational section $\beta(D)$. Its order of vanishing at s is given by $\text{ord}_s \beta(D) = \deg \omega|_D$.*

Proof. The statements on $\langle D, D' \rangle$ follow from general properties of the Deligne pairing, cf. [6] [19]. If D is vertical we obtain via the identification $\langle D, \omega \rangle = \langle D, D \otimes \omega \rangle - \langle D, D \rangle$ a canonical rational section $\beta(D)$ of $\langle D, \omega \rangle$ by letting it correspond to $Ad \otimes \langle \alpha(D, D) \rangle^{\otimes -1}$ in $\langle D, D \otimes \omega \rangle - \langle D, D \rangle$. The adjunction formula gives $\text{ord}_s \beta(D) = \deg \omega|_D$. \square

Let $(X \rightarrow S = \text{Spec } R, \sigma)$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 2$ over a discrete valuation ring. Assume X is regular and let $P \in X(S)$ be a section of $X \rightarrow S$. Then we define Φ_P to be the unique divisor Φ on X with $(\Phi, P) = 0$ and with $((2g - 2)P - \omega + \Phi, C) = 0$ for every irreducible component C in the special fiber of $X \rightarrow S$.

Proposition 4.7. *Assume that R is not of characteristic 2 and that all Weierstrass points of the generic fiber are rational. Then Γ can be written as a sum of images of sections of $X \rightarrow S$ and we have the formula*

$$\text{ord}_s \xi = - \sum_{P \in \Gamma(S)} \Phi_P^2 + (4g - 2)(g + 1) \text{ord}_s \Delta + 8 \deg \omega|_E$$

for the order of vanishing of ξ at the closed point s of S .

Proof. By the proof of Proposition 4.4 we have for each $P \in \Gamma(S)$ a canonical trivializing section s_P of the line bundle

$$\omega - (2g - 2)O_X(P) + \rho^*\langle P, P \rangle^{\otimes 2g-1}$$

restricted to the generic fiber of $X \rightarrow S$. This section extends uniquely to a rational section s_P of this same line bundle over X . Let's denote by Φ'_P its divisor. Then Φ'_P is supported on the special fiber of $X \rightarrow S$ and we have $P^*\Phi'_P$ trivial on S since the pullback of the above line bundle along P is $\langle P, P \otimes \omega \rangle$ which is canonically trivial by adjunction. One verifies easily that Φ'_P is equal to the Φ_P defined before the proposition. Hence we have a canonical isomorphism

$$\omega - (2g - 2)O_X(P) + \rho^*\langle P, P \rangle^{\otimes 2g-1} = O_X(\Phi_P).$$

Pairing with ω yields a canonical isomorphism

$$-4g(g - 1)\langle P, \omega \rangle + \langle \omega, \omega \rangle = \langle \Phi_P, \omega \rangle.$$

Pairing with Φ_P yields

$$\langle \Phi_P, \omega \rangle = \langle \Phi_P, \Phi_P \rangle.$$

Combining we find

$$-4g(g - 1)\langle P, \omega \rangle + \langle \omega, \omega \rangle = \langle \Phi_P, \Phi_P \rangle$$

and summing over $P \in \Gamma(S)$ gives

$$2\langle 4W - (g + 1)\omega + 4E, \omega \rangle = - \sum_{P \in \Gamma(S)} \langle \Phi_P, \Phi_P \rangle + 8\langle E, \omega \rangle.$$

The rational section ξ of $2\langle 4W - (g + 1)\omega + 4E, \omega \rangle \otimes (4g - 2)(g + 1)\delta$ corresponds then to the obvious linear combination of the sections $\alpha(\Phi_P, \Phi_P)$ and $\beta(E)$ provided by Lemma 4.6, and the canonical section of δ . By taking valuations on left and right one finds the required formula, using the identities from Lemma 4.6. \square

Example 4.2. Consider the semistable curve $X \rightarrow S$ of Example 4.1. Solving the equation $(2P - \omega + \Phi, C) = 0$ for $C = A, B, D$ and demanding that $(\Phi, P) = 0$ one finds Φ_P and hence Φ_P^2 for all $P \in \Gamma(S)$. The results are in the following table:

P	Φ_P	Φ_P^2
P_1, P_2, P_3	$-B - D$	-1
P_4	$-A$	-1
P_5, P_6	$-2A - B$	-3

We read off that $-\sum_{P \in \Gamma(S)} \Phi_P^2 = 10$.

