Global existence of positive mild solutions for a class of kinetic chemotaxis models

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Abstract

We consider mild solutions of a particular class of kinetic chemotaxis models in general physical dimensions. The models consist of a hyperbolic transport equation that is non-linearly and non-locally coupled to a linear reaction-diffusion equation through kernel operators. Under certain conditions on these kernel operators, we prove global existence of positive mild solutions, generalizing results by Hwang, Kang and Stevens [Siam J. Math. Anal. **36** , No. 4, 1177-1199]. Our approach uses the standard bootstrapping argument with a priori estimates, but keeps track comprehensively of all norms of the solution that can be shown to remain bounded on its maximal interval of existence using only the available estimates. We believe that this approach is useful when more complex coupled systems of mixed type are considered.

1 Introduction

Chemotaxis is a process in biology in which organisms, like bacteria, amoebae or leukocytes, adapt their movement behaviour to external chemical signals that they sense in their environment. The interplay of signal detection, movement and signal relay results in interesting collective behaviour like movement towards nutrients, away from toxic compounds or patterns in cell density (aggregation). These processes have attracted much attention over the past decades and various types of mathematical models have been devised to cover aspects of the dynamics at different space and time scales (e.g. [15, 16, 17, 18, 6, 20, 21, 22]).

At microscopic level for example, stochastic 'velocity jump' models have been considered (e.g. [2, 20, 22]), which closely represent the 'run-and-tumble' type of movement behaviour has been observed for flagellated bacteria like *E. coli* [3, 4]. A mesoscopic description of the dynamics of a population of (non-interacting) chemotactically moving cells in $\mathbb{R}^n$ with a signal $S = S(x,t)$ present at time t, which represents the concentration of a chemotactic chemical compound, is given by the evolution of the position-velocity distribution $f = f(x,v,t)$ of cells at position $x \in \mathbb{R}^n$ with velocity $v \in V$ at time t according to

$$\partial_t f(t) = -v \cdot \nabla_x f(t) + T(S(t), f(t)) \tag{1}$$

(e.g. [6, 10, 20]). The *turning operator* $T(S(t), f(t))$ depends on the *function* S and is often taken linear in f, of the form (suppressing time dependence):

$$T(S,f)(x,v) = -\lambda[S](x,v) \cdot f(x,v) + \mathcal{T}[S]f(x,v), \tag{2}$$

where the function $\lambda[S]$ is the *turning rate* and is given by

$$\lambda[S](x,v) := \int_V T[S](x,v',v)d\mu(v'), \tag{3}$$

and $\mathcal{T}[S]$ is a kernel operator defined by

$$\mathcal{T}[S]f(x,v) := \int_V T[S](x,v,v')f(x,v')d\mu(v'). \tag{4}$$

Equations (1)–(4) constitute a so-called *kinetic model for chemotaxis.* Note that it is a linear Boltzmann equation from kinetic theory. We assume that the velocity space V is a compact subset of $\mathbb{R}^n$ (reflecting a maximum speed) that in general carries a Radon measure μ. We shall consider here the case where V as subset of $\mathbb{R}^n$ has non-zero Lebesgue measure and μ is the restriction of Lebesgue measure to V. In the sequel we omit μ from notation and simply write dv.

$T[S](x,v,v')$ is called the *turning kernel.* It represents the probability of changing velocity from v' to v at position x, given the global signal S at that time. The biological peculiarities of the interplay between external chemotactic signal and the organism's movement behaviour are encoded in a particular functional dependence of the turning kernel on the global signal S. Various functional forms have been considered, see [6, 7, 21, 17].

Macroscopic equations for the cell density

$$\rho(x,t) := \int_V f(x,v,t)\,dv,$$

have been derived both formally and rigorously using singular perturbation techniques. These yield appropriate reaction-diffusion-advection equations of Patlak-Keller-Segal type (e.g. [6, 7, 21]) or mixed hyperbolic-parabolic models, e.g. the Cataneo system coupled to a signal equation [8, 10, 14].

1.1 Scope and layout of the paper

In this paper we extend global existence results from [6, 17] for mild solutions to the initial value problem for the system consisting of the kinetic chemotaxis equation (1)–(4) for f, coupled to a *linear* reaction-diffusion equation for the signal S:

$$\tau\partial_t S = D\Delta S + \alpha\rho - \beta S, \qquad (\alpha, \beta \geq 0,\ \tau, D > 0), \tag{5}$$

where ρ is the spatial cell distribution. This linear equation in combination with (1) has been studied most so far [11, 13, 15, 16, 17], also in the context of Keller-Segel type of chemotaxis models. In various studies of the kinetic chemotaxis model, (5) has been replaced by its quasi-steady state approximation with $\beta = 0$ or $\beta > 0$ [6, 17]:

$$-\Delta S = \alpha\rho - \beta S, \qquad (\alpha > 0).$$

In this paper we will focus on the fully dynamic system given by (1)-(5) however.

Crucial assumption in [6] and [17] are estimates on the turning kernel of the form

$$\begin{aligned} 0 \leq T[S](x,v,v') \leq C_1 + &C_2 S(x+\epsilon v,t) + C_3 S(x-\epsilon v',t) \\ &+ C_4|\nabla S(x+\epsilon v,t)| + C_5|\nabla S(x-\epsilon v',t)|, \end{aligned} \tag{6}$$

for any $S(\cdot,t) \in W^{1,\infty}(\mathbb{R}^n)$. In this paper (among others) we extend their result to arbitrary physical dimensions (which is of course mainly of mathematical interest) and by allowing that each of the last four terms in (6) to be taken to a power $\nu \geq 1$, possibly different for each term, though not too large (see Assumption *(AT1)* and *(AT2)* below for a precise formulation). Moreover, we allow for terms of the form $S(x - \epsilon v - \epsilon' v', t)$ in our fundamental estimate of the turning kernel. That is, simultaneous shifts by v and v' are possible. [6, 17] claim that their methods of proof unfortunately fail in that generality. However, we will show that the main ideas of their proof can still be employed in that setting by taking a well-suited functional analytic viewpoint. Moreover, the bootstrap argument used in [17] essentially also works when the exponents $\nu > 1$ are added. The technical problem is, that the closer the exponents approach a critical maximal value, the more (but finite) steps are required to reach an a priori estimate for $S(t)$ in $W^{k,\infty}(\mathbb{R}^n)$, which is needed to conclude global existence. In case that all exponents equal 1, only two steps are needed.

We believe that our view on this problem of 'bookkeeping' of availability and deduceability of various uniform upper bounds for norms of the soltions $f(t)$ and $S(t)$ on their maximal interval of existence is of separate interest. In this view the problem of proving global existence focusses on the derivation of properties of two set-valued sequences. Geometric observations may guide the road to proof. In this process we implicitly use the (in this case simple) coupling structure of the system. In future we will examine this approach for proving global existence also for coupled systems of mixed type with more complex coupling networks.

In Section 2.1 we summarise the functional analytic framework that has been set-up in [11] to define mild solutions and prove their local existence, uniqueness and continuous dependence on parameters. In Section 2.2 we state our main global existence results, after which we devote the full subsequent section to their technical proof (Section 3). Our approach to obtain the necessary a priori estimates is inspired by [17]. The use of our functional analytic framework allowed us to derive the more general new global existence results of Section 2.2. Section 3.2.4 concludes the exposition by showing that the results from [17] in the two-dimensional case can be uobtained and understood in our framework as well, using a comparable argument.

1.2 Notational conventions

$\mathbb{N} = \{1, 2, 3, ...\}$ is the set of positive integers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any set S, $\mathcal{P}(S)$ denotes the powerset of S. If (Ω, Σ, μ) is a measure space, we denote by $L^0(\Omega)$ the vector space of equivalence classes of measurable functions on

Ω, where the equivalence is equality almost everywhere with respect to μ. For $1 \le p \le \infty$, p^* will denote the exponent conjugate to p, i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$.

We denote by $W^{k,p}(\mathbb{R}^n)$ the usual Sobolev space of L^p-functions whose distributional derivatives up to order k are also L^p-functions. Here $\mathbb{R}^n$ carries Lebesgue measure m. We use multi-index notation for partial differentiation on $\mathbb{R}^n$, i.e., $\partial_j := \frac{\partial}{\partial x_j}$ and $D^\mu := \prod_{j=1}^n \partial_j^{\mu_j}$ for $\mu = (\mu_1, ..., \mu_n) \in \mathbb{N}_0^n, |\mu| := \sum_{j=1}^n |\mu_j|$. $C_0(\mathbb{R}^n)$ is the Banach space of continuous functions on $\mathbb{R}^n$ that vanish at infinity. We denote by $C_0^k(\mathbb{R}^n)$ the space of functions f on $\mathbb{R}^n$ for which $D^\mu f$ is in $C_0(\mathbb{R}^n)$ for all μ with $|\mu| \le k$. When $f \in L^2(\mathbb{R}^n)$, we denote by $\hat{f}$ the Fourier transform of f, i.e., $\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$. When $f, g \in L^0(\mathbb{R}^n)$, we denote by $f * g$ the convolution of f and g, i.e., $f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y)\, dy$, for all x such that the integral exists.

2 Set-up and main results

We first define the functional analytic framework, following [11], and summarise the available results on local well-posedness for the corresponding mild solutions, when viewing (1)-(5) as a system of abstract equations in suitable Banach spaces, using semigroup theory. Then we shall formulate the main results on global existence in this framework and comment on them.

