On two notions of complexity of algebraic numbers

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Abstract. *We derive new, improved lower bounds for the block complexity of an irrational algebraic number and for the number of digit changes in the b-ary expansion of an irrational algebraic number. To this end, we apply a version of the Quantitative Subspace Theorem by Evertse and Schlickewei [14], Theorem 2.1.*

1. Introduction

Throughout the present paper, b always denotes an integer ≥ 2 and ξ is a real number with $0 < \xi < 1$. There exists a unique infinite sequence $\mathbf{a} = (a_j)_{j \geq 1}$ of integers from $\{0, 1, \ldots, b-1\}$, called the b-ary expansion of ξ, such that

$$\xi = \sum_{j \geq 1} \frac{a_j}{b^j},$$

and $\mathbf{a}$ does not terminate in an infinite string of the digit $b-1$. Clearly, the sequence $\mathbf{a}$ is ultimately periodic if, and only if, ξ is rational. With a slight abuse of notation, we also denote by $\mathbf{a}$ the infinite word $a_1 a_2 \ldots$ To measure the complexity of ξ, we measure the complexity of $\mathbf{a}$. Among the different ways to do this, two notions of complexity have been recently studied. A first one, namely the block complexity, consists in counting the number $p(n, \xi, b) = p(n, \mathbf{a})$ of distinct blocks of length n occurring in the word $\mathbf{a}$, that is,

$$p(n, \xi, b) = \mathrm{Card}\,\{a_{k+1} a_{k+2} \ldots a_{k+n} : k \geq 0\}.$$

A second one deals with the asymptotic behaviour of the number of digit changes in $\mathbf{a}$. The function nbdc, 'number of digit changes', introduced in [8], is defined by

$$\mathrm{nbdc}(n, \xi, b) = \mathrm{Card}\,\{1 \leq k \leq n : a_k \neq a_{k+1}\}, \quad \text{for } n \geq 1.$$

Suppose from now on that ξ is algebraic and irrational. Non-trivial lower bounds for $p(n, \xi, b)$ and $\mathrm{nbdc}(n, \xi, b)$ were obtained in [1, 8] by means of transcendence criteria that

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ultimately depend on the Schmidt Subspace Theorem [24] or on the Quantitative Roth Theorem [23, 16]. Respectively, it is known that

$$\lim_{n\to+\infty} \frac{p(n,\xi,b)}{n} = +\infty \tag{1.1}$$

and

$$\mathrm{nbdc}(n,\xi,b) \geq 3\,(\log n)^{1+1/(\omega(b)+4)} \cdot (\log\log n)^{-1/4}, \tag{1.2}$$

for every sufficiently large n, where $\omega(\ell)$ counts the number of distinct prime factors of the integer ℓ.

Both (1.1) and (1.2) are very far from what can be expected if one believes that, regarding these notions of complexity, algebraic irrational numbers behave like almost all real numbers (in the sense of the Lebesgue measure). Thus, it is widely believed that the functions $n \mapsto p(n,\xi,b)$ and $n \mapsto \mathrm{nbdc}(n,\xi,b)$ should grow, respectively, exponentially in n and linearly in n.

The main purpose of the present paper is to improve (1.2) for all n and (1.1) for infinitely many n. Our results imply that

$$p(n,\xi,b) \geq n(\log n)^{0.09} \quad \text{for infinitely many } n \tag{1.3}$$

and

$$\mathrm{nbdc}(n,\xi,b) \geq c(d)\,(\log n)^{3/2} \cdot (\log\log n)^{-1/2}$$

for every sufficiently large n, where $c(d)$ is a constant depending only on the degree d of ξ. In particular, we have been able to remove the dependence on b in (1.2).

The new ingredient in the proof of (1.3) is the use of a quantitative version of the Subspace Theorem, while (1.1) was established by means of a standard qualitative version of the Subspace Theorem. Originally, quantitative versions of the Subspace Theorem were stated for a single inequality with a product of linear forms, and then the resulting upper bound for the number of subspaces depended on the number of places involved. Instead, we use a version for systems of inequalities each involving one linear form giving an upper bound for the number of subspaces independent of the number of places. In fact, for many applications, the version for systems of inequalities suffices, and it leads to much better results when many non-Archimedean places are involved.

Our paper is organized as follows. We begin by stating and discussing our result upon (1.1) in Section 2 and that upon (1.2) in Section 3. Then, we state in Section 4 our main auxiliary tool, namely the Quantitative Parametric Subspace Theorem from [14]. We have included an improvement of the two-dimensional case of the latter which is needed for our improvement upon (1.2); the proof of this improvement is included in an appendix at the end of our paper. This Quantitative Parametric Subspace Theorem is a statement about classes of twisted heights parametrized by a parameter Q, and one can deduce from this suitable versions of the Quantitative Subspace Theorem, dealing with (systems of) Diophantine inequalities. In Section 5 we deduce a quantitative result for systems of inequalities (Theorem 5.1) fine-tuned for the applications in our present paper. In the particular case where we have only two unknowns we obtain a sharper quantitative version of a Ridout type theorem (Corollary 5.2). The proof of Theorem 2.1 splits in Sections 6 and 7, and that of Theorem 3.1 is given in Section 8. Finally, further applications of our results are discussed in Section 9.

2. Block complexity of b-ary expansions of algebraic numbers

We keep the notation from the Introduction. Recall that the real number ξ is called *normal in base* b if, for any positive integer n, each one of the b^n words of length n on the alphabet $\{0, 1, \ldots, b-1\}$ occurs in the b-ary expansion of ξ with the same frequency $1/b^n$. The first explicit example of a number normal in base 10, namely the number

$$0.1234567891011121314\ldots,$$

whose sequence of digits is the concatenation of the sequence of all positive integers ranged in increasing order, was given in 1933 by Champernowne [10]. It follows from the Borel–Cantelli lemma that almost all real numbers (in the sense of the Lebesgue measure) are normal in every integer base, but proving that a specific number, like e, π or $\sqrt{2}$ is normal in some base remains a challenging open problem. However, it is believed that every real irrational algebraic number is normal in every integer base. This problem, which was first formulated by Émile Borel [7], is likely to be very difficult.

Assume from now on that ξ is algebraic and irrational. In particular, $\mathbf{a}$ is not ultimately periodic. By a result of Morse and Hedlund [18, 19], every infinite word $\mathbf{w}$ that is not ultimately periodic satisfies $p(n, \mathbf{w}) \geq n+1$ for $n \geq 1$. Consequently, $p(n, \xi, b) \geq n+1$ holds for every positive integer n. This lower bound was subsequently improved upon in 1997 by Ferenczi and Mauduit [15], who applied a non-Archimedean extension of Roth's Theorem established by Ridout [21] to show (see also [4]) that

$$\lim_{n\to+\infty} \big(p(n, \xi, b) - n\big) = +\infty.$$

Then, a new combinatorial transcendence criterion proved with the help of the Schmidt Subspace Theorem by Adamczewski, Bugeaud, and Luca [2] enabled Adamczewski and Bugeaud [1] to establish that

$$\lim_{n\to+\infty} \frac{p(n, \xi, b)}{n} = +\infty. \tag{2.1}$$

By combining ideas from [9] with a suitable version of the Quantitative Subspace Theorem, we are able to prove the following concerning (2.1).

Theorem 2.1. *Let $b \geq 2$ be an integer and ξ an algebraic irrational number with $0 < \xi < 1$. Then, for any real number η such that $\eta < 1/11$, we have*

$$\limsup_{n\to+\infty} \frac{p(n, \xi, b)}{n(\log n)^{\eta}} = +\infty. \tag{2.2}$$

Ideas from [9] combined with Theorem 3.1 from [14] allow us to prove a weaker version of Theorem 2.1, namely to conclude that (2.2) holds for any η smaller than $1/(4\omega(b)+15)$. The key point for removing the dependence on b is the use of Theorem 5.1 below, and more precisely the fact that the exponent on ε^{-1} in (5.9) does not depend of the cardinality of the set of places S.

Remark that Theorem 2.1 does not follow from (2.1). Indeed, there exist infinite words $\mathbf{w}$ having a complexity function p satisfying

$$\lim_{n\to+\infty} \frac{p(n,\mathbf{w})}{n} = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{p(n,\mathbf{w})}{n(\log\log n)} < +\infty. \tag{2.3}$$

In particular, there exist morphic words satisfying (2.3). We refer the reader to [1] for the definition of a morphic word. An open question posed in [1] asked whether the b-ary expansion of an irrational algebraic number can be a morphic word. Theorem 2.1 above allows us to make a small step towards a negative answer. Indeed, by a result of Pansiot [20], the complexity of a morphic word that is not ultimately periodic is either of order n, $n\log\log n$, $n\log n$, or n^2. It immediately follows from Theorem 2.1 that, regardless of the base b, if the b-ary expansion of an irrational algebraic number is generated by a morphism, then the complexity of this morphism is either of order $n\log n$, or of order n^2. However, by using combinatorical properties of morphic words and the transcendence criterion from [2], Albert [3], on page 59 of his thesis, was able to show a stronger result, namely to prove that, regardless of the base b, if the b-ary expansion of an irrational algebraic number is generated by a morphism, then its complexity is of order n^2.

Note that our method yields the existence of a positive δ such that

$$\limsup_{n\to+\infty} \frac{p(n,\xi,b)\cdot(\log\log n)^{\delta}}{n(\log n)^{1/11}} = +\infty. \tag{2.4}$$

In order to avoid painful technical details, we decided not to give a proof of (2.4).

3. Digit changes in b-ary expansions of algebraic numbers

Our next result is a new lower bound for the number of digit changes in b-ary expansions of irrational algebraic numbers.

Theorem 3.1. *Let $b \geq 2$ be an integer. Let ξ be an irrational, real algebraic number ξ of degree d. There exist an effectively computable absolute constant c_1 and an effectively computable constant $c_2(\xi,b)$, depending only on ξ and b, such that*

$$\mathrm{nbdc}(n,\xi,b) \geq c_1 \frac{(\log n)^{3/2}}{(\log\log n)^{1/2}\,(\log 6d)^{1/2}}$$

for every integer $n \geq c_2(\xi,b)$.

Theorem 3.1 improves upon Theorem 1 from [8], where the exponent of $(\log n)$ depends on b and tends to 1 as the number of prime factors of b tends to infinity. This improvement is a consequence of the use of the two-dimensional case of Theorem 5.1 (dealing with systems of inequalities) instead of a result of Locher [16] (dealing with one inequality with a product of linear forms).

Theorem 3.1 allows us to improve upon straightforwardly many of the results from [8]. We restrict our attention to Section 7 from [8], that is, to the study of the gap series

$$\xi_{\mathbf{n},b} = \sum_{j\geq 1} b^{-n_j}$$

for a given integer $b \geq 2$ and a non-decreasing sequence of positive integers $\mathbf{n} = (n_j)_{j\geq 1}$. As mentioned in [8], it easily follows from Ridout's Theorem [21] that the assumption

$$\limsup_{j\to+\infty} \frac{n_{j+1}}{n_j} > 1$$

implies the transcendence of $\xi_{\mathbf{n},b}$, see e.g. Satz 7 from Schneider's monograph [25].

In particular, for any positive real number ε, the real number $\xi_{\mathbf{n},b}$ is transcendental when $n_j = 2^{[\varepsilon j]}$, where $[\,\cdot\,]$ denotes the integer part function. A much sharper statement, that improves Corollary 4 from [8], follows at once from Theorem 3.1.

Corollary 3.2. *Let $b \geq 2$ be an integer. For any real number $\eta > 2/3$, the sum of the series*

$$\sum_{j\geq 1} b^{-n_j}, \quad \text{where } n_j = 2^{[j^\eta]} \text{ for } j \geq 1,$$

is transcendental.

