GEOMETRY OF CONFIGURATION SPACES OF TENSEGRITIES

FRANCK DORAY [1], OLEG KARPENKOV [2] AND JAN SCHEPERS [1]

Abstract. Consider a graph G with n vertices. In this paper we study geometric conditions for an n-tuple of points in $\mathbb{R}^d$ to admit a tensegrity with underlying graph G. We introduce and investigate a natural stratification, depending on G, of the configuration space of all n-tuples in $\mathbb{R}^d$. In particular we find surgeries on graphs that give relations between different strata. Based on numerous examples we give a description of geometric conditions defining the strata for plane tensegrities, we conjecture that the list of such conditions is sufficient to describe any stratum. We conclude the paper with particular examples of strata for tensegrities in the plane with a small number of vertices.

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1. Introduction

In his paper [9] J. C. Maxwell made one of the first approaches to the study of equilibrium states for frames under the action of static forces. He noted that the frames together with the forces give rise to reciprocal figures. In the second half of the twentieth century the artist K. Snelson built many surprising sculptures consisting of cables and bars that are actually such frames in equilibrium, see [14]. R. Buckminster Fuller introduced the name "tensegrity" for these constructions, combining the words "tension" and "integrity". A nice overview of the history of tensegrity constructions is made by R. Motro in his book [10].

In mathematics, tensegrities were investigated in several papers. In [12] B. Roth and W. Whiteley and in [3] R. Connelly and W. Whiteley studied rigidity and flexibility of tensegrities, see also the survey about rigidity in [18].

N. L. White and W. Whiteley started in [17] the investigation of geometric realizability conditions for a tensegrity with prescribed bars and cables. In the preprint [6] M. de Guzmán describes other examples of geometric conditions for tensegrities.

Tensegrities have a wide range of applications in different branches of science and in architecture. For instance they are used in the study of viruses [2], cells [5], for construction of deployable mechanisms [13, 16], etc.

We focus on the following important question. Suppose a graph G is given. *Is the graph G realizable as a tensegrity for a general configuration of its vertices?* We develop a new technique to study this question. We introduce special operations (*surgeries*) that change the graph in a certain way but preserve the property to be (not to be) realizable as a tensegrity.

Let n be the number of vertices of G. Consider the configuration space of all n-tuples of points in $\mathbb{R}^d$. In this paper we define a stratification of the configuration space such, that each stratum corresponds to a certain set of admissible tensegrities associated to G. Suppose that one wants to obtain a construction with some edges of G replaced by struts and the others by cables, then he/she should take a configuration in a specific stratum of the stratification.

In this paper we prove that all the strata are semialgebraic sets, and therefore a notion of dimension is well-defined for them. This allows to generalize the previous question: *what is the minimal codimension of the strata in the configuration space that contains n-tuples of points admitting a tensegrity with underlying graph G?* Our technique of surgeries on graphs also gives the first answers in this case. In particular we obtain the list of all 6, 7, and 8 vertex tensegrities in the plane that are realizable for codimension 1 strata. We note that the complete answers to the above questions are not known to the authors.

N. L. White and W. Whiteley [17] and M. de Guzmán and D. Orden [7, 8] have found the geometric conditions of realizability of plane tensegrities with 6 vertices and of some other particular cases. We continue the investigation for other graphs (see Subsection 6.2). In all the observed examples the strata are defined by certain systems of geometric conditions. It turns out that all these geometric conditions are obtained from elementary ones:

— *two points coincide*;

— *three points are on a line*;
— five points a, b, c, d, e satisfy: *e is the intersection point of the lines passing through points a and b and points c and d respectively.*
We conjecture that for plane tensegrities any stratum can be defined by certain geometric conditions (see Section 5).

This paper is organized as follows. We start in Section 2 with general definitions. In Subsection 2.1 we describe the configuration space of tensegrities associated to a given graph as a fibration over the affine space of all frameworks. We introduce a natural stratification on the space of all frameworks in Subsection 2.2. We prove that all strata are semialgebraic sets and therefore the strata have well-defined dimensions. In Section 3 we study the dimension of solutions for graphs on general configurations of points in $\mathbb{R}^d$. Later in this section we calculate the dimensions in the simplest cases, and formulate general open questions. In Section 4 we study surgeries on graphs and frameworks that induce isomorphisms of the spaces of self-stresses for the frameworks. We give general definitions related to systems of geometric conditions for plane tensegrities in Section 5. We conjecture that any stratum is a dense subset of the solution of one of such systems. Finally in Section 6 we give particular examples of graphs and their strata for tensegrities in the plane. We study the dimension of the space of self-stresses in Subsection 6.1 and give tables of geometric conditions for codimension 1 strata for graphs with 8 vertices and less in Subsection 6.2.

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2. General definitions

2.1. **Configuration spaces of tensegrities.** Recall a slightly modified definition of a framework from [8].

Definition 2.1. Fix a positive integer d. Let $G = (V, E)$ be an arbitrary graph without loops and multiple edges. Let it have n vertices.

- A *framework* $G(P)$ in $\mathbb{R}^d$ is a map of the graph G with vertices $v_1, \ldots, v_n$ on a finite point configuration $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$ with straight edges, such that $G(P)(v_i) = p_i$ for $i = 1, \ldots, n$.
- A *stress* w on a framework is an assignment of real scalars $w_{i,j}$ (called *tensions*) to its edges p_ip_j. We also put $w_{i,j} = 0$ if there is no edge between the corresponding vertices. Observe that $w_{i,j} = w_{j,i}$, since they refer to the same edge.
- A stress w is called a *self-stress* if, in addition, the following equilibrium condition is fulfilled at every vertex p_i:

$$\sum_{\{j|j\neq i\}} w_{i,j}\overline{p_ip_j} = 0.$$

By $\overline{p_ip_j}$ we denote the vector from the point p_i to the point p_j.

- A couple $(G(P), w)$ is called a *tensegrity* if w is a self-stress for the framework $G(P)$.

Remark 2.2. Tensegrities are self-tensional equilibrium frameworks. For instance, any framework for the two vertex graph without edges is always a tensegrity, although it is not rigid. For more information about rigidity of tensegrities we refer to a paper of B. Roth and W. Whiteley [12].

Denote by $W(n)$ the linear space of dimension n^2 of all tensions $w_{i,j}$. Consider a framework $G(P)$ and denote by $W(G, P)$ the subset of $W(n)$ of all possible self-stressed tensions for $G(P)$. By definition of self-stressed tensions, the set $W(G, P)$ is a linear subspace of $W(n)$.

The *configuration space of tensegrities* corresponding to the graph G is the set

$$\big\{(G(P), w) \mid P \in (\mathbb{R}^d)^n, w \in W(G, P)\big\},$$

we denote it by $\Omega(G)$. The set $\{G(P) \mid P \in (\mathbb{R}^d)^n\}$ is said to be the *base of the configuration space*, we denote it by $B_d(G)$. If we forget about the edges between the points in all the frameworks, then we get natural bijections between $\Omega(G)$ and a subset of $(\mathbb{R}^d)^n \times W(n)$ and between $B_d(G)$ and $(\mathbb{R}^d)^n$. Later on we actually identify the last two pairs of sets. The bijections induce natural topologies on $\Omega(G)$ and $B_d(G)$.

