INTEGER CONJUGACY CLASSES OF $SL(3, \mathbb{Z})$ AND HESSENBERG MATRICES.

OLEG KARPENKOV

ABSTRACT. In this paper we study the problem of description of conjugacy classes in the group $SL(n, \mathbb{Z})$. Gauss Reduction Theory gives the answer for the case $n = 2$, for $n \geq 3$ the problem is still open. We introduce a new approach to this problem based on reduction to reduced Hessenberg matrices. An important tool used in our approach is to determine minima of Markoff-Davenport characteristics at the vertices of Klein-Voronoi continued fractions. Mostly, we work in the case of three-dimensional matrices having a real and two complex-conjugate eigenvalues, nevertheless, the techniques shown in the paper can be applied both to the totally real case and to the multidimensional case.

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Date: 1 November 2007.
Key words and phrases. integer matrix reduction theory, Hessenberg matrices, Markoff-Davenport characteristic, Klein-Voronoi continued fractions, convex hulls.
The work is partially supported by RFBR NSh-4719.2006.1 grant, by RFBR 05-01-02805-CNRSL_a grant, NWO-DIAMANT 613.009.001 grant, and by RFBR grant 05-01-01012a.

1. Introduction

This paper is dedicated to the classical problem of description of conjugacy classes in the group $SL(n,\mathbb{Z})$. The conjugacy problem is solved only for $n=2$ and there is no any clear description of conjugacy classes for the case $n \geq 3$. In this paper we make first steps to understand the structure of the set of conjugacy classes in $SL(n,\mathbb{Z})$. Our main results are in the study of $SL(3,\mathbb{Z})$ conjugacy classes, nevertheless, the observed three-dimensional phenomena seems to be common in the cases of higher dimensions.

Unlike the case of $SL(n,\mathbb{C})$-matrices, the conjugacy classes of $SL(n,\mathbb{Z})$-matrices with all distinct eigenvalues are not uniquely determined by the coefficients of characteristic polynomials. Therefore, it is not possible to enumerate such classes by Jordan canonical forms. We propose to replace Jordan canonical forms by reduced perfect Hessenberg matrices. Still these matrices do not give the complete invariant of conjugacy classes, nevertheless the study of $SL(2,\mathbb{Z})$ and $SL(3,\mathbb{Z})$ cases shows that the set of Hessenberg matrices is a "nice approximation" of the set of conjugacy classes. For instance, for the set of $SL(3,\mathbb{Z})$-matrices with irreducible characteristic polynomial over rational numbers having one real and two complex roots we obtain the following: perfect Hessenberg matrices distinguish conjugacy classes asymptotically (see more precisely in Theorem 5.8 below).

The solution to the problem for $SL(2,\mathbb{Z})$ is a subject of Gauss Reduction Theory (see, for instance, in [20] or in [14]). The idea of this theory is to find special reduced matrices in each conjugacy class. The number of such matrices in a conjugacy class equals to the length of a minimal even period of ordinary continued fraction associated to the conjugacy class. We put together old constrictions of Klein's polyhedron (introduced in [15], [16]) and Voronoi's continued fraction (described in [23]) and the ideas of J. A. Buchmann (see in [3]) to obtain the general definition of geometric Klein-Voronoi multidimensional continued fraction. In our study of $SL(3,\mathbb{Z})$ conjugacy classes we work with periods of Klein-Voronoi continued fractions.

Integer notation. A point (vector) is said to be *integer* if all its coordinates are integers. A segment is said to be *integer* if its endpoints are integer. An *integer length* of an integer segment is the number of integer points contained in the interior of the segment plus one. An m-dimensional plane is said to be *integer* if the integer vectors contained in the plane generate the Abelian group of rank m. A linear transformation is said to be *integer* if it preserves lattice of integer points. Two matrices are said to be *integer conjugate* if they are conjugate and the transformation matrix corresponds to an integer linear transformation.

Hessenberg matrices and Hessenberg complexity. A matrix M in the group $SL(n,\mathbb{R})$ of the form

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

is called an (*upper*) *Hessenberg* matrix (such matrices were first studied by K. Hessenberg in [10]). The integer

$$\prod_{j=1}^{n-1} |a_{j+1,j}|^{n-j}$$

is called the *Hessenberg complexity* of the matrix M and denoted by $\varsigma(M)$.

Further, we use the following notation. Let $M = (a_{i,j})$ be a Hessenberg matrix with positive Hessenberg complexity. We say that the matrix M is of *Hessenberg type*

$$\langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \cdots | a_{n-1,1}, a_{n-1,2}, \ldots, a_{n-1,n} \rangle.$$

We denote the Hessenberg matrix M by

$$H_{\langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \cdots | a_{n-1,1}, a_{n-1,2}, \ldots, a_{n-1,n} \rangle}(a_{n,1}, \ldots, a_{n,n}).$$

Definition 1.1. An integer Hessenberg matrix in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial is said to be *perfect* if for any couple of integers (i,j) satisfying $1 \le i < j+1 \le n$ the following inequalities hold: $0 \le a_{i,j} < a_{j+1,j}$.

In other words all elements of all the rows except the last one of a perfect Hessenberg matrix are nonnegative integers, the maximal elements in these rows are the lowest nonzero ones (i.e., $a_{j+1,j}$, $j = 1, \ldots, n-1$).

A perfect integer Hessenberg matrix with $a_{2,1} = \cdots = a_{n,n-1} = 1$ is called *Frobenius* matrix. The elements of the last row of Frobenius matrix are the coefficients of the characteristic polynomial multiplied alternatively by ± 1, and all the remaining elements are zeroes.

Note that the set of values of Hessenberg complexity for perfect Hessenberg matrices of $SL(n,\mathbb{Z})$ for $n \ge 2$ coincides with the set of positive integers. The integer Hessenberg matrix has the unit complexity iff it is Frobenius.

Reduction problem. We study matrices of $SL(n,\mathbb{Z})$ with irreducible characteristic polynomials. Any such matrix is conjugate to a perfect Hessenberg matrix with positive Hessenberg complexity (see Proposition 2.1 below). Actually, there are infinitely many perfect Hessenberg matrices integer conjugate to a given one.

Definition 1.2. A perfect Hessenberg matrix W is said to be *reduced* if for any perfect Hessenberg matrix M integer conjugate to W we have

$$\varsigma(W) \leq \varsigma(M).$$

Note that it is possible to have two reduced perfect Hessenberg matrices conjugate to each other, see Example 5.19. In this case such matrices are of the same Hessenberg complexity.

Mostly, we work in $SL(3,\mathbb{Z})$. The case of $SL(3,\mathbb{Z})$-matrices with reduced characteristic polynomials over the field of rational numbers can be reduced to the case of $SL(2,\mathbb{Z})$ matrices. Let us now consider $SL(3,\mathbb{Z})$-matrices with irreducible characteristic polynomials. There are two essentially different subsets of such matrices. First of them consists of all matrices with all real eigenvalues; we call the matrices of this set the *real spectra matrices* or *RS-matirces*, for short. The second set consists of the matrices with a real and two complex conjugate eigenvalues; we call the matrices of this set the *non-real spectra matrices* or *NRS-matirces*, for short.

The main goal of this article is to investigate families of perfect Hessenberg NRS-matrices in $SL(3,\mathbb{Z})$ with fixed Hessenberg types.

It is easy to show that any NRS-matrix is conjugate to infinitely many Hessenberg matrices. Nevertheless, experiments show, that *for any fixed Hessenberg type there exist only finitely many NRS-matrices that are not perfect.* The similar statement does not hold for RS-matrices (with all real eigenvalues). These allow us to conclude that the results of the experiments are really unexpected.

Example 1.3. Let us consider matrices of the Hessenberg type $\langle 0,1|1,0,2\rangle$. All matrices of that type form a two-parametric family

$$H_{\langle 0,1|1,0,2\rangle}(m,n) = \begin{pmatrix} 0 & 1 & n+1 \\ 1 & 0 & m \\ 0 & 2 & 2n+1 \end{pmatrix}$$

with integer parameters m and n. The discriminant of the matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$ equals

$$-44 - 44n^2 - 56mn - 32n^3 + 32m^3 + 16m^2n^2 + 16mn^2 + 16m^2n - 56n - 8m + 52m^2.$$

The set of matrices with negative discriminant for the given family coincides with the union of integer solutions of the following inequalities:

$$2m \leq -n^2 - n - 2 \quad \text{and} \quad 2n \leq m^2 + m.$$

In Figure 1 we show the family of Hessenberg operators of type $\langle 0,1|1,0,2\rangle$. The square in the intersection of the m-th column and the n-th row corresponds to the matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$. Black squares correspond to the matrices with reducible characteristic polynomials. Light gray squares correspond to the RS-matrices. Dark gray squares correspond to the nonreduced NRS-matrices. Finally, white squares form the set of reduced NRS-Hessenberg matrices.

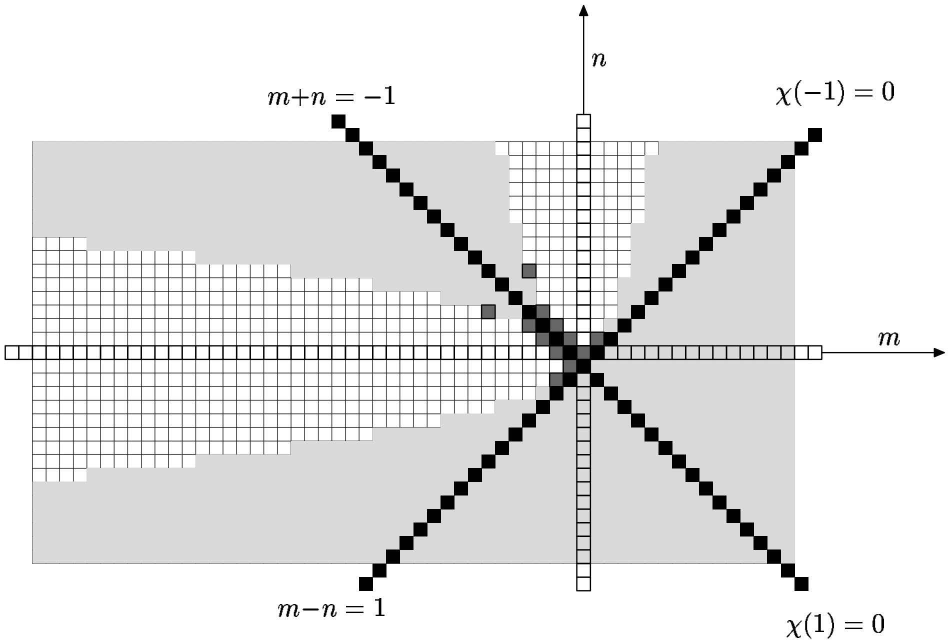


FIGURE 1. The family of matrices of Hessenberg type $\langle 0,1|1,0,2\rangle$.

There are only 12 dark gray squares corresponding to the nonreduced Hessenberg NRS-matrices. We conjecture that there is no other nonreduced NRS-matrices in the family of matrices $H_{\langle 0,1|1,0,2\rangle}(m,n)$. Further experiments (see in Subsections 5.4 and 5.5) show that the same holds for families of some other Hessenberg types. That leads to the following problem.

Problem 1. Let us fix some perfect Hessenberg type of three-dimensional matrices. Is it true that almost all (except a finite number) perfect Hessenberg NRS-matrices of a given Hessenberg type are reduced? Find this number.

If the answer to the question of the problem is negative then we have an additional problem: *find all Hessenberg types for which the answer to the question is positive.*

We conjecture that the answer to Problem 1 is positive. Meanwhile, it is not proven even in the simplest cases of matrices with Hessenberg complexity equal to 2. Note that there are only two types of such matrices: $\langle 0,1|1,0,2\rangle$ and $\langle 0,1|1,1,2\rangle$.

Here we show that *any family of Hessenberg NRS-matrices of the given Hessenberg type "looks like" the union of the convex hulls of two parabolas* (see Theorem 5.4), as it happens in the case of matrices of Hessenberg type $\langle 0,1|1,0,2\rangle$ in Example 1.3. For any of these two parabolas consider a unique integer asymptotic direction. For instance, in the case of Hessenberg type $\langle 0,1|1,0,2\rangle$ we have the parabolas with asymptotic directions corresponding to the vectors $(1,0)$, and $(0,1)$. We prove that *for the sequence of $SL(3,\mathbb{Z})$-matrices of any ray with vertex at an NRS-matrix and and the asymptotic direction we have only reduced matrices starting from some moment* (see Theorem 5.8).

Description of the paper. We are giving a small introduction in Section 1. In Section 2 we study basic properties of Hessenberg matrices of arbitrary dimensions. In particular, we discuss the question of existence and finiteness of reduced Hessenberg operators integer conjugate to the given one. In this section we also investigate families of Hessenberg operators with a given Hessenberg type. Further, in Section 3 we study the Markoff-Davenport characteristics of Dirichlet orbits (orbits for the action of the Dirichlet group, see also in [2]) and the relation to the Hessenberg complexity. In Section 4 we define multidimensional Klein-Voronoi continued fractions, that are further used in study of families of matrices with given Hessenberg type. Section 5 contains main results of the work related to the $SL(3,\mathbb{Z})$ case. In particular, we show that in any integer ray consisting only of Hessenberg NRS-matrices only finitely many matrices are not reduced. We also illustrate the results with examples and formulate new questions. Finally, in Section 6 we say a few words about four dimensional case.

Acknowledgment. The author is grateful to V. I. Arnold for constant attention to this work, to H. W. Lenstra and E. I. Pavlovskaya for useful remarks, and Mathematisch Instituut of Universiteit Leiden for the hospitality and excellent working conditions.

