$(\mathrm{GL}(n+1,\mathbb{R}),\,\mathrm{GL}(n,\mathbb{R}))$ IS A GENERALIZED GELFAND PAIR

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Abstract

Denote by $G = \mathrm{GL}(n+1,\mathbb{R})$ the group of invertible $(n+1)\times(n+1)$ matrices with real entries, acting on $\mathbb{R}^{n+1}$ in the usual way, and let $H_1 = \mathrm{GL}(n,\mathbb{R})$ be the stabilizer of the first unit vector e_0. Let $H_0 = \mathrm{GL}(1,\mathbb{R})$ and set $H = H_0 \times H_1$. It is known that the pair $(G,\,H)$ is a generalized Gelfand pair. Define a character χ of H by $\chi(h) = \chi(h_0h_1) = \chi_0(h_0)$ where χ_0 is a unitary character of H_0 ($h_0 \in H_0$, $h_1 \in H_1$). Let σ be the anti-involution on G given by $\sigma(g) = {}^tg$. In this note we show that any distribution T on G satisfying $T(h_1gh_2) = \chi(h_1h_2)\,T(g)$ ($g \in G$; $h_1, h_2 \in H$) is invariant under the anti-involution σ. This result implies that $(G,\,H_1)$ is a generalized Gelfand pair.

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1 Introduction

Let G be the general linear group of order (n+1) over $\mathbb{R}$. So G is the group of $(n+1)\times(n+1)$ matrices with real entries and non-vanishing determinant. Let $H_1 \simeq \mathrm{GL}(n,\mathbb{R})$ be the stabilizer in G of the first unit vector e_0 of $\mathbb{R}^{n+1}$, G acting in the ususal way on $\mathbb{R}^{n+1}$. We are going to show:

Theorem 1 *The pair $(G,\,H_1)$ is a generalized Gelfand pair.*

Let us recall the definition of generalized Gelfand pair (see [7]). Let $(\pi,\,\mathcal{H})$ be any irreducible unitary representation of G on the Hilbert space $\mathcal{H}$. Then $(G,\,H_1)$ is said to be a generalized Gelfand pair if $\dim \mathcal{H}_{-\infty}^{H_1} \le 1$. Here $\mathcal{H}_{-\infty}^{H_1}$ is the space of H_1-fixed vectors in $\mathcal{H}_{-\infty}$, the (continuous) anti-dual of the space $\mathcal{H}_\infty$ of C^∞-vectors. Equivalently one has: any unitary representation of G which can be realized on the space $D'(G/H_1)$ of distributions on G/H_1 decomposes multiplicity free in the sense that the commutant of $\pi(G)$ is an abelian algebra.

Though it is known that the pair $(\mathrm{GL}(2,\mathbb{R}), \mathrm{GL}(1,\mathbb{R}))$ is a generalized Gelfand pair, which is most easily seen by identifying this pair with the pair $(\mathrm{SO}(1,2), \mathrm{SO}(1,1))$ by letting $\mathrm{GL}(2,\mathbb{R})$ act on the Lie algebra of the group $\mathrm{SL}(2,\mathbb{R})$ through the adjoint action, and recalling that the latter pair is a generalized Gelfand pair, see [6], we shall not exclude the case $n = 1$ here.

2 A criterion

We shall formulate here a criterion (Theorem 2) which implies Theorem 1.

Let H be the subgroup $H = H_0 \times H_1$ of G where $H_0 \simeq \mathrm{GL}(1,\mathbb{R}) \simeq \mathbb{R}^*$. So any element of H has the form

$$h = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}$$

with $h_0 \in \mathbb{R}^*$, $h_1 \in H_1$. Let χ_0 be a (continuous) unitary character of $\mathbb{R}^*$ and define the character χ of H by $\chi(h) = \chi_0(h_0)$.