Let π be the natural projection of $\Omega(G)$ to the base $B_d(G)$. This defines the structure of a fibration. For a given framework $G(P)$ of the base we call the set $W(G, P)$ *the linear fiber* at the point P (or at the framework $G(P)$) of the configuration space.

Consider a self-stress w for the framework $G(P)$. We say that the edge p_ip_j is a *cable* if $w_{i,j} < 0$ and a *strut* if $w_{i,j} > 0$.

Remark 2.3. The definitions of struts and cables come from the following physical interpretation. Suppose we would like to construct the lightest possible tensegrity structure on a given framework and with a given self-stress using heavy struts and relatively light cables. Then we should replace the edges with positive $w_{i,j}$ with struts, and the edges with negative $w_{i,j}$ with cables. Such constructions would be the lightest possible.

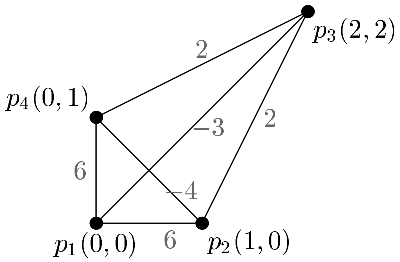
Denote by "sgn" the sign function over $\mathbb{R}$.

Definition 2.4. Consider a framework $G(P)$ and one of its self-stresses w. The $n \times n$ matrix $(\mathrm{sgn}(w_{i,j}))$ is called the *strut-cable matrix* of the stress w and denoted by $\mathrm{sgn}(w)$.

Let us give one example of a strut-cable matrix.

Example 2.5. Consider a configuration of four points in the plane: $p_1(0, 0)$, $p_2(1, 0)$, $p_3(2, 2)$, $p_4(0, 1)$ and a self-stress w as on the picture: $w_{1,2} = 6$, $w_{1,3} = -3$, $w_{1,4} = 6$,

$w_{2,3}=2$, $w_{2,4}=-4$, $w_{3,4}=2$. Then we have:



$$\operatorname{sgn}(w)=\begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

2.2. **Stratification of the base of a configuration space of tensegrities.** Suppose we have some framework $G(P)$ and we want to find the lightest cable-strut construction on it, as explained in Remark 2.3. Then the following questions arise. *Which edges can be replaced by cables, and which by struts? What is the geometric position of points in the configurations for which given edges may be replaced by cables and the others by struts?* The questions lead to the following definition.

Definition 2.6. A linear fiber $W(G,P_1)$ is said to be *equivalent* to a linear fiber $W(G,P_2)$ if there exists a homeomorphism ξ between $W(G,P_1)$ and $W(G,P_2)$, such that for any self-stress w in $W(G,P_1)$ the self-stress $\xi(w)$ satisfies

$$\operatorname{sgn}\big(\xi(w)\big)=\operatorname{sgn}\big(w\big).$$

The described equivalence relation gives us a stratification of the base $B_d(G)=(\mathbb{R}^d)^n$. A *stratum* is by definition a maximal connected set of points with equivalent linear fibers. Once we have proven Theorem 2.8, by general theory of semialgebraic sets (see for instance [1]) it follows that all strata are path-connected.

Example 2.7. We describe the stratification of $B_1(K_3)=\mathbb{R}^3$ for the complete graph K_3 on three vertices. The point (x_1,x_2,x_3) in $\mathbb{R}^3$ corresponds to the framework with vertices $p_1=(x_1)$, $p_2=(x_2)$, and $p_3=(x_3)$. The stratification consists of 13 strata. There is 1 one-dimensional stratum, and there are 6 two-dimensional and 6 three-dimensional strata.

The one-dimensional stratum consists of frameworks with all vertices coinciding. It is defined by the equations $x_1=x_2=x_3$. The dimension of the fiber at a point of this stratum is 3.

Any of the two-dimensional strata consists of frameworks with exactly two vertices coinciding. The strata are the connected components of the complement to the line $x_1=x_2=x_3$ in the union of the three planes $x_1=x_2$, $x_1=x_3$, and $x_2=x_3$. The dimension of the fiber at a point of any of these strata is 2.

Any of the three-dimensional strata consists of frameworks with distinct vertices. The strata are the connected components of the complement in $\mathbb{R}^3$ to the union of the three planes $x_1=x_2$, $x_1=x_3$, and $x_2=x_3$. The dimension of the fiber at a point of any of these strata is 1.

In general we have the following theorem.

Theorem 2.8. *Any stratum is a semialgebraic set.*

For the definition and basic properties of semialgebraic sets we refer the reader to [1].

We need two preliminary lemmas for the proof of the theorem, but first we introduce the following notation.

Let M be an arbitrary symmetric $n \times n$-matrix with zeroes on the diagonal and all the other entries belonging to $\{-1, 0, 1\}$. Let i be an integer with $0 \leq i \leq n^2$. We say that a couple (M, i) is a *stratum symbol.*

For an arbitrary framework $G(P)$ we denote by $W_M(G, P)$ the set of all self-stresses with strut-cable matrix $\text{sgn}(w)$ equal to M. The closure of $W_M(G, P)$ is a pointed polyhedral cone with vertex at the origin. The set $W_M(G, P)$ is homeomorphic to an open k-dimensional disc, we call k the dimension of $W_M(G, P)$ and denote it by $\dim(W_M(G, P))$.

For any stratum symbol (M, i) we denote by $\Xi(M, i)$ the set

$$\{(G(P), w) \mid w \in W(G, P), \text{sgn}(w) = M, \dim(W_M(G, P)) = i\} \subset \Omega(G).$$

Lemma 2.9. *For any stratum symbol (M, i), the subset $\pi(\Xi(M, i))$ of the base $B_d(G)$ is either empty or it is a semialgebraic set.*

Proof. The set $\Xi(M, i)$ is a semialgebraic set since it is defined by a system of equations and inequalities in the coordinates of the vertices and the tensions of the following three types:
a) quadratic equilibrium condition equations;
b) linear equations or inequalities specifying if the coordinate values $w_{i,j}$ are zeroes, positive, or negative reals;
c) algebraic equations and inequalities defining respectively $\dim(W_M(G, P)) \leq i$ and $\dim(W_M(G, P)) \geq i$. Note that $\dim(W_M(G, P))$ is equal to the dimension of the linear space spanned by $W_M(G, P)$.

Let us make a small remark about item (c). At each framework we take the system of equilibrium conditions and equations of type $w_{i,j} = 0$ in the variables $w_{i,j}$. This system consists of the *equalities* of items (a) and (b). It is linear in the variables $w_{i,j}$. The coefficients of the corresponding matrix depend linearly on the coordinates of the framework vertices. The equations and inequalities of item (c) are defined by some determinants of submatrices of this matrix being equal or not equal to zero. Therefore, they are algebraic.