2. Basic properties of Hessenberg matrices

In this section we study some basic properties of Hessenberg matrices. First, we show in Subsections 2.1 and 2.2 that for any $SL(n,\mathbb{Z})$-matrix with irreducible characteristic polynomial there exists at least one reduced Hessenberg matrix integer conjugate to the given one. In the majority of the observed examples such matrix is unique. Nevertheless, in some cases that is not true. In Subsections 2.2 we prove that a Hessenberg matrix is uniquely defined by its characteristic polynomial and Hessenberg type. This implies that the number of reduced Hessenberg matrices conjugate to a given one is always finite. In Subsection 2.3 we describe a structure of a family of Hessenberg matrices with a given Hessenberg type.

2.1. Reduction to Hessenberg matrices.

Proposition 2.1. *Any matrix in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial is integer conjugate to a reduced Hessenberg matrix with positive Hessenberg complexity.*

We use the following two lemmas to prove the proposition.

Lemma 2.2. *Any matrix in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial is integer conjugate to a Hessenberg matrix with positive Hessenberg complexity.*

Proof. Let M be a matrix in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial, and A be a linear operator with matrix M in some fixed integer basis. Take any integer vector v of unite integer length and consider a set of vector spaces

$$V_i = \operatorname{Span}\big(v, A(v), A^2(v), \ldots, A^{i-1}(v)\big),$$

for $i = 1, \ldots, n$ (here we denote the span of vectors $v_1, \ldots, v_m$ by $\operatorname{Span}(v_1, \ldots, v_m)$). Since for any integer j the vector $A^j(v)$ is integer, the spaces V_i are integer. Since the characteristic polynomial of A is irreducible, the dimension of V_i equals i and the set of all spaces V_i forms a complete flag in $\mathbb{R}^n$.

Let us describe the following basis of $\mathbb{R}^n$, we denote it by $\{g_i\}$. We choose v as g_1. For any $i>1$ we chose an integer vector $g_i \in V_i$ such that g_i together with all vectors of V_{i-1} generate an integer lattice in V_i. Notice that the choice of g_i is not unique. The vectors $g_1,\ldots,g_n$ form a basis of $\mathbb{R}^n = V_n$. Let $\hat{M} = (\hat{a}_{i,j})$ be the matrix of the operator A in the basis $\{g_i\}$. By construction, this matrix is Hessenberg.

Since the basis vectors $g_1,\ldots,g_n$ generate the integer lattice, the matrix $\hat{M}$ is integer conjugate to the matrix M.

Since the characteristic polynomial of A is irreducible, the integer spaces V_i are not invariant subspaces of A. Hence, the integers $\hat{a}_{i+1,i}$ are non-zero for $i=1,\ldots,n-1$. Therefore, the Hessenberg complexity of $\hat{M}$ is positive. This concludes the proof of Lemma 2.2. □

Lemma 2.3. *Any Hessenberg matrix in $SL(n,\mathbb{Z})$ with positive Hessenberg complexity is integer conjugate to a reduced Hessenberg matrix with positive Hessenberg complexity.*

Proof. Let $M = (a_{i,j})$ be a Hessenberg matrix in $SL(n,\mathbb{Z})$ with positive Hessenberg complexity. Suppose also, that A is an operator with matrix M in some integer basis $\{e_i\}$. Now we construct inductively a reduced Hessenberg matrix that is conjugate to M.

We put $\hat{e}_1$ to be equal to e_1.

Choose $c_{1,1}$ and $\hat{a}_{1,1}$ such that

$$A(\hat{e}_1) = |a_{2,1}|(\mathrm{sign}(a_{2,1})e_2 + c_{1,1}\hat{e}_1) + \hat{a}_{1,1}e_1 \quad \text{and} \quad 0 \le \hat{a}_{1,1} < |a_{2,1}|.$$

Let

$$\hat{e}_2 = \mathrm{sign}(a_{2,1})e_2 + c_{1,1}\hat{e}_1.$$

Suppose, that for some k we have constructed $\hat{e}_i$ for all $i \le k$ and $\hat{a}_{i,j}$ for all $i \le j \le k-1$. Let us construct $\hat{e}_{k+1}$ and $\hat{a}_{i,k}$ for $i=1,\ldots,k$. Choose $c_{i,k}$ and $\hat{a}_{i,k}$ for $i=1,\ldots,k$ such that

$$A(\hat{e}_k) = |a_{k+1,k}|\left(\mathrm{sign}(a_{k+1,k})e_{k+1} + \sum_{i=1}^{k} c_{i,k}\hat{e}_i\right)$$

and $0 \le \hat{a}_{i,k} < a_{k+1,k}$ for $i=1,\ldots,k$. We put

$$\hat{e}_{k+1} = \mathrm{sign}(a_{k+1,k})e_{k+1} + \sum_{i=1}^{k} c_{i,k}\hat{e}_i$$

and calculate the coefficients $\hat{a}_{i,k}$ from the expression for $A(\hat{e}_k)$ in the system of linearly independent vectors $\hat{e}_1,\ldots,\hat{e}_{k+1}$.

The matrix $\hat{M}$ of the operator A in the basis $\{\hat{e}_i\}$ is of Hessenberg type

$$\left\langle \hat{a}_{1,1}, |a_{2,1}| \middle| \hat{a}_{1,2}, \hat{a}_{2,2}, |a_{3,2}| \middle| \cdots \middle| \hat{a}_{1,n-1}, \ldots, \hat{a}_{n-1,n-1}, |a_{n,n-1}| \right\rangle.$$

By the definition, $\hat{M}$ is a perfect Hessenberg matrix.

From the construction it follows that the matrices M and $\hat{M}$ are integer conjugate. So there exists at least one perfect Hessenberg matrix integer conjugate to M. Since the set of values of Hessenberg complexity is discrete and bounded from below, there exist a reduced Hessenberg matrix among the perfect Hessenberg matrices integer conjugate to M. □

Proof of Proposition 2.1. It immediately follows from Lemmas 2.2 and 2.3. □

Corollary 2.4. *Let A be an $SL(n,\mathbb{Z})$-operator and v be any integer vector of unit integer length. Then there exists a unique integer basis $\{g_i\}$ such that*

1) $g_1 = v$;

2) $g_i \in V_i$ *where* $V_i = \operatorname{Span}\big(v, A(v), A^2(v), \ldots, A^{i-1}(v)\big)$;

3) *the matrix M of the operator A in the basis $\{g_i\}$ is perfect Hessenberg.*

Proof. The statement follows directly from the algorithms of Lemmas 2.2 and 2.3. □

2.2. On identification of Hessenberg matrix.

Proposition 2.5. *Any Hessenberg matrix with positive Hessenberg complexity is uniquely defined by its Hessenberg type and the characteristic polynomial.* □

The following lemma implies Proposition 2.5.

Lemma 2.6. *The last row in the Hessenberg matrix with positive Hessenberg complexity is uniquely defined by its Hessenberg type and the coefficients of the characteristic polynomial.*

Proof. Suppose, that we know all the coefficients of the characteristic polynomial and the Hessenberg type for some Hessenberg matrix $M = (a_{i,j})$. Then the elements $a_{i,j}$ for $1 \le i \le n$, $1 \le j \le n$ are uniquely defined by the Hessenberg type of M. Let the characteristic polynomial of M be

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

Direct calculations show, that for any k the coefficient c_k is a polynomial in $a_{i,j}$ variables that does not depend on $a_{1,n}, \ldots, a_{k,n}$. The unique monomial for c_k containing $a_{k+1,n}$ is

$$\left(\prod_{j=k+1}^{n-1} a_{j+1,j}\right) a_{k+1,n}.$$

Since the Hessenberg complexity of M is nonzero, we have uniquely defined expressions for $a_{k+1,n}$ where $k = n-1, n-2, \ldots, 0$. This concludes the proof of the lemma. □

Corollary 2.7. *For any $SL(n,\mathbb{Z})$-matrix M with irreducible characteristic polynomial there exist a finitely many reduced Hessenberg matrices integer conjugate to the given one with bounded from above Hessenberg complexity.*

Proof. The existence of reduced Hessenberg matrices integer conjugate to M follows from Proposition 2.1. Let the Hessenberg complexity of that reduced matrix equal c. Since the conjugate matrices have the same characteristic polynomial and by Proposition 2.5, there exists at most one Hessenberg matrix of a given Hessenberg type conjugate to M. The number of Hessenberg types with Hessenberg complexity equal to c is finite. Thus, there is only a finite number of reduced Hessenberg matrices integer conjugate to M. □

2.3. **A family of Hessenberg matrices with given Hessenberg type.** In this subsection we describe families of $SL(n,\mathbb{Z})$-matrices with given Hessenberg type.

We start with two important definitions of integer lattice geometry.

The *integer volume* of a simplex σ with integer vertices is the index of the sublattice generated by the edges of σ in the lattice of all integer vectors in the plane spanned by σ.

Take an integer vector v and a k-dimensional plane π containing the integer sublattice of rang k such that v in π. The *integer distance* from v to π is the index of the sublattice generated by the integer vectors of the set $\pi \cup \{u\}$ to the whole integer lattice of the $(k+1)$-dimensional plane spanning v and π.

Consider a vector space $\mathbb{R}^n$ with an integer basis $\{e_j\}$. Let us associate to the given Hessenberg type

$$\Omega = \langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \cdots | a_{n-1,1}, \ldots, a_{n-1,2}, a_{n-1,n} \rangle$$

vertices $v_k(\Omega) = (a_{k,1}, \ldots, a_{k,k+1}, 0, \ldots, 0)$ for $k = 1, \ldots, n-1$. Let O be the origin. Denote by $\sigma(\Omega)$ the $(n-1)$-dimensional simplex with vertices O, $v_1, \ldots, v_{n-1}$.

Proposition 2.8. *A Hessenberg matrix $M = H_\Omega(\lambda_1, \ldots, \lambda_n)$ is in $SL(n,\mathbb{Z})$ iff the following conditions hold (in the integer lattice of the integer basis $\{e_i\}$):*

i) *the integer volume of $\sigma(\Omega)$ equals one;*

ii) *the integer distance from the point $(\lambda_1, \ldots, \lambda_n)$ in the basis $\{e_i\}$ to the integer hyperplane containing $\sigma(\Omega)$ equals one.*

Proof. Let A be an operator with matrix M in the basis $\{e_i\}$.

If M is in $SL(n,\mathbb{Z})$, then A preserves all integer volumes and integer distances. Since the integer volume of the simplex with vertices

$$O, O+e_1, \ldots, O+e_{n-1}$$

equals one, the volume of the image $\sigma(\Omega)$ equals one. Since the integer distance from the point $O+e_n$ to the plane spanned by the vectors $e_1, \ldots, e_{n-1}$ equals one, the integer distance from the point $(\lambda_1, \ldots, \lambda_n)$ to the integer hyperplane containing $\sigma(\Omega)$ also equals one.

Suppose, that conditions i) and ii) hold. Then, the operator A takes the integer lattice (generated by vectors $e_1, \ldots, e_n$) to itself bijectively. Therefore, M is in $SL(n,\mathbb{Z})$ □

Denote by $H(\Omega)$ the set of all Hessenberg matrices in $SL(n,\mathbb{Z})$ of the Hessenberg type Ω. For $k = 1, \ldots, n-1$ we denote by $M_k(\Omega)$ the matrix with zero first $n-1$ columns and the last one equals to the coordinates of the vector

$$(a_{k,1}, \ldots, a_{k,2}, a_{k,k+1}, 0, \ldots, 0).$$

Corollary 2.9. *Let Ω be a Hessenberg type satisfying condition i) of Proposition 2.8, and M_0 be in $H(\Omega)$. Then*

$$H(\Omega) = \left\{ M_0 + \sum_{i=1}^{n-1} c_i M_i(\Omega) \middle| c_1, \ldots, c_n \in \mathbb{Z} \right\}.$$

Proof. The statement follows directly from Proposition 2.8. □

3. Dirichlet orbits and Markoff-Davenport characteristics

The study of Markoff-Davenport characteristic is closely related to the theory of minima of absolute values of homogeneous forms with integer coefficients in n-variables of degree n. One of the first works on minima of such forms was written by A. Markoff [22] for the case $n = 2$ for the forms decomposable into the product of two real linear forms. Further, H. Davenport in series of works [4], [5], [6], [7], and [8] made the first steps for the case of decomposable forms for $n = 3$.

In this section we give the definition of Markoff-Davenport characteristic. In Subsection 3.2 we show a relation between values of Markoff-Davenport characteristic and the Hessenberg complexities of Hessenberg matrices of an operator. In Subsection 3.3 we show that Markoff-Davenport characteristic is an absolute value of some homogeneous form of degree equal to the rank of the operator.

3.1. **Dirichlet orbits.** Consider any operator A of $SL(n, \mathbb{Z})$ with irreducible characteristic polynomial over the field of rational numbers.

Definition 3.1. The group of all $SL(n, \mathbb{Z})$-operators commuting with A with positive eigenvectors is called *the Dirichlet group* and denoted by $\Xi(A)$.

Example 3.2. Consider an operator $A_0 = A_{\langle 0,1|0,0,1|0,0,0,1\rangle}(1, -4, 1, 4)$. The Dirichlet group $\Xi(A_0)$ is isomorphic to $\mathbb{Z}^3$ and it is generated by the operators A_0^2, $(E - A_0)^2$, and $A_0^3 + A_0$.

By Dirichlet unity theorem the Dirichlet group $\Xi(A)$ is homomorphic to some free Abelian group. An orbit of the Dirichlet group consisting of integer points is said to be a *Dirichlet orbit.*

3.2. **Markoff-Davenport characteristic.** Let us start with the definition.