To establish Corollary 3.2, it is enough to check that the number of positive integers j such that $2^{[j^\eta]} \leq N$ is less than some absolute constant times $(\log N)^{1/\eta}$, and to apply Theorem 3.1 to conclude. Stronger transcendence results for the gap series $\xi_{\mathbf{n},2}$ follow from [5, 22], including the fact that Corollary 3.2 holds for any positive η when $b = 2$.

Further results are given in Section 9.

4. The Quantitative Parametric Subspace Theorem

We fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$; all algebraic number fields occurring henceforth will be subfields of $\overline{\mathbf{Q}}$.

We introduce the necessary absolute values. The set of places $M_{\mathbf{Q}}$ of $\mathbf{Q}$ may be identified with $\{\infty\} \cup \{\text{primes}\}$. We denote by $|\cdot|_\infty$ the ordinary (Archimedean) absolute value on $\mathbf{Q}$ and for a prime p we denote by $|\cdot|_p$ the p-adic absolute value, normalized such that $|p|_p = p^{-1}$.

Let $\mathbf{K}$ be an algebraic number field. We denote by $M_{\mathbf{K}}$ the set of places (equivalence classes of non-trivial absolute values) of $\mathbf{K}$. The completion of $\mathbf{K}$ at a place v is denoted by $\mathbf{K}_v$. Given a place $v \in M_{\mathbf{K}}$, we denote by p_v the place in $M_{\mathbf{Q}}$ lying below v. We choose the absolute value $|\cdot|_v$ in v in such a way that the restriction of $|\cdot|_v$ to $\mathbf{Q}$ is $|\cdot|_{p_v}$. Further, we define the normalized absolute value $\|\cdot\|_v$ by

$$\|\cdot\|_v := |\cdot|_v^{d(v)} \quad \text{where } d(v) := \frac{[\mathbf{K}_v : \mathbf{Q}_{p_v}]}{[\mathbf{K} : \mathbf{Q}]}. \tag{4.1}$$

These absolute values satisfy the product formula

$$\prod_{v \in M_{\mathbf{K}}} \|x\|_v = 1, \quad \text{for } x \in \mathbf{K}^*.$$

Further, they satisfy the extension formula: Suppose that $\mathbf{E}$ is a finite extension of $\mathbf{K}$ and normalized absolute values $\|\cdot\|_w$ ($w \in M_{\mathbf{E}}$) are defined in precisely the same manner as those for $\mathbf{K}$. Then if $w \in M_{\mathbf{E}}$ and $v \in M_{\mathbf{K}}$ is the place below w, we have

$$\|x\|_w = \|x\|_v^{d(w|v)} \text{ for } x \in \mathbf{K}, \quad \text{where } d(w|v) := \frac{[\mathbf{E}_w : \mathbf{K}_v]}{[\mathbf{E} : \mathbf{K}]}. \tag{4.2}$$

Notice that

$$\sum_{w|v} d(w|v) = 1 \tag{4.3}$$

where by '$w|v$' we indicate that w runs through all places of $\mathbf{E}$ lying above v.

Let again $\mathbf{K}$ be an algebraic number field, and n an integer ≥ 2. Let $\mathcal{L} = (L_{iv} : v \in M_{\mathbf{K}},\ i = 1, \dots, n)$ be a tuple of linear forms with the following properties:

$$L_{iv} \in \mathbf{K}[X_1, \dots, X_n] \text{ for } v \in M_{\mathbf{K}},\ i = 1, \dots, n, \tag{4.4}$$

$$L_{1v} = X_1, \dots, L_{nv} = X_n \text{ for all but finitely many } v \in M_{\mathbf{K}}, \tag{4.5}$$

$$\det(L_{1v}, \dots, L_{nv}) = 1 \text{ for } v \in M_{\mathbf{K}}, \tag{4.6}$$

$$\mathrm{Card}\Big(\bigcup_{v \in M_{\mathbf{K}}} \{L_{1v}, \dots, L_{nv}\}\Big) \leq r. \tag{4.7}$$

Further, we define

$$\mathcal{H} = \mathcal{H}(\mathcal{L}) = \prod_{v \in M_{\mathbf{K}}} \max_{1 \leq i_1 < \cdots < i_n \leq s} \|\det(L_{i_1}, \dots, L_{i_n})\|_v \tag{4.8}$$

where we have written $\{L_1, \dots, L_s\}$ for $\bigcup_{v \in M_{\mathbf{K}}} \{L_{1v}, \dots, L_{nv}\}$.

Let $\mathbf{c} = (c_{iv} : v \in M_{\mathbf{K}},\ i = 1, \dots, n)$ be a tuple of reals with the following properties:

$$c_{1v} = \cdots = c_{nv} = 0 \text{ for all but finitely many } v \in M_{\mathbf{K}}, \tag{4.9}$$

$$\sum_{v \in M_{\mathbf{K}}} \sum_{i=1}^{n} c_{iv} = 0, \tag{4.10}$$

$$\sum_{v \in M_{\mathbf{K}}} \max(c_{1v}, \dots, c_{nv}) \leq 1. \tag{4.11}$$

Finally, for any finite extension $\mathbf{E}$ of $\mathbf{K}$ and any place $w \in M_{\mathbf{E}}$ we define

$$L_{iw} = L_{iv},\ c_{iw} = d(w|v)c_{iv} \quad \text{for } i = 1, \ldots, n, \tag{4.12}$$

where v is the place of $M_{\mathbf{K}}$ lying below w and $d(w|v)$ is given by (4.2).

We define a so-called twisted height $H_{Q,\mathcal{L},\mathbf{c}}$ on $\overline{\mathbf{Q}}^n$ as follows. For $\mathbf{x} \in \mathbf{K}^n$ define

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) := \prod_{v \in M_{\mathbf{K}}} \max_{1 \le i \le n} \|L_{iv}(\mathbf{x})\|_v Q^{-c_{iv}}.$$

More generally, for $\mathbf{x} \in \overline{\mathbf{Q}}^n$ take any finite extension $\mathbf{E}$ of $\mathbf{K}$ with $\mathbf{x} \in \mathbf{E}^n$ and put

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) := \prod_{w \in M_{\mathbf{E}}} \max_{1 \le i \le n} \|L_{iw}(\mathbf{x})\|_w Q^{-c_{iw}}. \tag{4.13}$$

Using (4.12), (4.2), (4.3), and basic properties of degrees of field extensions, one easily shows that this does not depend on the choice of $\mathbf{E}$.

Proposition 4.1. *Let n be an integer ≥ 2, let $\mathcal{L} = (L_{iv} : v \in M_{\mathbf{K}}, i = 1, \ldots, n)$ be a tuple of linear forms satisfying (4.4)–(4.7) and $\mathbf{c} = (c_{iv} : v \in M_{\mathbf{K}}, i = 1, \ldots, n)$ a tuple of reals satisfying (4.9)–(4.11). Further, let $0 < \delta \le 1$.*
Then there are proper linear subspaces $T_1, \ldots, T_{t_1}$ of $\overline{\mathbf{Q}}^n$, all defined over $\mathbf{K}$, with

$$t_1 = t_1(n, r, \delta) = \begin{cases} 4^{(n+8)^2}\delta^{-n-4}\log(2r)\log\log(2r) & \text{if } n \ge 3, \\ 2^{25}\delta^{-3}\log(2r)\log\left(\delta^{-1}\log(2r)\right) & \text{if } n = 2 \end{cases}$$

such that the following holds: for every real Q with

$$Q > \max\left(\mathcal{H}^{1/\binom{r}{n}}, n^{2/\delta}\right)$$

there is a subspace $T_i \in \{T_1, \ldots, T_{t_1}\}$ such that

$$\{\mathbf{x} \in \overline{\mathbf{Q}}^n : H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \le Q^{-\delta}\} \subset T_i\,.$$

For $n \ge 3$ this is precisely Theorem 2.1 of [14], while for $n = 2$ this is an improvement of this theorem. This improvement can be obtained by combining some lemmata from [14] with more precise computations in the case $n = 2$. We give more details in the appendix at the end of the present paper.

5. Systems of inequalities

For every place $p \in M_{\mathbf{Q}} = \{\infty\} \cup \{\text{primes}\}$ we choose an extension of $|\cdot|_p$ to $\overline{\mathbf{Q}}$ which we denote also by $|\cdot|_p$. For a linear form $L = \sum_{i=1}^n \alpha_i X_i$ with coefficients in $\overline{\mathbf{Q}}$ we define the following: We denote by $\mathbf{Q}(L)$ the field generated by the coefficients of L, i.e., $\mathbf{Q}(L) := \mathbf{Q}(\alpha_1, \dots, \alpha_n)$; for any map σ from $\mathbf{Q}(L)$ to any other field we define $\sigma(L) := \sum_{i=1}^{n} \sigma(\alpha_i) X_i$; and the inhomogeneous height of L is given by $H^*(L) := \prod_{v \in M_{\mathbf{K}}} \max(1, \|\alpha_1\|_v, \dots, \|\alpha_n\|_v)$, where $\mathbf{K}$ is any number field containing $\mathbf{Q}(L)$. Further, we put $\|L\|_v := \max(\|\alpha_1\|_v, \dots, \|\alpha_n\|_v)$ for $v \in M_{\mathbf{K}}$.

Let n be an integer with $n \geq 2$, ε a real with $\varepsilon > 0$ and $S = \{\infty, p_1, \dots, p_t\}$ a finite subset of $M_{\mathbf{Q}}$ containing the infinite place. Further, let L_{ip} ($p \in S$, $i = 1, \dots, n$) be linear forms in $X_1, \dots, X_n$ with coefficients in $\overline{\mathbf{Q}}$ such that

$$\det(L_{1p}, \dots, L_{np}) = 1 \text{ for } p \in S, \tag{5.1}$$

$$\mathrm{Card}\Big(\bigcup_{p \in S} \{L_{1p}, \dots, L_{np}\} \Big) \leq R, \tag{5.2}$$

$$[\mathbf{Q}(L_{ip}) : \mathbf{Q}] \leq D \text{ for } p \in S,\ i = 1, \dots, n, \tag{5.3}$$

$$H^*(L_{ip}) \leq H \text{ for } p \in S,\ i = 1, \dots, n, \tag{5.4}$$

and e_{ip} ($p \in S$, $i = 1, \dots, n$) be reals satisfying

$$e_{i\infty} \leq 1 \ (i = 1, \dots, n), \quad e_{ip} \leq 0 \ (p \in S \setminus \{\infty\},\ i = 1, \dots, n), \tag{5.5}$$

$$\sum_{p \in S} \sum_{i=1}^{n} e_{ip} = -\varepsilon. \tag{5.6}$$

Finally let Ψ be a function from $\mathbf{Z}^n$ to $\mathbf{R}_{\geq 0}$. We consider the system of inequalities

$$\begin{aligned} |L_{ip}(\mathbf{x})|_p \leq \Psi(\mathbf{x})^{e_{ip}} \ (p \in S,\ i = 1, \dots, n) \\ \text{in } \mathbf{x} \in \mathbf{Z}^n \text{ with } \Psi(\mathbf{x}) \neq 0. \end{aligned} \tag{5.7}$$

Theorem 5.1. *The set of solutions of (5.7) with*

$$\Psi(\mathbf{x}) > \max\left(2H, n^{2n/\varepsilon}\right) \tag{5.8}$$

is contained in the union of at most

$$\begin{cases} 8^{(n+9)^2} (1 + \varepsilon^{-1})^{n+4} \log(2RD) \log\log(2RD) & \text{if } n \geq 3 \\ 2^{32} (1 + \varepsilon^{-1})^3 \log(2RD) \log\left((1 + \varepsilon^{-1}) \log(2RD)\right) & \text{if } n = 2 \end{cases} \tag{5.9}$$

proper linear subspaces of $\mathbf{Q}^n$.