Since by the Tarski-Seidenberg theorem any projection of a semialgebraic set is semialgebraic, the set $\pi(\Xi(M, i))$ is semialgebraic. □

Denote by $S(G, P)$ the set of all stratum symbols (M, i) that are realized by the point $G(P)$, in other words

$$S(G, P) = \{(M, i) \mid G(P) \in \pi(\Xi(M, i))\}.$$

Lemma 2.10. *Let $G(P_1)$ and $G(P_2)$ be two frameworks. Then $S(G, P_1) = S(G, P_2)$ if and only if the linear fiber $W(G, P_1)$ is equivalent to the linear fiber $W(G, P_2)$.*

Proof. Let the linear fiber at the point $G(P_1)$ be equivalent to the linear fiber at the point $G(P_2)$ then by definition we have

$$S(G, P_1) = S(G, P_2).$$

Suppose now that $S(G,P_1) = S(G,P_2)$. Let us denote by $\overline{W(G,P_i)}$ the one point compactification of the fiber $W(G,P_i)$ for $i = 1,2$. So $\overline{W(G,P_i)}$ is homeomorphic to a sphere of dimension $\dim W(G,P_i)$.

For any point P and any M the set $W_M(G,P)$ is a convex cone homeomorphic to an open disc of dimension $\dim(W_M(G,P))$. So, for any point P we have a natural CW-decomposition of $\overline{W(G,P)}$ with cells $W_M(G,P)$ and the new one point cell.

A cell $W_{M'}(G,P_1)$ is adjacent to a cell $W_{M''}(G,P_1)$ iff the cell $W_{M'}(G,P_2)$ is adjacent to the cell $W_{M''}(G,P_2)$. This is true, since the couples of cells corresponding to M' and to M'' are defined by the same sets of equations and inequalities of type "$>$", and the closures of $W_{M'}(G,P_i)$ for $i = 1,2$ are defined by the system defining $W_{M'}(G,P_i)$ with all "$>$" in the inequalities replaced by "$\geq$".

Therefore, there exists a homeomorphism of $\overline{W(G,P_1)}$ and $\overline{W(G,P_2)}$, sending all the cells $W_M(G,P_1)$ to the corresponding cells $W_M(G,P_2)$. We leave the proof of this statement as an exercise for the reader, this can be done by inductively constructing the homeomorphism on the k-skeletons of the CW-complexes.

Hence, the linear fibers $W(G,P_1)$ and $W(G,P_2)$ are equivalent. □

Proof of Theorem 2.8. Let us prove the theorem for a stratum containing some point P. Consider any point P' in the stratum. By definition, $W(G,P)$ is equivalent to the space $W(G,P')$, and hence by Lemma 2.10, we have $S(G,P) = S(G,P')$.

Consider the following set

$$\bigcap_{(M,i)\in S(G,P)} \pi(\Xi(M,i)) \setminus \Big(\bigcup_{(M,i)\notin S(G,P)} \pi(\Xi(M,i))\Big),$$

we denote it $\Sigma(P)$. So $\Sigma(P)$ is the set of frameworks $G(P')$ for which $S(G,P') = S(G,P)$. By Lemma 2.9 all the sets $\pi(\Xi(M,i))$ are semialgebraic. Therefore, the set $\Sigma(P)$ is semialgebraic. Denote by $\Sigma'(P)$ the connected component of $\Sigma(P)$ that contains the point P. Since the set $\Sigma(P)$ is semialgebraic, the set $\Sigma'(P)$ is also semialgebraic, see [1].

Let us show that $\Sigma'(P)$ is the stratum of $B_d(G)$ containing the point P. First, the set $\Sigma'(P)$ is contained in the stratum. This holds since $\Sigma'(P)$ is connected and consists of points with equivalent sets $S(G,P)$. And hence by Lemma 2.10 all the points of $\Sigma'(P)$ have equivalent linear fibers $W(G,P)$. Secondly, the stratum is contained in the space $\Sigma'(P)$. This holds since the stratum is connected and consists of points with equivalent linear fibers $W(G,P)$. Thus by Lemma 2.10 all the points of the stratum have equivalent sets $S(G,P)$.

As we have shown, the stratum containing P coincides with $\Sigma'(P)$ and hence it is semialgebraic. □

From the above proof it follows that the total number of strata is finite.

3. On the tensegrity d-characteristic of graphs

In this section we study the dimension of the linear fiber for graphs on a general point configuration in $\mathbb{R}^d$. We give a natural definition of the tensegrity d-characteristic of a

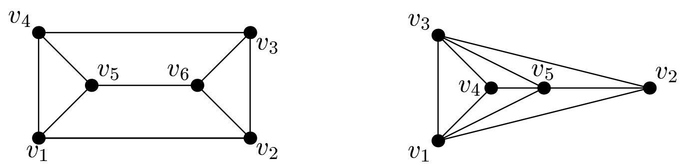


FIGURE 1. A graph with zero tensegrity 2-characteristic (the left one) and a graph whose 2-characteristic equals 2 (the right one).

graph and calculate it for the simplest graphs. In addition we formulate general open questions for further investigation.

3.1. **Definition and basic properties of the tensegrity d-characteristic.** Note that for any two points P_1 and P_2 of the same stratum S of the space $B_d(G)$ for a graph G we have

$$\dim(W(G, P_1)) = \dim(W(G, P_2)).$$

Denote this number by $\dim(G, S)$. Denote also by $\operatorname{codim}(S)$ the integer

$$\dim(B_d(G)) - \dim(S).$$

Consider a graph G with at least one edge. We call the integer

$$\min\{\operatorname{codim} S \mid S \text{ is a stratum of } B_d(G),\ \dim(G, S) > 0\}$$

the *codimension* of G and denote it by $\operatorname{codim}_d(G)$.

Definition 3.1. We call the integer

$$\begin{cases} 1 - \operatorname{codim}_d(G), \text{ if } \operatorname{codim}_d(G) > 0 \\ \max\big\{ \dim(W(G, P)) \,\big|\, G(P) \text{ contained in a codimension zero stratum}\big\}, \text{ otherwise} \end{cases}$$

the *tensegrity d-characteristic* of the graph G (or the *d-TC* of G for short), and denote it by $\tau_d(G)$.

Example 3.2. Consider the two graphs shown on Figure 1. The left one is a graph of codimension 1 in the plane, it can be realized as a tensegrity iff either the two triangles are in perspective position or the points of one of the two triples (v_1, v_4, v_5) or (v_2, v_3, v_6) lie on a line (for more details see [8]), so its 2-TC is zero. The graph on the right has a twodimensional space of self-stresses for a general position plane framework, and hence its 2-TC equals two (we show this later in Proposition 6.1).

Proposition 3.3. *Let S_1 and S_2 be two strata of codimension 0. Let $G(P_1)$ and $G(P_2)$ be two points of the strata S_1 and S_2 respectively. Then the following holds:*

$$\dim(W(G, P_1)) = \dim(W(G, P_2)).$$

Proof. The equilibrium conditions give a linear system of equations in the variables $w_{i,j}$, at each framework linearly depending on the coordinates of the vertices. The dimension of the solution space is determined by the rank of the matrix of this system. The subset of $B_d(G)$ where the rank is not maximal is an algebraic subset of positive codimension.

By definition, this set does not have elements in the strata S_1 and S_2. This yields the statement of the proposition. □

Corollary 3.4. *Let G be a graph. If $\tau_d(G) \geq 0$ then for every framework $G(P)$ in a codimension 0 stratum we have* $\dim W(G,P) = \tau_d(G)$. □

3.2. **Atoms and atom decomposition.** In this subsection we recall a definition and some results of M. de Guzmán and D. Orden [8] that we use later.