Definition 3.3. Let A be an $SL(n, \mathbb{Z})$-operator. The *Markoff-Davenport characteristic* (or *MD-characteristic*, for short) of an operator A is a functional defined on the space of all vectors by the following rule: at any vector v it takes the nonoriented integer volume of the pyramid with vertex at the origin and base with vertices v, $A(v)$, …, $A^{n-1}(v)$. We denote the MD-characteristic at vector v by $\Delta(A|v)$.

Further in Subsection 3.3 we show that MD-characteristic is a functional coinciding to the absolute value of some homogeneous form of degree equal to the rank of the operator.

Remark 3.4. The MD-characteristic naturally defines a functional over the set of Dirichlet orbits for the given operator, since the MD-characteristics at any two vectors of the same Dirichlet orbit coincide.

Proposition 3.5. *Consider an operator A with Hessenberg matrix M in some integer basis $\{e_i\}$. Then the Hessenberg complexity $\varsigma(M)$ equals the value of MD-characteristic $\Delta(A|e_1)$.* □

Proof. Suppose, that the Hessenberg type of the matrix M is

$$\big\langle a_{1,1}, a_{1,2} \big| a_{2,1}, a_{2,2}, a_{2,3} \big| \cdots \big| a_{n-1,1}, \ldots, a_{n-1,2}, a_{n-1,n} \big\rangle.$$

Denote by V_k the plane Span $\big(v, A(v), A^2(v), \ldots, A^{k-1}(v)\big)$.

Let us inductively show, that $A^k(e_1) = \left(\prod_{i=1}^k a_{i,i+1}\right) e_{k+1} + v_k$ where v_k is in V_k.

We have $A(e_1) = a_{1,2}e_2 + a_{1,1}e_1$ as the base of induction.

Suppose, that the statement holds for $k = m$, i.e., $A^m(e_1) = (\prod_{i=1}^m a_{i,i+1}) e_{m+1} + v_m$ and v_m is in V_m. Let us show the statement for $m+1$. Since M is Hessenberg, $A(v_m)$ is in V_{m+1}. Therefore, we have

$$\begin{aligned} A^{m+1}(e_1) &= A\left(\left(\prod_{i=1}^m a_{i,i+1}\right) e_{m+1}\right) + A(v_m) = \\ & \left(\prod_{i=1}^{m+1} a_{i,i+1}\right) e_{m+1} + \left(A(v_m) + \left(\prod_{i=1}^m a_{i,i+1}\right)\big(A(e_{m+1}) - a_{m+1,m+2}e_{m+2}\big)\right). \end{aligned}$$

The second additive in the last expression is in V_{m+1}. We have proven our statement. Therefore,

$$\Delta(A|e_1) = \prod_{i=1}^{n-1} |a_{i+1,i}|^{n-i} = \varsigma(M).$$

This concludes the proof of the lemma. □

The proposition implies that for any operator A and any integer d, there exist only finitely many Dirichlet orbits with MD-characteristic equal to d.

3.3. **Homogeneous forms associated to $SL(n,\mathbb{Z})$-operators.** Let $\{e_i\}$ be an integer basis of $\mathbb{R}^n$. Consider any $SL(n,\mathbb{Z})$-operator A with irreducible characteristic polynomial. Suppose, that it has k real roots $r_1, \ldots, r_k$ and $2l$ complex conjugate roots $c_1, \bar{c}_1, \ldots, c_l, \bar{c}_l$, here $k + 2l = n$. Let us now define a new basis of vectors $g_1, \ldots, g_{k+2l}$ in the following way. For $i = 1, \ldots, k$ we choose g_i to be some eigenvector corresponding to the eigenvalue r_i. For $j = 1, \ldots, l$ we choose g_{k+2j-1} and g_{k+2j} to be the real and the imaginary parts of some complex eigenvector corresponding to the eigenvalue c_j. Consider the system of coordinates

$$OX_1X_2\ldots X_kY_1Z_1Y_2Z_2\ldots Y_lZ_l$$

corresponding to the basis $\{g_i\}$.

A form

$$\alpha\left(\prod_{i=1}^k x_i \prod_{j=1}^l (y_j^2 + z_j^2)\right)$$

with nonzero α is said to be *associated* to the operator A.

Theorem 3.6. *Let A be an $SL(n,\mathbb{Z})$-operator with irreducible characteristic polynomial. Then the MD-characteristic of A is an absolute value of some form associated to A.*

Proof. Let us consider the formulas of MD-characteristic of A in the eigen-basis of vectors

$$g_1, \ldots, g_k, g_{k+1}+Ig_{k+2}, g_{k+1}-Ig_{k+2}, \ldots, g_{k+2l-1}+Ig_{k+2l}, g_{k+2l-1}-Ig_{k+2l}$$

in $\mathbb{C}^n$, where $I = \sqrt{-1}$. Let the coordinates in this eigen-basis be $\{t_i\}$.

Then for any vector $v=(t_1,\ldots,t_n)$ we have

$$A^j(x)=(r_1^jt_1,\ldots r_k^jt_k,c_1^jt_{k+1},\overline{c}_1^jt_{k+2},\ldots c_l^jt_{k+2l-1},\overline{c}_l^jt_{k+2l}).$$

Therefore,

$$\Delta(A|(t_1,\ldots,t_n))=\alpha\left|\prod_{i=1}^{k}t_i\prod_{j=1}^{l}(t_{k+2j-1}t_{t+2j})\right|=\frac{\alpha}{4^l}\left|\prod_{i=1}^{k}x_i\prod_{j=1}^{l}(y_j^2+z_j^2)\right|$$

Simple calculations show that $\alpha\neq 0$. □

4. Multidimensional continued fractions in the sense of Klein-Voronoi

Here we describe multidimensional continued fractions in the sense of Klein-Voronoi. We use these continued fractions to find the minimal value of the MD-characteristic for a given operator on the lattice of integer points except the origin.

In 1839 C. Hermite [9] posed a problem of generalizing ordinary continued fractions to the higher-dimensional case. Since that there were given many different definitions generalizing different properties of ordinary continued fractions. A nice geometrical generalization of ordinary continued fraction for a particular totally-real case was made by F. Klein in [15] and [16]. We refer for a nice description, properties, and examples of multidimensional continued fractions in the sense of Klein to the books by V. I. Arnold [1] and G. Lachaud [19] and papers by E. I. Korkina [18], M. L. Kontsevich and Yu. M. Suhov [17], and the author [11], [12]. Approximately at the same time of the works by F. Klein G. Voronoi introduced a nice geometric algorithmic definition for both complex and totally real three-dimensional cases in his dissertation [23]. In [3] J. A. Buchmann generalized Voronoi's algorithm making it more convenient for computation of fundamental units in orders.

We use ideas of J. A. Buchmann to define *multidimensional continued fraction in the sense of Klein-Voronoi* for all the cases. Note that in our definitions Klein's multidimensional continued fraction is a totally real case of continued fractions in the sense of Klein-Voronoi.

4.1. **General definitions.** Consider any real operator A of $SL(n,\mathbb{R})$ whose roots are all distinct. Suppose, that it has k real roots $r_1,\ldots,r_k$ and $2l$ complex conjugate roots $c_1,\overline{c}_1,\ldots,c_l,\overline{c}_l$, here $k+2l=n$.

Denote by $T^l(A)$ the set of all real operators commuting with A such that their real eigenvalues are all unit and the absolute values for all complex eigenvalues equal one. Actually, $T^l(A)$ is an abelian group with operation of matrix multiplication.

For a vector v in $\mathbb{R}^n$ we denote by $T_A(v)$ the orbit of v with respect of the action of the group of operators $T^l(A)$. If v is in general position with respect to the operator A (i.e. it does not lie in invariant planes of A), then $T_A(v)$ is homeomorphic to the l-dimensional torus. For a vector of an invariant plane of A the orbit $T_A(v)$ is also homeomorphic to a torus of positive dimension not greater than l, or to a point.

For instance, if v is a real eigenvector, then $T_A(v) = \{v\}$. The second example: if v is in a real hyperplane spanned by two complex conjugate eigenvectors, then $T_A(v)$ is an ellipse.

As before, let g_i be a real eigenvector with eigenvalue r_i for $i = 1, \ldots, k$; g_{k+2j-1} and g_{k+2j} be vectors corresponding to the real and imaginary parts of some complex eigenvector with eigenvalue c_j for $j = 1, \ldots, l$. Again we consider the coordinate system corresponding to the basis $\{g_i\}$:

$$OX_1X_2 \ldots X_kY_1Z_1Y_2Z_2 \ldots Y_lZ_l.$$

Denote by π the $(k+l)$-dimensional plane $OX_1X_2 \ldots X_kY_1Y_2 \ldots Y_l$. Let π_+ be the cone in the plane π defined by the equations $y_i \geq 0$ for $i = 1, \ldots, l$. For any v the orbit $T_A(v)$ intersects the cone π_+ in a unique point.

Definition 4.1. A point p in the cone π_+ is said to be *π-integer* if the orbit $T_A(p)$ contains at least one integer point.

Consider all (real) hyperplanes invariant under the action of the operator A. There are exactly k such hyperplanes. In the above coordinates the i-th of them is defined by the equation $x_i = 0$.

The complement to the union of all invariant hyperplanes in the cone π_+ consists of 2^k arcwise connected components. Consider one of them.

Definition 4.2. The convex hull of all π-integer points except the origin contained in the given arcwise connected component is called a *factor-sail* of the operator A. The set of all factor-sails is said to be the *factor-continued fraction* for the operator A.
The union of all orbits $T_A(*)$ in $\mathbb{R}^n$ represented by the points in the factor-sail is called the *sail* of the operator A. The set of all sails is said to be the *continued fraction* for the operator A (in the sense of Klein-Voronoi).

The intersection of the factor-sail with a hyperplane in π is said to be an m-dimensional face of the factor-sail if it is homeomorphic to the m-dimensional disc.

The union of all orbits in $\mathbb{R}^n$ represented by points in some face of the factor-sail is called the *orbit-face* of the operator A.

Integer points of the sail are said to be *vertices* of this sail.

4.2. **Algebraic continued fractions.** Consider now an operator A in the group $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial. Suppose, that it has k real roots $r_1, \ldots, r_k$ and $2l$ complex conjugate roots: $c_1, \bar{c}_1, \ldots, c_l, \bar{c}_l$, where $k + 2l = n$. In the simplest possible cases $k+l = 1$ any factor-sail of A is a point. If $k+l > 1$, than any factor-sail of A is an infinite polyhedral surface homeomorphic to $\mathbb{R}^{k+l-1}$.

The Dirichlet group $\Xi(A)$ defined in Subsection 3.1 takes any sail of A to itself. By Dirichlet unit theorem the group $\Xi(A)$ is homomorphic to $\mathbb{Z}^{k+l-1}$ and its action on any sail is free. The quotient of a sail by the action of $\Xi(A)$ is homeomorphic to the $(n-1)$-dimensional torus. By a *fundamental domain* of the sail we mean a collection of open orbit-faces such that for any $\Xi(A)$-orbit of orbit-faces of the sail there exists a unique representative in the collection.

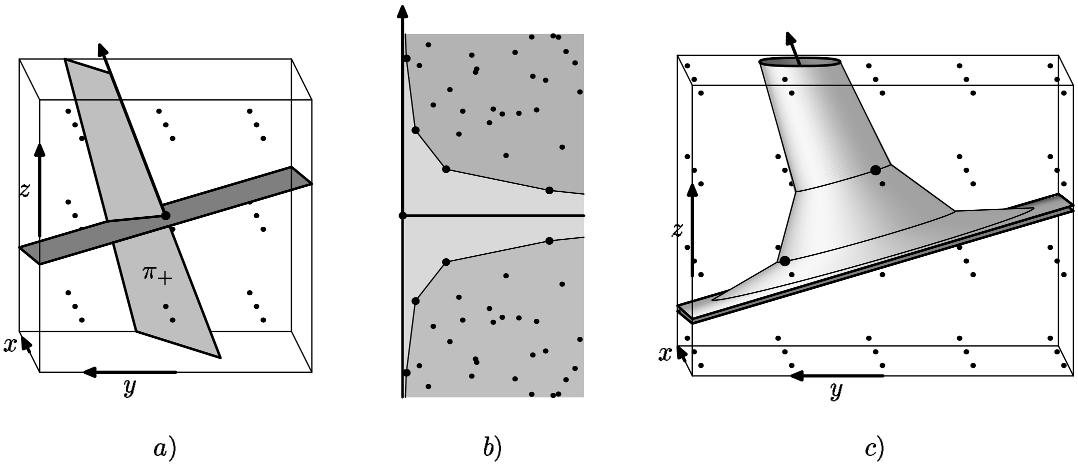


FIGURE 2. A tree-dimensional example: a) the cone π_+ and the eigenplane; b) the continued factor-fraction; c) the continued fraction.

Example 4.3. Let us study an operator A with a Frobenius matrix

$$H_{\langle 0,1|0,0,1\rangle}(1,3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

This operator has one real and two complex conjugate eigenvalues. Therefore, the cone π_+ for A is a two-dimensional half-plane. In Figure 2a the halfplane π_+ is colored in light gray and the invariant plane corresponding to the couple of complex eigenvectors is in dark gray. The vector shown in Figure 2a with endpoint at the origin is an eigenvector of A.

In Figure 2b we show the cone π_+. The invariant plane separates π_+ onto two parts. The dots on π_+ are the π-integer points. The boundaries of the convex hulls in each part of π_+ are two factor-sails. Actually, the sail corresponding to one factor is taken to the sail corresponding to the other by the operator $-E$, where E is an identity operator of $\mathbb{R}^3$.

Finally, in Figure 2c we show one of the sails. Tree orbit-vertices shown in the figure corresponds to the vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. You can see the large dark points $(0,1,0)$ and $(0,0,1)$ lying on the corresponding orbit-vertices.