2.1 Functional analytic set-up

We define a linear semigroup $(T_\Phi(t))_{t\ge 0}$ on the space of functions on $\mathbb{R}^n \times V$ by means of

$$T_\Phi(t) f(x, v) := f(x - vt, v).$$

It induces a semigroup of linear maps on the vector space $L^0(\mathbb{R}^n \times V)$. Note that $(T_\Phi(t))_{t\ge 0}$ is actually a group. For all $f \in L^p(\mathbb{R}^n \times V)$ with $1 \le p \le \infty$ we have for all $t \ge 0$:

$$\|T_\Phi(t) f\|_p = \|f\|_p.$$

It is easy to show that the group $(T_\Phi(t))_{t\ge 0}$ is strongly continuous on $L^p(\mathbb{R}^n \times V)$ for $1 \le p < \infty$. Furthermore, the space of Schwarz functions $\mathcal{S}(\mathbb{R}^n \times V)$ is a core for the infinitesimal generator A of $(T_\Phi(t))_{t\ge 0}$ in these spaces and $A = -v \cdot \nabla_x$ in the sense of distributions.

We divide in (5) both sides by τ, which gives us

$$\partial_t S = d\Delta S + \tilde{\alpha}\rho - \tilde{\beta} S,$$

where $d := \frac{D}{\tau}$, $\tilde{\alpha} := \frac{\alpha}{\tau}$ and $\tilde{\beta} := \frac{\beta}{\tau}$. Let $(T_d(t))_{t\geq 0}$ be the heat semigroup associated to $d\Delta$. It may be realised by means of

$$T_d(t)f = h_d(.,t) * f, \quad \text{for } t > 0,$$

where the heat kernel h_d is given by

$$h_d(x,t) = (4\pi dt)^{-n/2} e^{-|x|^2/4dt}.$$

It is strongly continuous and non-expansive on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ as well as on $C_0(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ is a core for the infinitisemal generator A of $(T_d(t))_{t\geq 0}$ in these spaces and $A = d\Delta$ in the sense of distributions (see [9], Section II.2.12, p.69). Furthermore, for all f in $W^{k,p}(\mathbb{R}^n)$ or in $C_0^k(\mathbb{R}^n)$, with $k \in \mathbb{N}$, we have $D^\mu(T_d(t)f) = T_d(t)(D^\mu f)$, for all multi-indices μ, with $|\mu| \leq k$. Hence $(T_d(t))_{t\geq 0}$ leaves $W^{k,p}(\mathbb{R}^n)$ and $C_0^k(\mathbb{R}^n)$ invariant and the restriction of $(T_d(t))_{t\geq 0}$ to $W^{k,p}(\mathbb{R}^n)$ and $C_0^k(\mathbb{R}^n)$ is strongly continuous.

Let us define the following state spaces and mappings:

$$X := X_0 \oplus X_1, \qquad Y := Y_0 \oplus Y_1, \qquad T(t) := T_\Phi(t) \oplus T_d(t),$$
$$F(f \oplus S) := F_0(f \oplus S) \oplus F_1(f \oplus S),$$

where

$$X_0 = Y_0 = L^{q_0}(\mathbb{R}^n \times V) \cap L^{q_0'}(\mathbb{R}^n \times V),$$

and

$$X_1 := W^{k,q_1}(\mathbb{R}^n) \cap C_0^k(\mathbb{R}^n), \qquad Y_1 := L^{q_1}(\mathbb{R}^n) \cap L^{q_1'}(\mathbb{R}^n)$$

and

$$F_0(f \oplus S) := \mathcal{T}[S]f, \qquad F_1(f \oplus S) := \tilde{\alpha}\rho - \tilde{\beta}S.$$

We make the following assumption on the dependence of the turning kernel on the signal S:

(AT1) The map $S \to T[S]$ is defined on $W^{k,\infty}(\mathbb{R}^n)$, maps this space into $L^\infty(\mathbb{R}^n \times V \times V)$ and is locally Lipschitz continuous as a map from $W^{k,\infty}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n \times V \times V)$.

This assumption assures that the turning operator given by (2), (3) and (4) is a well-defined bounded linear operator on $L^p(\mathbb{R}^n \times V)$, for all $1 \leq p < \infty$ (see [11, Corollary 4.10]), and that F_0 is locally Lipschitz continuous.

We now write our model in terms of the Variation of Constants Formula in X as follows:

$$f \oplus S(t) = T(t)(f_0 \oplus S_0) + \int_0^t T(t-s)F(s, f \oplus S(s))\, ds, \tag{7}$$

with initial values $f(0) = f_0 \in X_0$ and $S(0) = S_0 \in X_1$. We call $f \oplus S$ a *mild solution* in X to (7) on $[0, T]$ if $f \oplus S : [0, T] \to X$ is continuous and satisfies (7) for any $t \in [0, T]$.

From [11, Theorem 4.13] we get local existence of mild solutions to (7) under certain conditions. Positivity of these solutions for positive initial conditions can be shown [12] under positivity conditions on $T[S]$. We summarise these results in the following theorem:

Theorem 2.1. *([11, Theorem 4.13])*
Let $k \in \{0, 1\}$ and let (q_i, q_i'), $(i = 0, 1)$ be such that

$$1 \leq q_0 \leq q_1 \leq q_1' \leq q_0' < \infty, \qquad q_i' > \tfrac{n}{2-k}, \ (i = 0, 1). \tag{8}$$

Let $X, Y, (T(t))_{t\geq 0}$ and F be defined as above, and assume that (AT1) holds. Then mild solutions to (7) exist local in time, are unique and depend (locally Lipschitz) continuously on initial data. Furthermore, assume that $T[S] \geq 0$ for all $S \geq 0$. Then any mild solution to (7) with positive initial values remains positive on its maximal interval of existence.

2.2 Conditions for global existence

In addition to *(AT1)* we make an assumption on the map $S \to T[S]$, which is more general than that made in [6, 17] (compare with (6)):

(AT2) The map $S \to T[S]$ satisfies for any $S \in W^{k,\infty}(\mathbb{R}^n)$, $S \geq 0$:

$$0 \leq T[S](x, v, v') \leq C + \sum_{\substack{|\mu|\leq k \\ 1\leq i\leq d}} C_{\mu,i} |D^\mu S(x - \epsilon_{\mu,i} v - \epsilon'_{\mu,i} v')|^{\nu_{\mu,i}},$$

for almost all $(x, v, v') \in \mathbb{R}^n \times V \times V$, where $\nu_{\mu,i} \geq 0$, $C_{\mu,i} \geq 0$, $C \geq 0$, $d \in \mathbb{N}$ and $\epsilon_{\mu,i}, \epsilon'_{\mu,i} \in \mathbb{R}$: $|\epsilon_{\mu,i}| + |\epsilon'_{\mu,i}| \neq 0$.

If this assumption is satisfied, we define for $0 \leq j \leq k$:

$$\nu_j := \max(\{\nu_{\mu,i} | \epsilon'_{\mu,i} = 0, \ C_{\mu,i} > 0, \ 1 \leq i \leq d, \ |\mu| = j\} \cup \{0\}), \tag{9}$$

$$\nu_j^* := \max(\{\nu_{\mu,i} | \epsilon'_{\mu,i} \neq 0, \ C_{\mu,i} > 0, \ 1 \leq i \leq d, \ |\mu| = j\} \cup \{0\}). \tag{10}$$

Our main result is the following:

Theorem 2.2. *Let $n \geq 1$, $k \in \{0, 1\}$ and let (q_i, q_i'), $(i = 0, 1)$, satisfy the conditions of Theorem 2.1 with the additional constraints, that*

$$q_0 = 1 \quad \textit{and} \quad 1 - \tfrac{2-k}{n} < \tfrac{1}{q_1} \leq 1. \tag{11}$$

Assume that the turning kernel satisfies (AT1). Then the following holds:

(i) For $n = 1$ and $k = 0$: if the turning kernel is positive, then any mild solution in X to (7) *with positive initial values remains positive and exists globally.*

(ii) For $\frac{2-k}{n} \leq 1$: if the turning kernel also satisfies (AT2), where ν_j and ν_j^*, defined by (9) *and* (10)*, satisfy*

$$\max_{0\leq j\leq k}(1 - \tfrac{2-j}{n})\nu_j \;+\; \max_{0\leq j\leq k}(1 - \tfrac{2-j}{n})\nu_j^* < 1 \tag{12}$$

$$\max_{0\leq j\leq k}(1 - \tfrac{2-j}{n})\nu_j^* < 1 - \tfrac{1}{q_0'}, \tag{13}$$

then any mild solution in X to (7) *with positive initial values remains positive and exists globally.*

Theorem 2.2 gives a conclusion on global existence in the case of $n = 2$, $k = 1$ only for $\nu_1 + \nu_1^* < 2$ (and ν_0, ν_0^* arbitrary, positive). In [17] the authors prove that (positive) mild solutions also exist globally if the turning kernel satisfies *(AT2)* with $\nu_1 = \nu_1^* = 1$. We extend their arguments in our framework, yielding the result:

Theorem 2.3. *Let $n = 2$, $k = 1$. Let (q_i, q_i'), $(i = 0, 1)$, satisfy the conditions of Theorem 2.1 and the additional conditions $q_0 = 1$ and $1 \leq q_1 < 2$. If the turning kernel satisfies (AT2) with $\nu_0, \nu_0^* \geq 0$ arbitrary and either*

$$\nu_1 + \nu_1^* < 2 \quad \textit{and} \quad \tfrac{1}{2}\nu_1^* < 1 - \tfrac{1}{q_0'}$$

or

$$\nu_1 + \nu_1^* = 2 \quad \textit{and} \quad 1 \leq \nu_1 < 2,$$

then the mild solution to (7) remains positive for any positive initial value and exists globally in X.

We shall prove the additional claim (i.e. when $\nu_1 + \nu_1^* = 2$) in Section 3.2.4.

2.3 Some preliminary remarks

Hillen and Stevens [13, Corollary 3.2] proved (among others) global existence of mild solutions in the case $n = 1$, under the assumption that the turning kernel is positive and essentially uniformly bounded, i.e., there exists an $M > 0$ such that $\|T[S]\|_\infty \leq M$ for all $S \in W^{k,\infty}(\mathbb{R})$. A similar conclusion can be drawn in general physical dimension n using Lemma 3.3 below.