Remark 4.3. It can be proved that in general $-\Phi_P$ is effective for $P \in \Gamma(S)$. The canonical isomorphism $\langle \Phi_P, \omega \rangle = \langle \Phi_P, \Phi_P \rangle$ from the proof of Proposition 4.7 is then just the one obtained from adjunction on $-\Phi_P$.

Also the left hand side of the isomorphism in Proposition 2.5 has a canonical section.

Proposition 4.8. *Let $(X \rightarrow S, \sigma)$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 2$. Then the line bundle $(8g + 4)\lambda$ contains a canonical global section Λ . Its formation commutes with any base change preserving generic smoothness, and it has support on $S - S^o$. If $S = \text{Spec } k$ with k not of characteristic 2 and $y^2 + ay = b$ is an equation for $X \rightarrow S$ with $a, b \in k[x]$ and $f = a^2 + 4b$ separable of degree $2g + 2$ we have*

$$\Lambda = \left(2^{-(4g+4)} \cdot D \right)^g \cdot \left(\frac{dx}{2y+a} \wedge \dots \wedge \frac{x^{g-1}dx}{2y+a} \right)^{\otimes 8g+4},$$

where D is the discriminant of f .

Proof. The fact that Λ as given defines at least a rational section of $(8g+4)\lambda$ independent of the choice of equation follows from [14], Proposition 2.2 and [18], Proposition 2.7. The fact that Λ actually extends to a global section supported on $S - S^o$ follows from [14], Theorem 3.1 in the case that 2 is invertible on S , and [18], Théorème 1.1 in the general case. \square

Proposition 4.9. *Let $(X \rightarrow S, \sigma)$ be a generically smooth stable hyperelliptic curve of genus $g \geq 2$ with S regular. Assume that $\Gamma \rightarrow S$ is generically étale and that $X \rightarrow S$ is smooth over those s in S with $\kappa(s)$ of characteristic 2. Then, under the square of the canonical isomorphism of Proposition 2.5, the section $\Lambda^{\otimes 6g-2}$ is sent to ξ , up to a sign.*

Proof. We may obtain any $X \rightarrow S$ satisfying the assumptions of the proposition by base change from the universal stable hyperelliptic curve over the stack $\overline{\mathcal{H}}_g^o$ introduced in (4.1). By Propositions 4.4 and 4.8 we may obtain the rational sections ξ resp. Λ for $X \rightarrow S$ by pulling back the rational sections ξ resp. Λ for the universal curve over $\overline{\mathcal{H}}_g^o$. Note that the section ξ exists for the universal case since, as we remarked above, $\overline{\mathcal{H}}_g^o$ is smooth over $\text{Spec } \mathbb{Z}$. It suffices therefore by commutativity with base change of the isomorphism from Proposition 2.5 to prove the proposition for the universal case. Consider the rational sections $\Lambda^{\otimes 6g-2}$ and ξ on $S = \overline{\mathcal{H}}_g^o$, and take their respective divisors. By Propositions 4.4 and 4.8 both divisors are supported on the locus of isomorphism classes of singular curves in $S = \overline{\mathcal{H}}_g^o$. Hence their difference, which is a principal divisor, is the divisor of

a rational function f on $\overline{\mathcal{H}}_g^o$ whose restriction to \mathcal{H}_g gives an element of $H^0(\mathcal{H}_g, \mathbb{G}_m)$. According to [13], Proposition 7.3 (which is based on Lemma 2.1 of [11]) we have $f = \pm 1$. The proposition follows. \square

Corollary 4.10. *Let $(X \rightarrow S = \text{Spec } R, \sigma)$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 2$ over a discrete valuation ring which is not of characteristic 2. Assume that X is regular and that $X \rightarrow S$ has smooth reduction if the residue characteristic is 2. Assume furthermore that all Weierstrass points on the generic fiber are rational. Then Γ can be written as a sum of images of sections of $X \rightarrow S$ and the formula*

$$(3g - 1) \text{ord}_s \Lambda = -\frac{1}{2} \sum_{P \in \Gamma(S)} \Phi_P^2 + (2g - 1)(g + 1) \text{ord}_s \Delta + 4 \deg \omega|_E$$

holds.