The Dirichlet group in our example is homeomorphic to $\mathbb{Z}$ and it is generated by A. The operator A takes the point $(1,0,0)$ and its orbit-vertex to the point $(0,1,0)$ and the corresponding orbit-vertex. Therefore, a fundamental domain of the operator A contains one orbit-vertex and one vertex edge. For instance, we can choose the orbit-vertex corresponding to the point $(1,0,0)$ and the orbit-edge corresponding to the "tube" connecting orbit-vectors for the points $(1,0,0)$ and $(0,1,0)$.

4.3. **Fundamental domains of the sails and reduced Hessenberg matrices.** Let us show how to use Klein-Voronoi continued fractions in the study of reduced Hessenberg matrices.

Theorem 4.4. *For any sail of the multidimensional continued fraction in the sense of Klein-Voronoi of the operator A we choose one of its fundamental domains. The union of all these fundamental domains contains vertices at which the MD-characteristic $\Delta(A|(\lambda_1,\ldots,\lambda_n))$ attains the minimal values over the integer points of $\mathbb{R}^n$ except the origin.*

Remark 4.5. Any reduced Hessenberg matrix for the operator A can be constructed starting from some vertex in a fundamental domain of the multidimensional continued fraction in the sense of Klein-Voronoi. One should construct the Klein-Voronoi continued fraction for the operator A. Then one takes all vertices of the chosen fundamental domains for all the sails for A, and finds a vertex v with minimal value of the MD-characteristic (note that the set of such points is finite). By Corollary 2.4 we reconstruct a basis $\{e_i\}$ where $e_1 = v$ in which the operator A has a perfect Hessenberg matrix M. By Proposition 3.5 the Hessenberg complexity of any Hessenberg matrix M' in basis $\{e_i'\}$ equals to MD-characteristic $\Delta(A|e_1')$. By Theorem 4.4 the vector v is one of the absolute minima of the MD-characteristic over the integer points of $\mathbb{R}^n$ except the origin. Therefore, the perfect Hessenberg matrix M is reduced.

Proof. Let A be an $SL(n,\mathbb{Z})$-operator with irreducible characteristic polynomial. By Theorem 3.6 there exists a nonzero constant α such that MD-characteristic at any point in the system of coordinates $OX_1X_2\ldots X_kY_1Z_1Y_2Z_2\ldots Y_lZ_l$ is

$$F(x_1,\ldots,x_k,y_1,z_1,\ldots,y_l,z_l) = \alpha \left| \prod_{i=1}^{k} x_i \prod_{i=1}^{l} (y_i^2+z_i^2) \right|$$

for some positive α. Suppose, that the minimal absolute value of F on the set of integer points except the origin equals m_0.

Choose the coordinates $OX_1\ldots X_kY_1Y_2,\ldots,Y_l$ in the cone π_+. Consider a projection of $\mathbb{R}^n$ to the cone along the $T_A(v)$ orbits. Since we project along the $T_A(v)$ orbits on which the MD-characteristic is constant, the projection of the MD-characteristic is well-defined. In the chosen coordinates of π_+ it is written as follows:

$$\alpha\Big|\prod_{j=1}^{k} x_j \prod_{j=1}^{l} y_j^2\Big|.$$

The obtained function is convex in any orthant of the cone π_+. Let the minimal value of the MD-characteristic equals m. Then for any π-integer point v except the origin the value of the function is not less than m. Therefore, the m-th level of this function intersects the convex hull of all π-integer points of some open orthant in its vertices. By definition, these vertices are the vertices of some factor-sail, and the corresponding integer points are the integer points of the sail. Since the value of the MD-characteristic is invariant under the Dirichlet group action, any fundamental domain of the corresponding sail of the continued fraction contains vertices at which the MD-characteristic attains the minimal value over the integer points of $\mathbb{R}^n$ except the origin. □

5. Three-dimensional algebraic matrices

In this section we work with algebraic $SL(3,\mathbb{Z})$-matrices with irreducible characteristic polynomials. We start in Subsection 5.1 with formulation of the statement in two-dimensional case. In Subsection 5.2 we describe some properties of the subset of NRS-matrices in the family of all algebraic matrices with given Hessenberg type. Further in Subsection 5.3 we use Klein-Voronoi continued fractions to show that in any ray consisting of Hessenberg NRS-matrices only finitely many matrices are not reduced. In Subsection 5.4 we illustrate the obtained results in a few examples. Finally, in Subsection 5.5 we make a brief analysis of all the observed examples and formulate related questions for further study.

5.1. **A few words about two-dimensional case.** First, we recall a situation in two-dimensional case. For a Hessenberg type $\Omega = \langle a_{1,1}, a_{2,1}\rangle$ we consider a family $H(\Omega)$ of Hessenberg matrices

$$H_{\langle a_{1,1},a_{1,2}\rangle}(m) = \begin{pmatrix} a_{1,1} & a_{1,2}+ma_{1,1} \\ a_{2,1} & a_{2,2}+ma_{2,1} \end{pmatrix},$$

where $0 \le a_{1,1} < a_{1,2}$; $a_{1,1}$ and $a_{1,2}$ are relatively prime. Without loose of generality we also choose $a_{2,2}$ satisfying $0 < a_{2,2} \le a_{2,1}$. Now $a_{1,2}$ and $a_{2,2}$ are uniquely defined by $a_{1,1}$ and $a_{2,1}$.

Note that the Hessenberg complexity of $H_{\langle a_{1,1},a_{1,2}\rangle}(m)$ equals $a_{2,1}$.

The following statement holds in the two-dimensional case.

Statement 5.1. *For any Hessenberg type $\Omega = \langle a_{1,1}, a_{1,2}\rangle$ the family $H(\Omega)$ contains*

i) finitely many matrices with reducible over the field of rational numbers characteristic polynomial;

i) finitely many matrices with two complex conjugate eigenvalues;

ii) finitely many nonreduced matrices with all real eigenvalues. □

In other words, almost all matrices in H_Ω are reduced and has two real eigenvalues.

Remark. We omit the proof that needs additional definitions and statements. For instance, the proof can be easily deduced from Theorem 6 of [13].

Example 5.2. Consider the matrices of the Hessenberg type $\langle 2,5\rangle$. Let

$$H_{\langle 2,5\rangle}(m) = \begin{pmatrix} 2 & 1+2m \\ 5 & 3+5m \end{pmatrix}.$$

The Hessenberg complexity of all these matrices is 5. There are no matrices with reducible characteristic polynomial in the family. The matrix with $m=-1$ has non-real spectrum, and the matrices with $m=-3,-2,0,1$ are nonreduced. All the rest matrices of that type are reduced, have two real eigenvalues, and with irreducible characteristic polynomial.

For further information about the two-dimensional case we refer the reader, for instance, to the works [20], [21], and [13].

5.2. **Asymptotic structure of the subset of perfect Hessenberg NRS-matrices.** Note that an $SL(3,\mathbb{Z})$-matrix has a reducible characteristic polynomial iff one of its eigenvalues equals to ± 1.

In Proposition 2.1 we have shown that any $SL(3,\mathbb{Z})$-matrix is integer conjugate to one of the reduced Hessenberg matrices of the form:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & a_{3,2} & a_{3,3} \end{pmatrix},$$

where $0 \le a_{1,2} < a_{2,1}$, $0 \le a_{1,2} < a_{3,2}$, and $0 \le a_{2,2} < a_{3,2}$.

Consider the family $H(\Omega)$ for a Hessenberg type $\Omega = \langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2}\rangle$. Denote the subset of all NRS-matrices in $H(\Omega)$ by $El(\Omega)$.

For the given Hessenberg type Ω we choose the integers $a_{1,3}$, $a_{2,3}$, and $a_{3,3}$ such that the determinant of the matrix $M_0 = (a_{i,j})$ is unit. As we have shown in Corollary 2.9 the set $H(\Omega)$ is a family

$$\left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,1}m + a_{1,2}n + a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,1}m + a_{2,2}n + a_{2,3} \\ 0 & a_{3,2} & a_{3,2}n + a_{3,3} \end{pmatrix} \middle| \, m \in \mathbb{Z}, n \in \mathbb{Z} \right\}.$$

By $H_\Omega(m_0, n_0)$ we denote the matrix with integer parameters $m = m_0$ and $n = n_0$ in the family $H(\Omega)$. Further in the proofs we use also matrices $H_\Omega(m_0, n_0)$ with real parameters, i.e., the matrices in the real affine two-dimensional plane spanned by $H(\Omega)$ in the real affine space of all matrices. We denote by OMN the coordinate system corresponding to the parameters (m,n), the origin O here corresponds to the matrix $H_\Omega(0,0)$.

Denote the discriminant of the characteristic polynomial of $H_\Omega(m,n)$ by $\mathcal{D}_\Omega(m,n)$. So the set $El(\Omega)$ coincides with the set of integer solutions of the inequality

$$\mathcal{D}_\Omega(m,n) < 0$$

in variables m and n.

Example 5.3. We show in Figure 3 the subset of NRS-matrices $El(\Omega)$ for the Hessenberg type $\Omega = \langle 0,1|0,0,1\rangle$. For this example we fix $(a_{1,3}, a_{2,3}, a_{3,3}) = (0,0,1)$.

In both examples shown in Figure 3 and in Figure 1 on page 5 we can see that the set $El(\Omega)$ "looks like" the set of integer points in the union of the convex hulls of two parabolas.

Let us formulate a precise statement. Suppose, that the characteristic polynomial of the matrix $H_\Omega(0,0) = (a_{i,j})$ in the variable t equals

$$-t^3 + b_1 t^2 - b_2 t + b_3.$$

In the case of $SL(3,\mathbb{Z})$ we have $b_3 = 1$, nevertheless we continue to write b_3 for possible use for matrices with distinct determinants.

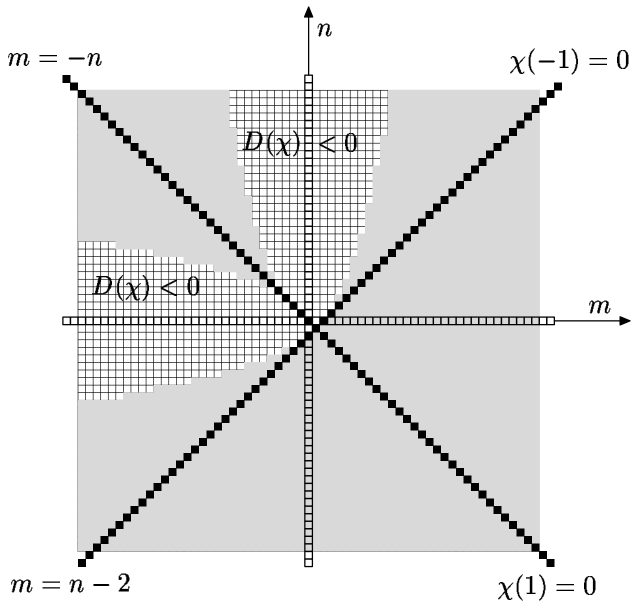


FIGURE 3. The family of matrices of Hessenberg type $\langle 0,1|0,0,1\rangle$.

For a family $H_\Omega(m,n)$ we define the following two quadratic functions

$$p_{1,\Omega}(m,n) = m - \alpha_1 n^2 - \beta_1 n - \gamma_1;$$
$$p_{2,\Omega}(m,n) = \frac{n}{a_{2,1}} - \alpha_2\Big(\frac{a_{2,1}m - a_{1,1}n}{a_{2,1}}\Big)^2 - \beta_2\Big(\frac{a_{2,1}m - a_{1,1}n}{a_{2,1}}\Big) - \gamma_2,$$

where

$$\begin{cases} \alpha_1 = -\dfrac{a_{3,2}}{4a_{2,1}} \\ \beta_1 = \dfrac{a_{1,1} - a_{2,2} - a_{3,3}}{2a_{2,1}} \\ \gamma_1 = \dfrac{4b_2 - b_1^2}{4a_{2,1}a_{3,2}} \end{cases}; \qquad \begin{cases} \alpha_2 = \dfrac{a_{3,2}a_{2,1}}{4b_3} \\ \beta_2 = -\dfrac{b_2}{2b_3} \\ \gamma_2 = \dfrac{b_2^2 - 4b_1b_3}{4a_{2,1}a_{3,2}b_3} \end{cases}.$$

Denote by $B_R(O)$ an interior of the circle of radius R centered at the origin $(0,0)$ in the real plane OMN of the family $H_\Omega(m,n)$. We denote also

$$\Lambda_\varepsilon = \{(m,n) | (p_{1,\Omega}(m,n) - \varepsilon)(p_{2,\Omega}(m,n) - \varepsilon) < 0\}.$$

Theorem 5.4. *For any positive ε there exists a positive R such that in the complement to the $B_R(O)$ the following inclusions hold*

$$\Lambda_\varepsilon \subset El(\Omega) \subset \Lambda_{-\varepsilon}.$$

Before to start the proof we make the following remark. The set $El(\Omega)$ is defined by the inequality

$$\mathcal{D}_\Omega(m,n) < 0.$$

In the left part of the inequality there is a polynomial of degree 4 in variables m and n. Note that the product $16a_{2,1}^2a_{3,2}^2b_3\big(p_{1,\Omega}(m,n)p_{2,\Omega}(m,n)\big)$ is a good approximation to $\mathcal{D}_\Omega(m,n)$ at infinity: the polynomial

$$\mathcal{D}_\Omega(m,n) - 16a_{2,1}^2a_{3,2}^2b_3\big(p_{1,\Omega}(m,n)p_{2,\Omega}(m,n)\big)$$

is a polynomial of degree 2 in variables m and n.