Theorem 2.2 essentially says, that the exponents in *(AT2)* of zero and first-order derivatives must be balanced. For example, superlinear growth of $T[S]$ in terms of $S(x - \epsilon v)$ is allowed when growth in terms of $\nabla S(x - \epsilon v)$ is reduced. Note that some of the ν_j or ν_j^* may be (slightly) larger than 1. So in any physical space dimension $n \geq 1$ some of the $\nu_{\mu,j}$ may satisfy $\nu_{\mu,j} > 1$.

Same cases deserve particular attention. According to Theorem 2.2 *(i)*, if $n = 1$ and $k = 0$, then there is global existence of positive mild solutions for any positive turning kernel that satisfies *(AT1)*, In case $n = 1$, $k = 1$ or $n = 2$, $k = 0$, then $\frac{2-k}{n} = 1$. This implies that there are no constraints on ν_k and ν_k^* (apart from being nonnegative) for positive solutions to exist globally. A similar situation is encountered in Theorem 2.3. There, no stringent conditions on ν_0 or ν_0^* need to be imposed when $n = 2$, $k = 1$.

Theorem 2.2 generalises some of the results by Hwang, Kang and Stevens [17]. They make the assumption that the turning kernel satisfies (6) for all $S \in W^{1,\infty}(\mathbb{R}^n)$. Global existence is proven for $n = 2, k = 1$ in that generality, while for $n = 3, k = 1$ they assume that either $C_2 = C_4 = 0$ or $C_3 = C_5 = 0$ in (6) [17, Theorem 3.12, Remark 3.8].

In our approach, *both $\epsilon_{\mu,i}$ and $\epsilon'_{\mu,i}$ may be non-zero in the same term, and of arbitrary sign.* This extends results in [5, 6, 17]. Moreover, we prove global existence for mild solutions for (positive) initial conditions $f_0 \in L^1 \cap L^{q'_0}(\mathbb{R}^n \times V)$ for an unbounded range of finite q'_0, instead of the slightly more restictive $f_0 \in L^1 \cap L^\infty$. In view of [11], Theorem 6.3, the latter result is a direct corollary of the former.

3 Proofs

Let us fix (q_i, q'_i), $(i = 0, 1)$, that satsify the conditions of the local existence theorem, Theorem 2.1, and the additional constraints (11). We assume that the turning kernel satisfies *(AT1)* and *(AT2)*. Throughout this section we also fix initial data $f_0 \geq 0$ in X_0 and $S_0 \geq 0$ in X_1. Let $[0, \tau)$ be the maximal interval of existence of the corresponding mild solution in X. This solution has components $f(t)$ and $S(t)$ in X_0 and X_1 respectively.

If $\tau < \infty$, then necessarily

$$\limsup_{t \uparrow \tau} \|f(t) \oplus S(t)\|_X = +\infty.$$

We use the standard approach of using various a priori estimates in order to show that for finite τ both functions $t \mapsto \|f(t)\|_{X_0}$ and $t \mapsto \|S(t)\|_{X_1}$ must remain uniformly bounded on $[0, \tau)$. By contradiction it then follows that positive mild solutions in X exist globally (but may blow-up at infinity).

We use a non-standard approach in the presentation and handling of the various a priori estimates. Allthough we developed it when studying the particular problem that is the subject of this paper, we expect it to have value beyond, in the treatment of global existence of mild solutions for more complex coupled systems of mixed type. In fact, it allows to translate divers norm estimates on the solutions that come available through different mathematical techniques and various continuous embeddings of Banach spaces into one unifying framework that makes these results more easily accessible for reasoning.

Let us now present this framework and the proofs of the main results.

3.1 Framework

Let $\mathcal{N}_f$ and $\mathcal{N}_S$ be sets of norms on X_0 and X_1 respectively, such that for all $\|\cdot\| \in \mathcal{N}_f$ and $|\cdot| \in \mathcal{N}_S$ and all $t \in [0, \tau)$, $\|f(t)\|$ and $|S(t)|$ are finite. In our case, because of the continuous embeddings

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), \quad L^p \cap L^{p'}(\Omega, \mu) \hookrightarrow L^r(\Omega, \mu), \qquad p \leq r \leq p', \tag{14}$$

and the local existence result Theorem 2.1, we take

$$\begin{aligned} \mathcal{N}_f &= \{\, \|\cdot\|_{L^r} \,|\, 1 \leq r \leq q_0' \}, \\ \mathcal{N}_S &= \{\, \|\cdot\|_{W^{j,r}} |\, q_1 \leq r \leq \infty,\ 0 \leq j \leq k \}. \end{aligned}$$

We could identify $\mathcal{N}_f$ with $[1, q_0'] \subset \mathbb{R}$ and $\mathcal{N}_S$ with $\{0, k\} \times [q_1, \infty]$ (recall that $k = 0$ or 1), but it turns out to be more convenient to identify $\|\cdot\|_{L^r}$ with $\frac{1}{r}$. Accordingly,

$$\mathcal{N}_f \simeq [\tfrac{1}{q_0'}, 1], \qquad \pi \mapsto \|\cdot\|_{L^{1/\pi}(\mathbb{R}^n \times V)}, \tag{15}$$

$$\mathcal{N}_S \simeq \{0, k\} \times [0, \tfrac{1}{q_1}], \qquad (j, \rho) \mapsto \|\cdot\|_{W^{j,1/\rho}(\mathbb{R}^n)}. \tag{16}$$

In general the argument will start with a subset $\mathcal{N}_f^0$ of $\mathcal{N}_f$ (or $\mathcal{N}_S^0 \subset \mathcal{N}_S$, depending on the particular problem) of norms $\|\cdot\|$ for which $t \mapsto \|f(t)\|$ is known to be uniformly bounded on $[0, \tau)$ a priori. In our case, we have by Fubini's Theorem

$$\int_{\mathbb{R}^n \times V} -\lambda[S]f + \mathcal{T}[S]f \, dx \, dv = 0,$$

which implies that $\|f(t)\|_1 = \|f_0\|_1$, for $t \in [0, \tau)$ ('*Conservation of Mass*'), since $f(t) \geq 0$ for all t by Theorem 2.1. Thus we may take $\mathcal{N}_f^0 = \{1\}$.

Existence of various a priori estimates and embeddings of the form (14) are represented through *inference maps*

$$\Psi_{\alpha,\beta} : \mathcal{P}(\mathcal{N}_\alpha) \to \mathcal{P}(\mathcal{N}_\beta), \qquad \alpha, \beta \in \{f, S\}.$$

Their interpretation is as follows. If for example $A \subset \mathcal{N}_f$ is a set of norms $\|\cdot\|$ for which it has been shown that $t \mapsto \|f(t)\|$ is uniformly bounded on $[0, \tau)$, then $\Psi_{f,f}(A)$ represents a set of norms on f that also yield uniform bounds that can be derived from A. Similarly, $\Psi_{f,S}$ represents the set of norms $\|\cdot\|$ for which one can show that $t \mapsto \|S(t)\|$ is uniformly bounded on $[0, \tau)$, given uniform boundedness of the norms on f in A and the available a priori estimates.

The maps $\Psi_{\alpha,\beta}$ should satisfy:

(AΨ1) $\Psi_{\alpha,\beta}(\emptyset) = \emptyset$ *for all* α, β.

(AΨ2) *For any* α, $\Psi_{\alpha,\alpha}(A) \supset A$ *for all* $A \subset \mathcal{N}_\alpha$.

(AΨ3) *For any* α *and* β, $\Psi_{\alpha,\beta}$ *is increasing, i.e. if* $A, B \subset \mathcal{N}_\alpha$ *and* $A \subset B$, *then* $\Psi_{\alpha,\beta}(A) \subset \Psi_{\alpha,\beta}(B)$.

The third assumption represents that once uniform estimates have been proven with limited a priori knowledge, these results cannot become invalid when more a priori knowledge is available.

The continuous embeddings (14) are encoded into

$$\Psi_{f,f}(A) := \{\pi \in \mathcal{N}_f \,|\, \exists \pi_1, \pi_2 \in A : \ \pi_1 \leq \pi \leq \pi_2\}, \tag{17}$$

$$\Psi_{S,S}(B) := \{(j, \rho) \in \mathcal{N}_S \,|\, \exists (i_m, \rho_m) \in B : \ j = \min(i_1, i_2), \ \rho_1 \leq \rho \leq \rho_2\}, \tag{18}$$

defined for all $A \in \mathcal{P}(\mathcal{N}_f)$ and $B \in \mathcal{P}(\mathcal{N}_S)$ respectively. Note that $\Psi_{f,f}(A)$ is the smallest interval containing A. Clearly, $\Psi_{f,f}$ and $\Psi_{S,S}$ satisfy the assumptions (AΨ1)-(AΨ3).

In Section 3.2.1 and Section 3.2.3 we will prove the following a priori estimates:

Proposition 3.1. *Given* $1 \leq p \leq q_0'$ *and a uniform upper bound for* $t \mapsto \|f(t)\|_p$ *on* $[0, \tau)$, *then there exists a uniform upper bound for* $t \mapsto \|S(t)\|_{W^{j,r}}$ *on* $[0, \tau)$, *for all* j *and* r *satisfying*

$$0 \leq j \leq k, \quad \tfrac{1}{p} - \tfrac{2-j}{n} < \tfrac{1}{r} \leq \tfrac{1}{p} \quad \textit{and} \quad q_1 \leq r \leq \infty.$$

Proposition 3.2. *Suppose that the turning kernel satisfies (AT2). Given r_j such that $q_1 \le r_j < \infty$ and a uniform upper bound for $t \mapsto \|S(t)\|_{W^{j,r_j}}$ on $[0,\tau)$ for $0 \le j \le k$, then there exists a uniform upper bound for $t \mapsto \|f(t)\|_p$ on $[0,\tau)$, for all p that satisfy*

$$1 \le p \le q_0', \quad \textit{and} \quad 0 \le \nu_j p \le r_j,\ 0 \le \nu_j^* p^* \le r_j$$

for all $0 \le j \le k$.