Proof. We check whether the sections $\Lambda^{\otimes 6g-2}$ and ξ correspond, at least up to a sign, under the square of the isomorphism of Proposition 2.5. This would give the corollary by invoking Proposition 4.7. Let $X' \rightarrow S$ be the stable model of $X \rightarrow S$ and denote by ω' its relative dualizing sheaf, by W' the closure of its generic Weierstrass divisor, and by E' its residual divisor. Write

$$\mathcal{L} = 4W' - (g + 1)\omega' + 4E', \mathcal{M} = \omega', \mathcal{L}' = 4W' - (g + 1)\omega' + 4E', \mathcal{M}' = \omega'.$$

Let λ' and δ' be the line bundles on S occurring in the Noether formula for $X' \rightarrow S$. We have sections Λ' of $(8g + 4)\lambda'$ and ξ' of the square of $\langle \mathcal{L}', \mathcal{M}' \rangle \otimes (2g - 1)(g + 1)\delta'$. By Proposition 4.9 the sections $(6g - 2)\Lambda'$ and ξ' correspond, up to a sign, under the isomorphism of Proposition 2.5. On the other hand, we have a canonical isomorphism $\lambda = \lambda'$ inducing an identification of Λ with Λ' . We would be done therefore if we could show that under the resulting identification of $\langle \mathcal{L}, \mathcal{M} \rangle \otimes (2g - 1)(g + 1)\delta$ with $\langle \mathcal{L}', \mathcal{M}' \rangle \otimes (2g - 1)(g + 1)\delta'$ the section ξ is sent to ξ' . Note first of all that we have an identification $\delta' = \delta$ sending the canonical section of δ' to the canonical section of δ . Hence we are reduced to showing that the bundles $\langle \mathcal{L}, \mathcal{M} \rangle$ and $\langle \mathcal{L}', \mathcal{M}' \rangle$ are canonically identified. But by property (viii) of the Deligne pairing one has a canonical S -morphism $\pi: X \rightarrow X'$ and a canonical isomorphism $\langle \pi^* \mathcal{L}', \pi^* \mathcal{M}' \rangle = \langle \mathcal{L}', \mathcal{M}' \rangle$. Furthermore one has $\pi^* \omega' = \omega$ and $\pi^* \mathcal{L}' = \mathcal{L} + D$ for some effective divisor D supported on the exceptional fibers of π . Since D is supported on the exceptional fibers, the line bundle $\langle D, \omega \rangle$ is canonically trivialized by $\beta(D)$. Thus we have identifications

$$\langle \mathcal{L}, \mathcal{M} \rangle = \langle \mathcal{L}, \mathcal{M} \rangle \otimes \langle D, \omega \rangle = \langle \mathcal{L}, \mathcal{M} \rangle \otimes \langle D, \mathcal{M} \rangle = \langle \mathcal{L} + D, \mathcal{M} \rangle = \langle \pi^* \mathcal{L}', \pi^* \mathcal{M}' \rangle = \langle \mathcal{L}', \mathcal{M}' \rangle$$

which is what we wanted to prove. \square

Example 4.4. We verify Corollary 4.10 for the curve $X \rightarrow S$ of Example 4.1. As $g = 2$ in this case, the formula reduces to

$$5 \operatorname{ord}_s \Lambda = -\frac{1}{2} \sum_{P \in \Gamma(S)} \Phi_P^2 + 9 \operatorname{ord}_s \Delta + 4 \deg \omega|_E.$$

We have seen in Example 4.1 that $E = A + B + 2D$ which gives $\deg \omega|_E = 2$. From Example 4.2 we recall that $-\sum_{P \in \Gamma(S)} \Phi_P^2 = 10$. Note that $\operatorname{ord}_s \Delta = 3$. Finally, one has $\operatorname{ord}_s \Lambda = 8$ by the table in §5 of [25]. The formula checks.

Remark 4.5. In some sense our formula can be seen as a complement to the formula in Theorem 3.1 of [14] which also gives an expression for $\operatorname{ord}_s \Lambda$ in terms of combinatorial data determined by the special fiber. We do not know if the condition of smooth reduction in residue characteristic 2 can be removed in Corollary 4.10.

(4.5) We end this section by proving that under a weak assumption the formation of both W and E commutes with base change.