Let us denote by $d(h_0)$ the diagonal matrix in G with entries h_0 ($h_0 \in \mathbb{R}^*$). Clearly $d(h_0)$ belongs to the center of G,

If $(\pi, \mathcal{H})$ is an irreducible unitary representation then by Schur's Lemma $\pi(d(h_0))$ acts as a scalar, say $\chi_0(h_0)$. Therefore the vectors in $\mathcal{H}_{-\infty}^{H_1}$ transform in the same way under $\pi_{-\infty}(d(h_0))$. This leads to the following criterion.

Theorem 2 *Let χ_0 be a continuous unitary character of $\mathbb{R}^*$. Any distribution T on G satisfying*

$$T(h_1 g h_2) = \chi(h_1 h_2)\, T(g) \quad (g \in G;\ h_1, h_2 \in H)$$

is invariant under the anti-involution of G defined by $\sigma(g) = {}^t g$.

This theorem implies Theorem 1, see [6]. Notice that this anti-involution differs from the one considered in [6]. This is a crucial fact for the proof.

Theorem 2 has been proved in [1]. We give a quite different proof of this theorem. Though in our context we need Theorem 2 only for positive-definite distributions, we prove it in the general case. Our proof is short and has nice applications for the proof of the Plancherel formula for G/H_1, see [4, 7],

3 Proof of Theorem 2

Let X_1 be the algebraic manifold, contained in $\mathbb{R}_*^{n+1} \times \mathbb{R}_*^{n+1}$, defined by

$$X_1 = \{x, y) : \langle x,\, y\rangle = x_0 y_0 + x_1 y_1 + \cdots + x_n y_n = 1\}.$$

if $x = (x_0, x_1, \ldots, x_n),\ y = (y_0, y_1, \ldots, y_n)$.

Then G acts on X_1 by

$$g \cdot (x, y) = (g \cdot x,\ {}^t g^{-1} \cdot y)$$

$(g \in G, (x,y) \in X_1)$. Clearly $X_1 \simeq G/H_1$, the isomorphism being given by $p_1 : g \mapsto g \cdot (e_0,\, e_0)$ $(g \in G)$, where again e_0 is the first standard unit vector in $\mathbb{R}^{n+1}$.

Let $D(X_1)$ denote the space of complex-valued C^∞-functions on X_1 with compact support. The left G-action on X_1 induces a representation U of G on $D(X_1)$ and by taking the inverse adjoint a representation U of G on $D'(X_1)$. We define

$$D'(X_1, \chi) = \{T \in D'(X_1) \,:\, U_h T = \chi(h) T \ (h \in H)\}.$$

This space is naturally isomorphic to the space considered in Theorem 2, the space of distributions T on G that satisfy the (formal) transformation rule

$$T(h_1 g h_2) = \chi(h_1 h_2) T(g) \quad (g \in G;\, h_1, h_2 \in H).$$

Let us introduce a map ξ which describes the H_1-orbits on X_1. Let $x_0,\, y_0$ be the first coordinates of x and y respectively, where $(x,\, y) \in X_1$. Consider the map $\xi :\, X_1 \to \mathbb{R}^2$ given by $\xi(x,\, y) = (x_0,\, y_0)$. It has the following properties:

- ξ is H_1-invariant,
- ξ is real analytic,
- $\xi(x,y) = (s,\, t)$ is an H_1-orbit on X_1 if $st \neq 1$,

If $st = 1$ there are 4 H_1-orbits (3 orbits if $n = 1$) inside $\xi(x,y) = (s,t)$: $(se_0,\, te_0)$; $\{(x,y) \,:\, x = se_0 + \overline{x},\, y = te_0 + \overline{y}; \overline{x},\, \overline{y} \in \mathbb{R}^n_*,\, \langle \overline{x},\, \overline{y}\rangle = 0\}$; $\{(x,y) \,:\, x = se_0 + \overline{x},\, y = te_0,\, \overline{x} \in \mathbb{R}^n_*\}$, $\{(x,y) \,:\, x = se_0,\, y = te_0 + \overline{y},\, \overline{y} \in \mathbb{R}^n_*\}$.