Consider a point configuration P of d+2 points in general position in $\mathbb{R}^d$. Throughout this subsection 'general position' means that no d+1 of them are contained in a hyperplane. An *atom* in $\mathbb{R}^d$ is a tensegrity $(K_{d+2}(P), w)$, where K_{d+2} is the complete graph on d+2 vertices and where w is a nonzero self-stress.

According to [8, Section 2] the linear fiber $W(K_{d+2}, P)$ is one-dimensional for P in general position, in particular this implies $\tau_d(K_{d+2}) = 1$. In addition the tension on every edge in the atom is nonzero. A more general statement holds.

Lemma 3.5. [8, Lemma 2.2] *Let $G(P)$ be a framework on a point configuration P in $\mathbb{R}^d$ in general position. Let $p \in P$. Given a nonzero self-stress on $G(P)$, then either at least d+1 of the edges incident to p receive nonzero tension, or all of them have zero tension.*

M. de Guzmán and D. Orden showed that one can consider atoms as the building blocks of tensegrity structures. First, we explain how to add tensegrities. Let $T = (G(P), w)$ and $T' = (G'(P'), w')$ be two tensegrities. We define $T + T'$ as follows. The framework of $T + T'$ is $G(P) \cup G'(P')$, we take the union of vertices and edges. The tension on a common edge $p_ip_j = p'_kp'_l$ is defined as $w_{i,j} + w'_{k,l}$ and on an edge appearing exactly in one of the original frameworks we put the original tension. It is easy to see that the defined stress is a self-stress, so $T + T'$ is a tensegrity.

Theorem 3.6. [8, Theorem 3.2] *Every tensegrity $(G(P), w)$ with a general position point configuration P and $w_{i,j} \neq 0$ on all edges of G is a finite sum of atoms. This decomposition is not unique in general.*

3.3. **Calculation of tensegrity d-characteristic in the simplest cases.** We start this subsection with the formulation of a problem, we do not know the complete solution of it.

Problem 1. Give a general formula for $\tau_d(G)$ in terms of the combinatorics of the graph.

Let us calculate the d-TC for a complete graph, this will give us the maximal value of the d-TC for fixed number of vertices n and dimension d.

Proposition 3.7. *For any positive integers n and d satisfying $n \geq d$+2, we have*

$$\tau_d(K_n) = \frac{(n-d-1)(n-d)}{2}.$$

Proof. We work by induction on n. For $n = d+2$ the d-TC equals 1, as mentioned above. For $n > d$+2 we choose any point configuration P on n points such that no d+1 of them lie in a hyperplane. Take $p \in P$. Any tensegrity $(K_n(P), w)$ can be decomposed as a

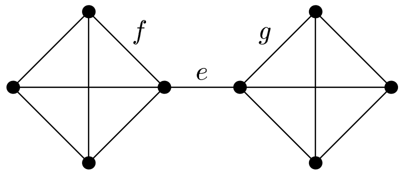


FIGURE 2. The tension on the edge e is always zero.

sum of $n-d-1$ atoms with p as vertex and a tensegrity on $P \setminus \{p\}$ with underlying graph K_{n-1}. Indeed, we can use such atoms to cancel the given tensions on $n-d-1$ edges at p. Then there are only d edges left, so by Lemma 3.5 the tensions on these edges equal zero. We conclude by induction that

$$\tau_d(K_n) = \tau_d(K_{n-1}) + n - d - 1 = \frac{(n-d-1)(n-d)}{2}.$$

□

Now we show how the d-TC behaves when we remove an edge of the graph.

Proposition 3.8. *Let G be some graph satisfying $\tau_d(G) > 1$. Let a graph G' be obtained from the graph G by erasing one edge. Then*

$$\tau_d(G) - \tau_d(G') \in \{0, 1\}.$$

Proof. Erasing one edge is equivalent to adding a new linear equation $w_{i,j} = 0$ to the linear system defining the space $W(G, P)$ for the graph G (for any point P). This implies that the space of solutions coincides with $W(G, P)$ or it is a hyperplane in $W(G, P)$.

So, first, $\tau_d(G') \leq \tau_d(G)$.

Secondly, since $\tau_d(G) = \dim(W(G, P_0))$ for some framework P_0 of a codimension 0 stratum for G (and therefore it belongs to a codimension 0 stratum for G'), then

$$\tau_d(G') = \dim(W(G', P_0)) \geq \dim(W(G, P_0)) - 1 = \tau_d(G) - 1.$$

This completes the proof. □

As we show in the example below, erasing an edge does not always reduce the tensegrity characteristic.

Example 3.9. Consider the graph shown in Figure 2. Assume that this graph underlies a tensegrity. Then we can add an atom on the four leftmost vertices, to cancel the tension on edge f for instance. This automatically cancels the tensions on the edges connecting the four leftmost vertices by Lemma 3.5. We can do the same on the right. So the tension on e is zero as well. Therefore the tension on e was zero from the beginning and hence deleting e does not change the 2-TC. In Example 6.3 we give a less trivial example of this phenomenon.

Let us formulate two general corollaries of Proposition 3.8.

Corollary 3.10. *Let G be a graph on n vertices and $m \in \mathbb{Z}_{>0}$. If G has*

$$m + \frac{n(n-1)}{2} - \tau_d(K_n) = m + dn - \frac{d^2+d}{2}$$

edges, then $\tau_d(G) \geq m$.

Proof. Combine Proposition 3.7 and Proposition 3.8. □

The following corollary is useful for the calculation of the tensegrity d-characteristic. In Subsection 6.1 we use it to calculate all the tensegrity 2-characteristics for sufficiently connected graphs with less than 8 vertices.

Corollary 3.11. *Let G be a graph on n vertices with $\tau_d(G) \geq 0$. Assume that G has*

$$dn - \frac{d^2+d}{2} + \tau_d(G)$$

edges. Then for any graph H that can be obtained from G by adding N edges we have

$$\tau_d(H) = \tau_d(G) + N.$$

Proof. We delete $\tau_d(K_n) - \tau_d(G) - N$ edges from K_n to reach H. If the d-TC does not drop by 1 at one of these steps, then we apply Proposition 3.8 an additional N times to H to reach G. This leads to a wrong value of $\tau_d(G)$. So the d-TC drops by one in each of the first $\tau_d(K_n) - \tau_d(G) - N$ steps and the formula for $\tau_d(H)$ follows. □

Example 3.12. A *pseudo-triangle* is a planar polygon with exactly three vertices at which the angles are less than π. Let G be a planar graph with n vertices and k edges that admits a pseudo-triangular embedding $G(P)$ in the plane, i.e. a non-crossing embedding such that the outer face is convex and all interior faces are pseudo-triangles. It is obvious that a pseudo-triangular embedding $G(P)$ belongs to a codimension 0 stratum of $B_2(G)$. By Lemma 2 of [11] we find that

— $\tau_2(G) = k - (2n-3)$ if $k - (2n-3) \geq 1$,
— $\tau_2(G) \leq 0$ if $k - (2n-3) = 0$.

(Note that for pseudo-triangular embeddings we always have $k \geq 2n-3$.)

4. Surgeries on graphs that preserve the dimension of the fibers

In this section we describe operations that one can perform on a graph without changing the dimensions of the corresponding fibers for the frameworks. We refer to such operations as *surgeries*. The first type of surgeries is for general dimension, while the other two are restricted to dimension $d = 2$. We do not know other similar operations that are not compositions of the surgeries described below.