Remark. Let $\|\cdot\|$ be any vector norm on $\mathbf{Z}^n$. Then for the solutions $\mathbf{x}$ of (5.7) we have, in view of (5.5),

$$\|\mathbf{x}\| \ll \max_{1 \leq i \leq n} |L_{i\infty}(\mathbf{x})| \ll \Psi(\mathbf{x}).$$

So it would not have been a substantial restriction if in the formulation of Theorem 5.1 we had restricted the function Ψ to vector norms. But for applications it is convenient to allow other functions for Ψ.

We deduce from Theorem 5.1 a quantitative Ridout type theorem. Let S_1, S_2 be finite, possibly empty sets of prime numbers, put $S := \{\infty\} \cup S_1 \cup S_2$, let $\xi \in \overline{\mathbf{Q}}$ be an algebraic number, let $\varepsilon > 0$, and let f_p $(p \in S)$ be reals such that

$$f_p \geq 0 \text{ for } p \in S, \quad \sum_{p \in S} f_p = 2 + \varepsilon. \tag{5.10}$$

We consider the system of inequalities

$$\left. \begin{cases} |\xi - \frac{x}{y}| \leq y^{-f_\infty}, \\ |x|_p \leq y^{-f_p} \ (p \in S_1) \\ |y|_p \leq y^{-f_p} \ (p \in S_2) \end{cases} \right\} \text{in } (x, y) \in \mathbf{Z}^2 \text{ with } y > 0. \tag{5.11}$$

Define the height of ξ by $H(\xi) := \prod_{v \in M_{\mathbf{K}}} \max(1, \|\xi\|_v)$ where $\mathbf{K}$ is any algebraic number field with $\xi \in \mathbf{K}$. Suppose that ξ has degree d.

Corollary 5.2. *The set of solutions of (5.11) with*

$$y > \max\left(2H(\xi), 2^{4/\varepsilon}\right) \tag{5.12}$$

is contained in the union of at most

$$2^{32}(1 + \varepsilon^{-1})^3 \log(6d) \log\left((1 + \varepsilon^{-1}) \log(6d)\right) \tag{5.13}$$

one-dimensional linear subspaces of $\mathbf{Q}^2$.

To obtain Corollary 5.2 one simply has to apply Theorem 5.1 with $n = 2$, $S = \{\infty\} \cup S_1 \cup S_2$ and with

$$\begin{aligned} &L_{1\infty} = X_1 - \xi X_2,\ L_{2\infty} = X_2, \\ &L_{1p} = X_1,\ L_{2p} = X_2 \text{ for } p \in S_1 \cup S_2, \\ &e_{1\infty} = 1 - f_\infty,\ e_{2\infty} = 1, \\ &e_{1p} = -f_p,\ e_{2p} = 0 \text{ for } p \in S_1, \\ &e_{1p} = 0,\ e_{2p} = -f_p \text{ for } p \in S_2, \\ &\Psi(\mathbf{x}) = |x_2| \text{ for } \mathbf{x} = (x_1, x_2) \in \mathbf{Z}^2. \end{aligned}$$

It is straightforward to verify that (5.1) is satisfied, and that (5.2), (5.3), (5.4) are satisfied with $R = 3$, $D = d$, $H = H(\xi)$, respectively. Further, it follows at once from (5.10) that (5.5) and (5.6) are satisfied.

Proof of Theorem 5.1. Let $\mathbf{K}$ be a finite normal extension of $\mathbf{Q}$, containing the coefficients of L_{ip} as well as the conjugates over $\mathbf{Q}$ of these coefficients, for $p \in S$, $i = 1, \ldots, n$. For $v \in M_{\mathbf{K}}$ we put $d(v) := [\mathbf{K}_v : \mathbf{Q}_{p_v}]$ where p_v is the place of $\mathbf{Q}$ below v, and

$$s(v) = d(v) \text{ if } v \text{ is Archimedean}, \quad s(v) = 0 \text{ if } v \text{ is non-Archimedean}.$$

Recall that every $|\cdot|_p$ ($p \in M_{\mathbf{Q}}$) has been extended to $\overline{\mathbf{Q}}$ so in particular to $\mathbf{K}$. For every $v \in M_{\mathbf{K}}$ there is an automorphism σ_v of $\mathbf{K}$ such that $|\sigma_v(\cdot)|_p$ represents v. So by (4.1) we have

$$\|x\|_v = |\sigma_v(x)|_{p_v}^{d(v)} \text{ for } x \in \mathbf{K}. \tag{5.14}$$

Let T denote the set of places of $\mathbf{K}$ lying above the places in S. Define linear forms L_{iv} and reals e_{iv} ($v \in M_{\mathbf{K}}$, $i = 1, \dots, n$) by

$$L_{iv} = \sigma_v^{-1}(L_{i,p_v}) \ (v \in T), \quad L_{iv} = X_i \ (v \in M_{\mathbf{K}} \setminus T) \tag{5.15}$$

and

$$e_{iv} = d(v) e_{i,p_v} \ (v \in T), \quad e_{iv} = 0 \ (v \in M_{\mathbf{K}} \setminus T), \tag{5.16}$$

respectively. Then system (5.7) can be rewritten as

$$\begin{aligned} &\|L_{iv}(\mathbf{x})\|_v \le \Psi(\mathbf{x})^{e_{iv}} \ (v \in M_{\mathbf{K}},\ i = 1, \dots, n) \\ &\qquad\qquad\qquad\qquad \text{in } \mathbf{x} \in \mathbf{Z}^n \text{ with } \Psi(\mathbf{x}) \ne 0. \end{aligned} \tag{5.17}$$

Notice that in view of (5.17), (5.5), (5.6), and $\sum_{v|p} d(v) = 1$ for $p \in M_{\mathbf{Q}}$ we have

$$e_{iv} \le s(v) \ (i = 1, \dots, n), \quad \sum_{v \in M_{\mathbf{K}}} \sum_{i=1}^{n} e_{iv} \le -\varepsilon. \tag{5.18}$$

Further, by (5.2), (5.15),

$$\text{Card} \bigcup_{v \in M_{\mathbf{K}}} \{L_{1v}, \dots, L_{nv}\} \le r := n + DR. \tag{5.19}$$

Now define

$$\delta := \frac{\varepsilon}{n + \varepsilon}, \tag{5.20}$$

let $\mathcal{L} = (L_{iv} : v \in M_{\mathbf{K}},\ i = 1, \dots, n)$, and define the tuple of reals $\mathbf{c} = (c_{iv} : v \in M_{\mathbf{K}},\ i = 1, \dots, n)$ by

$$c_{iv} := \big(1 + (\varepsilon/n)\big)^{-1} \Big(e_{iv} - \frac{1}{n} \sum_{j=1}^{n} e_{jv}\Big). \tag{5.21}$$

Let $\mathcal{H} = \mathcal{H}(\mathcal{L})$ be the quantity defined by (4.8) and $H_{Q,\mathcal{L},\mathbf{c}}$ the twisted height defined by (4.13). We want to apply Proposition 4.1, and to this end we have to verify the conditions (4.4)–(4.7) and (4.9)–(4.11). Condition (4.4) is obvious. (5.1) and (5.15) imply (4.5),(4.6), while (4.7) is (5.18). Condition (4.9) is satisfied in view of (5.16), (5.20), while (4.10) follows at once from (5.21). To verify (4.11), observe that by (5.21), (5.18) we have

$$\begin{aligned} &\sum_{v \in M_{\mathbf{K}}} \max(c_{1v}, \dots, c_{nv}) \\ &\qquad \le \Big(1 + \frac{\varepsilon}{n}\Big)^{-1} \Big(\sum_{v \in M_{\mathbf{K}}} s(v) - \frac{1}{n} \sum_{v \in M_{\mathbf{K}}} \sum_{j=1}^{n} e_{jv} \Big) \\ &\qquad = \Big(1 + \frac{\varepsilon}{n}\Big)^{-1} \Big(1 + \frac{\varepsilon}{n}\Big) = 1. \end{aligned}$$

The following lemma connects system (5.7) to Proposition 4.1.

Lemma 5.3. *Let* $\mathbf{x}$ *be a solution of (5.7) with (5.8). Put*

$$Q := \Psi(\mathbf{x})^{1+\varepsilon/n}.$$

Then

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \le Q^{-\delta}, \tag{5.22}$$

$$Q \ge \max\left(\mathcal{H}^{1/\binom{r}{n}}, n^{2/\delta}\right). \tag{5.23}$$

Proof. As observed above, $\mathbf{x}$ satisfies (5.17). In combination with (5.21) this yields

$$\begin{aligned}\|L_{iv}(\mathbf{x})\|_v Q^{-c_{iv}} &= \|L_{iv}(\mathbf{x})\|_v \cdot \Psi(\mathbf{x})^{-e_{iv}} \cdot \Psi(\mathbf{x})^{\frac{1}{n}\sum_{j=1}^n e_{jv}} \\ &\le \Psi(\mathbf{x})^{\frac{1}{n}\sum_{j=1}^n e_{jv}}\end{aligned}$$

for $v \in M_{\mathbf{K}}$, $i = 1, \dots, n$. By taking the product over $v \in M_{\mathbf{K}}$ and using (5.18), (5.20) we obtain

$$\begin{aligned}H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) &= \prod_{v \in M_{\mathbf{K}}} \max_{1\le i\le n} \|L_{iv}(\mathbf{x})\|_v Q^{-c_{iv}} \\ &\le \Psi(\mathbf{x})^{-\varepsilon/n} = Q^{-\delta}.\end{aligned}$$

This proves (5.22).

To prove (5.23), write $\bigcup_{v\in M_{\mathbf{K}}}\{L_{1v}, \dots, L_{nv}\} = \{L_1, \dots, L_s\}$. Then $s \le r$ by (5.19). By (5.4), (5.15) we have $H^*(L_{iv}) \le H$ for $v \in M_{\mathbf{K}}$, $i = 1, \dots, n$. By applying e.g., Hadamard's inequality for the Archimedean places and the ultrametric inequality for the non-Archimedean places, we obtain for $i_1, \dots, i_n \in \{1, \dots, s\}$, $v \in M_{\mathbf{K}}$,

$$\begin{aligned}\|\det(L_{i_1}, \dots, L_{i_n})\|_v &\le (n^{n/2})^{s(v)} \prod_{j=1}^n \|L_{i_j}\|_v \\ &\le (n^{n/2})^{s(v)} \prod_{i=1}^s \max(1, \|L_i\|_v),\end{aligned}$$

hence

$$\mathcal{H} \le n^{n/2} \prod_{i=1}^r H^*(L_i) \le n^{n/2} H^r.$$

Together with (5.19), (5.20) this implies

$$\begin{aligned}\max\left(\mathcal{H}^{1/\binom{r}{n}}, n^{2/\delta}\right) &\le \max\left(n^{n/2\binom{r}{n}} H^{r/\binom{r}{n}}, n^{2(n+\varepsilon)/\varepsilon}\right) \\ &\le \max\left(2H, n^{2n/\varepsilon}\right)^{1+\varepsilon/n}.\end{aligned}$$

So if $\mathbf{x}$ satisfies (5.8), then $Q = \Psi(\mathbf{x})^{1+\varepsilon/n}$ satisfies (5.23). This proves Lemma 5.3. $\square$

We apply Proposition 4.1 with the values of r, δ given by (5.19), (5.20), i.e., $r = n + DR$ and $\delta = \frac{\varepsilon}{n+\varepsilon}$. It is straightforward to show that for these choices of r, δ the quantity t_1 from Proposition 4.1 is bounded above by the quantity in (5.9). By Proposition 4.1, there are proper linear subspaces $T_1, \ldots, T_{t_1}$ of $\overline{\mathbf{Q}}^n$ such that for every Q with (5.23) there is $T_i \in \{T_1, \ldots, T_{t_1}\}$ with

$$\{\mathbf{x} \in \overline{\mathbf{Q}}^n : H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \leq Q^{-\delta}\} \subset T_i.$$

Now Lemma 5.3 implies that the solutions $\mathbf{x}$ of (5.7) with (5.8) lie in $\cup_{i=1}^{t_1}(T_i \cap \mathbf{Q}^n)$. Theorem 5.1 follows. □

6. A combinatorial lemma for the proof of Theorem 2.1

In this section, we establish the following lemma.