Lemma 5.5. *The curve $\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ is contained in the domain defined by the inequalities:*

$$\begin{cases} (m^2-4n+3)(n^2+4m+3) \geq 0 \\ (m^2-4n-3)(n^2+4m-3) - 72 \leq 0 \end{cases}$$

Remark. Lemma 5.5 implies that the curve $\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ is contained in some tubular neighborhood of the curve

$$(m^2-4n)(n^2+4m) = 0.$$

Proof. Note that

$$\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) = (m^2-4n)(n^2+4m) - 2mn - 27.$$

Thus, we have

$$\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) - (m^2-4n+3)(n^2+4m+3) = -2(n-3)^2 - 2(m+3)^2 - (n+m)^2 \leq 0,$$

and

$$\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) - (m^2-4n-3)(n^2+4m-3) + 72 = 2(n-3)^2 + 2(m+3)^2 + (n-m)^2 \geq 0.$$

Therefore, the curve $\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ is contained in the domain defined in the lemma. □

Lemma 5.6. *For any $\Omega = \langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32}\rangle$ there exists a (not necessary integer) affine transformation of the plane OMN taking the curve $\mathcal{D}_\Omega(m,n) = 0$ to the curve $\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$.*

Proof. Let $H_\Omega(0,0) = (a_{i,j})$. Note that a matrix $H_\Omega(m,n)$ is rational conjugate to the matrix

$$H_{\langle 0,1|0,0,1\rangle}(a_{23}a_{32} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{11}a_{22} + a_{21}a_{32}m - a_{11}a_{32}n,\ a_{11} + a_{22} + a_{33} + a_{32}n)$$

by the matrix

$$X_\Omega = \begin{pmatrix} 1 & a_{1,1} & a_{1,1}^2 + a_{1,2}a_{2,1} \\ 0 & a_{2,1} & a_{1,1}a_{2,1} + a_{2,1}a_{2,2} \\ 0 & 0 & a_{2,1}a_{3,2} \end{pmatrix}.$$

Since both matrices

$$H_\Omega(m,n) \quad \text{and} \quad X_\Omega^{-1}\big(H_\Omega(m,n)\big)X_\Omega$$

have the equivalent characteristic polynomials, their discriminants coincide. Therefore, the curve $\mathcal{D}_\Omega = 0$ is mapped to the curve $\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ bijectively.

In OMN coordinates this map corresponds to the following affine transformation

$$\begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} a_{21}a_{32}m - a_{11}a_{32}n \\ a_{32}n \end{pmatrix} + \begin{pmatrix} a_{23}a_{32} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{11}a_{22} \\ a_{11} + a_{22} + a_{33} \end{pmatrix}.$$

This completes the proof of the lemma. □

Proof of Theorem 5.4. Consider a family of matrices $H_\Omega(-p_{1,\Omega}(0,t)+\varepsilon, t)$ with real parameter t. Direct calculations show that for $\varepsilon \neq 0$ the discriminant of the matrices for this family is a polynomial of the forth degree in variable t, and

$$\mathcal{D}_\Omega(-p_{1,\Omega}(0,t)+\varepsilon,t) = \frac{1}{4}a_{2,1}a_{3,2}^5\varepsilon t^4 + O(t^3).$$

Therefore, there exists a neighborhood of infinity with respect to the variable t such that the function $\mathcal{D}_\Omega(-p_{1,\Omega}(0,t)+\varepsilon,t)$ is positive for positive ε in the neighborhood, and negative for negative ε.

Hence for a given ε there exists a sufficiently large $N_1 = N_1(\varepsilon)$ such that for any $t > N_1$ there exists a solution of the equation $\mathcal{D}_\Omega(m,n) = 0$ at the segment with endpoints

$$\big(-p_{1,\Omega}(0,t)+\varepsilon,t\big) \quad \text{and} \quad \big(-p_{1,\Omega}(0,t)-\varepsilon,t\big)$$

of the plane OMN.

Now we examine the family in variable t for the second parabola:

$$H_\Omega\left(t - a_{1,1}p_{2,\Omega}(t,0) - \frac{a_{1,1}}{\sqrt{a_{1,1}^2+a_{2,1}^2}}\varepsilon,\ -a_{2,1}p_{2,\Omega}(t,0) - \frac{a_{2,1}}{\sqrt{a_{1,1}^2+a_{2,1}^2}}\varepsilon\right).$$

By the same reasons, for a given ε there exists a sufficiently large $N_2 = N_2(\varepsilon)$ such that for any $t > N_2$ there exists a solution of the equation $\mathcal{D}_\Omega(m,n) = 0$ at the segment with endpoints

$$\begin{array}{c} \big(t - a_{1,1}p_{2,\Omega}(t,0) - \frac{a_{1,1}}{\sqrt{a_{1,1}^2+a_{2,1}^2}}\varepsilon,\ -a_{2,1}p_{2,\Omega}(t,0) - \frac{a_{2,1}}{\sqrt{a_{1,1}^2+a_{2,1}^2}}\varepsilon\big) \quad \text{and} \\ \big(t - a_{1,1}p_{2,\Omega}(t,0) + \frac{a_{1,1}}{\sqrt{a_{1,1}^2+a_{2,1}^2}}\varepsilon,\ -a_{2,1}p_{2,\Omega}(t,0) + \frac{a_{2,1}}{\sqrt{a_{1,1}^2+a_{2,1}^2}}\varepsilon\big) \end{array}$$

of the plane OMN.

We have shown that for any of the four branches of the two parabolas defined by $p_{1,\Omega}$ and $p_{2,\Omega}$ there exists (at least) one branch of $\mathcal{D}_\Omega(m,n) = 0$ contained in the ε-tube of the chosen parabolic branch if we are far enough from the origin.

In Lemma 5.5 we have obtained that $\mathcal{D}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ is contained in some tubular neighborhood of $p_{1,\langle 0,1|0,0,1\rangle}(m,n)p_{2,\langle 0,1|0,0,1\rangle}(m,n) = 0$. Then by Lemma 5.6 we have, that the curve $\mathcal{D}_\Omega(m,n) = 0$ is contained in some tubular neighborhood of the curve $p_{1,\Omega}(m,n)p_{2,\Omega}(m,n) = 0$ outside some ball centered at the origin. Finally, by Viet Theorem, the intersection of the curve $\mathcal{D}_\Omega(m,n) = 0$ with each of the parallel lines

$$\ell_t : \quad \frac{a_{1,1}+a_{2,1}}{a_{2,1}}n - m = t$$

contains at most four points. Therefore, there exists sufficiently large T such that for any $t \geq T$ the intersection of the curve $\mathcal{D}_\Omega(m,n) = 0$ and ℓ_t contains exactly four points corresponding to the branches of the parabolas $p_{1,\Omega}(m,n)=0$ and $p_{2,\Omega}(m,n)=0$ lying in $\Lambda_{-\varepsilon}\setminus\Lambda_{\varepsilon}$.

Hence, there exists $R=R(\varepsilon,N_1,N_2,T)$ such that in the complement to the ball $B_R(O)$ we have

$$\Lambda_\varepsilon\subset El(\Omega)\subset\Lambda_{-\varepsilon}.$$

The proof of Theorem 5.4 is completed. □

5.3. **Asymptotic behaviour of Klein-Voronoi continued fractions for the NRS-case.** In the previous subsection we have shown that for any Hessenberg type $\Omega=\langle a_{1,1},a_{2,1}|a_{1,2},a_{2,2},a_{3,2}\rangle$ the set of Hessenberg perfect NRS-matrices $El(\Omega)$ almost coincides with the union of the convex hulls of two parabolas. We say that the set $El(\Omega)$ has "two asymptotic directions" that corresponds to the symmetry lines of the parabolas. These directions are defined by the vectors $(-1,0)$ and $(a_{1,1},a_{2,1})$. To be precise, an integer ray in OMN is said to be *an NRS-ray* if all its integer points correspond to reduced Hessenberg NRS-matrices. A direction is said to be *asymptotic* for the set $El(\Omega)$ if there exists an NRS-ray with this direction.

Theorem 5.4 implies the following proposition.

Proposition 5.7. *There are exactly two asymptotic directions for the set $El(\Omega)$, they are defined by the vectors $(-1,0)$ and $(a_{1,1},a_{2,1})$.* □

We will use the following notation. First, we fix some basis $\{e_1,e_2,e_3\}$ in $\mathbb{R}^3$. Let $\Omega=\langle a_{1,1},a_{2,1}|a_{1,2},a_{2,2},a_{3,2}\rangle$ be a Hessenberg type. For any couple of real numbers (m,n) and nonnegative t we denote the operators $R^{1,\Omega}_{m,n}(t)$ and $R^{2,\Omega}_{m,n}(t)$ with matrices:

$$H_\Omega(m-t,n)\quad\text{and}\quad H_\Omega(m+a_{1,1}t,n+a_{2,1}t)$$

respectively. So $R^{1,\Omega}_{m,n}$ and $R^{1,\Omega}_{m,n}$ are two families of operators with nonnegative parameter t, corresponding to two rays of matrices in the plane H_Ω. The directions of these rays are asymptotic: $(-1,0)$ and $(a_{1,1},a_{2,1})$ respectively.

Now we are ready to formulate and to prove the following theorem.

Theorem 5.8. On assymptotics for NRS-rays. *Any NRS-ray for the set $El(\Omega)$ contains only finitely many non-reduced matrices. Any such ray contains only finitely many reduced matrices integer conjugate to some other reduced matrices.*

Example 5.9. Any NRS-ray for the Hessenberg type $\langle 0,1|0,0,1\rangle$ contains only reduced perfect matrices. Experiments show that any NRS-ray for $\langle 0,1|1,0,2\rangle$ contains at most one non-reduced matrix.

Proof. First, we prove the theorem for NRS-rays with asymptotic direction $(-1,0)$.

Let us begin with the case of real matrices of Hessenberg type $\Omega_0=\langle 0,1|0,0,1\rangle$. Such matrices form a family $H(\Omega_0)$ with real parameters m and n as before:

$$H_{\langle 0,1|0,0,1\rangle}(m,n)=\begin{pmatrix}0&0&1\\1&0&m\\0&1&n\end{pmatrix}.$$

Lemma 5.10. *Let $R^{1,\Omega}_{m,n}$ be a family of operators having NRS-matrices with nonnegative integer parameter t. Then for any $\varepsilon > 0$ there exists a positive constant such that for any t greater than this constant the convex hull of the union of two orbit-vertices*

$$T_{R^{1,\Omega_0}_{m,n}(t)}(1,0,0) \quad \textit{and} \quad T_{R^{1,\Omega_0}_{m,n}(t)}(0,1,0)$$

is contained in the ε-tubular neighborhood of the convex hull of three points $(1,0,0)$, $(0,1,0)$, $(0,-1,0)$.

Proof. Let us find the asymptotics of eigenvectors and eigenplanes for operators $R^{1,\Omega_0}_{m,n}(t)$ while t tends to $+\infty$. Denote any real eigenvector of $R^{1,\Omega_0}_{m,n}(t)$ by $e(t)$. We have

$$e(t) = \mu\big((1,0,0) + O(t^{-1})\big)$$

for some nonzero real μ.

Consider the unique invariant real plane of the operator $R^{1,\Omega_0}_{m,n}(t)$ (it corresponds to the couple of complex conjugate eigenvalues). Note that this plane is a union of all closed orbits of $R^{1,\Omega_0}_{m,n}(t)$. Any such orbit is an ellipse with axes $\lambda g_{\max}(t)$ and $\lambda g_{\min}(t)$ for some positive real number λ, where

$$\begin{aligned} g_{\max}(t) &= (0,t,0) + O(1), \\ g_{\min}(t) &= (0,0,t^{1/2}) + O(t^{-1/2}). \end{aligned}$$

Actually, the vectors $g_{\max}(t) \pm I g_{\min}(t)$ are two complex eigenvectors of $R^{1,\Omega_0}_{m,n}(t)$. For the ratio of the lengths of maximal and minimal axes of any orbit we have the following asymptotic estimate:

$$\frac{\lambda|g_{\max}(t)|}{\lambda|g_{\min}(t)|} = |t|^{1/2} + O(|t|^{-1/2}).$$

Since

$$(1,0,0) - \frac{1}{\mu}e(t) = O(|t|^{-1}),$$

the minimal axis of the orbit-vertex $T_{R^{1,\Omega_0}_{m,n}(t)}(1,0,0)$ is asymptotically not greater than $O(t^{-1})$. Therefore, the length of the maximal axis is asymptotically not greater than some function $O(|t|^{-1/2})$. Hence, the orbit of the point $(1,0,0)$ is contained in the $(C_1|t|^{-1/2})$-ball of the point $(1,0,0)$, where C_1 is a constant that does not depend on t.

We have

$$(0,1,0) - \frac{1}{t}g_{\max}(t) = O(|t|^{-1}).$$

Therefore, the length of the maximal axis of the orbit-vertex $T_{R^{1,\Omega_0}_{m,n}(t)}(1,0,0)$ is asymptotically not greater than some $1 + O(t^{-1/2})$. Hence, the length of the minimal axis is asymptotically not greater than some $O(|t|^{-1/2})$. This implies, that the orbit of the point $(0,1,0)$ is contained in the $(C_2|t|^{-1/2})$-tubular neighborhood of the segment with vertices $(0,1,0)$ and $(0,-1,0)$, where C_2 is a constant that does not depend on t.

Therefore, the convex hull of the union of two orbit-vertices

$$T_{R^{1,\Omega_0}_{m,n}(t)}(1,0,0) \quad \text{and} \quad T_{R^{1,\Omega_0}_{m,n}(t)}(0,1,0)$$

is contained in the $(C_3|t|^{-1/2})$-tubular neighborhood of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,-1,0)$, where $C_3=\max(C_1,C_2)$. This concludes the proof of the lemma. □

Let us now formulate a similar statement for the general case of Hessenberg operators.