Proposition 4.11. *Let $(X \rightarrow S, \sigma)$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 2$ with S regular. Assume that Γ is a Cartier divisor on X and that σ does not act as the identity on any irreducible component of the fibers. Then the formation of Γ commutes with any base change preserving generic smoothness and regularity of the base.*

Proof. The proposition follows once we prove that $\Gamma \rightarrow S$ is finite and flat of degree $2g + 2$. Indeed then let $T \rightarrow S$ be a morphism with T regular. Denote by Γ^* the pullback of Γ over T , and let Γ_* be the scheme Γ for $X_T \rightarrow T$. By base change from $\Gamma \rightarrow S$ we have Γ^* finite and flat of degree $2g + 2$ over T . On the other hand Γ_* is clearly a closed subscheme of Γ^* . But both are locally free over T of the same rank, hence the closed immersion $\Gamma_* \rightarrow \Gamma^*$ is an isomorphism. In order to prove that $\Gamma \rightarrow S$ is finite, note that by assumption $\Gamma \rightarrow S$ is at least quasi-finite. Since Γ is closed in X it is also proper over S . It follows that Γ is finite over S . To prove finally that $\Gamma \rightarrow S$ is flat, use the assumption that Γ is a Cartier divisor on X . Then Γ , being locally generated by one element in a Cohen-Macaulay scheme, is itself Cohen-Macaulay. Together with the finiteness of $\Gamma \rightarrow S$ and the regularity of S this leads by [17] Theorem 23.1 to the conclusion that $\Gamma \rightarrow S$ is flat. \square

Corollary 4.12. *Let $(X \rightarrow S, \sigma)$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 2$ with S regular. Assume that the Weierstrass divisor W is contained in X^{sm} and that σ does not act as the identity on any irreducible component of the fibers. Then the formation of both W and E commutes with any base change preserving generic smoothness and regularity of the base.*

Proof. It follows from Proposition 4.3 that $W^o = g(g-1)/2 \cdot \Gamma^o$. By taking closures on left and right we find $W = g(g-1)/2 \cdot \Gamma$. As W is contained in X^{sm} we find that Γ is a Cartier divisor on X . Hence by the proposition the formation of Γ commutes with any base change as indicated. As a consequence so does the formation of W by the formula $W = g(g-1)/2 \cdot \Gamma$. Since the formation of Wr commutes with any base change as indicated, we find in the end that the same holds for the divisor E . \square

5. CURVES OVER NUMBER FIELDS

In this section we study arithmetic aspects of Proposition 2.5. It turns out that the norm of the isomorphism from that proposition is closely related to an invariant T introduced in [12]. As a corollary we find an explicit closed formula for the Faltings stable height of a curve over a number field.

(5.1) Let $\rho: X \rightarrow S$ be a semistable curve of genus $g \geq 1$, generically smooth, with S regular. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be admissible hermitian line bundles on X in the sense of [2]. For example, one of them could be the relative dualizing sheaf ω equipped with its canonical Arakelov metric, notation $\overline{\omega}$. As is shown in [6], if s is a point of the complex manifold $S(\mathbb{C})$ then the fiber of $\langle \mathcal{L}, \mathcal{M} \rangle$ at s has a canonical norm derived from the admissible structures on $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$. One of the properties of this norm is that if S is the spectrum of the ring of integers O_K of a number field K and X is regular, the arithmetic degree of the metrized line bundle $\langle \mathcal{L}, \mathcal{M} \rangle$ for any two admissible hermitian line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on X is equal to the Arakelov intersection product $(\overline{\mathcal{L}}, \overline{\mathcal{M}})$ of $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$. A further property is that the isomorphisms listed in (i)-(viii) of Section 2 are isometries. Here, the determinant of cohomology $\det R\rho_* \mathcal{L}$ of an admissible hermitian line bundle $\overline{\mathcal{L}}$ is metrized in the sense of Faltings [9], Theorem 1. In particular $\lambda = \det R\rho_* \omega$ carries the Hodge metric coming from the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_{\mathfrak{X}} \omega \wedge \overline{\eta}$ on the vector space of holomorphic 1-forms on a compact and connected Riemann surface \mathfrak{X} . We put a norm

on $\delta = O_S(\Delta)$ by giving the canonical section unit norm. Our goal is to derive a formula for the arithmetic degree $\deg \lambda$ of λ in the case $S = \text{Spec } O_K$, using Proposition 2.5.

(5.2) We begin by recalling the invariant T from [12]. The basic ingredients are the functions $\|\vartheta\|$ and $\|J\|$ introduced in [9] and [4], [10], respectively. Let g be a positive integer, let \mathbb{H}_g be the generalized Siegel upper half plane of complex symmetric $g \times g$ -matrices with positive definite imaginary part, and let \mathfrak{X} be a compact and connected Riemann surface of genus g . Let $\tau \in \mathbb{H}_g$ be a period matrix associated to a symplectic basis of $H_1(\mathfrak{X}, \mathbb{Z})$ and consider the analytic jacobian $\text{Jac}(\mathfrak{X}) = \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g)$ associated to τ . On \mathbb{C}^g one has a theta function $\vartheta(z; \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i {}^t n \tau n + 2\pi i {}^t n z)$, giving rise to an effective Cartier divisor Θ_0 and a line bundle $O(\Theta_0)$ on $\text{Jac}(\mathfrak{X})$.