- ξ is submersive at all $(x,y) \in X_1$ with at least one non-vanishing coordinate x_i $(i \neq 0)$ or y_j $(j \neq 0)$.

Define also a map $Q :\, X_1 \to \mathbb{R}$ by

$$Q(x,y) = x_0 y_0.$$

We define the following open subsets of X_1:

$$\begin{aligned} X_1^0 &= \{(x,y) \in X_1 : Q(x,y) < 1\} \\ X_1^1 &= \{(x,y) \in X_1 : Q(x,y) > 0\}. \end{aligned}$$

The map Q is left H-invariant, hence both sets are. Therefore we may define for $j = 0, 1$

$$D'(X_1^j, \chi) = \{T \in D'(X_1^j) : U_h T = \chi(h) T \ (h \in H)\}.$$

Since $X_1^0 \cup X_1^1 = X_1$, the map $T \mapsto (T|_{X_1^0},\, T|_{X_1^1})$ defines a linear bijection from $D'(X_1, \chi)$ onto the set of pairs $(T_0,\, T_1) \in D'(X_1^0, \chi) \times D'(X_1^1, \chi)$ satisfying the matching condition

$$T_0|_{X_1^0 \cap X_1^1} = T_1|_{X_1^0 \cap X_1^1}.$$

We shall study such pairs of distributions. It is sufficient to study the subspaces $D'(X_1^j, \chi)$ separately, since X_1^j $(j = 0, 1)$ are clearly σ-invariant.

The spaces are treated with different methods. We first deal with the space $D'(X_1^1,\chi)$ and use a now classical ingredient, see Faraut [3] and Kosters-van Dijk [4], with applying results by Tengstrand. For this purpose we introduce the space $X = X_1/\sim$ where points (x_1,y_1) and (x_2,y_2) are identified if $(x_1,y_1) = (\lambda x, \lambda^{-1} y_2)$ for some $\lambda \in \mathbb{R}^*$. Let $(x,y) \mapsto \widetilde{(x,y)}$ be the natural projection of X_1 onto X. Observe that $X \simeq G/H$. Set $p(g) = \widetilde{p_1(g)}$ $(g \in G)$. Clearly Q is well-defined on X. In addition the point $\widetilde{(e_0,\, e_0)}$ is an isolated non-degenerate critical point of Q on X. Set $X^1 = \widetilde{X_1^1}$. Notice that for the map $\xi(x,y) = (x_0,\, y_0)$ both x_0 and y_0 do not vanish on X_1^1. Therefore one readily sees that multiplication by $\chi_0 \circ \xi_1$ where $\xi(x,y) = (\xi_1(x,y),\, \xi_2(x,y))$ $((x,y) \in X_1)$ induces a bijection $D'(X^1)^H \to D'(X_1^1,\chi)$. Here $D'(X^1)^H$ denotes the space of H-invariant distribution on X^1. We have the following results (see [4]). There is a map $M : f \mapsto M_f$ which is surjective from $D(X^1)$ onto a space $\mathcal{H}_\eta$, defined by

$$M_f(t) = \int_X f(z)\, \delta(Q(z) - t)\, dz$$

where dz is a G-invariant measure measure on X. One calls $M_f(t)$ the average of f over the surface $Q(z) = t$. The space $\mathcal{H}_\eta$ consists of functions φ on $(0,\infty)$ of the form

$$\varphi(t) = \varphi_0(t) + \eta(t)\varphi_1(t)$$

with $\varphi_0,\, \varphi_1 \in D((0,\infty))$ and η the "singularity function"

$$\eta(t) = \begin{cases} Y(1-t)\,(1-t)^{n-1} & \text{if } n \text{ is even} \\ (t-1)^{n-1} \log|t-1| & \text{if } n \text{ is odd} \end{cases}$$

with Y the Heaviside function: $Y(t) = 1$ of $t \geq 0$, $Y(t) = 0$ if $t < 0$.