Corollary 5.11. *Let $R^{1,\Omega}_{m,n}$ be a family of operators having NRS-matrices with nonnegative integer parameter t. Then for any $\varepsilon>0$ there exists a positive constant such that for any t greater than this constant the convex hull of the union of two orbit-vertices*

$$T_{R^{1,\Omega_0}_{m,n}(t)}(1,0,0) \quad \text{and} \quad T_{R^{1,\Omega_0}_{m,n}(t)}(a_{1,1},a_{2,1},0)$$

is contained in the ε-tubular neighborhood of the convex hull of three points $(1,0,0)$, $(a_{1,1},a_{2,1},0)$, $(-a_{1,1},-a_{2,1},0)$.

Proof. Denote $\Omega=\langle a_{1,1},a_{2,1}|a_{1,2},a_{2,2},a_{3,2}\rangle$ and choose

$$X=\begin{pmatrix} a_{2,1}a_{3,2} & -a_{3,2}a_{1,1} & a_{1,1}a_{2,2}-a_{2,1}a_{1,2} \\ 0 & a_{3,2} & -a_{1,1}-a_{2,2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Direct calculation shows that

$$XH_\Omega(m-t,n)X^{-1}=H_{\langle 0,1|0,0,1\rangle}\left(l_1(m,n)-\frac{t}{a_{2,1}a_{3,2}},l_2(n_0)\right),$$

where l_1 and l_2 are linear functions with coefficients depending only on $a_{1,1}$, $a_{2,1}$, $a_{1,2}$, $a_{2,2}$, and $a_{3,2}$.

Therefore, the family $R^{1,\Omega}_{m,n}$ after the described change of coordinates and a homothety is taken to the family of matrices $R^{1,\Omega_0}_{\tilde m,\tilde n}$ of the type $\langle 0,1|0,0,1\rangle$ for certain $\tilde m$ and $\tilde n$.

Lemma 5.10 implies the following. For any $\varepsilon>0$ there exists a positive constant such that for any t greater than this constant the convex hull of the union of two orbit-vertices

$$T_{R^{1,\Omega_0}_{\tilde m,\tilde n}(t)}(1,0,0) \quad \text{and} \quad T_{R^{1,\Omega_0}_{\tilde m,\tilde n}(t)}(0,1,0)$$

is contained in the ε-tubular neighborhood of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,-1,0)$.

Now if we reformulate the last statement for the family of operators in old coordinates, then we get the statement of the corollary. □

In the algebraic case we have the following statement.

Proposition 5.12. *Let (m,n) be a couple of integers such that the ray $R^{1,\Omega}_{m,n}$ contains only operators having NRS-matrices for integer nonnegative parameter t. Then there exists a positive constant such that for any integer t greater than this constant there exists a fundamental domain for a Klein-Voronoi sail of the operator $R^{1,\Omega}_{m,n}(t)$ such that all (integer) orbit-vertices of this fundamental domain are contained in the set of all integer orbits corresponding to the integer points in the convex hull of three points $(1,0,0)$, $(a_{1,1},a_{2,1},0)$, and $(-a_{1,1},-a_{2,1},0)$.*

We use the following general lemma on continued fractions.

Take an operator A having an NRS-matrix in $SL(3,\mathbb{Z})$ and any integer point x distinct to the origin. Denote by $\Gamma_A^0(p)$ the convex hull of the union of two orbits corresponding to the points p and $A(p)$. For any integer k we denote by $\Gamma_A^k(p)$ the set $A^k(\Gamma_A^0(x))$. Now let

$$\Gamma_A(p) = \bigcup_{k\in\mathbb{Z}} \Gamma_A^k(p).$$

Lemma 5.13. *Let A be an operator in $SL(3,\mathbb{Z})$ having an NRS-matrix and p — any integer point distinct to the origin. Then one of the Klein-Voronoi sails for A is contained in the set $\Gamma_A(p)$.*

Proof. Note that the set $\Gamma_A(p)$ is a union of orbits. Let us project $\Gamma_A(p)$ to the halfplane π_+, see in Figure 2 above. The set $\Gamma_A(p)$ projects to the closure of the complement of the convex hull for the points $\pi(A^k(p))$ for all integer k in the angle defined by eigenspaces. Since all the points $A^k(p)$ are integer, their convex hull is contained in the convex hull of all points corresponding to integer orbits in the angle. Hence $\pi(\Gamma_A(p))$ contains the projection of the sail. Therefore, the set $\Gamma_A(p)$ contains one of the sails. □

Corollary 5.14. *Let A be an operator in $SL(3,\mathbb{Z})$ having an NRS-matrix and p — an integer point distinct to the origin. Then there exists a fundamental domain for one of the Klein-Voronoi sails for an operator A with all (integer) orbit-vertices contained in the set $\Gamma_A^0(p)$.*

Proof. Note that $\Gamma_A^0(p)$ is a fundamental domain of $\Gamma_A(p)$ for the action of the Dirichlet group $\Xi(A)$. Therefore, $\Gamma_A(p)$ contains all orbits of orbit-vertices for the action of $\Xi(A)$. □

Proof of Proposition 5.12. We note that the operator $R_{m,n}^{1,\Omega}(t)$ takes the point $(1,0,0)$ to the point $(a_{1,1}, a_{2,1}, 0)$. Therefore, the convex hull of the union of two orbit-vertices

$$T_{R_{m,n}^{1,\Omega}(t)}(1,0,0) \quad \text{and} \quad T_{R_{m,n}^{1,\Omega}(t)}(a_{1,1}, a_{2,1}, 0)$$

(we denote it by $W(t)$) coincides with the set $\Gamma^0_{R_{m,n}^{1,\Omega}(t)}(1,0,0)$.

From Corollary 5.14 it follows that there exists a fundamental domain for a sail with all its orbit-vertices contained in $W(t)$. Choose a sufficiently small ε_0 such that the ε_0-tubular neighborhood of the triangle with vertices $(1,0,0)$, $(a_{1,1}, a_{2,1}, 0)$, and $(-a_{1,1}, -a_{2,1}, 0)$ does not contain integer points distinct to the points of the triangle. From Corollary 5.11 it follows that for a sufficiently large t the set $W(t)$ is contained in the ε_0-tubular neighborhood of the triangle. This implies the statement of Proposition 5.12. □

Now let us study the remaining case of the rays of matrices with asymptotic direction $(a_{1,1}, a_{2,1})$. We remind that $\Omega_0 = \langle 0,1|0,0,1\rangle$.

Lemma 5.15. *Let $R_{m,n}^{2,\Omega_0}$ be an NRS-ray. Then for any $\varepsilon > 0$ there exists a positive constant such that for any t greater than this constant the convex hull of the union of two orbit-vertices $T_{R_{m,n}^{2,\Omega_0}(t)}(1,0,0)$ and $T_{R_{m,n}^{2,\Omega_0}(t)}(0,1,0)$ is contained in the ε-tubular neighborhood of the convex hull of three points $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$.*

Proof. First, we note that the continued fractions for the operators A and A^{-1} coincide.

Secondly, the following holds:

$$H_{\langle 0,1|0,0,1\rangle}(m, n+t) = XH_{\langle 0,1|0,0,1\rangle}(-n-t, -m)X^{-1},$$

where

$$X = \begin{pmatrix} 0 & -1 & -n-t \\ -1 & 0 & -m \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, in the new coordinates we obtain the equivalent statement for the ray $R^{1,\Omega_0}_{-n,-m}(t)$. Now Lemma 5.15 follows directly from Lemma 5.10. □

Corollary 5.16. *Let $\Omega = \langle a_{1,1}, a_{2,1}|a_{1,2}, a_{2,2}, a_{3,2}\rangle$ and $R^{2,\Omega}_{m,n}(t)$ be a family of operators having NRS-matrices with nonnegative parameter t. Then for any $\varepsilon > 0$ there exists a positive constant such that for any t greater than this constant the convex hull of the union of two orbit-vertices*

$$T_{H_\Omega(m+a_{1,1}t, n+a_{2,1}t)}(1,0,0) \quad \textit{and} \quad T_{H_\Omega(m+a_{1,1}t, n+a_{2,1}t)}(a_{1,1}, a_{2,1}, 0)$$

is contained in the ε-tubular neighborhood of the triangle with vertices $(1,0,0)$, $(-1,0,0)$, and $(a_{1,1}, a_{2,1}, 0)$. □

Proposition 5.17. *Let (m,n) be a couple of integers, such that the family $R^{2,\Omega}_{m,n}(t)$ contains only operators having NRS-matrices. Then there exists a positive constant such that for any integer t greater than this constant there exists a fundamental domain for a Klein-Voronoi sail of the operator $R^{2}_{m,n}(t)$ such that all orbit-vertices of this fundamental domain are contained in the set of all integer orbits corresponding to the integer points in the convex hull of three points $(1,0,0)$, $(-1,0,0)$, and $(a_{1,1}, a_{2,1}, 0)$.* □

Remark We omit the proofs of Corollary 5.16 and Proposition 5.17, since they repeat the proofs of Corollary 5.11 and Proposition 5.12.

Now we prove Theorem 5.8.

Step 1. Let A be an operator with Hessenberg matrix M in $SL(3,\mathbb{Z})$. By Proposition 3.5 the Hessenberg complexity of the Hessenberg matrix M coincides with the MD-characteristic $\Delta(A|(1,0,0))$. Therefore, the Hessenberg matrix M is reduced if and only if the MD-characteristic of A attains the minimal possible absolute value on the integer lattice except the origin exactly at point $(1,0,0)$.

Step 2. By Theorem 4.4 we know that all minima of the aet of absolute values for the MD-characteristic of A are attained at integer points of the Klein-Voronoi sails for A.

Step 3. In the case of a three-dimensional operator A with two complex-conjugate eigenvalues, the continued fraction of A contains exactly two sails that are symmetric with respect to the transformation $-E$. Therefore, both sails of A has minima of the MD-characteristic on an integer lattice except the origin. This allows us to consider one sail. By Theorem 4.4 we can restrict the search of the minimal absolute value of the MD-characteristic to one of the fundamental domains of the chosen sail.

Step 4.1. The case of NRS-rays with asymptotic direction $(-1,0)$.

By Proposition 5.12 there exists a positive constant such that for any integer t greater than this constant all integer points of one of the fundamental domains for $R^{1,\Omega}_{m,n}(t)$ are contained in the convex hull of three points $(1,0,0)$, $(a_{1,1},a_{2,1},0)$, and $(-a_{1,1},-a_{2,1},0)$.

This triangle contains only finitely many integer points, all of them have the last coordinate equal to zero. The value of the MD-characteristic for the points of type $(x,y,0)$ equals:

$$(a_{2,1}x-a_{1,1}y)a_{3,2}^2y^2t+C,$$

where the constant C does not depend on t, it depends only on x, y, and the elements of the matrix $H_\Omega(m-t,n)$ for the operator $R^{1,\Omega}_{m,n}(t)$.

So for any point $(x,y,0)$ the MD-characteristic is linear with respect to the parameter t, and it increases with growth of t. The only exceptions are the points of type $\lambda(1,0,0)$ and $\mu(a_{1,1},a_{2,1},0)$ (for integers λ and μ). The values of MD-characteristic are constant in these points with respect to the parameter t.

Since there are finitely many integer points in the triangle $(1,0,0)$, $(a_{1,1},a_{2,1},0)$, and $(-a_{1,1},-a_{2,1},0)$, for sufficiently large t the MD-characteristic at points of the triangle attains the minima at $(1,0,0)$ or at $(a_{1,1},a_{2,1},0)$. Since $R^{1,\Omega}_{m,n}(t)$ takes the point $(1,0,0)$ to the point $(a_{1,1},a_{2,1},0)$, the values of the MD-characteristic at $(1,0,0)$ and at $(a_{1,1},a_{2,1},0)$ coincide.

Therefore, for sufficiently large m the matrix

$$H_{\langle a_{1,1},a_{2,1}|a_{1,2},a_{2,2},a_{3,2}\rangle}(m-t,n)$$

is always reduced and there are no other reduced matrices integer congruent to the given one. We have proved the statement of Theorem 5.8 for any NRS-ray with asymptotic direction $(-1,0)$.

Step 4.2. The case of NRS-rays with asymptotic direction $(a_{1,1},a_{2,1})$. This case is similar to the case of NRS-rays with asymptotic direction $(-1,0)$. We leave the details of the proof to the reader.

Proof of Theorem 5.8 is completed. □

5.4. **Examples of NRS-matrices for the given Hessenberg type.** In this subsection we bring together some examples of families $El(\Omega)$ for the Hessenberg types:

$$\langle 0,1|0,0,1\rangle,\quad \langle 0,1|1,0,2\rangle,\quad \langle 0,1|1,1,2\rangle,\quad \langle 0,1|1,0,3\rangle,\quad \text{and}\quad \langle 1,2|1,1,3\rangle.$$

5.4.1. *Hessenberg type* $\langle 0,1|0,0,1\rangle$. In Figure 3 on page 18 we show an example of the simplest family of Hessenberg perfect NRS-matrices $El(\langle 0,1|0,0,1\rangle)$. The Hessenberg complexity of all these matrices is one, and, therefore, they are all reduced.