Lemma 6.1. *Let $b \geq 2$ be an integer. Let c and u be positive real numbers. Let ξ be an irrational real number such that $0 < \xi < 1$ and*

$$p(n, \xi, b) \leq cn(\log n)^u, \quad \text{for } n \geq 1.$$

Then for every positive real number $v < u$, there exist integer sequences $(r_n)_{n\geq1}$, $(t_n)_{n\geq1}$, $(p_n)_{n\geq1}$ and a positive real number C depending only on c, u, v such that

$$|b^{t_n}\xi - b^{r_n}\xi - p_n| \leq \left(b^{t_n}\right)^{-(\log t_n)^{-v}},$$

$$0 \leq r_n < t_n, \quad 2t_n < t_{n+1}, \quad t_n \leq (2n)^{Cn}, \quad \text{for } n \geq 1. \tag{6.1}$$

Furthermore, b does not divide p_n if $r_n \geq 1$.

Proof. Let b and ξ be as in the statement of the lemma. Let $\mathbf{a}$ denote the b-ary expansion of ξ. Throughout this proof, $c_1, c_2, \ldots$ are positive constants depending only on c, u, v. The length of a finite word W, that is, the number of letters composing W, is denoted by $|W|$. The infinite word W^∞ is obtained by concatenation of infinitely many copies of the finite word W.

By assumption, the complexity function of $\mathbf{a}$ satisfies

$$p(n, \mathbf{a}) \leq c_1\, n(\log n)^u, \quad \text{for } n \geq 1.$$

Our aim is to show that there exists a (in some sense) 'dense' sequence of rational approximations to ξ with special properties.

Let $\ell \geq 2$ be an integer, and denote by $A(\ell)$ the prefix of $\mathbf{a}$ of length ℓ. By the *Schubfachprinzip*, there exist (possibly empty) words U_ℓ, V_ℓ, W_ℓ and X_ℓ such that

$$A(\ell) = U_\ell V_\ell W_\ell V_\ell X_\ell,$$

and

$$|V_\ell| \geq c_2\, \ell (\log \ell)^{-u}.$$

Set $r_\ell = |U_\ell|$ and $s_\ell = |V_\ell W_\ell|$. We choose the words U_ℓ, V_ℓ, W_ℓ and X_ℓ in such a way that $|V_\ell|$ is maximal and, among the corresponding factorisations of $A(\ell)$, such that $|U_\ell|$ is minimal. In particular, either U_ℓ is the empty word, or the last digits of U_ℓ and $V_\ell W_\ell$ are different.

If ξ_ℓ denotes the rational number with b-ary expansion $U_\ell (V_\ell W_\ell)^\infty$, then there exists an integer p_ℓ such that

$$\xi_\ell = \frac{p_\ell}{b^{r_\ell}(b^{s_\ell} - 1)}, \qquad |\xi - \xi_\ell| \leq b^{-r_\ell - s_\ell - |V_\ell|},$$

and b does not divide p_ℓ if $r_\ell \geq 1$.

Take $t_\ell = r_\ell + s_\ell$. Then

$$\ell \geq t_\ell \geq s_\ell \geq c_2 \ell (\log \ell)^{-u}. \tag{6.2}$$

Hence,

$$\begin{aligned} |b^{t_\ell}\xi - b^{r_\ell}\xi - p_\ell| &\leq b^{-c_2 \ell (\log \ell)^{-u}} \\ &\leq (b^{t_\ell})^{-c_3 (\log t_\ell)^{-u}}. \end{aligned}$$

We construct a sequence of positive integers $(\ell_k)_{k=1}^\infty$ such that for every $k \geq 1$,

$$|b^{t_{\ell_k}}\xi - b^{r_{\ell_k}}\xi - p_{\ell_k}| \leq (b^{t_{\ell_k}})^{-(\log t_{\ell_k})^{-v}}, \tag{6.3}$$

$$t_{\ell_{k+1}} > 2 t_{\ell_k}, \tag{6.4}$$

$$t_{\ell_k} \leq (2k)^{Ck}. \tag{6.5}$$

Then a slight change of notation establishes the lemma.

Let ℓ_1 be the smallest positive integer ℓ such that $c_3 (\log t_\ell)^u \geq (\log t_\ell)^v$. Further, for $k = 1, 2, \ldots$, let ℓ_{k+1} be the smallest positive integer ℓ such that $t_\ell > 2 t_{\ell_k}$. This sequence is well-defined by (6.2). It is clear that (6.3), (6.4) are satisfied.

To prove (6.5), observe that if ℓ is any integer with $c_2 \ell (\log \ell)^{-u} > 2\ell_k$ then, by (6.2), $t_\ell > 2\ell_k \geq 2 t_{\ell_k}$. This shows that there is a constant c_4 such that $\ell_{k+1} \leq c_4 \ell_k (\log \ell_k)^u$. Now an easy induction yields that there exists a constant C, depending only on c, u, v, such that $\ell_k \leq (2k)^{Ck}$ for $k \geq 1$. Invoking again (6.2) we obtain (6.5). □

7. Completion of the proof of Theorem 2.1

Let ξ be an algebraic irrational real number. Let v be a real number such that $0 < v < 1/11$. Define the positive real number η by

$$(11 + 2\eta)(v + \eta) + \eta = 1. \tag{7.1}$$

We assume that there exists a positive constant c such that the complexity function of ξ in base b satisfies

$$p(n, \xi, b) \leq cn(\log n)^{v+\eta} \quad \text{for } n \geq 1, \tag{7.2}$$

and we will derive a contradiction. Then Theorem 2.1 follows.

Let N be a sufficiently large integer. We will often use the fact that N is large, in order to absorb numerical constants.

Let $(r_n)_{n\geq 1}$, $(t_n)_{n\geq 1}$, and $(p_n)_{n\geq 1}$ be the sequences given by Lemma 6.1 applied with $u := v + \eta$. Set

$$\varepsilon = (\log t_N)^{-v}, \tag{7.3}$$

and observe that, in view of (6.1) and (7.3), we have

$$\varepsilon^{-1} = (\log t_N)^{v} \leq N^{v+\eta}. \tag{7.4}$$

For $n = 1, \ldots, N$, we have

$$|b^{t_n}\xi - b^{r_n}\xi - p_n| < (b^{t_n})^{-\varepsilon}. \tag{7.5}$$

Put

$$k := [2/\varepsilon] + 1. \tag{7.6}$$

For each $n = 1, \ldots, N$ there is $\ell \in \{0, 1, \ldots, k-1\}$ such that

$$\frac{\ell}{k} \leq \frac{r_n}{t_n} < \frac{\ell+1}{k}.$$

For the moment, we consider those $n \in \{1, \ldots, N\}$ such that

$$\frac{N}{2} \leq n \leq N, \quad \frac{\ell}{k} \leq \frac{r_n}{t_n} < \frac{\ell+1}{k}, \tag{7.7}$$

where $\ell \in \{0, 1, \ldots, k-1\}$ is fixed, and show that the vectors

$$\mathbf{x}_n := (b^{t_n}, b^{r_n}, p_n)$$

satisfy a system of inequalities to which Theorem 5.1 is applicable.

Let $S = \{\infty\} \cup \{p : p \mid b\}$ be the set of places on $\mathbf{Q}$ composed of the infinite place and the finite places corresponding to the prime divisors of b. We choose

$$\Psi(\mathbf{x}) = x_1 \text{ for } \mathbf{x} = (x_1, x_2, x_3) \in \mathbf{Z}^3.$$

We introduce the linear forms with real algebraic coefficients

$$L_{1\infty}(\mathbf{X}) = X_1, \quad L_{2\infty}(\mathbf{X}) = X_2, \quad L_{3\infty}(\mathbf{X}) = -\xi X_1 + \xi X_2 + X_3,$$

and, for every prime divisor p of b, we set

$$L_{1p}(\mathbf{X}) = X_1, \quad L_{2p}(\mathbf{X}) = X_2, \quad L_{3p}(\mathbf{X}) = X_3.$$

Set also

$$e_{1\infty} = 1, \quad e_{2\infty} = \frac{\ell+1}{k}, \quad e_{3\infty} = -\varepsilon,$$

and, for every prime divisor p of b,

$$e_{1p} = \frac{\log |b|_p}{\log p}, \quad e_{2p} = \frac{\log |b|_p}{\log p} \cdot \frac{\ell}{k}, \quad e_{3p} = 0.$$

Notice that

$$\begin{aligned} &\sum_{p\in S} \sum_{i=1}^{3} e_{ip} = -(\varepsilon - 1/k), \\ &e_{i\infty} \leq 1 \quad (i = 1,2,3), \\ &e_{ip} \leq 0 \quad (p \in S \setminus \{\infty\},\ i = 1,2,3). \end{aligned} \tag{7.8}$$

Furthermore,

$$\det(L_{1p}, L_{2p}, L_{3p}) = 1, \quad \text{for } p \in S. \tag{7.9}$$

Writing $d := [\mathbf{Q}(\xi) : \mathbf{Q}]$, we have

$$\begin{aligned} &\mathrm{Card} \bigcup_{p\in S} \{L_{1p}, L_{2p}, L_{3p}\} = 4, \\ &[\mathbf{Q}(L_{ip}) : \mathbf{Q}] \leq d, \quad \text{for } p \in S,\ i = 1,2,3. \end{aligned} \tag{7.10}$$

Further,

$$\max_{p\in S, i=1,2,3} H^*(L_{ip}) = H(\xi). \tag{7.11}$$

(7.8)–(7.11) imply that the linear forms L_{ip} and reals e_{ip} defined above satisfy the conditions (5.1)–(5.6) of Theorem 5.1 with $n = 3$, $R = 4$, $D = d$, $H = H(\xi)$.

It is clear from (7.5), (7.8) that for any integer n with (7.7) we have

$$|L_{ip}(\mathbf{x}_n)|_p \leq \Psi(\mathbf{x}_n)^{e_{ip}}, \quad \text{for } p \in S,\ i = 1,2,3.$$

Assuming that N is sufficiently large, we infer from (6.1), (7.4) that for every n with (7.7) we have

$$\Psi(\mathbf{x}_n) = b^{t_n} \geq 2^{2^{N/2}-1} > \max\{2H(\xi), 3^{6/(\varepsilon-1/k)}\}.$$

Now, Theorem 5.1 implies that the set of vectors $\mathbf{x}_n = (b^{t_n}, b^{r_n}, p_n)$ with n satisfying (7.7) is contained in the union of at most

$$A_1 := 8^{144}\Big(1 + (\varepsilon - 1/k)^{-1}\Big)^{-7} \log(8d) \log\log(8d)$$

proper linear subspaces of $\mathbf{Q}^3$. We now consider the vectors $\mathbf{x}_n$ with $N/2 \leq n \leq N$ and drop the condition $\ell/k \leq r_n/t_n < (\ell+1)/k$. Then by (7.6), for any sufficiently large N, the set of vectors $\mathbf{x}_n = (b^{t_n}, b^{r_n}, p_n)$ with

$$\frac{N}{2} \leq n \leq N,$$

lies in the union of at most

$$kA_1 \le (\varepsilon^{-1})^{8+\eta}$$

proper linear subspaces of $\mathbf{Q}^3$.

We claim that if N is sufficiently large, then any two-dimensional linear subspace of $\mathbf{Q}^3$ contains at most $(\varepsilon^{-1})^{3+\eta}$ vectors $\mathbf{x}_n$. Having achieved this, it follows by (7.1), (7.4) that

$$\frac{N}{2} \le (\varepsilon^{-1})^{8+\eta} \cdot (\varepsilon^{-1})^{3+\eta} \le N^{(11+2\eta)(v+\eta)} = N^{1-\eta},$$

which is clearly impossible if N is sufficiently large. Thus (7.2) leads to a contradiction.