These two sets of a priori estimates can be translated into the following definitions for $\Psi_{f,S}$ on singletons:

$$\Psi_{f,S}\big(\pi\big) := \{(j,\rho) \in \mathcal{N}_S |\ \pi - \tfrac{2-j}{n} < \rho \le \pi\} \tag{19}$$

and for $\Psi_{S,f}$, for $(j,\rho_j) \in \mathcal{N}_S$, $0 \le j \le k$:

$$\Psi_{S,f}\big(\{(0,\rho_0),(k,\rho_k)\}\big) := \{\pi \in \mathcal{N}_f |\ \nu_j\rho_j \le \pi \le 1 - \nu_j^*\rho_j,\ 0 \le j \le k\}$$

We extend these maps to subsets by means of

$$\Psi_{\alpha,\beta}(A) := \bigcup_{S \subset A} \Psi_{\alpha,\beta}(S),$$

where the union extends over subsets S that are either singletons (for $\Psi_{f,S}$) or sets of the form $\{(0,\rho_0),(k,\rho_k)\}$ (for $\Psi_{S,f}$). The resulting $\Psi_{\alpha,\beta}$ satisfy (AΨ1)-(AΨ3).

Given the initial set $\mathcal{N}_f^0$ of norms with uniform bound on f on $[0,\tau)$ and the maps $\Psi_{\alpha,\beta}$ we define iteratively an increasing sequences of sets $\mathcal{N}_f^i$ and $\mathcal{N}_S^i$ (with $\mathcal{N}_S^0 := \emptyset$):

$$\mathcal{N}_S^{i+1} := \Psi_{S,S}\big(\mathcal{N}_S^i \ \cup\ \Psi_{f,S}(\mathcal{N}_f^i)\big), \tag{20}$$
$$\mathcal{N}_f^{i+1} := \Psi_{f,f}\big(\mathcal{N}_f^i \ \cup\ \Psi_{S,f}(\mathcal{N}_S^i)\big). \tag{21}$$

Finally, put

$$\mathcal{N}_f^\infty := \bigcup_{i=0}^{\infty} \mathcal{N}_f^i, \qquad \mathcal{N}_S^\infty := \bigcup_{i=0}^{\infty} \mathcal{N}_S^i. \tag{22}$$

Then $\mathcal{N}_f^\infty$ contains all norms $\|\cdot\|$ in $\mathcal{N}_f$ for which one can show that $t \mapsto \|f(t)\|$ remains uniformly bounded on $[0,\tau)$, starting from $\mathcal{N}_f^0$ and using only the a priori estimates provided by Proposition 3.1 and Proposition 3.2 and the embeddings (14).

In our case, the objective is to show that

$$\{\tfrac{1}{q_0}, \tfrac{1}{q_0'}\} \subset \mathcal{N}_f^\infty, \quad \{(k, \tfrac{1}{q_1}), (k,0)\} \subset \mathcal{N}_S^\infty.$$

According to the following lemma and corollary it suffices to show that $(k,0) \in \mathcal{N}_S^\infty$. The lemma is also key to the proof of Proposition 3.2, together with some operator norm estimates that will be established in Section 3.2.2:

Lemma 3.3. *Under the conditions of Theorem 2.1, if the function* $t \mapsto \|\mathcal{T}[S(t)]\|_{\mathcal{L}(L^p(\mathbb{R}^n\times V))}$ *is uniformly bounded on* $[0,\tau)$ *for some* p *with* $q_0 \leq p \leq q_0'$*, then* $t \mapsto \|f(t)\|_p$ *is uniformly bounded on* $[0,\tau)$*.*

Proof. Since $f(t) \geq 0$, for $0 \leq t < \tau$, $T[S] \geq 0$ for all $S \in W^{k,\infty}(\mathbb{R}^n)$ and $(T_\Phi(t))_{t\geq 0}$ is a positive semigroup, we have

$$\begin{aligned} f(t) &= T_\Phi(t)f_0 + \int_0^t T_\Phi(t-s)\big(-\lambda[S(s)]f(s) + \mathcal{T}[S(s)]f(s)\big)\,ds \\ &\leq T_\Phi(t)f_0 + \int_0^t T_\Phi(t-s)(\mathcal{T}[S(s)]f(s))\,ds \qquad (23) \end{aligned}$$

in the Banach lattice $X_0 = L^{q_0} \cap L^{q_0'}(\mathbb{R}^n \times V)$. The order estimate (23) therefore yields for $q_0 \leq p \leq q_0'$ the norm estimate

$$\begin{aligned} \|f(t)\|_p &\leq \|f_0\|_p + \int_0^t \|\mathcal{T}[S(s)]f(s)\|_p\,ds \\ &\leq \|f_0\|_p + \int_0^t \|\mathcal{T}[S(s)]\|_{\mathcal{L}(L^p(\mathbb{R}^n\times V))}\|f(s)\|_p\,ds. \qquad (24) \end{aligned}$$

Here we used that the operators $T_\Phi(t)$ are non-expansive in L^p. By assumption, the operator norm of $\mathcal{T}[S(t)]$ can be bounded by a constant $C \geq 0$ for all $0 \leq t < \tau$. Gronwall's Lemma then yields

$$\|f(t)\|_p \leq \|f_0\|_p[1 + Cte^{Ct}] \leq \|f_0\|_p[1 + C\tau e^{C\tau}]$$

for all $t \in [0,\tau)$. □

Corollary 3.4. *Under the conditions of Theorem 2.1, if* $t \mapsto \|S(t)\|_{W^{k,\infty}}$ *is uniformly bounded on* $[0,\tau)$*, then* $t \mapsto \|f(t)\|_{X_0} + \|S(t)\|_{X_1}$ *is uniformly bounded on* $[0,\tau)$*.*

Proof. The map $S \to T[S] : W^{k,\infty}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n\times V\times V)$ is locally Lipschitz continuous by *(AT1)*. Because the collection $\{S(t)|t \in [0,\tau)\}$ is bounded in $W^{k,\infty}(\mathbb{R}^n)$ by assumption, there exists $M \geq 0$ such that $\|T[S(t)]\|_\infty \leq M$ for all $t \in [0,\tau)$. Then for all p with $\frac{1}{p} \in \mathcal{N}_f$,

$$\|\mathcal{T}[S(t)]\|_{\mathcal{L}(L^p(\mathbb{R}^n\times V))} \leq m(V)\|T[S(t)]\|_\infty \leq m(V)M =: C.$$

(cf. [11], Proposition 4.8). Lemma 3.3 yields uniform boundedness of $t \mapsto \|f(t)\|_{X_0}$. Uniform boundedness of $t \mapsto \|S(t)\|_{W^{k,q_1}}$ on $[0,\tau)$ is garanteed by condition (11) on q_1 and Proposition 3.1 with $p = 1$. $\square$

The crucial technical result that will allow us to conclude global existence is (note that only the case $n = 1$ and $k = 0$ is excluded):

Lemma 3.5. *Suppose that $n \geq 1$ and $k \in \{0, 1\}$ are such that $\frac{2-k}{n} \leq 1$ and that the turning kernel satisfies (AT1) and (AT2). Then the following statements are equivalent:*

(i) $\mathcal{N}_f^1$ properly contains $\mathcal{N}_f^0 = \{1\}$.

(ii) The parameters ν_j and ν_j^* satisfy

$$\max_{0 \leq j \leq k} (1 - \tfrac{2-j}{n})\nu_j \; + \; \max_{0 \leq j \leq k} (1 - \tfrac{2-j}{n})\nu_j^* < 1, \tag{25}$$

$$\max_{0 \leq j \leq k} (1 - \tfrac{2-j}{n})\nu_j^* < 1 - \tfrac{1}{q_0'}, \tag{26}$$

(iii) There exists $\epsilon > 0$ such that for all i and all $\pi \in \mathcal{N}_f^i$,

$$\max(\pi - \tfrac{\epsilon}{q_0'}, \tfrac{1}{q_0'}) \in \Psi_{S,f}\big(\Psi_{f,S}(\pi)\big). \tag{27}$$

(iv) Either $\mathcal{N}_f^{i+1}$ properly contains $\mathcal{N}_f^i$, or $\mathcal{N}_f^{i+1} = \mathcal{N}_f = \mathcal{N}_f^i$.

Proof. (i)⇔(ii): Recall (20) and (21). It is easily verified that

$$\begin{aligned}\mathcal{N}_S^1 &= \big(\{0\} \times (1 - \tfrac{2}{n}, 1] \cup \{1\} \times (1 - \tfrac{1}{n}, 1]\big) \cap \mathcal{N}_S. \\ &= \begin{cases} \{0\} \times [0, \frac{1}{q_0'}], & \text{if } n = 1, \; k = 0, \\ \bigcup_{0 \leq j \leq k} \{j\} \times (1 - \frac{2-j}{n}, \frac{1}{q_1}], & \text{otherwise.} \end{cases}\end{aligned} \tag{28}$$

Thus, under the stated conditions on n and k,

$$\begin{aligned}\Psi_{S,f}(\mathcal{N}_S^1) &= \bigcup_{(0,\rho_0),(k,\rho_k) \in \mathcal{N}_S^1} \{\pi \in \mathcal{N}_f | \nu_j \rho_j \leq \pi \leq 1 - \nu_j^* \rho_j, 0 \leq j \leq k\} \\ &= [\tfrac{1}{q_0'}, 1] \cap \bigcap_{0 \leq j \leq k} \big((1 - \tfrac{2-j}{n})\nu_j, 1 - (1 - \tfrac{2-j}{n})\nu_j^*\big) \\ &= [\tfrac{1}{q_0'}, 1] \cap \big(\max_{0 \leq j \leq k} (1 - \tfrac{2-j}{n})\nu_j, 1 - \max_{0 \leq j \leq k} (1 - \tfrac{2-j}{n})\nu_j^*\big).\end{aligned} \tag{29}$$

Because $\mathcal{N}_f^1$ properly contains $\mathcal{N}_f^0$, $\Psi_{S,f}(\mathcal{N}_S^1)$ must contain at least one element different from 1. From this and (29) we conclude that this can be the case if and only if

$$\max_{0\le j\le k}(1-\tfrac{2-j}{n})\nu_j < 1-\max_{0\le j\le k}(1-\tfrac{2-j}{n})\nu_j^*, \quad \text{and}$$
$$1-\max_{0\le j\le k}(1-\tfrac{2-j}{n})\nu_j^* > \tfrac{1}{q_0'},$$

which can easily be rewritten into (25) and (26).