The idea of surgeries is analogous to the idea of Reidemeister moves in knot theory. If two graphs are connected by a sequence of surgeries, then one obtains tensegrities for the first graph from tensegrities for the second graph and vice versa.

We essentially use surgeries to calculate the list of geometric conditions for the strata for (sufficiently connected) graphs with less than 9 vertices and with zero 2-TC in Subsection 6.2.

4.1. **General surgeries in arbitrary dimension.** For an edge e of a graph G we denote by G_e the graph obtained from G by removing e. Recall that a subgraph G' of a graph G is said to be *induced* if, for any pair of vertices v_i and v_j of G', v_iv_j is an edge of G' if and only if v_iv_j is an edge of G.

Denote by $\Sigma_d(G)$ the union of codimension zero strata in $B_d(G)$. Let G be a graph and H a subgraph. Consider the map that takes a framework for G to the framework for H by forgetting all the vertices and edges of G that are not in H. Denote by $\Sigma_d(G, H)$ the preimage of $\Sigma_d(H)$ for this map.

Proposition 4.1. *Let G be a graph and H an induced subgraph with $\tau_d(H) = 1$. Consider a configuration P_0 lying in $\Sigma_d(G, H)$. Suppose that there exists a self-stress on the framework $G(P_0)$ that has nonzero tensions for all edges of H and zero tensions on the other edges. Let e_1, e_2 be edges of H. Then for any $P \in \Sigma_d(G, H)$ we have*

$$W(G_{e_1}, P) \cong W(G_{e_2}, P).$$

The corresponding surgery takes the graph G_{e_1} to G_{e_2}, or vice versa.

Remark 4.2. We always have the inclusion $\Sigma_d(G) \subset \Sigma_d(G, H)$, this follows directly from the definition of the strata. Nevertheless the set $\Sigma_d(G, H)$ usually contains many strata of $B_d(G)$ of positive codimension. So Proposition 4.1 is applicable to all strata of codimension zero as well as to some strata of positive codimension.

For the proof of Proposition 4.1 we need the following lemma.

Lemma 4.3. *Let G be a graph with $\tau_d(G) = 1$ and e one of its edges. Suppose that there exists a configuration $P_0 \in \Sigma_d(G)$ and a nonzero self-stress w_0 such that $w_0(e) = 0$. Then for any tensegrity $(G(P), w)$ with $P \in \Sigma_d(G)$ we get $w(e) = 0$.*

Proof. Since $\tau_d(G) = 1$ and $P_0 \in \Sigma_d(G)$, any tensegrity $(G(P_0), w)$ satisfies the condition $w(e) = 0$. Therefore, any tensegrity with P in the same stratum as P_0 has zero tension at e. So the condition *always to have zero tension at e* defines a somewhere dense subset S in $B_d(G)$. Since the condition is defined by a solution of a certain linear system, S is dense in $B_d(G)$. It follows that $\Sigma_d(G)$ is a subset of S. □

Proof of Proposition 4.1. From Lemma 4.3 we have that for any configuration of $\Sigma_d(H)$ there exist a unique up to a scalar self-stress that is nonzero at each edge of H. The uniqueness follows from the fact that $\tau_d(H) = 1$. Hence for any configuration of $\Sigma_d(G, H)$ there exists a unique up to a scalar self-stress that is nonzero at each edge of H and zero at all other edges of G.

For any $P \in \Sigma_d(G, H)$ we obtain an isomorphism between $W(G_{e_1}, P)$ and $W(G_{e_2}, P)$ by adding the unique tensegrity on the underlying subgraph H of G that cancels the tension on e_2, considered as edge of G_{e_1}. □

In particular one can use atoms (i.e. $H = K_{d+2}$) in the above proposition.

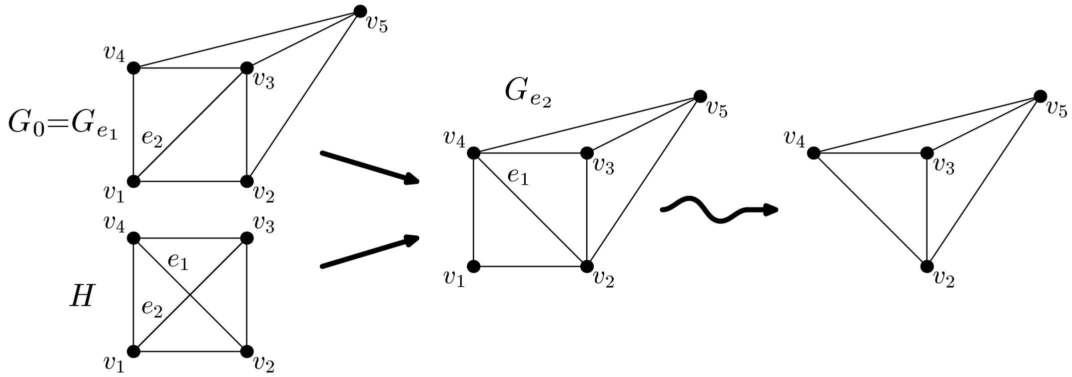


FIGURE 3. This shows that $\tau_2(G_0) = 1$.

Corollary 4.4. *In the notation and with the conditions of Proposition 4.1 we have: if either $\tau_d(G_{e_1}) > 0$ or $\tau_d(G_{e_2}) > 0$ then*

$$\tau_d(G_{e_1}) = \tau_d(G_{e_2}).$$

Proof. The statement follows directly from Proposition 4.1 and Corollary 3.4. $\square$

Let us show how to use the above corollary to compute the tensegrity characteristic.

Example 4.5. We calculate the 2-TC of the graph G_0 shown in Figure 3. Consider the atom H on the vertices v_1, v_2, v_3, and v_4 and let e_1, e_2 be the edges v_2v_4, v_1v_3 respectively. Denote by G the graph obtained from G_0 by adding the edge e_2. So the graph G_0 is actually G_{e_1}. By Corollary 3.10 we have $\tau_2(G_0) \geq 1$, and hence it is possible to apply Corollary 4.4. Consider the graph G_{e_2}, it is shown in Figure 3 in the middle. The degree of the vertex v_1 in this graph equals 2, so by Lemma 3.5 the tensions on its incoming edges equal zero if the points v_1, v_2, and v_4 are not on a line. After removing these two edges and the vertex v_1 we get the graph of an atom. Therefore,

$$\tau_2(G_0) = 1.$$

4.2. **Additional surgeries in dimension two.** In this subsection we study two surgeries on edges of plane frameworks that do not change the dimension of the fibers of the frameworks.

Surgery I. Consider a graph G and a framework $G(P)$. Let G contain the complete graph K_4 with vertices v_1, v_2, v_3, and v_4 as an induced subgraph. Suppose that the edges between v_1, v_2, v_3, v_4 and other vertices of G are as follows:
— pv_2 and qv_3 for unique vertices p and q;
— the edges pv_1 and qv_1;
— any set of edges from v_4.