So let T be a two-dimensional linear subspace of $\mathbf{Q}^3$, say given by an equation $z_1X_1 + z_2X_2 + z_3X_3 = 0$ where we may assume that z_1, z_2, z_3 are integers without a common prime divisor. Let

$$\mathcal{N} = \{i_1 < i_2 < \ldots < i_r\}$$

be the set of n with $N/2 \le n \le N$ and $\mathbf{x}_n \in T$. So we have to prove that $r \le (\varepsilon^{-1})^{3+\eta}$.

Recall that by Lemma 6.1, for every $n \ge 1$ we have either $r_n = 0$, or $r_n > 0$ and b does not divide p_n. Hence the vectors $\mathbf{x}_n$, $n \ge 1$, are pairwise non-collinear. So the exterior product of $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}$ must be a non-zero multiple of $\mathbf{z} = (z_1, z_2, z_3)$, and therefore

$$\max\{|z_1|, |z_2|, |z_3|\} \le 2b^{2t_{i_2}}. \tag{7.12}$$

By combining (7.5) with $z_1 b^{t_n} + z_2 b^{r_n} + z_3 p_n = 0$, eliminating b^{r_n}, it follows that for n in $\mathcal{N}$

$$\left|\frac{\xi(z_1+z_2)}{\xi z_3 - z_2} - \frac{-p_n}{b^{t_n}}\right| < \left|\frac{z_2}{\xi z_3 - z_2}\right| \cdot (b^{t_n})^{-1-\varepsilon}. \tag{7.13}$$

We want to apply Corollary 5.2 with $\xi(z_1+z_2)/(\xi z_3 - z_2)$ instead of ξ.

Recall that d denotes the degree of ξ. By (7.12), assuming that N is sufficiently large, we have

$$H\left(\frac{\xi(z_1+z_2)}{\xi z_3 - z_2}\right) \le 4b^{2t_{i_2}} H(\xi) \le b^{3t_{i_2}}.$$

Likewise,

$$\left|\frac{z_2}{\xi z_3 - z_2}\right| \le H\left(\frac{\xi(z_1+z_2)}{\xi z_3 - z_2}\right)^d \le b^{3dt_{i_2}}.$$

There is no loss of generality to assume that there is an integer $k \le r$ with

$$b^{t_{i_k}} \ge b^{(3dt_{i_2})^{2/\varepsilon}}. \tag{7.14}$$

Indeed, if there is no such k then we infer from (6.1) that

$$b^{t_{i_2}^{2^{r-3}}} \le b^{(3dt_{i_2})^{2/\varepsilon}} \le b^{t_{i_2}^{4/\varepsilon}},$$

hence

$$r \le 3 + \frac{\log(4/\varepsilon)}{\log 2},$$

which is stronger than what we have to prove. Letting k_0 be the smallest integer k with (7.14), we have

$$b^{t_{i_{k_0}}} \geq b^{(3dt_{i_2})^{2/\varepsilon}}, \quad k_0 \leq 4 + \frac{\log(4/\varepsilon)}{\log 2}. \tag{7.15}$$

Let $\mathcal{N}' = \{i_{k_0}, i_{k_0+1}, \ldots, i_r\}$. We divide this set further into

$$\mathcal{N}'' = \{n \in \mathcal{N}' : r_n \neq 0\}, \quad \mathcal{N}''' = \{n \in \mathcal{N}' : r_n = 0\}.$$

By (7.13) we have for n in $\mathcal{N}''$

$$\left| \frac{\xi(z_1 + z_2)}{\xi z_3 - z_2} - \frac{-p_n}{b^{t_n}} \right| < (b^{t_n})^{-1-\varepsilon/2}. \tag{7.16}$$

Let $S_1 = \emptyset$ and $S_2 = \{p : p \mid b\}$. Then for $\ell \in S_2$ we have

$$|b^{t_n}|_\ell \leq (b^{t_n})^{\log |b|_\ell/(\log b)}. \tag{7.17}$$

Lastly,

$$b^{t_n} \geq b^{(3dt_{i_2})^{2/\varepsilon}} \geq \max\left\{ H\left(\frac{\xi(z_1 + z_2)}{\xi z_3 - z_2} \right), 2^{4/\varepsilon} \right\}. \tag{7.18}$$

Now, (7.16), (7.17) and (7.18) imply that all the conditions of Corollary 5.2 are satisfied with $\varepsilon/2$ instead of ε and with

$$x = p_n, \ y = b^{t_n}, \quad f_\infty = 1 + \frac{\varepsilon}{2}, \quad f_\ell = -\frac{\log |b|_\ell}{\log b} \ \ (\ell \in S_2).$$

Notice that

$$f_\infty + \sum_{\ell \in S_2} f_\ell = 2 + \varepsilon/2,$$

and $f_\infty \geq 0$, $f_\ell \geq 0$ for $\ell \in S_2$. Consequently, the set of vectors (p_n, b^{t_n}), $n \in \mathcal{N}''$, lies in the union of at most

$$B(d, \varepsilon) := 2^{32}\left(1 + 2\varepsilon^{-1}\right)^3 \log(6d) \log\left((1 + 2\varepsilon^{-1}) \log(6d)\right) \tag{7.19}$$

one-dimensional linear subspaces of $\mathbf{Q}^2$. But the vectors (p_n, b^{t_n}), $n \in \mathcal{N}''$, are pairwise non-proportional, since b does not divide p_n for these values of n. Hence $\operatorname{Card} \mathcal{N}'' \leq B(d, \varepsilon)$.

To deal with $n \in \mathcal{N}'''$, we observe that by combining (7.5) again with $z_1 b^{t_n} + z_2 b^{r_n} + z_3 p_n = 0$, but now eliminating p_n, we obtain

$$\left| \frac{\xi z_3 + z_1}{\xi z_3 - z_2} - \frac{1}{b^{t_n}} \right| < \left| \frac{z_3}{\xi z_3 - z_2} \right| \cdot (b^{t_n})^{-1-\varepsilon}.$$

In precisely the same manner as above, one obtains that the pairs $(b^{t_n}, 1)$ lie in at most $B(d, \varepsilon)$ one-dimensional subspaces. Since these pairs are pairwise non-proportional, it follows that $\operatorname{Card} \mathcal{N}''' \leq B(d, \varepsilon)$.

By combining the above we obtain

$$\mathrm{Card}\,\mathcal{N} = r \le k_0 + \mathrm{Card}\,\mathcal{N}'' + \mathrm{Card}\,\mathcal{N}''' \le k_0 + 2B(d,\varepsilon).$$

In view of (7.15), (7.19), this is smaller than $(\varepsilon^{-1})^{3+\eta}$ for N sufficiently large. This proves the claim, hence Theorem 2.1. □

8. Proof of Theorem 3.1

We closely follow Section 4 of [8]. Assume without loss of generality that

$$\frac{b-1}{b} < \xi < 1.$$

Define the increasing sequence of positive integers $(n_j)_{j\ge1}$ by $a_1 = \ldots = a_{n_1}$, $a_{n_1} \ne a_{n_1+1}$ and $a_{n_j+1} = \ldots = a_{n_{j+1}}$, $a_{n_{j+1}} \ne a_{n_{j+1}+1}$ for $j \ge 1$. Observe that

$$\mathrm{nbdc}(n,\xi,b) = \max\{j : n_j \le n\}$$

for $n \ge n_1$, and that $n_j \ge j$ for $j \ge 1$. Define

$$\xi_j := \sum_{k=1}^{n_j} \frac{a_k}{b^k} + \sum_{k=n_j+1}^{+\infty} \frac{a_{n_j+1}}{b^k} = \sum_{k=1}^{n_j} \frac{a_k}{b^k} + \frac{a_{n_j+1}}{b^{n_j}(b-1)}.$$

Then,

$$\xi_j = \frac{P_j(b)}{b^{n_j}(b-1)},$$

where $P_j(X)$ is an integer polynomial of degree at most n_j whose constant coefficient $a_{n_j+1} - a_{n_j}$ is not divisible by b. That is, b does not divide $P_j(b)$. We have

$$|\xi - \xi_j| < \frac{1}{b^{n_{j+1}}},$$

and this can be rewritten as

$$\left| (b-1)\xi - \frac{P_j(b)}{b^{n_j}} \right| < \frac{b-1}{b^{n_{j+1}}}. \tag{8.1}$$

By Liouville's inequality,

$$\left| (b-1)\xi - \frac{P_j(b)}{b^{n_j}} \right| \ge \bigl(2H\bigl((b-1)\xi\bigr)b^{n_j}\bigr)^{-d},$$

so, if

$$n_j \ge U := 1 + 3H\bigl((b-1)\xi\bigr), \tag{8.2}$$

then

$$n_{j+1} \le 2dn_j. \tag{8.3}$$

In what follows, constants implied by the Vinogradov symbols $\ll$, $\gg$ are absolute. We need the following lemma.

Lemma 8.1. *Let $0 < \varepsilon \le 1$ and let j_1 denote the smallest j such that $n_j \ge \max\{U, 5/\varepsilon\}$. Then*

$$\mathrm{Card}\,\{j : j \ge j_1, n_{j+1}/n_j \ge 1 + 2\varepsilon\} \ll \log(6d)\varepsilon^{-3} \log\big(\varepsilon^{-1}\log(6d)\big).$$

Proof. For the integers j into consideration, we have

$$b^{n_j} > \max\big\{2H\big((b-1)\xi\big), 2^{4/\varepsilon}\big\}.$$

Further, by (8.1), $n_j \ge U$, we get

$$\left|(b-1)\xi - \frac{P_j(b)}{b^{n_j}}\right| < \frac{b-1}{(b^{n_j})^{1+2\varepsilon}} \le \frac{1}{(b^{n_j})^{1+\varepsilon}}. \tag{8.4}$$

Moreover, for every prime ℓ dividing b,

$$|b^{n_j}|_\ell \le \big(b^{n_j}\big)^{\log|b|_\ell/\log b}. \tag{8.5}$$

Since

$$1 + \varepsilon + \sum_{\ell | b} \frac{-\log|b|_\ell}{\log b} = 2 + \varepsilon,$$

Corollary 5.2 applied to (8.4), (8.5) yields that for the integers j into consideration the pairs $(P_j(b), b^{n_j})$ lie in

$$\ll \log(6d)\varepsilon^{-3} \log\big(\varepsilon^{-1}\log(6d)\big)$$

one-dimensional linear subspaces of $\mathbf{Q}^2$. But these pairs are non-proportional since b does not divide $P_j(b)$. The lemma follows. $\square$

Let j_0 be the smallest j such that $n_j \ge U$. Let J be an integer with

$$J > \max\big\{n_{j_0}^3, (4d)^6\big\}. \tag{8.6}$$

Let j_2 be the largest integer with

$$n_{j_2} \le 6dJ^{1/3}. \tag{8.7}$$

Then since $n_{j_2} \ge n_{j_0} \ge U$, we have

$$n_{j_2} \ge \frac{n_{j_2+1}}{2d} \ge 3J^{1/3}. \tag{8.8}$$

Now choose

$$\varepsilon_1 := \Big(\frac{\log(6d)\log J}{J}\Big)^{1/3} \tag{8.9}$$

and let k be any positive integer and $\varepsilon_2, \ldots, \varepsilon_{k-1}$ any reals such that

$$\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_{k-1} < \varepsilon_k := 1. \tag{8.10}$$

We infer from (8.8) that

$$n_{j_2} \geq \max\{U, 5/\varepsilon_h\}, \quad \text{for } h = 1, \ldots, k. \tag{8.11}$$