Consider next the set $\text{Pic}_{g-1}(\mathfrak{X})$ of divisor classes of degree $g-1$ on \mathfrak{X} . It comes with a special subset Θ given by classes of effective divisors. A fundamental classical result says that there is a bijection $\text{Pic}_{g-1}(\mathfrak{X}) \xrightarrow{\sim} \text{Jac}(\mathfrak{X})$ mapping Θ onto Θ_0 . As a result we can equip $\text{Pic}_{g-1}(\mathfrak{X})$ with the structure of a compact complex manifold, together with a Cartier divisor Θ and a line bundle $O(\Theta)$ on it. We fix this structure for the rest of the discussion.

The function ϑ is not well-defined on $\text{Pic}_{g-1}(\mathfrak{X})$ or $\text{Jac}(\mathfrak{X})$, but one can take

$$\|\vartheta\|(z; \tau) = (\det \text{Im } \tau)^{1/4} \exp(-\pi {}^t y (\text{Im } \tau)^{-1} y) |\vartheta(z; \tau)|$$

where $y = \text{Im } z$. This $\|\vartheta\|$ descends to a function on $\text{Jac}(\mathfrak{X})$ and by our identification $\text{Pic}_{g-1}(\mathfrak{X}) \xrightarrow{\sim} \text{Jac}(\mathfrak{X})$ we obtain $\|\vartheta\|$ as a function on $\text{Pic}_{g-1}(\mathfrak{X})$. This function is independent of the choice of τ .

For w_1, \dots, w_g column vectors in \mathbb{C}^g we put

$$J(w_1, \dots, w_g) := \det \left(\frac{\partial \vartheta}{\partial z_k}(w_l) \right)_{1 \leq k, l \leq g}$$

and

$$\|J\|(w_1, \dots, w_g) := (\det \text{Im } \tau)^{\frac{g+2}{4}} \exp(-\pi \sum_{k=1}^g {}^t y_k (\text{Im } \tau)^{-1} y_k) \cdot |J(w_1, \dots, w_g)|.$$

Here we write $y_k = \text{Im } w_k$ for $k = 1, \dots, g$. The function $\|J\|(w_1, \dots, w_g)$ depends only on the classes in $\text{Jac}(\mathfrak{X})$ of the vectors w_k . Let P_1, \dots, P_g be a set of g points on X . We take vectors $w_1, \dots, w_g \in \mathbb{C}^g$ such that for all $k = 1, \dots, g$ the class $[w_k] \in \text{Jac}(\mathfrak{X})$ corresponds to $[\sum_{l=1, l \neq k}^g P_l] \in \text{Pic}_{g-1}(\mathfrak{X})$ under the bijection $\text{Pic}_{g-1}(\mathfrak{X}) \leftrightarrow \text{Jac}(\mathfrak{X})$. We

then put $\|J\|(P_1, \dots, P_g) := \|J\|(w_1, \dots, w_g)$. The function $\|J\|$ on $\text{Sym}^g \mathfrak{X}$ does not depend on the choice of the period matrix τ .

Now let P_1, \dots, P_g, Q be generic points on \mathfrak{X} . It is proved in [12], Proposition 4.6 that the expression

$$T(\mathfrak{X}) = \left(\frac{\|\vartheta\|(P_1 + \dots + P_g - Q)}{\prod_{k=1}^g \|\vartheta\|(gP_k - Q)^{1/g}} \right)^{2g-2} \cdot \left(\frac{\prod_{k \neq l} \|\vartheta\|(gP_k - P_l)^{1/g}}{\|J\|(P_1, \dots, P_g)^2} \right) \cdot \prod_{R \in W} \prod_{k=1}^g \|\vartheta\|(gP_k - R)^{(g-1)/g^4}$$

is non-zero and independent of the choice of P_1, \dots, P_g, Q on \mathfrak{X} . Here W is the Weierstrass divisor of \mathfrak{X} . Hence T defines an invariant of \mathfrak{X} , called the T -invariant. Although its definition as given above is rather complicated, the invariant $T(\mathfrak{X})$ is an ‘‘elementary’’ invariant in the sense that it does not depend on normalization constants that derive from the normalization of certain potential functions on \mathfrak{X} .