5.4.2. *Hessenberg type* $\langle 0,1|1,0,2\rangle$. In Figure 1 on page 5 we show an example of the family of Hessenberg perfect NRS-matrices $El(\langle 0,1|1,0,2\rangle)$. We take

$$H_{\langle 0,1|1,0,2\rangle}(0,0)=\begin{pmatrix}0&1&1\\1&0&0\\0&2&1\end{pmatrix}$$

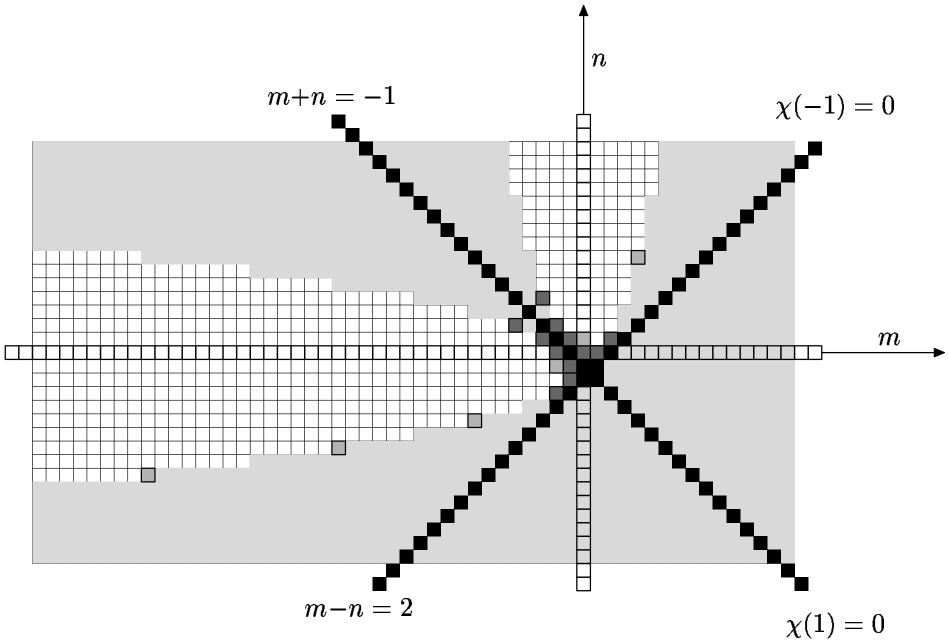


FIGURE 4. The family of Hessenberg matrices $H_{\langle 0,1|1,1,2\rangle}(m,n)$.

as the origin. The Hessenberg complexity of these matrices equals 2. Experiments shows that only 12 of such matrices are non-reduced. It is conjectured that all others Hessenberg matrices of $El\big(\langle 0,1|1,0,2\rangle\big)$ are reduced.

In Figures 4, 5, and 6 the dark gray squares corresponds to non-reduced operators. We also fill with gray the reduced Hessenberg matrices that are n-th powers (where $n \geq 2$) of some integer matrices.

5.4.3. *Hessenberg type* $\langle 0,1|1,1,2\rangle$. In Figure 4 we show an example of the family of Hessenberg perfect NRS-matrices $El\big(\langle 0,1|1,1,2\rangle\big)$. We take

$$H_{\langle 0,1|1,1,2\rangle}(0,0)=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

The Hessenberg complexity of these matrices equals 2. We have found only 12 non-reduced matrices in the family. It is conjectured that all other Hessenberg matrices of $El\big(\langle 0,1|1,1,2\rangle\big)$ are reduced.

5.4.4. *Hessenberg type* $\langle 0,1|1,0,3\rangle$. In Figure 5 we show an example of the family of Hessenberg perfect NRS-matrices $El\big(\langle 0,1|1,0,3\rangle\big)$. We take

$$H_{\langle 0,1|1,0,3\rangle}(0,0)=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

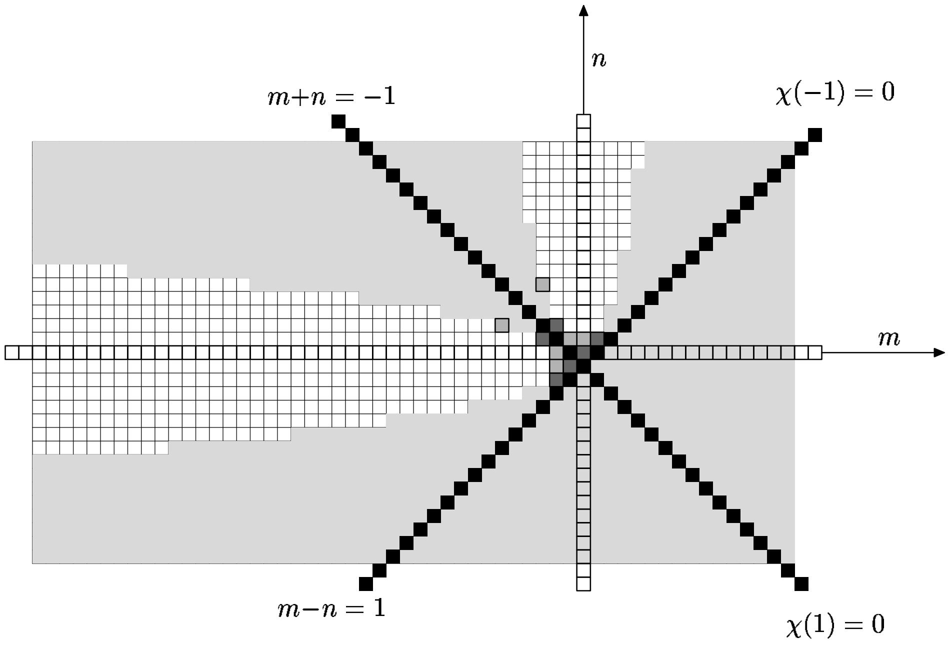


FIGURE 5. The family of Hessenberg matrices $H_{\langle 0,1|1,0,3\rangle}(m,n)$.

The Hessenberg complexity of these matrices equals 3. We have found only 6 non-reduced matrices in the family. It is conjectured that all other Hessenberg matrices of $El\big(\langle 0,1|1,0,3\rangle\big)$ are reduced.

5.4.5. *Hessenberg type* $\langle 1,2|1,1,3\rangle$. We conclude (in Figure 6) a more complicated example of a family of Hessenberg perfect NRS-matrices $El\big(\langle 1,2|1,1,3\rangle\big)$. We take

$$H_{\langle 1,2|1,1,3\rangle}(0,0)=\begin{pmatrix} 1 & 1 & 0\\ 2 & 1 & 0\\ 0 & 3 & -1\end{pmatrix}.$$

The Hessenberg complexity of these matrices equals 12. We have found only 27 non-reduced matrices in the family. It is conjectured that all other Hessenberg matrices of $El\big(\langle 1,2|1,1,3\rangle\big)$ are reduced.

5.5. **Some questions related to the experiments.** As we have shown in Theorem 5.8 the number of non-reduced matrices in NRS-rays is always finite. Actually, from the experiments we conjecture that for some (probably for all) Hessenberg types the sets of non-reduced perfect NRS-matrices of these types are finite.

Let us remind the questions of Problem 1: *Is it true that for any Hessenberg type there exist only finitely many non-reduced perfect matrices? For which Hessenberg types the number of such matrices is finite?*

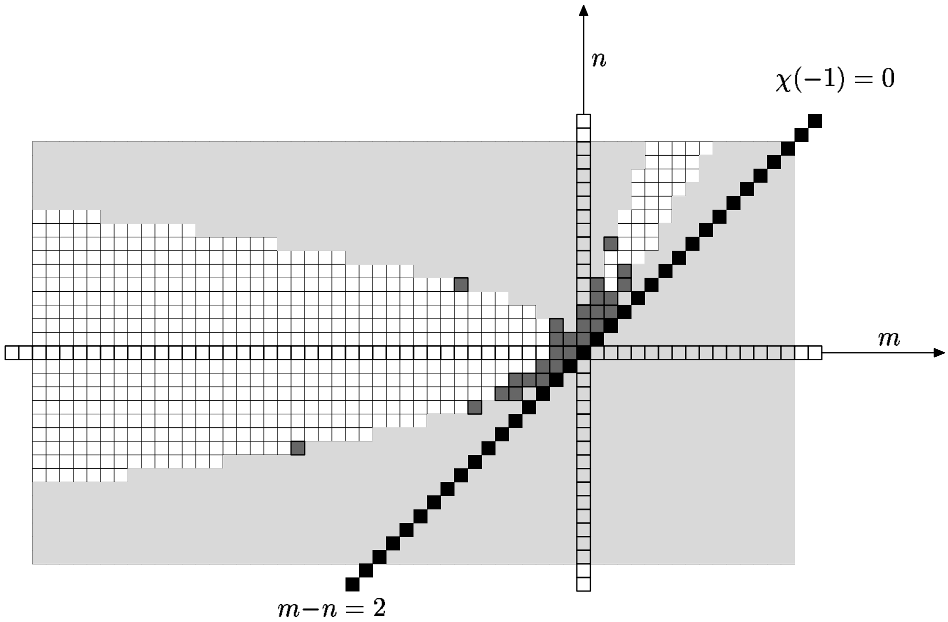


FIGURE 6. The family of Hessenberg matrices $H_{\langle 1,2|1,1,3\rangle}(m,n)$.

Consider the family $El\big(\langle 0,1|1,1,2\rangle\big)$ shown in Figure 4. Note that any operator with matrix

$$H_{\langle 0,1|1,1,2\rangle}(-2t^2,-2t-1) \quad \text{for } t>0$$

and any operator with matrix

$$H_{\langle 0,1|1,1,2\rangle}(2u,2u^2-1) \quad \text{for } u>0$$

is a square of some other $SL(3,\mathbb{Z})$-operator. All these operators are contained in two parabolas at the "boundary" of the family $El\big(\langle 0,1|1,1,2\rangle\big)$. We guess that a similar situation is possible for the case of non-reduced operators. In the case of negative answer to Problem 1 it is interesting to know the answer to the following question.

Problem 2. Is it true that for any Hessenberg type the set of non-reduced perfect matrices is contained in a finite number of parabolas? How many infinite series do we have for a given Hessenberg type?

Here we show a small table of conjectured numbers of non-reduced perfect matrices for given Hessenberg types. We list all Hessenberg types of Hessenberg complexity less than 5. The Hessenberg complexity of Ω is denoted by '$\varsigma(\Omega)$', the conjectured number of non-reduced Hessenberg NRS-matrices is denoted by '$\#(\Omega)$'. The conjectured number is verified only in the simplest case of Frobenius operators (i.e., of Hessenberg type $\langle 0,1|0,0,1\rangle$). It is also interesting to *study the asymptotics of the number of reduced operators with respect to the growth of the Hessenberg complexity.*

Ω	$\langle 0,1\|0,0,1\rangle$	$\langle 0,1\|1,0,2\rangle$	$\langle 0,1\|1,1,2\rangle$	$\langle 0,1\|1,0,3\rangle$	$\langle 0,1\|1,1,3\rangle$	$\langle 0,1\|1,2,3\rangle$
$\varsigma(\Omega)$	1	2	2	3	3	3
$\#(\Omega)$	0	12	12	6	10	10

Ω	$\langle 0,1\|2,0,3\rangle$	$\langle 0,1\|2,1,3\rangle$	$\langle 0,1\|2,2,3\rangle$	$\langle 1,2\|0,0,1\rangle$	$\langle 0,1\|1,0,4\rangle$	$\langle 0,1\|1,1,4\rangle$
$\varsigma(\Omega)$	3	3	3	4	4	4
$\#(\Omega)$	14	10	10	94	6	8

Ω	$\langle 0,1\|1,2,4\rangle$	$\langle 0,1\|1,3,4\rangle$	$\langle 0,1\|3,0,4\rangle$	$\langle 0,1\|3,1,4\rangle$	$\langle 0,1\|3,2,4\rangle$	$\langle 0,1\|3,3,4\rangle$
$\varsigma(\Omega)$	4	4	4	4	4	4
$\#(\Omega)$	10	8	10	12	8	8

5.6. **Two examples of couples of Hessenberg matrices.**

Example 5.18. The following two matrices

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 3 & 8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 & 5 \\ 1 & 1 & 2 \\ 0 & 3 & 7 \end{pmatrix}$$

are not integer conjugate but have the same Hessenberg complexity equal to 3 and equivalent characteristic polynomials.

Example 5.19. The matrices M_1 M_2, where

$$M_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 3 & 5 \end{pmatrix} \qquad \text{and} \quad M_2 = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix}$$

are both reduced Hessenberg matrices (with Hessenberg complexity equal to 3) of the same operator A in two different integer bases. The reason for that is the following. The sail of A in the basis of the matrix M_1 contains two different integer vertices $p_1 = (-1,0,0)$ and $p_2 = (0,1,-1)$ in a fundamental domain that are not in the same Dirichlet orbit.

5.7. **A small remark about totally real three-dimensional case.** We study the totally real case (of RS-matrices) in a particular example of the family of Hessenberg matrices $H_{\langle 0,1|1,0,2\rangle}(m,n)$:

$$H_{\langle 0,1|1,0,2\rangle}(m,n) = \begin{pmatrix} 0 & 1 & n+1 \\ 1 & 0 & m \\ 0 & 2 & 2n+1 \end{pmatrix}.$$

By definition, the Hessenberg complexity of the matrices of the family equals two. This implies that a Hessenberg matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$ is reduced iff it is not conjugate to some matrix of the unit Hessenberg complexity, i.e., to some matrix of Hessenberg type $\langle 0,1|0,0,1\rangle$.

By Proposition 3.5, the matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$ is reduced iff the set of absolute values of the MD-characteristic of the operator with matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$ does not attain the value 1 at integer points.

In Figure 7 we study the matrices $H_{\langle 0,1|1,0,2\rangle}(m,n)$ with

$$-20 \le m,n \le 20.$$

The square in the intersection of the m-th column and n-th row corresponds to the matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$. The square is colored in black if the corresponding matrix is reducible over the field of rationals. The square is colored in gray if the matrix is irreducible and there exists an integer vector (x,y,z) with the coordinates satisfying

$$-1000 \le x,y,z \le 1000,$$

such that the MD-characteristic of the operator with matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$ attains the value 1 at (x,y,z). All the rest squares are white.