Let $\mathcal{S}_0 = \{j_2, j_2+1, \ldots, J\}$ and, for $h = 1, \ldots, k$, let $\mathcal{S}_h$ denote the set of positive integers j such that $j_2 \leq j \leq J$ and $n_{j+1} \geq (1+2\varepsilon_h)n_j$. Further, let T_h be the cardinality of $\mathcal{S}_h$ for $h = 1, \ldots, k$. Obviously, $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \ldots \supset \mathcal{S}_k$ and

$$\mathcal{S}_0 = (\mathcal{S}_0 \setminus \mathcal{S}_1) \cup (\mathcal{S}_1 \setminus \mathcal{S}_2) \cup \ldots \cup (\mathcal{S}_{k-1} \setminus \mathcal{S}_k) \cup \mathcal{S}_k.$$

Now,

$$\begin{aligned}
\frac{n_J}{n_{j_2}} &= \frac{n_J}{n_{J-1}} \times \frac{n_{J-1}}{n_{J-2}} \times \ldots \times \frac{n_{j_2+1}}{n_{j_2}} \\
&= \prod_{h=0}^{k-1} \Bigg(\prod_{j \in \mathcal{S}_h \setminus \mathcal{S}_{h+1}} \frac{n_{j+1}}{n_j} \Bigg) \Bigg(\prod_{j \in \mathcal{S}_h} \frac{n_{j+1}}{n_j} \Bigg) \\
&\leq (1+2\varepsilon_1)^J \prod_{h=1}^{k-1} (1+2\varepsilon_{h+1})^{T_h - T_{h+1}} \, (2d)^{T_k},
\end{aligned}$$

where in the last estimate we have used (8.11) and (8.3). Taking logarithms, we get

$$\begin{aligned}
\log(n_J/n_{j_2}) &\leq 2\varepsilon_1 J + 2\sum_{h=1}^{k-1} \varepsilon_{h+1}(T_h - T_{h+1}) + T_k \log(2d) \\
&\leq 2\varepsilon_1 J + 2\varepsilon_2 T_1 + 2\sum_{h=2}^{k-1} (\varepsilon_{h+1} - \varepsilon_h) T_h - 2T_k + T_k \log(2d).
\end{aligned}$$

In view of (8.11), we can apply Lemma 8.1, and obtain that

$$T_h \ll \log(6d) \varepsilon_h^{-3} \log\big(\varepsilon_h^{-1} \log(6d)\big)$$

for $h = 1, \ldots, k$. This gives

$$\begin{aligned}
\log(n_J/n_{j_2}) \ll \varepsilon_1 J &+ \log(6d)\varepsilon_2 \varepsilon_1^{-3} \log\big(\varepsilon_1^{-1}\log(6d)\big) \\
&+ \log(6d) \sum_{h=2}^{k-1} \varepsilon_h^{-3} \log\big(\varepsilon_h^{-1}\log(6d)\big) \cdot (\varepsilon_{h+1} - \varepsilon_h) \\
&+ \big(\log(6d)\big)^2 \log\log(6d).
\end{aligned}$$

Now, let k tend to infinity and $\max_{1 \leq h \leq k-1} (\varepsilon_{h+1} - \varepsilon_h)$ tend to zero. Then the sum converges to a Riemann integral, and, after a short computation, using that in view of (8.6), (8.9) we have $\varepsilon_1^{-1} \gg d$, we get

$$\log(n_J/n_{j_2}) \ll \varepsilon_1 J + \log(6d)\varepsilon_1^{-2} \log(\varepsilon_1^{-1}).$$

By (8.6) and (8.7), we have $n_{j_2} \le J^{1/2} \le n_J^{1/2}$, so $n_J/n_{j_2} \ge n_J^{1/2}$. Inserting our choice (8.9) for ε_1 and using (8.6), we get

$$\log n_J \ll J^{2/3}(\log J)^{1/3}\big(\log(6d)\big)^{1/3},$$

i.e.,

$$J \gg (\log n_J)^{3/2}(\log\log n_J)^{-1/2}\big(\log(6d)\big)^{-1/2}.$$

This proves Theorem 3.1. □

9. Final remarks

We deduce from Corollary 5.2 an improvement of an extension due to Mahler [17] of a theorem of Cugiani [11], see [9] for further references on the Cugiani–Mahler Theorem.

Let S_1, S_2 be finite, possibly empty sets of prime numbers, put $S := \{\infty\} \cup S_1 \cup S_2$, let $\xi \in \overline{\mathbf{Q}}$ be an algebraic number, let $\varepsilon > 0$, and let f_p $(p \in S)$ be reals such that

$$f_p \ge 0 \text{ for } p \in S, \quad \sum_{p\in S} f_p = 2.$$

Let $\varepsilon : \mathbf{Z}_{\ge 1} \to \mathbf{R}_{>0}$ be a non-increasing function. We consider the system of inequalities

$$\left\{\begin{array}{l} |\xi - \frac{x}{y}| \le y^{-f_\infty - \varepsilon(y)}, \\ |x|_p \le y^{-f_p} \ (p \in S_1) \\ |y|_p \le y^{-f_p} \ (p \in S_2) \end{array}\right\} \text{in } (x,y) \in \mathbf{Z}^2 \text{ with } y > 0 \text{ and } \gcd(x,y) = 1. \tag{9.1}$$

Arguing as in [9], we get the following improvement of Theorem 1 on page 169 of [17], that we state without proof. For a positive integer m, we denote by $\exp_m$ the mth iterate of the exponential function and by $\log_m$ the function that coincides with the mth iterate of the logarithm function on $[\exp_m 1, +\infty)$ and that takes the value 1 on $(-\infty, \exp_m 1]$.

Theorem 9.1. *Keep the above notation. Let m be a positive integer, and c be a positive real number. Set*

$$\varepsilon(y) = c\,(\log_{m+1} y)^{-1/3}\,(\log_{m+2} y), \quad \text{for } y \ge 1.$$

Let $(x_j/y_j)_{j\ge1}$ be the sequence of reduced rational solutions of (9.1) ordered such that $1 \le y_1 < y_2 < \dots$ Then either the sequence $(x_j/y_j)_{j\ge1}$ is finite or

$$\limsup_{j\to+\infty} \frac{\log_m y_{j+1}}{\log_m y_j} = +\infty.$$

Theorem 9.1 improves upon Mahler's result, which deals only with the case $m = 1$ and involves the very slowly decreasing function $y \mapsto (\log_3 y)^{-1/2}$.

Theorem 9.1 can be compared with Theorem 2 from [9] that deals with products of linear forms and involves a function ε that depends on the cardinality of $S_1 \cup S_2$. Note that Theorem 6.5.10 from Chapter 6 of the monograph of Bombieri and Gubler [6], given without proof, deals also with products of linear forms, but the function ε occurring there does not involve the cardinality of $S_1 \cup S_2$.

We can then proceed exactly as Mahler did ([17], Theorem 3, page 178) to construct new explicit examples of transcendental numbers.

Theorem 9.2. *Let $b \geq 2$ be an integer. Let θ be a real number with $0 < \theta < 1$. Let $\mathbf{n} = (n_j)_{j \geq 1}$ be an increasing sequence of positive integers satisfying $n_1 \geq 3$ and*

$$n_{j+1} \geq \left(1 + \frac{\log \log n_j}{(\log n_j)^{1/3}}\right) n_j, \quad (j \geq 1).$$

Let $(a_j)_{j \geq 1}$ be a sequence of positive integers prime to b such that

$$a_{j+1} \leq b^{\theta(n_{j+1} - n_j)}, \quad j \geq 1.$$

Then the real number

$$\xi = \sum_{j \geq 1} a_j \, b^{-n_j}$$

is transcendental.

We omit the proof of Theorem 9.2, which follows from Theorem 9.1 with $m = 1$.

It is of interest to note that Theorem 9.2 yields Corollary 3.2 only for $\eta > 3/4$. We would have obtained the same result by taking $k = 1$ in (8.10). It is precisely the introduction of the parameter k there that allows us to get in Theorem 3.1 the exponent of $(\log n)$ equal to $3/2$ and not to $4/3$.

APPENDIX

A quantitative two-dimensional Parametric Subspace Theorem

We give a proof of the two-dimensional case of Proposition 4.1. We keep the notation and assumptions from Section 4, except that we assume $n = 2$. As before, $\mathbf{K}$ is an algebraic number field. We recall the notation from Section 4, but now specialized to $n = 2$. Thus, $\mathcal{L} = (L_{iv} : v \in M_{\mathbf{K}}, i = 1, 2)$ is a tuple of linear forms satisfying

$$L_{iv} \in \mathbf{K}[X_1, X_2] \text{ for } v \in M_{\mathbf{K}}, i = 1, 2, \tag{A.1}$$

$$L_{1v} = X_1,\ L_{2v} = X_2 \text{ for all but finitely many } v \in M_{\mathbf{K}}, \tag{A.2}$$

$$\det(L_{1v}, L_{2v}) = 1 \text{ for } v \in M_{\mathbf{K}}, \tag{A.3}$$

$$\mathrm{Card}\Big(\bigcup_{v \in M_{\mathbf{K}}} \{L_{1v}, L_{2v}\} \Big) \leq r \tag{A.4}$$

and $\mathbf{c} = (c_{iv} : v \in M_{\mathbf{K}}, i = 1, 2)$ is a tuple of reals satisfying

$$c_{1v} = c_{2v} = 0 \text{ for all but finitely many } v \in M_{\mathbf{K}}, \tag{A.5}$$

$$\sum_{v \in M_{\mathbf{K}}} \sum_{i=1}^{2} c_{iv} = 0, \tag{A.6}$$

$$\sum_{v \in M_{\mathbf{K}}} \max(c_{1v}, c_{2v}) \leq 1. \tag{A.7}$$

We define

$$\mathcal{H} = \mathcal{H}(\mathcal{L}) := \prod_{v \in M_{\mathbf{K}}} \max_{1 \le i_1 < i_2 \le s} \|\det(L_{i_1}, L_{i_2})\|_v \tag{A.8}$$

where we have written $\{L_1, \ldots, L_s\}$ for $\bigcup_{v \in M_{\mathbf{K}}} \{L_{1v}, L_{2v}\}$. Finally, for any finite extension $\mathbf{E}$ of $\mathbf{K}$ and any place $w \in M_{\mathbf{E}}$ we define

$$L_{iw} = L_{iv},\ c_{iw} = d(w|v) c_{iv} \quad \text{for } i = 1, 2, \tag{A.9}$$

where v is the place of $M_{\mathbf{K}}$ lying below w and $d(w|v)$ is given by (4.2).

The twisted height $H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x})$ of $\mathbf{x} \in \overline{\mathbf{Q}}^2$ is defined by taking any finite extension $\mathbf{E}$ of $\mathbf{K}$ such that $\mathbf{x} \in \mathbf{E}^2$ and putting

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) := \prod_{w \in M_{\mathbf{E}}} \max_{i=1,2} \|L_{iw}(\mathbf{x})\|_w Q^{-c_{iw}}; \tag{A.10}$$

this does not depend on the choice of $\mathbf{E}$.

Proposition A.1. *Let $\mathcal{L} = (L_{iv} : v \in M_{\mathbf{K}},\ i = 1, 2)$ be a tuple of linear forms and $\mathbf{c} = (c_{iv} : v \in M_{\mathbf{K}},\ i = 1, 2)$ a tuple of reals satisfying (A.1)–(A.7). Further, let $0 < \delta \le 1$. Then there are one-dimensional linear subspaces $T_1, \ldots, T_{t_2}$ of $\overline{\mathbf{Q}}^2$, all defined over $\mathbf{K}$, with*

$$t_2 = t_2(r, \delta) = 2^{25} \delta^{-3} \log(2r) \log\left(\delta^{-1} \log(2r)\right) \tag{A.11}$$

such that the following holds: for every real Q with

$$Q > \max\left(\mathcal{H}^{\frac{2}{r(r-1)}}, 4^{1/\delta}\right) \tag{A.12}$$

there is a subspace $T_i \in \{T_1, \ldots, T_{t_2}\}$ such that

$$\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \le Q^{-\delta}\} \subset T_i\,. \tag{A.13}$$

The proof of Proposition A.1 is by combining some lemmata from [14], specialized to $n = 2$. We keep the notation and assumptions from above. By condition (A.4), there exists a 'family' (unordered sequence with possibly repetitions) of linear forms $\{L_1, \ldots, L_r\}$ such that L_{1v}, L_{2v} belong to this family for every $v \in M_{\mathbf{K}}$ and such that $L_1 = X_1$, $L_2 = X_2$. Now conditions (A.1)–(A.7) imply the conditions (5.12)–(5.17) on p. 36 of [14] with $n = 2$. These conditions are kept throughout [14] and so all arguments of [14] from p. 36 onwards are applicable in our situation. Since in what follows the tuples $\mathcal{L}$ and $\mathbf{c}$ will be fixed and only Q will vary, we will write H_Q for the twisted height $H_{Q,\mathcal{L},\mathbf{c}}$.