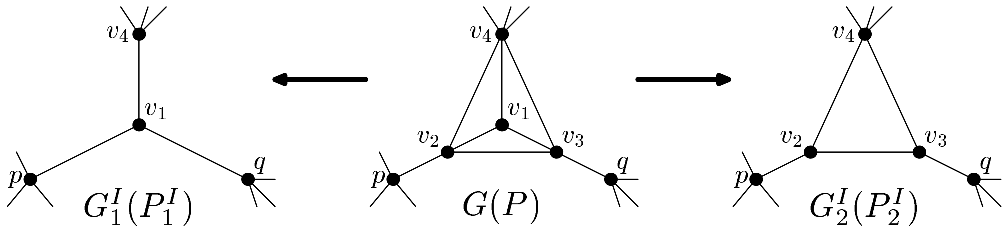


FIGURE 4. Surgery I.

In addition we require that the framework $G(P)$ has the triples of points (p, v_1, v_2) and (q, v_1, v_3) on one line. See Figure 4 in the middle.

Let us delete from the graph G the vertices v_2 and v_3 (the vertex v_1) with all edges adjacent to them. We denote the resulting graph by G_1^I (by G_2^I respectively). The corresponding framework is denoted by $G_1^I(P_1^I)$ (by $G_2^I(P_2^I)$ respectively). See Figure 4 on the left (on the right). Surgery I takes G_1^I to G_2^I or vice versa.

Proposition 4.6. *Consider the frameworks $G(P)$, $G_1^I(P_1^I)$, and $G_2^I(P_2^I)$ as above. If the triples of points (p, v_2, v_3), (q, v_2, v_3), (p, v_2, v_4), (q, v_3, v_4) and (v_2, v_3, v_4) are not on a line then we have*

$$W(G_1^I, P_1^I) \cong W(G_2^I, P_2^I).$$

Proof. We explain how to go from $W(G_2^I, P_2^I)$ to $W(G_1^I, P_1^I)$. The inverse map is simply given by the reverse construction. By the conditions the intersection point v_1 of pv_2 and qv_3 is uniquely defined and not on the lines through v_2 and v_4 or v_3 and v_4. We add the uniquely defined atom on v_1, v_2, v_3, v_4 to $G_2^I(P_2^I)$ that cancels the tension on v_2v_3. Since p, v_2, v_1 lie on one line, this surgery also cancels the tension on v_2v_4 and similarly for v_3v_4. Due to the equilibrium condition at v_2, we can replace the edges pv_2 and v_2v_1 with their tensions $w_{p,2}$ and $w_{2,1}$ by an edge pv_1 with tension $w_{p,1}$ defined by one of the following vector equations:

$$w_{p,2}\overline{pv_2} = w_{p,1}\overline{pv_1} = w_{2,1}\overline{v_2v_1}.$$

This uniquely defines a self-stress on $G_1^I(P_1^I)$. □

Corollary 4.7. *Assume that one of the following conditions holds:*

(1) $\tau_2(G_1^I) > 0$ *or* $\tau_2(G_2^I) > 0$.

(2) $\tau_2(G_1^I) = 0$ *and there is a codimension 1 stratum S of $B_2(G_1^I)$ such that*
— $\dim W(G_1^I, P) > 0$ *for a $G_1^I(P)$ in the stratum S,*
— *the stratum S is not contained in the subset of $B_2(G_1^I)$ of frameworks having one of the triples of points (p, v_1, q), (p, v_1, v_4), or (q, v_1, v_4) on one line.*

(3) $\tau_2(G_2^I) = 0$ *and there is a codimension 1 stratum S' of $B_2(G_2^I)$ such that*
— $\dim W(G_2^I, P') > 0$ *for a $G_2^I(P')$ in the stratum S',*
— *the stratum S' is not contained in the subset of $B_2(G_2^I)$ of frameworks having (p, v_2, v_3), (q, v_2, v_3), (p, v_2, v_4), (q, v_3, v_4), or (v_2, v_3, v_4) on one line.*

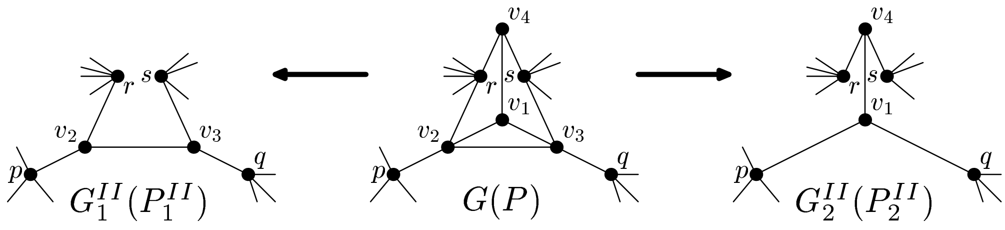


FIGURE 5. Surgery II.

Then

$$\tau_2(G_1^I) = \tau_2(G_2^I).$$

Proof. Let A be the subset of $B_2(G_2^I)$ of frameworks having (p, v_2, v_3), (q, v_2, v_3), (p, v_2, v_4), (q, v_3, v_4) or (v_2, v_3, v_4) on one line. Let B be the subset of $B_2(G_1^I)$ of frameworks having (p, v_1, q), (p, v_1, v_4) or (q, v_1, v_4) on one line. Note that A and B are of codimension 1. The proof of Proposition 4.6 gives a surjective map

$$\varphi : B_2(G_2^I) \setminus A \to B_2(G_1^I) \setminus B$$

inducing an isomorphism between the linear fibers above $G(P) \in B_2(G_2^I)\setminus A$ and $\varphi(G(P))$. Now in all the cases (1)—(3) the statement of the corollary follows directly from the definition of the tensegrity characteristic. □

Surgery II. Consider a graph G and a framework $G(P)$. Let G contain the complete graph K_4 with vertices v_1, v_2, v_3, and v_4 as an induced subgraph. Suppose that the set of edges between v_1, v_2, v_3, v_4 and other vertices of G is

$$\{pv_1, pv_2, qv_1, qv_3, rv_2, rv_4, sv_3, sv_4\},$$

for unique points p, q, r, s. In addition we require that the framework $G(P)$ has the triples of points

$$(p, v_1, v_2), \quad (q, v_1, v_3), \quad (r, v_2, v_4), \quad \text{and} \quad (s, v_3, v_4)$$

on one line. See Figure 5 in the middle.

Let us delete from the graph G the vertices v_1 and v_4 (v_2 and v_3) with all edges adjacent to them. We denote the resulting graph by G_1^{II} (by G_2^{II} respectively). The corresponding framework is denoted by $G_1^{II}(P_1^{II})$ (by $G_2^{II}(P_2^{II})$ respectively). See Figure 5 on the left (on the right). Surgery II takes G_1^{II} to G_2^{II} or vice versa.

The proofs of the proposition and corollary below are similar to the proofs of Proposition 4.6 and Corollary 4.7.