(ii)⇒(iii): Inequality (25) implies that there exists $\epsilon > 0$ such that

$$\max_{0\le j\le k}(1-\tfrac{2-j}{n})\nu_j < 1-\epsilon. \tag{30}$$

Let $\pi \in \mathcal{N}_f^i$. Then, similar to the derivation of (29), we have

$$\Psi_{S,f}\big(\Psi_{f,S}(\pi)\big) = [\tfrac{1}{q_0'},1] \cap \big(\max_{0\le j\le k}(\pi-\tfrac{2-j}{n})\nu_j, 1-\max_{0\le j\le k}(\pi-\tfrac{2-j}{n})\nu_j^*\big). \tag{31}$$

According to (26),

$$(\pi-\tfrac{2-j}{n})\nu_j^* \le (1-\tfrac{2-j}{n})\nu_j^* < 1-\tfrac{1}{q_0'}. \tag{32}$$

Hence

$$1-\max_{0\le j\le k}(\pi-\tfrac{2-j}{n})\nu_j^* > \tfrac{1}{q_0'}. \tag{33}$$

Moreover,

$$(\pi-\tfrac{2-j}{n})\nu_j \le \pi(1-\tfrac{2-j}{n})\nu_j < \pi(1-\epsilon) \le \pi-\tfrac{\epsilon}{q_0'}. \tag{34}$$

Because of assumption (25), (32) and a similar inequality for ν_j, we find that the second interval in the intersection in (31) is nonempty. Inequalities (33) and (34) now yield *(iii)*.

(iii)⇒(iv): Note that by properties (AΨ2) and (AΨ3) we have for any $\pi \in \mathcal{N}_f^i$:

$$\mathcal{N}_f^{i+1} \supset \Psi_{S,f}(\mathcal{N}_S^i) \supset \Psi_{S,f}\big(\Psi_{f,S}(\pi)\big). \tag{35}$$

If $\frac{1}{q_0'} \in \mathcal{N}_f^i$, then also $\frac{1}{q_0'} \in \mathcal{N}_f^{i+1}$ according to *(iii)* and (35). In that case $\mathcal{N}_f^i = \mathcal{N}_f = \mathcal{N}_f^{i+1}$. If $\frac{1}{q_0'} \notin \mathcal{N}_f^i$, then there exists $\pi \in \mathcal{N}_f^i$ such that $\max(\pi-\frac{\epsilon}{q_0'}, \frac{1}{q_0'})$ is an element of $\mathcal{N}_f^{i+1}$ that is not in $\mathcal{N}_f^i$. Here we use that $\mathcal{N}_f^i$ is an interval. The implication *(iv)⇒(i)* is trivial. □

Part *(iii)* and *(iv)* of Proposition 3.5 (in particular that ϵ can be chosen independently from the iteration number i) yields the following:

Corollary 3.6. $\mathcal{N}_f^\infty = \mathcal{N}_f$ *and consequently* $(k,0) \in \mathcal{N}_S^\infty$.

The proof of Theorem 2.2*(ii)* is now complete.

Proof of Theorem 2.2 (i).
If $n = 1$ and $k = 0$, then $1 - \frac{2-k}{n} < 0$. Proposition 3.1 yields a uniform upper bound for $t \mapsto \|S(t)\|_{W^{k,\infty}}$ on $[0,\tau)$ (under Assumption *(AT1)* on the turning kernel only). By Corollary 3.4 we have global existence of the mild solution. □

3.2 A priori estimates

We now turn our attention towards proving Proposition 3.1 and 3.2.

3.2.1 Estimates on S

The a priori estimates on S as formulated in Proposition 3.1 are based on the following result:

Proposition 3.7. *Let* $1 \leq p < \infty$*. If* $k \in \{0,1\}$ *and* $1 \leq r \leq \infty$ *are such that*

$$\tfrac{1}{p} - \tfrac{2-k}{n} < \tfrac{1}{r} \leq \tfrac{1}{p}, \tag{36}$$

then $\tilde{q} := [1 + \frac{n}{2}(\frac{1}{r} - \frac{1}{p}) - \frac{k}{2}]^{-1}$ *satisfies* $1 \leq \tilde{q} < \infty$ *and, as Bochner integral in* $L^p(\mathbb{R}^n)$,

$$\int_0^t T_d(t-s) f(s)\, ds \in W^{k,r}(\mathbb{R}^n),$$

for all $t > 0$ *and* $f \in L^q([0,T], L^p(\mathbb{R}^n))$*, whenever* $1 \leq \tilde{q} < q \leq \infty$*. If* $r = \infty$ *then the above integral is actually in* $C_0^k(\mathbb{R}^n) \subset W^{k,\infty}(\mathbb{R}^n)$*. Furthermore, for every* $t_0 > 0$ *there exists an* $M > 0$ *such that*

$$\| \int_0^t T_d(t-s) f(s)\, ds \|_{W^{k,r}} \leq M \|f\|_{L^q([0,t], L^p(\mathbb{R}^n))},$$

for all $0 < t \leq t_0$ *and* $f \in L^q([0,t], L^p(\mathbb{R}^n))$.

A proof can be found in [11, Proposition 4.5, Lemma 4.7].

Proof of Proposition 3.1.
The integral equation for the S component of (7),

$$S(t) = T_d(t)S_0 + \int_0^t T_d(t-s)(\alpha\rho(s) - \beta S(s))\,ds,$$

is equivalent to

$$S(t) = e^{-\beta t}T_d(t)S_0 + \int_0^t e^{-\beta(t-s)}T_d(t-s)(\alpha\rho(s))\,ds \tag{37}$$

(cf. [12]). By positivity of f, Minkowski's Inequality for integrals and Hölder's Inequality,

$$\|\rho(t)\|_p \le \int_V \|f(t)(\cdot,v)\|_{L^p(\mathbb{R}^n)}\,dv \le m(V)^{1/p^*}\ \|f(t)\|_p. \tag{38}$$

Because of the uniform upper bound for $t \mapsto \|f(t)\|_p$ on $[0,\tau)$ and (38), $t \mapsto \rho(t)$ can be considered as element of $L^\infty([0,\tau], L^p(\mathbb{R}^n)) \subset L^q([0,\tau], L^p(\mathbb{R}^n))$, for any $1 \le q < \infty$. Proposition 3.7 (taking $t_0 = \tau$), (38), the uniform bound for $t \mapsto \|f(t)\|_p$ on $[0,\tau)$ and (37) yield the desired uniform bound for $t \mapsto \|S(t)\|_{W^{j,r}}$ on $[0,\tau)$. □

3.2.2 Some crucial operator norm estimates

According to Lemma 3.3 it is important to obtain estimates for the operator norm of $\mathcal{T}[S]$ acting in $L^p(\mathbb{R}^n \times V)$ in terms of (various norms applied to) S. Derivation of such estimates is the concern of this section.

Proposition 3.8. *Let $1 \le p < \infty$ and $k \in L^0(\mathbb{R}^n \times V \times V)$ be such that*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left[\int_V \|k(x,v,.)\|^p_{L^{p^*}(V)}\,dv\right]^{\frac{1}{p}} < \infty. \tag{39}$$

Then

$$Kf(x,v) := \int_V k(x,v,v')f(x,v')\,dv'$$

defines a bounded linear operator on $L^p(\mathbb{R}^n \times V)$ and

$$\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n\times V))} \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left[\int_V \|k(x,v,.)\|^p_{L^{p^*}(V)}\,dv\right]^{\frac{1}{p}}.$$

Its proof is straighforward, using the identification of $L^p(\mathbb{R}^n \times V)$ with $L^p(\mathbb{R}^n, L^p(V))$, which holds for finite p.

The following lemma gives us a crucial inequality:

Lemma 3.9. *Assume that* $0 < m(V) < \infty$. *Let* $\epsilon \neq 0$, $1 \leq r < \infty$ *and* $0 \leq s \leq r$. *Then there exists a constant* $C = C_{s,r,\epsilon} > 0$ *such that for all* $S \in L^r(\mathbb{R}^n)$ *and almost all* $x \in \mathbb{R}^n$*:*

$$\int_V |S(x - \epsilon v)|^s \, dv \leq C_{s,r,\epsilon} \|S\|_r^s.$$

Proof. Let us write $\chi_{\epsilon V}$ for the measurable function on $\mathbb{R}^n$ that equals 1 on ϵV and 0 elsewhere. For $\epsilon > 0$ we have for almost all $x \in \mathbb{R}^n$:

$$\int_V |S(x - \epsilon v)|^s \, dv = \frac{1}{\epsilon^n} |S|^s * \chi_{\epsilon V}(x) \leq \frac{1}{\epsilon^n} \| |S|^s * \chi_{\epsilon V} \|_\infty.$$

Young's Inequality yields

$$\frac{1}{\epsilon^n} \| |S|^s * \chi_{\epsilon V} \|_\infty \leq \frac{1}{\epsilon^n} \| |S|^s \|_{\frac{r}{s}} \| \chi_{\epsilon V} \|_{(\frac{r}{s})^*} = C_{s,r,\epsilon} \|S\|_r^s,$$

where

$$C_{s,r,\epsilon} = \frac{1}{\epsilon^n} \| \chi_{\epsilon V} \|_{(\frac{r}{s})^*} = \frac{1}{\epsilon^n} \left[\epsilon^n \int_V dv \right]^{1 - \frac{s}{r}} = \epsilon^{-\frac{ns}{r}} m(V)^{1 - \frac{s}{r}}$$

is positive and finite.
If $\epsilon < 0$, define $\check{\epsilon} = -\epsilon > 0$. Let $\check{S}(x) := S(-x)$. Then

$$\int_V |S(x - \epsilon v)|^s \, dv = \int_V |\check{S}(-x - \check{\epsilon} v)|^s \, dv,$$

and by the previous arguments we get

$$\int_V |\check{S}(-x - \check{\epsilon} v)|^s \, dv \leq C_{s,r,\check{\epsilon}} \|\check{S}\|_r^s = C_{s,r,\epsilon} \|S\|_r^s.$$

□

We can use Proposition 3.8 and Lemma 3.9 to bound the operator norm of kernel operators of the particular form encountered in the terms of Assumption *(AT2)*:

Lemma 3.10. *Let $\epsilon, \epsilon' \in \mathbb{R} : |\epsilon| + |\epsilon'| \neq 0$, $\nu \geq 0, 1 \leq r < \infty$, $S \in L^r(\mathbb{R}^n)$ and $k(x,v,v') = |S(x - \epsilon v - \epsilon' v')|^\nu$. Furthermore, let $1 \leq p < \infty$ be such that either*

(i) $0 \leq \nu p^* \leq r$, when $\epsilon' \neq 0$, or

(ii) $0 \leq \nu p \leq r$, when $\epsilon' = 0$.