Let Q be a real with $Q \ge 1$. We define the "successive infima" $\lambda_1(Q)$, $\lambda_2(Q)$ of H_Q as follows: for $i = 1, 2$, $\lambda_i(Q)$ is the infimum of all reals $\lambda > 0$ such that $\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_Q(\mathbf{x}) \le \lambda\}$ contains at least i linearly independent points. Since we are working on the algebraic closure of $\mathbf{Q}$ and not on a given number field, these infima need not be assumed by H_Q.

In [14] (specialized to $n = 2$), $\lambda_1(Q)$, $\lambda_2(Q)$ were defined to be the successive infima of some sort of parallelepiped $\Pi(Q, \mathbf{c})$ defined over $\overline{\mathbf{Q}}$, and the lemmata in that paper were all formulated in terms of these infima. However, according to [14], Corollary 7.4, p. 53, applied with $n = 2$ and $\mathbf{A} = (Q^{c_{iv}},\ v \in M_{\mathbf{K}},\ i = 1, 2)$, the successive infima of $\Pi(Q, \mathbf{c})$ are equal to the successive infima of H_Q as defined above.

Lemma A.2. *Let $\delta > 0$, $Q \geq 1$.*

(i) $\frac{1}{2} \leq \lambda_1(Q)\lambda_2(Q) \leq 2$.

(ii) If there exists a non-zero $\mathbf{x} \in \overline{\mathbf{Q}}^2$ *with* $H_Q(\mathbf{x}) \leq Q^{-\delta}$ *then* $\lambda_1(Q) \leq Q^{-\delta}$ *and* $\lambda_2(Q) \geq \frac{1}{2}Q^{\delta}$.

Proof. Assertion (i) follows from [14], Corollary 7.6, p. 54. Assertion (ii) is then obvious. □

Lemma A.3. *(Gap Principle). Let $\delta > 0$, and let Q_0 be a real with $Q_0 > 4^{1/\delta}$. Then there is a unique, one-dimensional linear subspace T of $\overline{\mathbf{Q}}^2$ with the following property: for every Q with*

$$Q_0 \leq Q < Q_0^{1+\delta/2}$$

we have $\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_Q(\mathbf{x}) \leq Q^{-\delta}\} \subset T$.

Proof. Let T be the linear subspace of $\overline{\mathbf{Q}}^2$ spanned by all $\mathbf{x}$ such that $H_{Q_0}(\mathbf{x}) \leq Q_0^{-\delta/2}$. If $T \neq (\mathbf{0})$ then by Lemma A.2 we have $\lambda_1(Q_0) \leq Q_0^{-\delta/2}$ and $\lambda_2(Q_0) \geq \frac{1}{2}Q_0^{\delta/2}$, which by our assumption on Q_0 is strictly larger than $\lambda_1(Q_0)$. Hence T has dimension at most 1. So it suffices to prove that if $\mathbf{x} \in \overline{\mathbf{Q}}^2$ and Q are such that $Q_0 \leq Q < Q_0^{1+\delta/2}$ and $H_Q(\mathbf{x}) \leq Q^{-\delta}$, then $H_{Q_0}(\mathbf{x}) \leq Q_0^{-\delta/2}$.

To prove this, choose a finite extension $\mathbf{E}$ of $\mathbf{K}$ such that $\mathbf{x} \in \mathbf{E}^2$. Notice that by (A.7), (A.9), (4.3) we have $u := \sum_{w \in M_{\mathbf{E}}} \max(c_{1w}, c_{2w}) \leq 1$. For $w \in M_{\mathbf{E}}$ we have

$$\begin{aligned}
&\max\Big(\frac{\|L_{1w}(\mathbf{x})\|_w}{Q_0^{c_{1w}}}, \frac{\|L_{2w}(\mathbf{x})\|_w}{Q_0^{c_{2w}}}\Big) \\
&\quad \leq \max\Big(\frac{\|L_{1w}(\mathbf{x})\|_w}{Q^{c_{1w}}}, \frac{\|L_{2w}(\mathbf{x})\|_w}{Q^{c_{2w}}}\Big) \cdot \Big(\frac{Q}{Q_0}\Big)^{\max(c_{1w}, c_{2w})}.
\end{aligned}$$

So

$$\begin{aligned}
H_{Q_0}(\mathbf{x}) &\leq H_Q(\mathbf{x})\Big(\frac{Q}{Q_0}\Big)^u \\
&\leq Q^{-\delta} \cdot \frac{Q}{Q_0} \leq Q_0^{-\delta} Q_0^{\delta/2} = Q_0^{-\delta/2}.
\end{aligned}$$

□

Lemma A.4. *Let $\delta > 0$ and let A, B be reals with $4^{1/\delta} < A < B$. Then there are one-dimensional linear subspaces $T_1, \ldots, T_{t_3}$ of $\overline{\mathbf{Q}}^2$, with*

$$t_3 \leq 1 + \frac{\log(\log B / \log A)}{\log(1 + \delta/2)}$$

such that for every Q with $A \leq Q < B$ there is $T_i \in \{T_1, \ldots, T_{t_3}\}$ with

$$\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_Q(\mathbf{x}) \leq Q^{-\delta}\} \subset T_i.$$

Proof. Let k be the smallest integer with $A^{(1+\delta/2)^k} \geq B$. Apply Lemma A.3 with $Q_0 = A^{(1+\delta/2)^i}$ for $i = 0, \ldots, k-1$. $\square$

We define the Euclidean height $H_2(\mathbf{x})$ for $\mathbf{x} = (x_1, \ldots, x_m) \in \overline{\mathbf{Q}}^m$ as follows. Choose any number field $\mathbf{E}$ such that $\mathbf{x} \in \mathbf{E}^m$, define

$$\|\mathbf{x}\|_{w,2} := \left\{ \Big(\sum_{i=1}^{m} |x_i|_w^2 \Big)^{1/2} \right\}^{\frac{[\mathbf{E}_w:\mathbf{R}]}{[\mathbf{E}:\mathbf{Q}]}} \text{ if } w \text{ is Archimedean,}$$

$$\|\mathbf{x}\|_{w,2} := \max(\|x_1\|_w, \ldots, \|x_m\|_w) \text{ if } w \text{ is non-Archimedean,}$$

and put

$$H_2(\mathbf{x}) := \prod_{w \in M_{\mathbf{E}}} \|\mathbf{x}\|_{w,2}.$$

This is independent of the choice of $\mathbf{E}$. For a polynomial P with coefficients in $\overline{\mathbf{Q}}$, define $H_2(P) := H_2(\mathbf{p})$, where $\mathbf{p}$ is a vector consisting of the coefficients of P.

Lemma A.5. *Let $\delta > 0$. Consider the set of reals Q such that*

$$\text{there is } \mathbf{x}_Q \in \overline{\mathbf{Q}}^2 \setminus \{\mathbf{0}\} \text{ with } H_Q(\mathbf{x}_Q) \leq Q^{-\delta}, \tag{A.14}$$

$$Q^{\delta} > (2\mathcal{H})^{6\binom{r}{2}}. \tag{A.15}$$

Then one of the following two alternatives is true:

(i) For all Q under consideration we have $H_2(\mathbf{x}_Q) > Q^{\delta/3\binom{r}{2}}$;

(ii) There is a single one-dimensional linear subspace T_0 of $\overline{\mathbf{Q}}^2$ such that for all Q under consideration we have $\mathbf{x}_Q \in T_0$.

Proof. This is [14], p.80, Lemma 12.4 with $n = 2$. Condition (A.14) and Lemma A.2 imply $\lambda_1(Q) \leq Q^{-\delta}$ which is condition (12.37) of Lemma 12.4 of [14] with $n = 2$. Further, the quantity R in that lemma is $\leq \binom{r}{2}$ (see [14], p.75, Lemma 12.1). $\square$

Let m be a positive integer and $\mathbf{r} = (r_1, \ldots, r_m)$ a tuple of positive integers. We say that a polynomial is multihomogeneous of degree $\mathbf{r}$ in the blocks of variables $\mathbf{X}_1 = (X_{11}, X_{12}), \ldots, \mathbf{X}_m = (X_{m1}, X_{m2})$ if it can be expressed as a linear combination of monomials

$$\prod_{h=1}^{m} \prod_{k=1}^{2} X_{hk}^{i_{hk}} \quad \text{with } i_{h1} + i_{h2} = r_h \text{ for } h = 1, \ldots, m.$$

(Below, h will always index the block). Given points $\mathbf{x}_h = (x_{h1}, x_{h2})$ $(h = 1, \ldots, m)$ and a polynomial P which is multihomogeneous in $\mathbf{X}_1, \ldots, \mathbf{X}_m$, we write $P(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ for the value obtained by substituting x_{hk} for X_{hk} $(h = 1, \ldots, m,\ k = 1, 2)$.

The *index* of a polynomial P multihomogeneous in $\mathbf{X}_1, \ldots, \mathbf{X}_m$ with respect to points $\mathbf{x}_1, \ldots, \mathbf{x}_m$ and to a tuple of positive integers $\mathbf{r} = (r_1, \ldots, r_m)$, denoted by

$$\mathrm{Ind}(P; \mathbf{r}; \mathbf{x}_1, \ldots, \mathbf{x}_m),$$

is defined to be the smallest real σ with the following property: there is a tuple of non-negative integers $\mathbf{i} = (i_{hk} : h = 1, \dots, m,\ k = 1, 2)$ such that

$$\left(\prod_{h=1}^{m} \prod_{k=1}^{2} \left(\frac{\partial}{\partial X_{hk}} \right)^{i_{hk}} P \right)(\mathbf{x}_1, \dots, \mathbf{x}_m) \neq 0;$$
$$\sum_{h=1}^{m} \frac{i_{h1} + i_{h2}}{r_h} = \sigma \,.$$

For a field $\mathbf{F}$ and a tuple of positive integers $\mathbf{r} = (r_1, \dots, r_m)$, We denote by $\mathbf{F}[\mathbf{r}]$ the set of polynomials with coefficients in $\mathbf{F}$ which are multihomogeneous of degree $\mathbf{r}$ in $\mathbf{X}_1, \dots, \mathbf{X}_m$.

We define the constant $C(\mathbf{K}) := |D_{\mathbf{K}}|^{1/[\mathbf{K}:\mathbf{Q}]}$, where $D_{\mathbf{K}}$ denotes the discriminant of $\mathbf{K}$. In fact, the precise value of $C(\mathbf{K})$ is not of importance.