We restrict now to the cases $n > 1$, for the time being. Then we have in addition that the adjoint M' of M is injective from $\mathcal{H}'_\eta$ to $D'(X^1)$ with image $D'(X^1)^H$. The proof is non-standard due to the fact that $Q(x,y) = 1$ splits into 4 H-orbits on X again, and one has to exclude the contribution of the singular orbits. We refer to [4], Lemma 7.4. This being established, we may conclude that any bi-H-invariant distribution T on $p^{-1}(X^1) \subset G$ satisfies $T = T^\sigma$. Though this proof is standard, see [6], we repeat the argument again.

Fix Haar measures dg on G and dh on H in such a way that $dg = dzdh$. For $f \in D(G)$ set

$$f^0(z) = \int_H f(gh)\, dh \quad (z = p(g) \in X).$$

Given a bi-H-invariant distribution T on $p^{-1}(X^1) \subset G$ there is a unique H-invariant distribution T_1 on X^1 satisfying $\langle T,\, f\rangle = \langle T_1,\, f^0\rangle$ $(f \in D(p^{-1}(X^1)))$, and conversely. Extend the function Q from X to G by $Q(g) = Q(p(g)$. To show that T is σ-invariant, it is sufficient to show that

$$M_{[(f^\sigma)^0]} = M_{f^0}$$

for all $f \in D(p^{-1}(X^1))$. This is easily checked. For all continuous functions F on $(0,\infty)$ one has

$$\begin{aligned} \int_0^\infty F(t)\, M_{[(f^\sigma)^0]}(t)\, dt &= \int_{X^1} F(Q(z))\, (f^\sigma)^0(z)\, dz \\ = \int_G F(Q(g))\, f^\sigma(g)\, dg &= \int_G F(Q(\sigma(g)))\, f(g)\, dg. \end{aligned}$$

Since $Q(g) = Q(\sigma(g))$ $(g \in G)$ we get the result.

Since the function ξ_1, extended to G by

$$\xi_1(g) = \xi_1(p_1(g))$$

is also σ-invariant, we get that any distribution T on $p_1^{-1}(X_1^1)$ satisfying

$$T(h_1 g h_2) = \chi(h_1 h_2) T(g) \quad (g \in G;\ h_1, h_2 \in H)$$

is σ-invariant.

We now consider the other space $D'(X_1^0, \chi)$ (for all $n \geq 1$), the more difficult one. See for that purpose [4] (section 7, after Lemma 7.5) for the case $\chi = 1$. Here we use another method than in [4], a method without differential equations, and consider therefore the map ξ in more detail. Recall that ξ is a submersion from X_1^0 onto the set U of all points (s,t) in $\mathbb{R}^2$ given by $st < 1$. Its level sets are H_1-orbits on X_1^0. Moreover there are "good" local coordinates, such that Harish-Chandra's descent method can be applied (see e.g. [5]) to obtain

Lemma 1 *The natural pull-back map*

$$\xi^* : D'(U) \rightarrow D'(X_1^0)$$

is injective with image $D'(X_1^0)^{H_1}$.

Indeed, let the notations be as usual, so $H_1 \simeq GL(n)$ and let $M_1 \simeq GL(n-1)$. Define

$$A = \left\{ a_t = \begin{pmatrix} cosh\, t & 0 & sinh\, t \\ 0 & I & 0 \\ sinh\, t & 0 & cosh\, t \end{pmatrix} \right\}, \quad B = \left\{ b_\theta = \begin{pmatrix} cos\,\theta & 0 & sin\,\theta \\ 0 & I & 0 \\ -sin\,\theta & 0 & cos\,\theta \end{pmatrix} \right\}$$