Then there is a constant $C > 0$, depending on ν, p, r, ϵ and ϵ', such that

$$\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n \times V))} \leq C\|S\|_r^\nu,$$

where we define

$$Kf(x,v) := \int_V k(x,v,v') f(x,v')\, dv'. \tag{40}$$

Proof. First we assume that $\epsilon' \neq 0$. Then

$$\left[\int_V \|k(x,v,.)\|^p_{L^{p^*}(V)}\, dv\right]^{\frac{1}{p}} = \left[\int_V \left(\int_V |S(x - \epsilon v - \epsilon' v')|^{\nu p^*}\, dv'\right)^{\frac{p}{p^*}} dv\right]^{\frac{1}{p}}.$$

Since $0 \leq \nu p^* \leq r$, we can apply Lemma 3.9 to get a constant $C > 0$ depending on ν, p^*, r and ϵ' such that

$$\int_V |S(x - \epsilon v - \epsilon' v')|^{\nu p^*}\, dv' \leq C\|S\|_r^{\nu p^*}, \tag{41}$$

for almost all $x - \epsilon v \in \mathbb{R}^n$. If we define $\Phi : \mathbb{R}^n \times V \to \mathbb{R}^n$ by $\Phi(x,v) := x - \epsilon v$, Φ is a non-trivial, linear and surjective map. Lemma 2.5 in [11] claims, that if $N \subset \mathbb{R}^n$ is a null set, then $\Phi^{-1}(N) \subset \mathbb{R}^n \times V$ is a null set. Hence inequality (41) holds for almost all $(x,v) \in \mathbb{R}^n \times V$ and we obtain for almost all $x \in \mathbb{R}^n$:

$$\begin{aligned}\left[\int_V \left(\int_V |S(x - \epsilon v - \epsilon' v')|^{\nu' p^*}\, dv'\right)^{\frac{p}{p^*}} dv\right]^{\frac{1}{p}} &\leq \left[\int_V C\|S\|_r^{\nu p}\, dv'\right]^{\frac{1}{p}} \\ &= C^{\frac{1}{p}} m(V)^{\frac{1}{p}} \|S\|_r^\nu < \infty.\end{aligned}$$

Thus by Proposition 3.8 we get $\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n \times V))} \leq C'\|S\|_r^\nu$, where $C' = C^{\frac{1}{p}} m(V)^{\frac{1}{p}}$, a constant depending on ν, p, r and ϵ'.

If $\epsilon' = 0$, then necessarily $\epsilon \neq 0$ and we have

$$\begin{aligned}\left[\int_V \|k(x,v,.)\|^p_{L^{p^*}(V)}\, dv\right]^{\frac{1}{p}} &= \left[\int_V \left(\int_V |S(x - \epsilon v)|^{\nu p^*}\, dv'\right)^{\frac{p}{p^*}} dv\right]^{\frac{1}{p}} \\ &= m(V)^{\frac{1}{p^*}} \left[\int_V |S(x - \epsilon v)|^{\nu p}\, dv\right]^{\frac{1}{p}}.\end{aligned}$$

Again, by Lemma 3.9, we know that there is a constant C depending on ν, p, r and ϵ such that

$$\int_V |S(x-\epsilon v)|^{\nu p}\, dv \leq C\|S\|_r^{\nu p},$$

for almost every $x \in \mathbb{R}^n$. Hence

$$\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n\times V))} \leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left[\int_V \|k(x,v,.)\|^p_{L^{p^*}(V)}\, dv\right]^{\frac{1}{p}} \leq C'\|S\|_r^{\nu},$$

where $C' := C^{\frac{1}{p}}\mu(V)^{\frac{1}{p^*}}$, a constant depending on ν, p, r and ϵ. □

3.2.3 A priori estimates on f in general dimensions

We now consider the proof of Proposition 3.2. Thus let the turning kernel satisfy Assumption *(AT2)* and suppose that we have uniform upper bounds for the functions $t \mapsto \|S(t)\|_{W^{j,r_j}}$, where r_j satisfy $q_1 \leq r_j < \infty$, $(0 \leq j \leq k)$. Recall definitions (9) and (10) for ν_j and ν_j^*.

Proof of Proposition 3.2.
If the r_j, $(0 \leq j \leq k)$, are such that there exists $1 \leq p < \infty$ such that

$$0 \leq \nu_j p \leq r_j \quad \text{and} \quad 0 \leq \nu_j^* p^* \leq r_j$$

for $0 \leq j \leq k$, then Lemma 3.10 shows that the kernel operator $K_{\mu,i}[S]$ associated to any $S \in L^{r_0} \cap W^{1,r_1}(\mathbb{R}^n)$ by means of

$$k_{\mu,i}[S](x,v,v') := C_{\mu,i}|D^\mu S(x-\epsilon_{\mu,i}v-\epsilon'_{\mu,i}v')|^{\nu_{\mu,i}}$$

as in (40), is a bounded positive linear operator on $L^p(\mathbb{R}^n \times V)$.

Assumption *(AT2)* implies that for any *positive* $f \in L^p(\mathbb{R}^n \times V)$ and any $S \in L^{r_0} \cap W^{k,r_k} \cap W^{1,\infty}(\mathbb{R}^n)$:

$$0 \leq \mathcal{T}[S]f \leq Cf \;+ \sum_{|\mu|\leq k,\; 1\leq i\leq d} K_{\mu,i}[S]f.$$

Because $L^p(\mathbb{R}^n \times V)$ is a Banach lattice, this order estimate yields the norm estimate

$$\|\mathcal{T}[S]f\|_p \leq C\|f\|_p + \sum_{\mu,i}\|K_{\mu,i}[S]f\|_p. \tag{42}$$

Because the operator norm of a positive operator A in a Banach lattice equals the supremum of $\|Ax\|$ over all *positive* x with $\|x\| \leq 1$, we derive from (42) and Proposition 3.10 that there exist constants $C' > 0$ and $\hat{C} > 0$, not depending on S, such that

$$\begin{aligned}\|\mathcal{T}[S]\|_{\mathcal{L}(L^p(\mathbb{R}^n\times V))} &\leq C'\big(1 + \sum_{\substack{0\leq j\leq k \\ |\mu|=j,\ 1\leq i\leq d}} \|D^\mu S\|_{r_j}^{\nu_{\mu,i}}\big) \\ &\leq \hat{C}\big(1 + \sum_{0\leq j\leq k} \|S\|_{W^{j,r_j}}^{\max(\nu_j,\nu_j^*)}\big). \qquad (43)\end{aligned}$$

Uniform bounds for $t \mapsto \|S(t)\|_{W^{j,r_j}}$ on $[0,\tau)$, $(0 \leq j \leq k)$, thus yield a uniform upper bound for $t \mapsto \|\mathcal{T}[S(t)]\|_{\mathcal{L}(L^p)}$. Lemma 3.3 now gives the result. □

3.2.4 An additional estimate on f in dimension two

In this entire section, the physical dimension n equals 2 and $k = 1$.

We prove Theorem 2.3 in this section, i.e. we can conclude global existence of positive mild solutions not only when $\nu_1 + \nu_1^* < 2$ (and $\nu_0, \nu_0^* \geq 0$ arbitrary), but also when $\nu_1 + \nu_1^* = 2$ and $1 \leq \nu_1 < 2$. We modified and extended the approach in [17], where the case $\nu_1 = \nu_1^* = 1$ has been proven. The key point there is, that an estimate of $\|\nabla S(t)\|_2$ in terms of $\ln(\|f(\cdot)\|_2 + 1)$ is available. This inequality can only be derived for $n = 2$. This additional a priori estimate allows to conclude, using a generalised Gronwall's Lemma, Lemma 3.14, that $t \mapsto \|f(t)\|_2$ remains uniformly bounded on $[0,\tau)$. We prove that this log-type estimate for $\|\nabla S(t)\|_2$ still holds, but now in terms of $\|f(\cdot)\|_{2/\nu_1}$.