Lemma A.6. *Suppose that $0 < \delta \leq 1$, let θ be a real with*

$$0 < \theta \leq \frac{\delta}{80}, \tag{A.16}$$

m an integer with

$$m > 4\theta^{-2} \log(2r) \tag{A.17}$$

and $\mathbf{r} = (r_1, \dots, r_m)$ a tuple of positive integers, and put $q := r_1 + \cdots + r_m$.
Suppose that there exist positive reals $Q_1, \dots, Q_m$ and non-zero points $\mathbf{x}_1, \dots, \mathbf{x}_m$ in $\overline{\mathbf{Q}}^2$ such that

$$r_1 \log Q_1 \leq r_h \log Q_h \leq (1+\theta) r_1 \log Q_1 \quad (h = 1, \dots, m), \tag{A.18}$$
$$H_{Q_h}(\mathbf{x}_h) \leq Q_h^{-\delta} \quad (h = 1, \dots, m), \tag{A.19}$$
$$Q_h^{\delta} > C(\mathbf{K})^{5/4q} \cdot 2^{50} \mathcal{H}^5 \theta^{-5/2}. \tag{A.20}$$

Then there is a non-zero polynomial $P \in K[\mathbf{r}]$ such that

$$\mathrm{Ind}(P; \mathbf{r}; \mathbf{x}_1, \dots, \mathbf{x}_m) \geq m\theta, \tag{A.21}$$
$$H_2(P) \leq C(\mathbf{K})^{1/2} \cdot 2^{3m} (12\mathcal{H})^q. \tag{A.22}$$

Proof. This is [14], Lemma 15.1, p. 89, with $n = 2$. The space $V_{[h]}(Q_h)$ in that lemma is in our situation precisely the space spanned by $\mathbf{x}_h$ for $h = 1, \dots, m$. Inequality (A.17) comes from (14.7) on [14], p. 83; later it is assumed that $s = r$ (see (14.10) on [14], p.85). Inequality (A.22) comes from the inequality at the bottom of p. 87 of [14]. The construction of the polynomial P is by means of a now standard argument, based on the Bombieri–Vaaler Siegel's Lemma. □

Lemma A.7. *(Roth's Lemma) Let $0 < \theta \leq 1$. Let m be an integer with $m \geq 2$ and $\mathbf{r} = (r_1, \dots, r_m)$ a tuple of positive integers such that*

$$\frac{r_h}{r_{h+1}} \geq \frac{2m^2}{\theta} \quad (h = 1, \dots, m-1). \tag{A.23}$$

Further, let P be a non-zero polynomial in $\overline{\mathbf{Q}}[\mathbf{r}]$ and $\mathbf{x}_1, \dots, \mathbf{x}_m$ non-zero points in $\overline{\mathbf{Q}}^2$ such that

$$H_2(\mathbf{x}_h)^{r_h} \geq \big(e^q H_2(P)\big)^{(3m^2/\theta)^m} \quad (h = 1, \dots, m) \tag{A.24}$$

where $e = 2.7182\dots$, $q = r_1 + \dots + r_m$. Then

$$\mathrm{Ind}(P; \mathbf{r}; \mathbf{x}_1, \dots, \mathbf{x}_m) < m\theta\,. \tag{A.25}$$

Proof. This is the case $n = 2$ of [13], Lemma 24. It is an immediate consequence of [12], Theorem 3. □

We keep our assumption $0 < \delta \leq 1$ and define the integer

$$m := 1 + [25600 \cdot \delta^{-2} \log(2r)]. \tag{A.26}$$

Put

$$C := (36\mathcal{H})^{m\cdot(240m^2/\delta)^m \cdot 3\binom{r}{2}/\delta}. \tag{A.27}$$

Denote by $\mathcal{S}$ the set of reals Q such that

$$Q \geq C; \quad \text{there is } \mathbf{x} \in \overline{\mathbf{Q}}^2 \setminus \{\mathbf{0}\} \text{ with } H_Q(\mathbf{x}) \leq Q^{-\delta}. \tag{A.28}$$

Lemma A.8. *One of the following two alternatives is true:*

(i) There is a single, one-dimensional linear subspace T_0 of $\overline{\mathbf{Q}}^2$ such that for every $Q \in \mathcal{S}$ we have $\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_Q(\mathbf{x}) \leq Q^{-\delta}\} \subset T_0$;

(ii) There are reals $Q_1, \dots, Q_{m-1}$ with $C \leq Q_1 < \dots < Q_{m-1}$ such that

$$\mathcal{S} \subset \bigcup_{h=1}^{m-1} [Q_h, Q_h^{162m^2/\delta}]. \tag{A.29}$$

Proof. We assume that neither of the alternatives (i) or (ii) is true. From this assumption, we will deduce that there are a tuple of positive integers $\mathbf{r} = (r_1, \dots, r_m)$, a non-zero polynomial $P \in \mathbf{K}[\mathbf{r}]$, and non-zero points $\mathbf{x}_1, \dots, \mathbf{x}_m \in \overline{\mathbf{Q}}^2$, satisfying both (A.21) and (A.25). This is obviously impossible.

By our assumption, there are reals $Q_1, \dots, Q_m \in \mathcal{S}$ with

$$\frac{\log Q_{h+1}}{\log Q_h} \geq \frac{162m^2}{\delta} \quad (h = 1, \dots, m-1), \tag{A.30}$$

and non-zero points $\mathbf{x}_1, \dots, \mathbf{x}_m \in \overline{\mathbf{Q}}^2$ with

$$H_{Q_h}(\mathbf{x}_h) \leq Q_h^{-\delta} \quad (h = 1, \dots, m). \tag{A.31}$$

Put $\theta := \delta/80$. First choose a positive integer s_1 such that $\theta s_1 \log Q_1 > \log Q_h$ for $h = 2, \ldots, m$. Then there are integers $s_2, \ldots, s_m$ such that

$$s_1 \log Q_1 \leq s_h \log Q_h \leq (1+\theta) s_1 \log Q_1 \quad (h = 1, \ldots, m).$$

Now take $r_h := t s_h$ $(h = 1, \ldots, m)$, $\mathbf{r} = (r_1, \ldots, r_m)$, where t is a positive integer, chosen large enough such that the right-hand side of (A.20) is smaller than C^δ and the right-hand side of (A.22) is smaller than $(13\mathcal{H})^q$, where $q = r_1 + \cdots + r_m$. Then conditions (A.18)–(A.20) of Lemma A.6 are satisfied, hence there exists a non-zero polynomial $P \in \mathbf{K}[\mathbf{r}]$ such that (A.21), (A.22) are satisfied. So we have in fact

$$H_2(P) \leq (13\mathcal{H})^q. \tag{A.32}$$

We now show that P, $\mathbf{r}$, $\mathbf{x}_1, \ldots, \mathbf{x}_m$ satisfy conditions (A.23), (A.24) of Lemma A.7. Then it follows that (A.25) holds, and we arrive at the contradiction we wanted.

In view of (A.30), (A.18) and $\theta = \frac{\delta}{80} \leq \frac{1}{80}$ we have

$$\begin{aligned}\frac{r_h}{r_{h+1}} &= \frac{r_h \log Q_h}{r_{h+1} \log Q_{h+1}} \cdot \frac{\log Q_{h+1}}{\log Q_h} \\ &\geq \frac{1}{1+\theta} \cdot \frac{162m^2}{\delta} \geq \frac{160m^2}{\delta} = \frac{2m^2}{\theta}\end{aligned}$$

for $h = 1, \ldots, m-1$, which is (A.23).

Our reals $Q \in \mathcal{S}$ satisfy conditions (A.14), (A.15) of Lemma A.5. Since we assumed that alternative (i) of Lemma A.8 is false, alternative (ii) of Lemma A.5 must be false. So alternative (i) of that lemma must be true. This implies in particular, that

$$H_2(\mathbf{x}_h) > Q_h^{\delta/2\binom{r}{2}} \quad (h = 1, \ldots, m).$$

By combining this with (A.18), (A.32), this implies

$$\begin{aligned}H_2(\mathbf{x}_h)^{r_h} &\geq (Q_h^{r_h})^{\delta/3\binom{r}{2}} \geq (Q_1^{r_1})^{\delta/3\binom{r}{2}} \geq C^{r_1 \delta/3\binom{r}{2}} \\ &\geq (36\mathcal{H})^{m r_1 (3m^2/\theta)^m} \geq (e^q H_2(P))^{(3m^2/\theta)^m}\end{aligned}$$

for $h = 1, \ldots, m$, which is (A.25). This completes our proof. □

Proof of Proposition A.1. First suppose that alternative (ii) of Lemma A.8 is true. By applying Lemma A.4 with $A = Q_h$, $B = Q_h^{162m^2/\delta}$ for $h = 1, \ldots, m-1$ we conclude the following:

There are one-dimensional linear subspaces $T_1, \ldots, T_{t_4}$ of $\overline{\mathbf{Q}}^2$, with

$$t_4 \leq (m-1)\Big\{1 + \frac{\log(162m^2/\delta)}{\log(1+\delta/2)}\Big\} \leq 5\delta^{-1} m \log(162m^2/\delta)$$

such that for every Q with

$$Q \geq C := (36\mathcal{H})^{(240m^2/\delta)^m \cdot 3m\binom{r}{2}/\delta}$$

there is $T_i \in \{T_1, \ldots, T_{t_4}\}$ with $\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_Q(\mathbf{x}) \leq Q^{-\delta}\} \subset T_i$. This holds true trivially also if alternative (i) of Lemma A.8 is true; so it holds true in all cases.

It remains to consider those values Q with

$$\max\left(\mathcal{H}^{1/\binom{r}{2}}, 4^{1/\delta}\right) =: C' < Q < C. \tag{A.33}$$

Notice that $C' \geq (36\mathcal{H})^{1/12\binom{r}{2}}$. Hence by Lemma A.4, there are one-dimensional linear subspaces $T'_1, \ldots, T'_{t_5}$ of $\overline{\mathbf{Q}}^2$, with

$$\begin{aligned} t_5 &\leq 1 + \frac{\log(\log C/\log C')}{\log(1+\delta/2)} \\ &\leq 5\delta^{-1}\Big(m\log(240m^2/\delta)\Big) + \log\left(3m\binom{r}{2}\right)\Big) + \log\left(12\binom{r}{2}\right)\Big) \\ &\leq 6\delta^{-1}m\log(240m^2/\delta) \end{aligned}$$

such that for every Q with (A.33) there is $T'_i \in \{T'_1, \ldots, T'_{t_5}\}$ with $\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_Q(\mathbf{x}) \leq Q^{-\delta}\} \subset T'_i$.

Collecting the above, we get that there are one-dimensional linear subspaces $T_1, \ldots, T_{t_2}$ of $\overline{\mathbf{Q}}^2$, with

$$t_2 \leq t_4 + t_5 \leq 11\delta^{-1}m\log(240m^2/\delta) \leq 33\delta^{-1}m\log m$$

such that for every $Q > C'$ there is $T_i \in \{T_1, \ldots, T_{t_2}\}$ with $\{\mathbf{x} \in \overline{\mathbf{Q}}^2 : H_Q(\mathbf{x}) \leq Q^{-\delta}\} \subset T_i$. Substituting (A.26) for m we obtain

$$\begin{aligned} t_2 &\leq 33\delta^{-1} \cdot 25601\delta^{-2}\log(2r)\log(25601\delta^{-2}\log(2r)) \\ &< 2^{25}\delta^{-3}\log(2r)\log\left(\delta^{-1}\log(2r)\right) \end{aligned}$$

which is the right-hand side of (A.11).

To finish the proof of Proposition A.1, it remains to show that the spaces $T_1, \ldots, T_{t_2}$ are defined over $\mathbf{K}$. Let Q be any real ≥ 1. Suppose that there are non-zero vectors $\mathbf{x} \in \overline{\mathbf{Q}}^2$ with $H_Q(\mathbf{x}) \leq Q^{-\delta}$, and that these vectors span a one-dimensional linear subspace T of $\overline{\mathbf{Q}}^2$. According to [14], Lemma 4.1, p.32, we have for any K-automorphism σ of $\overline{\mathbf{Q}}$ that $H_Q(\sigma(\mathbf{x})) = H_Q(\mathbf{x})$, where $\sigma(\mathbf{x})$ is obtained by applying σ to the coordinates of $\mathbf{x}$; hence $\sigma(\mathbf{x}) \in T$. This implies that T is defined over $\mathbf{K}$. □

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