Theorem 5.1. *Let X be a smooth proper curve of genus $g \geq 1$ over $S = \text{Spec } \mathbb{C}$. Then the norm of the isomorphism in Proposition 2.5 is equal to*

$$(2\pi)^{-4g(2g-1)(g+1)} \cdot T(\mathfrak{X})^{8g^2}.$$

Here \mathfrak{X} is the compact Riemann surface $X(\mathbb{C})$.

Proof. We follow the proof of Proposition 2.5, taking care of the norm at each step. Let $\delta(\mathfrak{X})$ and $S(\mathfrak{X})$ be the delta-invariant of Faltings [9] and the S -invariant defined in [12], respectively. According to [12], proof of Theorem 4.4 the isomorphism

$$4W + 4E \xrightarrow{\sim} 2g(g+1)\omega - 4\rho^*\lambda$$

stemming from Proposition 2.1 has norm

$$\left(S(\mathfrak{X})^{-1} e^{\delta(\mathfrak{X})/8} \right)^4,$$

hence so does the isomorphism

$$4W - (g+1)\omega + 4E \xrightarrow{\sim} (2g-1)(g+1)\omega - 4\rho^*\lambda.$$

It follows that the isomorphism

$$\langle 4W - (g+1)\omega + 4E, \omega \rangle \xrightarrow{\sim} (2g-1)(g+1)\langle \omega, \omega \rangle - (8g-8)\lambda$$

obtained by pairing with ω has norm

$$\left(S(\mathfrak{X})^{-1} e^{\delta(\mathfrak{X})/8} \right)^{8g-8}.$$

According to [9] [19] the (up to sign) canonical isomorphism $\langle \omega, \omega \rangle \xrightarrow{\sim} 12\lambda - \delta$ has norm

$$(2\pi)^{4g} e^{-\delta(\mathfrak{X})},$$

hence the isomorphism of Proposition 2.5 has norm

$$\left(S(\mathfrak{X})^{-1} e^{\delta(\mathfrak{X})/8} \right)^{-(8g-8)} \cdot \left((2\pi)^{4g} e^{-\delta(\mathfrak{X})} \right)^{-(2g-1)(g+1)}.$$

This simplifies to

$$(2\pi)^{-4g(2g-1)(g+1)} \cdot (e^{\delta(\mathfrak{X})/4})^{8g^2} \cdot S(\mathfrak{X})^{8g-8}.$$

As we have

$$e^{\delta(\mathfrak{X})/4} = S(\mathfrak{X})^{-(g-1)/g^2} \cdot T(\mathfrak{X})$$

by Theorem 4.4 of [12] we end up with norm

$$(2\pi)^{-4g(2g-1)(g+1)} \cdot T(\mathfrak{X})^{8g^2}$$

as required. □

Example 5.1. Assume that X is (hyper)elliptic of genus g . Put $n = \binom{2g}{g+1}$ and $r = \binom{2g+1}{g+1}$. We have seen in the proof of Proposition 4.4 that the square of the line bundle

$$\langle 4W - (g+1)\omega + 4E, \omega \rangle \otimes (2g-1)(g+1)\delta$$

on $S = \text{Spec } \mathbb{C}$ is canonically isomorphic to the trivial line bundle O_S . Going once more through the various steps in that proof, now taking care of the norm at each point, one finds that the canonical identification of the above line bundle with O_S is in fact an isometry. In other words ξ has unit norm. Combining with the fact (cf. [13], p. 11) that Λ has norm satisfying

$$\|\Lambda\|^n = (2\pi)^{4g^2 r} \|\Delta\|(\mathfrak{X})^g$$

where $\|\Delta\|(\mathfrak{X})$ is the Petersson norm of the modular discriminant of \mathfrak{X} , we find that the formula

$$T(\mathfrak{X}) = (2\pi)^{-2g} \cdot \|\Delta\|(\mathfrak{X})^{-\frac{3g-1}{8ng}}$$

holds.