Let throughout $h \in H_1/M_1$. For $Q(x) > 1$ choose local coordinates:

$$(h, s, t) \mapsto (h a_t s \cdot e_0, {}^t h^{-1} a_{-t} s^{-1} \cdot e_0)$$

For $0 < Q(x) < 1$ choose local coordinates

$$(h, \theta, s) \mapsto (h b_\theta s \cdot e_0, {}^t h^{-1} b_\theta s^{-1} \cdot e_0)$$

For $Q(x) < 0$ choose local coordinates

$$(h, s, t) \mapsto (h.a_t s \cdot e_n, {}^t h^{-1} a_{-t} s^{-1} \cdot e_n)$$

In a neighborhood of $Q(x) = 0$ choose local coordinates

$$(h, t, \theta) \mapsto (h a_t b_\theta \cdot e_n, {}^t h^{-1} a_{-t} b_\theta \cdot e_n)$$

In the latter case observe that that the map

$$(t, \theta) \mapsto (x, y)$$

given by

$$x = (h a_t b_\theta \cdot e_n, e_0) = cosh\, t\, sin\, \theta + sinh\, t\, cos\, \theta$$

and

$$y = ({}^t h^{-1} a_{-t} b_\theta \cdot e_n, e_0) = cosh\, t\, sin\, \theta - sinh\, t\, cos\, \theta$$

is a diffeomorphism and $dt d\theta = (2 + x^2 + y^2)^{-1} dx dy$.

It is now easy to show, as before, that any distribution T on $p^{-1}(X_1^0) \subset G$ satisfying

$$T(h_1 g h_2) = \chi(h_1 h_2) T(g) \quad (g \in G;\ h_1, h_2 \in H)$$

is again σ-invariant, since ξ (extended to G) is.

So we are left with the case $n = 1$ on X_1^1. We follow a similar, but more sophisticated reasoning. Let U be as before, the subset of $\mathbb{R}^2$ defined by the relation $st < 1$. Define a map $\zeta : X_1^1 \to U$ by $\zeta(x, y) = (x_1, y_1)$. Then ζ is submersive and parametrizes the H_0-orbits on X_1^1. So we may conclude, as above,

Lemma 2 *The natural pull-back map*

$$\zeta^* : D'(U) \to D'(X_1^0)$$

is injective with image $D'(X_1^1)^{H_0}$.

We have only to consider H-invariant distributions on X_1^1. One easily gets that ζ^* is bijective $D'(U)^L \to D'(X_1^1)^H$ where $L = \mathrm{S}(\mathrm{GL}(1, \mathbb{R}) \times \mathrm{GL}(1, \mathbb{R}))$.

Restricting a distribution $T \in D'(X_1^1)^H$ to the set of all (x, y) with $x_1 \neq 0$, we easily see that T only depends on the product $x_1 y_1$, so on $x_0 y_0 = 1 - x_1 y_1$, since $q(x, y) = x_1 y_1$ is submersive and parametrizes the H-orbits on X_1^1. Hence T is σ-invariant considered as a distribution on G. The same is true on $y_1 \neq 0$. Considering $T - T^\sigma$, we are left with a distribution on U with support at $(0, 0)$ and L-invariant, so of the form

$$\sum_l \alpha_l \frac{\partial^{2l}}{\partial x_1^l \partial y_1^l} \delta \quad \text{(finite sum)}$$

with complex constants α_l, so with a distribution depending only on the product $x_0 y_0$, so with a σ-invariant distribution on G, which must be equal to zero, since it is also σ-anti-invariant. Hence $T = T^\sigma$.

This completes the proof of Theorem 2.

Remark 1. Contrary to the treatment of H-invariant distributions T in [4, 6], we don't have to use in our note the assumption that T has to be spherical. This fact is due to another choice of the anti-involution.

Remark 2. An analogue of Theorem 2 for $(\mathrm{GL}(n + 1, F), \mathrm{GL}(n, F))$ with F any local field, can be derived in a similar way. See e.g. [2]. The hard case is $F = \mathbb{R}$.

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