Let us define $\varphi_s : \mathbb{R}^2 \to \mathbb{R} : \xi \mapsto |\xi|e^{-s|\xi|^2}$. Then it is easily verified that for all $s > 0$:

$$\|\varphi_s\|_p^p = \pi\Gamma\big(\tfrac{p}{2}+1\big)\cdot\big(\tfrac{1}{sp}\big)^{\frac{p}{2}+1} =: C_p^p \cdot s^{-1-\frac{p}{2}} \qquad (44)$$

for $1 \leq p < \infty$ and

$$\|\varphi_s\|_\infty \leq C_\infty \cdot s^{-\frac{1}{2}} \qquad (45)$$

for some $C_\infty > 0$. First we have

Lemma 3.11. *For any $1 < p \leq 2$ there is a constant $C > 0$, depending on d and p, such that for all $t \in [0,\tau)$,*

$$\|\nabla S(t)\|_2 \leq \|\nabla S_0\|_2 \ + \ \alpha C\, t^{\frac{1}{p^*}} \sup_{0\leq s\leq t} \|\rho_s\|_p.$$

Proof. The solution S satisfies (37). For the first term we have

$$\|\nabla T_d(t)S_0\|_2 = \|T_d(t)\nabla S_0\|_2 \leq \|\nabla S_0\|_2.$$

Let us write ρ_s instead of $\rho(s)$ and

$$S_1(t) := \int_0^t e^{-\beta s} T_d(s)\rho_{t-s}\, ds.$$

By Plancherel's equality we have for this second term

$$\|\nabla S_1(t)\|_2 = \|\widehat{\nabla S_1(t)}\|_2 = \||\xi \mapsto |\xi|\widehat{S_1(t)}(\xi)\|_2.$$

Furthermore, $T_d(s)\rho_{t-s} = h_d(\cdot, s) * \rho(\cdot, t-s)$ by definition, hence

$$\||\xi|\widehat{S_1(t)}\|_2 \leq 2\pi \int_0^t e^{-\beta s}\||\xi|\hat{h}_d(\cdot, s)\hat{\rho}_{t-s}\|_2\, ds.$$

Note that the Fourier transform of $h_d(\cdot, s)$ is given by $\hat{h}_d(\xi, s) = \frac{1}{2\pi}e^{-sd|\xi|^2}$. Thus we obtain for any $1 \leq q \leq \infty$, by Hölder's inequality,

$$\|\nabla S_1(t)\|_2 \leq \int_0^t e^{-\beta s}\|\varphi_{sd}\,\hat{\rho}_{t-s}\|_2\, ds \leq \int_0^t e^{-\beta s}\|\varphi_{sd}\|_{2q^*}\,\|\hat{\rho}_{t-s}\|_{2q}\, ds. \tag{46}$$

Because $1 \leq p \leq 2$, we can find $q \geq 1$ such that $2q = p^*$. Then $q^* = \frac{p}{2-p}$ and with this choice we have, according to (44),

$$\|\varphi_{sd}\|_{2q^*}\,\|\hat{\rho}_{t-s}\|_{2q} \leq C_{2q^*}\, s^{-\frac{1}{p}}\|\hat{\rho}_{t-s}\|_{p^*} \leq C_{2q^*}\, s^{-\frac{1}{p}}\|\rho_{t-s}\|_p. \tag{47}$$

Here we used that the Fourier transform on $L^1 \cap L^p(\mathbb{R}^2)$ extends to a continuous linear map from $L^p(\mathbb{R}^2)$ into $L^{p^*}(\mathbb{R}^2)$ (for $1 \leq p \leq 2$). Because $p > 1$, (46) and (47) yield

$$\|\nabla S_1(t)\|_2 \leq C_{2q^*} \sup_{0\leq s\leq t}\|\rho_{t-s}\|_p \int_0^t s^{-\frac{1}{p}} ds \leq Ct^{\frac{1}{p^*}} \sup_{0\leq s\leq t}\|\rho_s\|_p. \tag{48}$$

□

Allthough the lemma is of separate interest, it can readily be improved into the following crucial estimate, which generalises a result in [17] for $p = 2$:

Proposition 3.12. *For any $1 < p \leq 2$ and all $t \in [0, \tau)$,*

$$\|\nabla S(t)\|_2 \leq \|\nabla S_0\|_2 + C\, t^{\frac{1}{p^*}} + \tfrac{p^*\sqrt{\pi}}{2}\,\|\rho(0)\|_1 \ln\big(\sup_{0\leq s\leq t}\|\rho(s)\|_p + 1\big), \tag{49}$$

where C is the constant of Lemma 3.11.

Proof. The central idea is to choose $0 < r \leq t$ and observe that (46) and (47) yield

$$\|\nabla S_1(t)\|_2 \leq C\, r^{\frac{1}{p^*}} \sup_{0\leq s\leq r} \|\rho_s\|_p \; + C_{2q^*} \int_r^t e^{-\beta s} s^{-\frac{1}{2q^*}-\frac{1}{2}} \|\hat{\rho}_{t-s}\|_{2q}\, ds, \qquad (50)$$

for any $1 \leq q \leq \infty$. Because $r > 0$, the integrand in (50) is integrable when $q = \infty$. Making this choice for q, observing that the function $t \mapsto \sup_{0\leq s\leq t} \|\rho_s\|_p$ is nondecreasing and that $\|\rho_s\|_1 = \|\rho_0\|_1$, we obtain

$$\|\nabla S_1(t)\|_2 \leq C\, r^{\frac{1}{p^*}} \sup_{0\leq s\leq t} \|\rho_s\|_p \; + C_2\|\rho_0\|_1(\ln t - \ln r). \qquad (51)$$

Now choose

$$r = t \cdot \big[\sup_{0\leq s\leq t} \|\rho(s)\|_p + 1\big]^{-p^*}.$$

Then indeed $0 < r \leq t$. Substitution of r into (51) and observing that $C_2 = \frac{\sqrt{\pi}}{2}$ yields (49). □

This log-type estimate will allow us to abtain a uniform upper bound for $t \mapsto \|f(t)\|_p$ on $[0, \tau)$ for some $p > 1$. Starting point is:

Lemma 3.13. *Let the turning kernel satisfy (AT2) with $\nu_1 + \nu_1^* = 2$ and $\nu_1, \nu_1^* \leq 2$, $\nu_1 \neq 0$. Let $1 \leq r_0 < \infty$ be such that $r_0 \geq \frac{2}{\nu_1}\max(\nu_0, \nu_0^*)$. Then*

$$\|\mathcal{T}[S]\|_{\mathcal{L}(L^{2/\nu_1}(\mathbb{R}^n\times V))} \leq C(1 + \|S\|_{r_0}^{\max(\nu_0,\nu_0^*)} + \|S\|_{W^{1,2}}^{\max(\nu_1,\nu_1^*)}) \qquad (52)$$

for some constant $C > 0$.

Proof. $p = \frac{2}{\nu_1}$ is the unique solution to the inequalities $0 \leq \nu_1 p \leq 2$ and $0 \leq \nu_1^* p^* \leq 2$, given that $\nu_1 + \nu_1^* = 2$. In this case $p^* = \frac{2}{\nu_1^*}$. Because $\nu_1 \neq 0$, the conditions of Lemma 3.10 are satisfied and we can proceed as in the proof of Proposition 3.2 to deduce (52). □

Moreover, we will use

Lemma 3.14 (Generalised Gronwall's Lemma). *Let $z \in C([0, T))$, for some $0 < T < \infty$, such that $z \geq 1$ on $[0, T)$. If there are constants $a \geq 0$, $c \geq 0$ and a nondecreasing, nonnegative function $b \in C^1([0, T])$, such that*

$$z(t) \leq c \; + \; \big(a \ln z(t) + b(t)\big) \int_0^t z(s)\, ds$$

for all $t \in [0, T)$, then z is bounded on $[0, T)$.

Proof. Let $p(t) := 1 + \int_0^t z(s)\,ds$. Then $p \in C^1([0,T))$, $p, p' \geq 1$ on $[0,T)$ and

$$p'(t) \leq c \;+\; \big(a \ln p'(t) + b(t)\big) \int_0^t z(s)ds \leq (a \ln p'(t) + b(t))p(t) + c,$$

because $a \ln p'(t) + b(t) \geq 0$ by assumption. Hence we can apply Corollary A.2 to get an upper bound $M > 0$ for $p(t)$ on $[0,T)$. There is also an upper bound $N > 0$ for $b(t)$ on $[0,T)$, since b is continuous on $[0,T]$. Hence $z(t) \leq Ma \ln z(t) + MN + c$ for all $t \in [0,T)$. If $a = 0$, then z is bounded. If $a > 0$, we have $\frac{1}{Ma}\big(z(t) - MN - c\big) \leq \ln z(t)$. If $A > 0$ and $B \in \mathbb{R}$ and $Ax - B \leq \ln x$ for all x in some subset S of $(0,\infty)$, then S must be bounded. □

This lemma and the log-type estimate of Proposition 3.12 yield the following

Proposition 3.15. *Let $n = 2$, $k = 1$. Suppose that the turning kernel satisfies (AT2) with $\nu_1 \geq 1$, $\nu_1^* \leq 2$ and $\nu_1 + \nu_1^* = 2$, while $\nu_0, \nu_0^* \geq 0$ are arbitrary. Then $t \mapsto \|f(t)\|_{2/\nu_1}$ is uniformly bounded on $[0,\tau)$.*

Proof. Let $p := \frac{2}{\nu_1}$. By the assumptions on q_0' and ν_1, one has $\frac{1}{p} \in \mathcal{N}_f$ and $p^* = \frac{2}{\nu_1^*}$. Because of the conditions on ν_1 and ν_1^* and because $\{0\} \times (0, \frac{1}{q_1}] \subset \mathcal{N}_S^1$ according to (28), Lemma 3.13, (24) and Proposition 3.12 yield

$$\begin{aligned}\|f(t)\|_p &\leq \|f_0\|_p \;+\; C \int_0^t (1 + \|\nabla S(s)\|_2^{\max(\nu_1,\nu_1^*)})\, \|f(s)\|_p\, ds \\ &\leq \|f_0\|_p + \big(C_0' + C_1'\, t^{\frac{\nu_1}{p^*}} + C_2'\big[\ln(\sup_{0\leq s\leq t} \|\rho(s)\|_p + 1)\big]^{\nu_1}\big) \int_0^t \|f(s)\|_p\, ds.\end{aligned}$$

Observe that $[\ln x]^\nu \leq x^{\nu/e}$ for all $x \geq 1$ and $\nu > 0$. By using (38), that $\frac{\nu_1}{p^*} = \frac{\nu_1\nu_1^*}{2} = \frac{\nu_1(2-\nu_1)}{2} \leq \frac{1}{2}$ and the latter observation with $\nu_1 \leq 2 < e$, we obtain

$$\|f(t)\|_p \leq \|f_0\|_p + \big(C_3' + C_1'\, t^{\frac{1}{2}} + C_2' \ln(\sup_{0\leq s\leq t} \|f(s)\|_p + 1\big) \int_0^t \|f(s)\|_p\, ds. \tag{53}$$

Now define $z(t) := \sup_{0\leq s\leq t} \|f(s)\|_p + 1$ and $b(t) := C_1' t + C_1' + C_3'$. Then $b(t) \geq C_1' t^{1/2} + C_3'$ and (53) yields

$$z(t) \leq z(0) + \big(b(t) + C_2' \ln z(t)\big) \int_0^t z(s)ds.$$

Because $z \geq 1$ and b is a continuously differentiable, nondecreasing and nonnegative function on $[0,\tau]$, Lemma 3.14 gives that z, hence $t \mapsto \|f(t)\|_{2/\nu_1}$, is bounded on $[0,\tau)$. □

We can now finally turn to the proof of Theorem 2.3.