Proposition 4.8. *Consider the frameworks $G(P)$, $G_1^{II}(P_1^{II})$, and $G_2^{II}(P_2^{II})$ as above. If non of the triples of points (p, q, v_1), (p, v_1, v_4), (r, v_1, v_4), (q, v_1, v_4), (s, v_1, v_4), or (r, s, v_4) lie on a line then we have*

$$W(G_1^{II}, P_1^{II}) \cong W(G_2^{II}, P_2^{II}).$$

□

Corollary 4.9. *Assume that one of the following conditions holds:*
(1) $\tau_2(G_1^{II}) > 0$ *or* $\tau_2(G_2^{II}) > 0$.
(2) $\tau_2(G_1^{II}) = 0$ *and there is a codimension 1 stratum* S *of* $B_2(G_1^{II})$ *such that*
— $\dim W(G_1^{II}, P) > 0$ *for a* $G_1^{II}(P)$ *in the stratum* S,
— *the stratum* S *is not contained in the subset of* $B_2(G_1^{II})$ *of frameworks having* (p, v_2, v_3), (q, v_2, v_3), (p, v_2, r), (q, v_3, s), (r, v_2, v_3), *or* (s, v_2, v_3) *on one line.*
(3) $\tau_2(G_2^{II}) = 0$ *and there is a codimension 1 stratum* S' *of* $B_2(G_2^{II})$ *such that*
— $\dim W(G_2^{II}, P') > 0$ *for a* $G_2^{II}(P')$ *in the stratum* S',
— *the stratum* S' *is not contained in the subset of* $B_2(G_2^{II})$ *of frameworks having* (p, q, v_1), (p, v_1, v_4), (r, v_1, v_4), (q, v_1, v_4), (s, v_1, v_4), *or* (r, s, v_4) *on one line.*

Then

$$\tau_2(G_1^{II}) = \tau_2(G_2^{II}).$$

□

5. Geometric relations for strata and complexity of tensegrities in two-dimensional case

In all the observed examples of plane tensegrities with a given graph the strata for which a tensegrity is realizable are defined by certain geometric conditions on the points of the corresponding frameworks. In this section we study such geometric conditions. In Subsection 5.1 we describe an example of a geometric condition for a particular graph. Further, in Subsection 5.2 we give general definitions related to systems of geometric conditions. Finally, in Subsections 5.3 and 5.4 we formulate two open questions related to the geometric nature of tensegrity strata.

To avoid problems with describing annoying cases of parallel/nonparallel lines we extend the plane $\mathbb{R}^2$ to the projective space. It is convenient for us to consider the following model of the projective space: $\mathbb{R}P^2 = \mathbb{R}^2 \cup l_\infty$. The set of points l_∞ is the set of all "directions" in the plane. The set of lines of $\mathbb{R}P^2$ is the set of all plane lines (each plane line contains now a new point of l_∞ that is the direction of l) together with the line l_∞. Now any two lines intersect at exactly one point.

5.1. A simple example.

First, we study the graph shown in Figure 1 on the left, we denote it by G_0. In [17] N. L. White and W. Whiteley proved that the 2-TC of this graph is zero. They showed that there exists a nonzero tensegrity with graph G_0 and framework P iff the points of P satisfy one of the following three conditions:
i) the lines v_1v_2, v_3v_4, and v_5v_6 have a common nonempty intersection (in $\mathbb{R}P^2$);
ii) the vertices v_1, v_4, and v_5 are in one line;
iii) the vertices v_2, v_3, and v_6 are in one line.

We remind that the base $B(G_0)$ of the configuration space is $\mathbb{R}^{12}$ with coordinates $(x_1, y_1, \ldots, x_6, y_6)$, where (x_i, y_i) are the coordinates of v_i. Condition (i) defines a degree

4 hypersurface with equation

$$\det\begin{pmatrix} y_1-y_2 & y_3-y_4 & y_5-y_6 \\ x_2-x_1 & x_4-x_3 & x_6-x_5 \\ x_1y_2-x_2y_1 & x_3y_4-x_4y_3 & x_5y_6-x_6y_5 \end{pmatrix}=0.$$

and Conditions (*ii*) and (*iii*) define the conics

$$x_1y_4+x_4y_5+x_5y_1-x_1y_5-x_4y_1-x_5y_4=0, \quad \text{and}$$
$$x_2y_3+x_3y_6+x_6y_2-x_2y_6-x_3y_2-x_6y_3=0$$

respectively.

5.2. **Systems of geometric conditions.** Let us define three elementary geometric conditions. Consider an ordered subset $P=\{p_1,\dots,p_n\}$ of the projective plane.

2-point condition. We say that the subset P *satisfies the condition* $p_i=p_j$ if p_i coincides with p_j.

3-point condition. We say that the subset P *satisfies the condition*

$$p_i \nabla p_j \nabla p_k = 0$$

if the points p_i, p_j, and p_k are on a line.

5-point condition. We say that the subset P *satisfies the condition*

$$p_i=[p_j,p_{j'};p_k,p_{k'}]$$

if the four points $p_j,p_{j'},p_k$, and $p_{k'}$ are on a line and p_i also belongs to this line, or if $p_i=p_jp_{j'}\cap p_kp_{k'}$ otherwise. We say that $[p_j,p_{j'};p_k,p_{k'}]$ is the *intersection symbol* of the lines $p_jp_{j'}$ and $p_kp_{k'}$.

Note that we define the last condition in terms of closures, since $[p,q;r,s]$ is not defined for all 4-tuples, but for a dense subset.

Definition 5.1. Consider a system of elementary geometric conditions for ordered n-point subsets of $\mathbb{R}P^2$, and let $m\le n$.

— We say that the ordered n-point subset P of projective plane *satisfies* the system of elementary geometric conditions if P satisfies each of these conditions.

— We say that the ordered subset $\{p_1,\dots,p_m\}$ *satisfies conditionally* the system of elementary geometric conditions if there exist points $q_1,\dots,q_{n-m}$ such that the ordered set

$$\{p_1,\dots,p_m,q_1,\dots,q_{n-m}\}$$

satisfies the system. We call the number $n-m$ *the conditional number* of the system.

Example 5.2. The condition that *six points* $p_1,\dots,p_6$ *lie on a conic* is equivalent to the following geometric conditional system:

$$\begin{cases} q_1=[p_1,p_2;p_4,p_5] \\ q_2=[p_2,p_3;p_5,p_6] \\ q_3=[p_3,p_4;p_1,p_6] \\ q_1\nabla q_2\nabla q_3=0 \end{cases}.$$

This is a reformulation of Pascal's theorem. The conditional number is 3 here.

We can rewrite the system as follows, for short:

$$[p_1, p_2; p_4, p_5] \nabla [p_2, p_3; p_5, p_6] \nabla [p_3, p_4; p_1, p_6] = 0.$$

Example 5.3. The condition for six points $p_1, \dots, p_6$ that *the lines* p_1p_2*,* p_3p_4*, and* p_5p_6 *have a common point* is equivalent to the following geometric conditional system:

$$\begin{cases} q_1 = [p_1, p_2; p_3, p_4] \\ q_1 \nabla p_5 \nabla p_6 = 0 \end{cases},$$

or in a shorter form:

$$[p_1, p_2; p_3, p_4] \nabla p_5 \nabla p_6 = 0.$$

The conditional number of the system is 1.

5.3. **Conjecture on geometric structure of the strata.** For a given positive integer k and a graph G consider the set of all frameworks $G(P)$ at which the dimension of the fiber $W(G, P)$ is greater than or equal to k. We call this set the (G, k)*-stratum.* Since any (G, k)-stratum is a finite union of strata of the base $B_2(G)$, it is semialgebraic.