We apply Theorem 5.1 to the case of a semistable regular arithmetic surface $X \rightarrow S$ of genus $g \geq 1$ over the ring of integers of a number field K . Since S is a Dedekind scheme, if the Weierstrass points of the generic fiber of $X \rightarrow S$ are rational W can be written as a sum of sections, multiplicities allowed. If $P \in X(S)$ is a section of $X \rightarrow S$ we denote by $\hat{h}(P)$ the Néron-Tate height of the divisor class of $(2g-2)P - \omega$ in the jacobian of X_K . Further we denote by Φ_P any (classical) \mathbb{Q} -divisor Φ on X such that

$((2g - 2)P - \omega + \Phi, C) = 0$ for all irreducible components C of the fibers of $X \rightarrow S$. We remark that Φ_P is unique up to adding and subtracting rational multiples of fibers. Also note the similarity with the Φ_P defined before Proposition 4.7.

There is a formula for $\hat{h}(P)$ in Arakelov intersection theory.

Lemma 5.2. *(Cf. [24], Section 1.1) Let $X \rightarrow S = \text{Spec } O_K$ be a semistable regular arithmetic surface of genus $g \geq 1$ over the ring of integers of a number field K . Let $P \in X(S)$ be a section of $X \rightarrow S$. Then in Arakelov intersection theory, the relation*

$$-2[K : \mathbb{Q}]\hat{h}(P) = -4g(g - 1)(P, \bar{\omega}) + (\bar{\omega}, \bar{\omega}) - (\Phi_P, \Phi_P)$$

holds.

Proof. By the Hodge index theorem of Faltings-Hriljac (cf. [9], Section 5) one has

$$-2[K : \mathbb{Q}]\hat{h}(P) = ((2g - 2)P - \bar{\omega} + \Phi_P, (2g - 2)P - \bar{\omega} + \Phi_P),$$

or equivalently since $((2g - 2)P - \bar{\omega} + \Phi_P, \Phi_P) = 0$,

$$-2[K : \mathbb{Q}]\hat{h}(P) = ((2g - 2)P - \bar{\omega}, (2g - 2)P - \bar{\omega}) - (\Phi_P, \Phi_P).$$

The fact that adjunction is an isometry gives an equality of intersection numbers $(P, \bar{\omega}) = -(P, P)$. Using this, expanding the above formula gives the required result. \square

Corollary 5.3. *Let $X \rightarrow S = \text{Spec } O_K$ be a semistable regular arithmetic surface of genus $g \geq 2$ over the ring of integers of a number field K . Assume that the Weierstrass points of the generic fiber are rational. Then W can be written as a sum of images of sections and the formula*

$$\begin{aligned} (3g - 1)(8g + 4) \deg \lambda &= \frac{2[K : \mathbb{Q}]}{g(g - 1)} \sum_{P \in W(S)} \hat{h}(P) - \frac{1}{g(g - 1)} \sum_{P \in W(S)} (\Phi_P, \Phi_P) \\ &+ (2g - 1)(g + 1) \sum_{s \in S} \text{ord}_s \Delta \log \#\kappa(s) + 4(E, \bar{\omega}) \\ &- 4g(2g - 1)(g + 1)[K : \mathbb{Q}] \log(2\pi) + 8g^2 \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log T(X_\sigma) \end{aligned}$$

holds. Here for each complex embedding σ of K we denote by X_σ the Riemann surface corresponding to the complex curve $X \times_\sigma \mathbb{C}$, and the sum on the second line is over the closed points s of S .

Proof. Taking Arakelov degrees on left and right in Proposition 2.5 and applying Theorem 5.1 one finds

$$\begin{aligned} (3g-1)(8g+4) \deg \lambda &= 4(W, \bar{\omega}) - (g+1)(\bar{\omega}, \bar{\omega}) \\ &+ (2g-1)(g+1) \sum_{s \in S} \text{ord}_s \Delta \log \#\kappa(s) + 4(E, \bar{\omega}) \\ &- 4g(2g-1)(g+1)[K:\mathbb{Q}] \log(2\pi) + 8g^2 \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log T(X_\sigma). \end{aligned}$$

According to Lemma 5.2 we have, for each $P \in W(S)$,

$$4g(g-1)(P, \bar{\omega}) - (\bar{\omega}, \bar{\omega}) = 2[K:\mathbb{Q}] \hat{h}(P) - (\Phi_P, \Phi_P).$$

Taking the sum over P running through $W(S)$ and dividing by $g(g-1)$ we find the equality

$$4(W, \bar{\omega}) - (g+1)(\bar{\omega}, \bar{\omega}) = \frac{2[K:\mathbb{Q}]}{g(g-1)} \sum_{P \in W(S)} \hat{h}(P) - \frac{1}{g(g-1)} \sum_{P \in W(S)} (\Phi_P, \Phi_P).$$