Proof of Theorem 2.3.
According to Proposition 3.15 we may start the iteration of the sets $\mathcal{N}_f^i$ and $\mathcal{N}_S^i$ with $\mathcal{N}_f^0 = \{\frac{\nu_1}{2}, 1\}$ instead of $\{1\}$. We can now proceed in a way similar to the proof of Lemma 3.5.

It is easily checked that for $\frac{1}{2} \le \pi < 1$,

$$\Psi_{f,S}(\pi) = \{0\} \times [0, \tfrac{1}{q_1}] \,\cup\, \{1\} \times (\pi - \tfrac{1}{2}, \tfrac{1}{q_1}].$$

Thus $(0,0) \in \mathcal{N}_S^i$ for all $i \ge 1$, because $\frac{\nu_1}{2} < 1$. If $\frac{1}{q_0'} \in \mathcal{N}_f^i$ for some i, then $(1,0) \in \mathcal{N}_S^i$, because of the condition $q_0' > \frac{2-k}{n} = 2$ made in Theorem 2.1. In that case, Corollary 3.4 implies global existence in X.

Now let $\pi \in \mathcal{N}_f^i$. We may assume that $\pi \le \frac{\nu_1}{2} < 1$. Then because of $(0,0) \in \mathcal{N}_S^i$ and $\nu_1 + \nu_1^* = 2$,

$$\begin{aligned}\Psi_{S,f}(\Psi_{f,S}(\pi)) &= \bigcup_{(1,\rho_1)\in\Psi_{f,S}(\pi)} \{\pi' \in \mathcal{N}_f | \nu_1\rho_1 \le \pi' \le 1 - \nu_1^*\rho_1\} \\ &= [\tfrac{1}{q_0'}, 1] \,\cap\, \big(\nu_1(\pi - \tfrac{1}{2})\,,\, 1 - \nu_1^*(\pi - \tfrac{1}{2})\big) \\ &\supset [\tfrac{1}{q_0'}, 1] \,\cap\, \big(\nu_1(\tfrac{\nu_1}{2} - \tfrac{1}{2})\,,\, 1 - \nu_1^*(\tfrac{\nu_1}{2} - \tfrac{1}{2})\big) \end{aligned} \tag{54}$$

By assumption, $1 \le \nu_1 < 2$. Therefore $1 - \frac{1}{2}\nu_1^* = \frac{1}{2}\nu_1 \ge \frac{1}{2} > \frac{1}{q_0'}$ and

$$\nu_1(\tfrac{\nu_1}{2} - \tfrac{1}{2}) < \tfrac{\nu_1}{2} < 1 \quad \text{and} \quad 1 - \nu_1^*(\tfrac{\nu_1}{2} - \tfrac{1}{2}) > 1 - \tfrac{\nu_1^*}{2} > \tfrac{1}{q_0'}. \tag{55}$$

The relation $\nu_1 + \nu_1^* = 2$ gives

$$1 - \nu_1^*(\tfrac{\nu_1}{2} - \tfrac{1}{2}) = \nu_1(\tfrac{\nu_1}{2} - \tfrac{1}{2}) + 2 - \nu_1. \tag{56}$$

From $\nu_1 < 2$, (54), (55) and (56) we conclude that $\Psi_{S,f}(\Psi_{f,S}(\pi)) \ne \emptyset$. Moreover, we can choose $\epsilon > 0$ such that $\frac{1}{2}\nu_1 < 1 - \epsilon$. Then

$$\nu_1(\pi - \tfrac{1}{2}) \le \pi \cdot \tfrac{1}{2}\nu_1 < \pi(1 - \epsilon) \le \pi - \tfrac{\epsilon}{q_0'}.$$

Thus $\max(\pi - \frac{\epsilon}{q_0'}, \frac{1}{q_0'}) \in \mathcal{N}_f^{i+1}$. Because ϵ can be choosen independently from i, we find by iteration, that for some $m \in \mathbb{N}$, $\frac{1}{q_0'} \in \mathcal{N}_f^{i+k}$. Thus the solution exists globally in X. $\square$

A Generalisations of Gronwall's Lemma

For this we will need a nonstandard version of Gronwall's Lemma, that generalises [17, Lemma 3.1]:

Lemma A.1. *Let $0 < T < \infty$. Suppose that $y \in C^1([0,T))$ satisfies $y(t) \geq 1$ and $y'(t) > 0$ for all $t \in [0,T)$ and suppose that there exist a constant $a \geq 0$ and a function $b \in C([0,T])$ such that $b \geq 0$ on $[0,T]$ and*

$$y'(t) \leq ay(t) \ln y'(t) + b(t)y(t) \tag{57}$$

for all $t \in [0,T)$. Then for any $\delta > 1$ there is a constant $C_{\delta,a} \geq 0$, depending on δ and a, such that

$$y(t) \leq \Big[y(0) \exp\big(\delta \int_0^t aC_{\delta,a} + b(s)ds\big)\Big]^{1+\delta at \exp(\delta at)} \tag{58}$$

for all $t \in [0,T)$. If $a = 0$, then (58) is valid even with $\delta = 1$. In any case, y is bounded on $[0,T)$.

Proof. First rewrite (57) into

$$y' \leq ay \ln((\ln y)') + ay \ln y + by.$$

and substitute $z = \ln y$ to obtain $z' \leq a \ln z' + az + b$, on $[0,T)$. If $a = 0$, then

$$\ln y(t) \leq \ln y(0) + \int_0^t b(s)ds,$$

which yields (58) with $\delta = 1$. Next consider the case $a > 0$. Fix $\delta > 1$ and put $\epsilon := (1 - \frac{1}{\delta})^{-1}$. Then $\epsilon > 1$ and and there is a constant $C_{\delta,a} \geq 0$, depending on δ and a such that $\ln x \leq \frac{1}{\epsilon a}x + C_{\delta,a}$ for all $x > 0$, because $\frac{1}{\epsilon a} > 0$. This implies that $z' \leq \frac{1}{\epsilon}z' + aC_{\delta,a} + az + b$ and consequently $z' \leq \delta(az + aC + b)$ on $[0,T)$, or equivalently,

$$z(t) \leq z(0) + \delta aC_{\delta,a}t + \delta \int_0^t b(s)ds + \delta a \int_0^t z(s)ds.$$

Because $y \geq 1$, $z \geq 0$ and we can use Gronwall's Lemma to obtain

$$z(t) \leq \alpha(t)\big[1 + \delta ate^{\delta at}\big], \qquad \alpha(t) := z(0) + \delta aC_{\delta,a}t + \delta \int_0^t b(s)ds.$$

Since $y(t) = e^{z(t)}$, the result follows. □

From this lemma we can derive the following corollaries:

Corollary A.2. *Let $y \in C^1([0,T))$, for some $0 < T < \infty$, such that $y \geq 1$ and $y' \geq 1$ on $[0,T)$. If there are constants $a \geq 0$, $c \geq 0$ and a nondecreasing nonnegative function $b \in C^1([0,T])$ such that*

$$y'(t) \leq ay(t)\ln y'(t) + b(t)y(t) + c \tag{59}$$

for all $t \in [0,T)$, then y is bounded on $[0,T)$.

Proof. Define $w := (b+1)y + c$. Then $w \geq y \geq 1$ and w is differentiable, with $w' = b'y + (b+1)y' \geq y'$. Thus $\ln y' \leq \ln(w')$. Then

$$\begin{aligned} w' = b'y + (b+1)y' \;&\leq\; b'y + (b+1)(ay\ln y' + by + c) \\ &= ay(b+1)\ln y' + b'y + (b+1)(by+c). \end{aligned}$$

Using $y, y' \geq 1$, $c \geq 0$, $b \geq 0$ and $0 \leq \ln y' \leq \ln w'$, we get

$$y(b+1)\ln y' \;\leq\; y(b+1)\ln w' \;\leq w\ln w'.$$

Thus, because $b' \geq 0$, $y \leq w$ and $a \geq 0$, we finally obtain

$$w' \leq aw\ln w' + (b'+b+1)w.$$

Because $w \geq y \geq 1$ and $w' \geq y' \geq 1$ we can apply Lemma A.1 and find that w is bounded on $[0,T)$. Because $y \leq w$, we get the result. □

Corollary A.3 (Generalised Gronwall's Lemma). *Let $z \in C([0,T))$, for some $0 < T < \infty$, such that $z \geq 1$ on $[0,T)$. If there are constants $a \geq 0$, $c \geq 0$ and a nondecreasing, nonnegative function $b \in C^1([0,T])$, such that*

$$z(t) \leq c \;+\; \big(a\ln z(t) + b(t)\big)\int_0^t z(s)\,ds$$

for all $t \in [0,T)$, then z is bounded on $[0,T)$.

Its proof has been given in Section 3.2.4.

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