If a square is gray, then the corresponding matrix is not reduced. If a square is white, then we cannot conclude whether the matrix is reduced or not (since the integer vector with MD-characteristic equal to 1 may have coordinates with absolute values greater than 1000).

From Figure 7 the *NRS-domain* can be easily visualized. This domain is almost completely consists of white squares. We draw a boundary broken line between the NRS-squares and the other squares with gray.

Remark 5.20. We have checked explicitly all the periods for all sails of the continued fractions for operators with matrices $H_{\langle 0,1|1,0,2\rangle}(m,n)$ with

$$-10 \le m,n \le 10.$$

These matrices are contained inside the black square in Figure 7. It turns out that the exact result for this set completely coincides with the above approximation. If the square is gray, then the corresponding matrix is not reduced. If the square is white, then the matrix is reduced.

Statement 5.21. *If an integer $m+n$ is odd, then the matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$ is reduced.*

Proof. Note that the MD-characteristic at a vector (x,y,z) for the operator with Hessenberg matrix $H_{\langle 0,1|1,0,2\rangle}(m,n)$ equals

$$\begin{aligned}&\big| -2x^3-(4n+2)x^2y-(4n^2+4m+4n)x^2z+(4m+2)xy^2+\\&(4mn+2m+6n+6)xyz+(-2m^2+4n^2+2m+6n+2)xz^2-2y^3-\\&(2m+2n+2)y^2z-(2mn+2n+2)yz^2+(m^2-n^2-2n-1)z^3\big|.\end{aligned}$$

From this expression it follows that the MD-characteristic with odd $m+n$ takes only even values at any integer point (x,y,z). □

Remark 5.22. At this moment we can say nothing about asymptotic behavior of RS-matrices. From one hand Statement 5.21 implies that there exist rays of reduced operators. From the other hand all matrices corresponding to integer points of the lines

$1)m=n;\quad 2)m=n+2;\quad 3)m=-n;\quad 4)m=-n-2;\quad 5)n=3m-4;\quad 6)m=3n+6$

are reduced (we do note state that the list of such lines is complete).

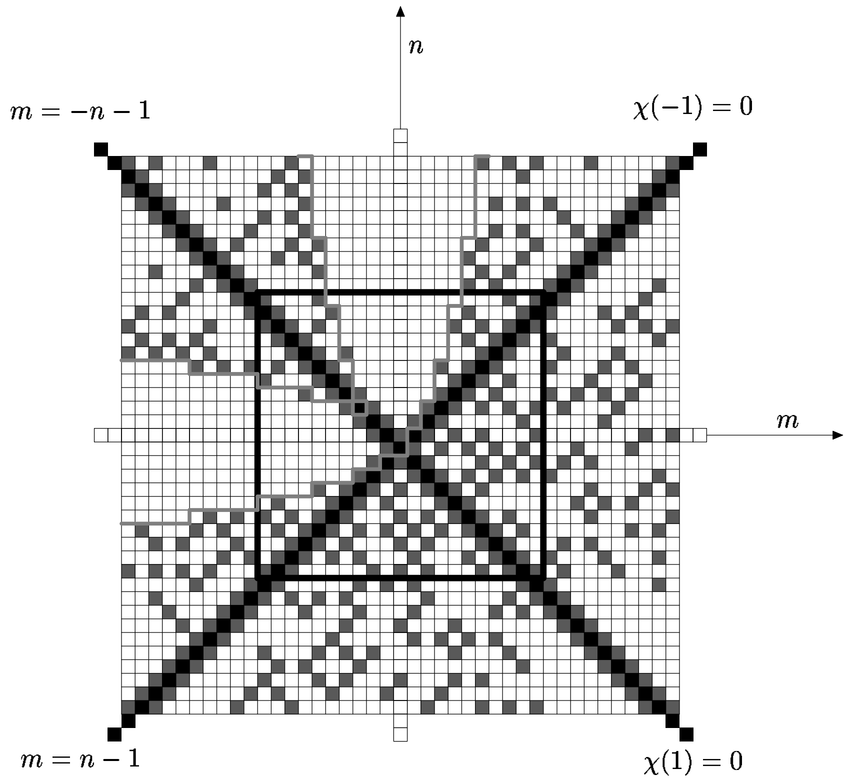


FIGURE 7. The family of matrices of Hessenberg type $\langle 0,1|1,0,2\rangle$.

So the following problems arise. Denote by S_p the square

$$\{(m,n)| -p \le m,n \le p\}.$$

For a fixed Hessenberg type Ω denote by $\Theta_\Omega(p)$ the ratio of the number of reduced matrices of type Ω corresponding to integer points of S_p to the number of all integer points of S_p (i.e., to $(2p+1)^2$).

Problem 3. For any Hessenberg type Ω find all limit points for the sequence $\Theta_\Omega(p)$, $p=1,2,3,\ldots$ Is it true that there exists a limit for this sequence?

Problem 4. Is it true that for any Hessenberg type Ω it holds

$$\lim_{p\to+\infty} \Theta_\Omega(p) = 1?$$

Give the upper and the lover estimates for the limit points of the sequence.

6. On the structure of families of Hessenberg matrices in four-dimensional case

As we have already studied for the case of three-dimensional matrices, the families of NRS-matrices for distinct Hessenberg types has similar structures (see in the proof of

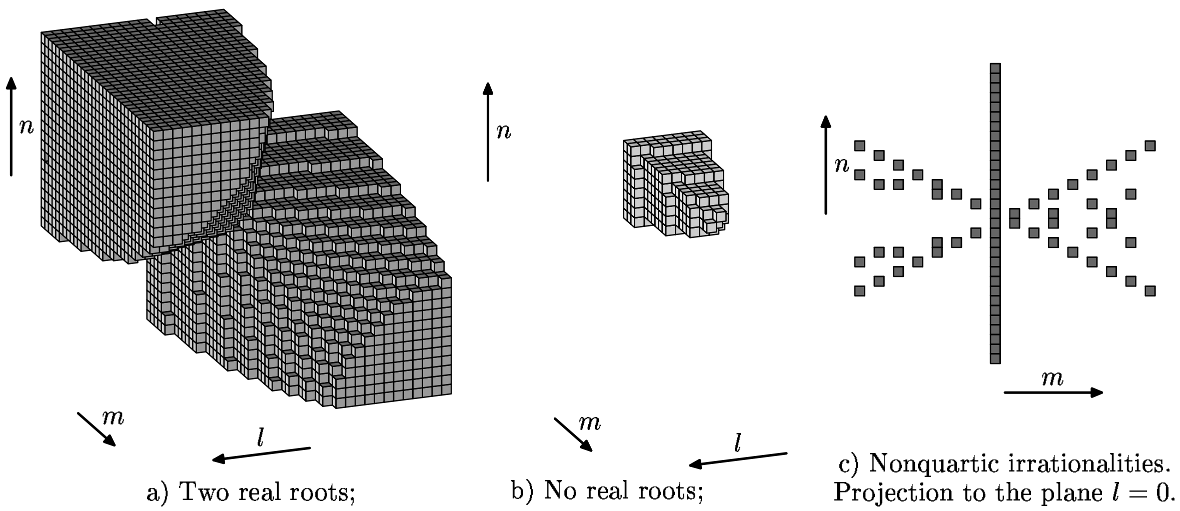


FIGURE 8. The family of matrices of the Hessenberg type $\langle 0,1|0,0,1|1,3,1,4\rangle$.

Corollary 5.11). For instance, they are contained in the union of two parabolas. The reason for that is the $SL(3,\mathbb{Q})$-congruence of all the families with expanded integer parameters to rational parameters. In multidimensional case the same reason works.

In this section we take a first glance at $SL(4,\mathbb{Z})$, studying a family of Hessenberg matrices of Hessenberg type $\langle 0,1|0,0,1|1,3,1,4\rangle$ of Hessenberg complexity 4. Let

$$H_{\langle 0,1|0,0,1|1,3,1,4\rangle}(l,m,n) = \begin{pmatrix} 0 & 0 & 1 & n \\ 1 & 0 & 3 & 3n+l+1 \\ 0 & 1 & 1 & n+m \\ 0 & 0 & 4 & 4n+1 \end{pmatrix}.$$

In this section we write $H_{\langle 0,1|0,0,1|1,3,1,4\rangle}(l,m,n)$ as $H(l,m,n)$, for short.

In Figure 8 we show the family of matrices $H(l,m,n)$ with integer parameters l,m,n satisfying

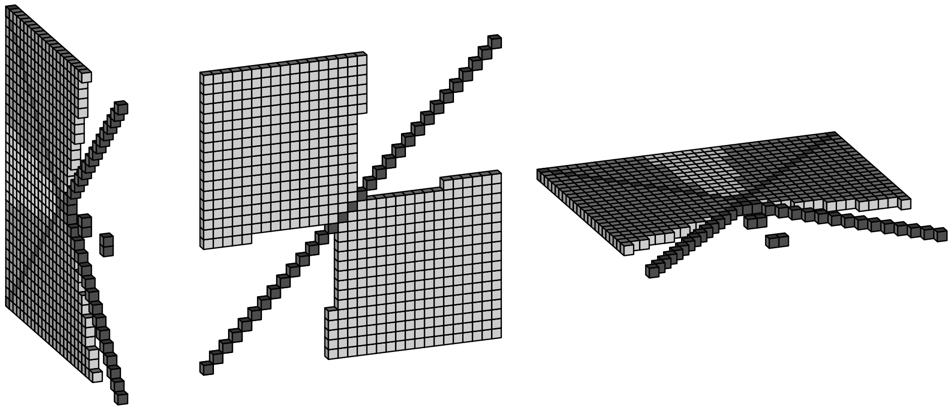
$$-15 \le l,m,n \le 15.$$

In the figure to any operator $H(l,m,n)$ we associate the cube with unit edges parallel to the axes and with center at point (l,m,n). Gray cubes of Figure 8a) correspond to the matrices with two complex and two real eigenvalues. Light-gray cubes of Figure 8b) correspond to the matrices with two distinct couples of complex-conjugate eigenvalues.

As before, we study the set of matrices with irreducible characteristic polynomial over rational numbers. That gives us the following restrictions on admissible triples (l,m,n).

First, the characteristic polynomial does not have rational roots. This is equivalent to the following: the integers 1 and -1 are not the roots of the polynomial. The characteristic polynomial for the operator $H(l,m,n)$ in t is

$$t^4 + (-4n-2)t^3 + (-4m-2)t^2 + (2-4l)t + 1.$$



a) $l = 0$; b) $m = 0$; c) $n = 0$.

FIGURE 9. Three coordinate sections of the family of matrices $H(l, m, n)$.

Then, 1 (or -1) is a root of the characteristic polynomial iff $n+m+l = 0$ (or $n-m-l = 0$ respectively).

Secondly, the characteristic polynomial is not a product of two polynomials of degree 2 with rational coefficients. There are exactly two series of such decomposable polynomials:

$$(t^2 + at + 1)(t^2 + bt + 1); \qquad (t^2 + at - 1)(t^2 + bt - 1),$$

where a and b are integers. The characteristic polynomial of the matrix $A(l, m, n)$ with integer parameters (m, l, n) is decomposable into two polynomials of degree 2 with rational coefficients iff at least one of the following systems has an integer solution in (a, b) variables:

$$\left\{\begin{array}{rcl} (-b-4l+2)b+2 &=& -4m-2 \\ -b-4l+2 &=& a \\ l-n-1 &=& 0 \end{array}\right. ; \qquad \left\{\begin{array}{rcl} (-b+4l-2)b-2 &=& -4m-2 \\ -b+4l-2 &=& a \\ l+n &=& 0 \end{array}\right. .$$

All the solutions of the above systems correspond to the cubes with centers lying on the plane $l-n-1 = 0$ and $l+n = 0$ respectively. The projection along the l-axis to the plane $l = 0$ of all cubes corresponding to the integer solutions of the first system coincides with the projection of the cubes of integer solutions of the second system. In Figure 8c) the projections of these cubes are dark-gray squares.

We do not worry that the discriminant of the characteristic polynomial is nonzero, since it is zero only if the characteristic polynomial is reducible over rational numbers.

In Figure 9 we show three coordinate planes for the families of matrices $H(l, m, n)$. The light-gray cubes correspond to the matrices with four complex eigenvalues and irreducible characteristic polynomial. The gray cubes correspond to the irreducible matrices with two real and two complex eigenvalues and irreducible characteristic polynomial. The dark gray cubes correspond to matrices with reducible over rational numbers characteristic polynomials. Finally, we do not draw the cubes for matrices with four real roots.

Here we formulate some questions concerning to the four dimensional case.

Problem 5. Study the asymptotic behaviour for the families of reduced Hessenberg operators for fixed Hessenberg types.

The answers to Problem 5 are interesting for the subfamilies of the matrices with four real eigenvalues, with two real and two complex eigenvalues, and with four complex eigenvalues.

It is also interesting to know the answers to the following questions:

Q.1. Is it true that there is only one asymptotic direction for the family of Hessenberg matrices with all four non-real eigenvalues for a fixed Hessenberg type?

Q.2. What is the dimension of the family of asymptotic directions for the case of Hessenberg matrices with two real and two complex eigenvalues for a fixed Hessenberg type?

Q.3. Is it true that the set of asymptotic directions of Hessenberg matrices with all real eigenvalues is everywhere dense in the space of all asymptotic directions with the natural topology?

Q.4. Is it true that for any Hessenberg type there exist only finitely many non-reduced Hessenberg matrices with two real and two complex eigenvalues (with four complex eigenvalues) for a fixed Hessenberg type?

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E-mail address, Oleg Karpenkov: karpenk@mccme.ru

Mathematisch Instituut, Universiteit Leiden, P.O. Box 9512, 2300 RA Leiden, The Netherlands