The formula follows. □

Apart from the term involving $\hat{h}(P)$ the formula in the corollary looks as if we computed $\deg \lambda$ by explicitly taking a canonical non-zero rational section ζ of a power of λ on S and then calculating the arithmetic degree of that section. We know however that since λ is ample on the moduli space of smooth complex curves of genus g every section ζ of a power of λ has a zero locus which prevents us from having $\log \|\zeta\|$ well-defined everywhere. Thus, a formula for $\deg \lambda$ of that type using a fixed ζ is not possible. In contrast, remark that over the hyperelliptic locus, where a global nowhere vanishing section Λ of a power of λ exists, the term involving $\hat{h}(P)$ is zero and the resulting formula is just the one obtained by taking the arithmetic degree of the pullback under the moduli map of this Λ .

Another feature of our formula is that most terms in it can be easily bounded from below. We have $\hat{h}(P) \geq 0$ and $-(\Phi_P, \Phi_P) \geq 0$ for all $P \in W(S)$, we have $\text{ord}_s \Delta \geq 0$ for all closed points s of S , and we have $(E, \bar{\omega}) \geq 0$ since E is effective and vertical and the fibers of $X \rightarrow S$ do not contain exceptional curves. More in particular we have

Corollary 5.4. *Let \mathcal{X} be a smooth proper curve of genus $g \geq 1$ over a number field K . Denote by $h_F(\mathcal{X})$ its stable Faltings height. Then the inequality*

$$(3g-1)(8g+4)h_F(\mathcal{X}) \geq -4g(2g-1)(g+1) \log(2\pi) + 8g^2 \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log T(\mathcal{X}_\sigma)$$

holds. Here for each complex embedding σ of K we denote by \mathcal{X}_σ the Riemann surface corresponding to the complex curve $\mathcal{X} \times_\sigma \mathbb{C}$. Equality is attained if \mathcal{X} is a hyperelliptic curve with potentially everywhere good reduction.

Proof. The case $g = 1$ follows directly from [9], Theorem 7 and Example 5.1. Assume therefore that $g \geq 2$. Let L be a finite extension of K over which all Weierstrass points become rational and over which \mathcal{X} acquires semistable reduction. Let $X \rightarrow \text{Spec } O_L$ be the minimal regular model of \mathcal{X} over L . By definition of the Faltings stable height one has $h_F(\mathcal{X}) = (1/[L : \mathbb{Q}]) \cdot \deg \lambda$ and with the lower bounds just mentioned we get the required estimate immediately from Corollary 5.3. Equality holds in $-(\Phi_P, \Phi_P) \geq 0$ and $\text{ord}_s \Delta \geq 0$ if \mathcal{X} has potentially everywhere good reduction. If in addition \mathcal{X} is hyperelliptic then the divisor class of $(2g - 2)P - \omega$ is trivial for P a Weierstrass point, yielding $\hat{h}(P) = 0$ in the formula. Lastly by Proposition 4.1 the divisor E is empty in this case. Hence we obtain the second statement. \square

It is perhaps interesting to compare the above lower bound for the Faltings stable height of curves with known lower bounds for the Faltings stable heights of abelian varieties. For example J.-B. Bost (cited in [3], p. 1452) has proved that if \mathcal{A} is an abelian variety of dimension $g \geq 1$ over a number field K , then the inequality

$$h_F(\mathcal{A}) \geq -g \log(2\pi) - \frac{2}{[K : \mathbb{Q}]} \sum_{\sigma: K \hookrightarrow \mathbb{C}} \int_{\mathcal{A}_\sigma} \log \|\vartheta\|_\sigma(x) d\mu_\sigma(x)$$

holds, where for each complex embedding σ of K we denote by \mathcal{A}_σ the complex points of the abelian variety $\mathcal{A} \times_\sigma \mathbb{C}$ and by μ_σ the Haar measure of \mathcal{A}_σ of volume 1.

Another aspect of our lower bound is that if the jacobian $\text{Jac}(\mathcal{X})$ of a curve \mathcal{X} of genus $g \geq 2$ over a number field is isomorphic (as an abelian variety) to a power E^g of an elliptic curve E , our formula forces a lower bound on $h_F(E)$. Thus, as soon as good lower bounds are known for the function $T(\mathfrak{X})$ on the moduli space of Riemann surfaces of genus g , one might have a means of showing that certain E^g cannot be isomorphic to a jacobian.

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