Definition 5.4. Let G be a graph and k be a positive integer. The (G, k)-stratum is said to be *geometric* if it is a finite union of the sets of conditional solutions of systems of geometric conditions (in these systems $p_1, \dots, p_m$ correspond to the vertices of the graph).

Conjecture 2. For any graph G and integer k the (G, k)-stratum is geometric.

The conjecture is checked for all the graphs with seven and fewer vertices, see Section 6 for the techniques.

Problem 3. Find analogous elementary geometric conditions in the three- (higher-) dimensional case.

We refer to [17] for examples of geometric conditions in dimension 3.

5.4. **Complexity of the strata.** We end this section with a discussion of the complexity of geometric (G, k)-strata.

A geometric (G, k)-stratum is defined by some union of the conditional solutions of systems of geometric conditions. Each system in this union has its own conditional number. Take the maximal among all the conditional numbers in the union. We call the minimal number among such maximal numbers for all the unions of systems defining the same (G, k)-stratum the *geometric complexity* of the (G, k)-stratum.

Example 5.5. The geometric complexity of $(G_0, 1)$ stratum for the graph G_0 described in Subsection 5.1 and shown in Figure 1 on the left equals 3.

Problem 4. Find the asymptotics of the maximal complexity of geometric (G, k)-strata with bounded number of vertices k while k tends to infinity.

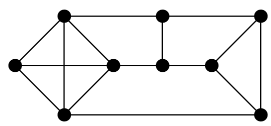


FIGURE 6. A graph G with 9 vertices, 15 edges and $\tau_2(G) = 1$.

6. Plane tensegrities with a small number of vertices

In this section we work in the two-dimensional case (unless otherwise stated). In Subsection 6.1 we study the 2-TC of graphs. In particular, we calculate the 2-TC for sufficiently connected graphs with seven or less vertices. In Subsection 6.2 we give a list of geometric conditions for realizability of tensegrities in the plane for graphs with zero 2-TC.

6.1. On the tensegrity 2-characteristic of graphs.

Recall the following definitions from graph theory. Let G be a graph. The *vertex connectivity* $\kappa(G)$ is the minimal number of vertices whose deletion disconnects G. The *edge connectivity* $\lambda(G)$ is the minimal number of edges whose deletion disconnects G. It is well known that $\kappa(G) \leq \lambda(G)$.

For general dimension d, let $G(P)$ be a framework in $\mathbb{R}^d$ with underlying graph G. If $\kappa(G) < d$ or $\lambda(G) < d + 1$ then $G(P)$ consists of two or more pieces that can rotate with respect to each other. So for us the most interesting graphs are those with $\kappa(G) \geq d$ and $\lambda(G) \geq d+1$.

Proposition 6.1. *Let G be a 2-vertex and 3-edge connected graph with k edges and n vertices. If $n \leq 7$, then*

$$\tau_2(G) = k - 2n + 3.$$

Remark 6.2. In particular we have equality in Corollary 3.10 under the conditions of Proposition 6.1. The formula of Proposition 6.1 holds for many graphs in general, see for instance Example 3.12. It does not always hold for graphs with 9 vertices as the example below shows.

Example 6.3. Let G be the graph with 9 vertices and 15 edges as in Figure 6. If we use the formula of Proposition 6.1, then we have $\tau_2(G) = 0$. Nevertheless, G contains K_4 as an induced subgraph. Hence for any framework $G(P)$ the dimension of $W(G, P)$ is at least 1 (we put zero tensions on all edges not belonging to K_4 and choose a nonzero self-stress on K_4). So $\tau_2(G) \geq 1$. In fact it is not hard to prove that $\tau_2(G) = 1$. This in particular implies that the tensions on all edges not belonging to K_4 are zero for a framework in a codimension zero stratum.

Notice that the graph G of Example 6.3 is not a *Laman graph*, i.e. a graph with $2n-3$ edges, where n is the number of vertices, for which each subset of $m \geq 2$ vertices spans at most $2m-3$ edges. Theorem 1.1 of [4] shows that every planar Laman graph H can be embedded as a pseudo-triangulation and hence $\tau_2(H) \leq 0$ by Example 3.12. We suspect that equality holds here, and more generally for all Laman graphs.

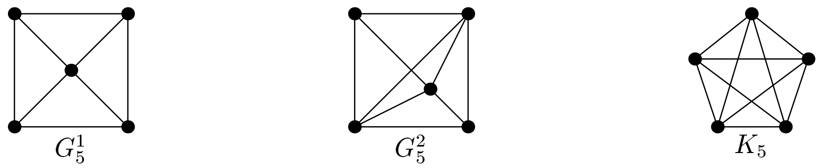


FIGURE 7. The three possible graphs with five vertices, $\kappa \geq 2$ and $\lambda \geq 3$.

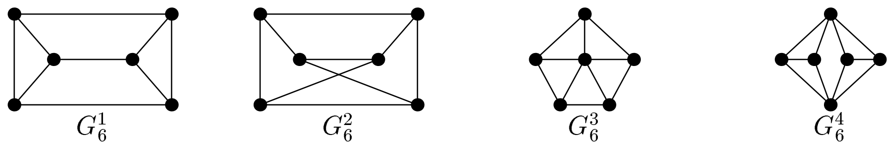


FIGURE 8. The four graphs with six vertices, $\kappa \geq 2$, $\lambda \geq 3$ and a minimal number of edges.

Proof of Proposition 6.1. We use a classification argument.

Four vertices. For the complete graph K_4 we have $\tau_2(K_4) = 1 = 6-8+3$. There are no other graphs satisfying the conditions of the proposition.

Five vertices. There are three possibilities, we show them in Figure 7. From Proposition 3.7 we know that $\tau_2(K_5) = 3 = 10 - 10 + 3$ and in Example 4.5 we have seen that $\tau_2(G_5^1) = 1 = 8 - 10 + 3$. To see that $\tau_2(G_5^2) = 2$ we apply Corollary 3.11.

Six vertices. From the classification of graphs on six vertices (see for instance [15]) we know that any such 2-vertex and 3-edge connected graph can be obtained by adding edges to one of the four graphs shown in Figure 8. By Corollary 3.11 it suffices to check the formula of the proposition for them.

Note that G_6^1 and G_6^2 have 9 edges. They both have zero 2-TC $(9-12+3 = 0)$. Indeed, in Subsection 5.1 we mentioned that $B_2(G_6^1)$ has codimension 1 strata with nontrivial linear fiber. As it is stated in [17] the graph G_6^2 underlies a tensegrity if and only if the six points lie on a conic, which is also a codimension 1 condition. Note that G_6^2 is the complete bipartite graph $K_{3,3}$. For G_6^3 we proceed as follows. From Corollary 3.10 it follows that $\tau_2(G_6^3) \geq 1$. Then we use Proposition 4.1 in the same way as in Example 4.5 to show that

$$\tau_2(G_6^3) = \tau_2(G_5^1) = 1, \quad \text{and again} \quad 10 - 12 + 3 = 1,$$

see Figure 9.

It is easy to see that the same argument works to show that $\tau_2(G_6^4) = 1$.

Seven vertices. From the classification of graphs with seven vertices (see [15]) we get that all 2-vertex and 3-edge connected graphs on seven vertices can be obtained by adding edges to one of the seven graphs shown in Figure 10. By Corollary 3.11 it suffices again to check these graphs.

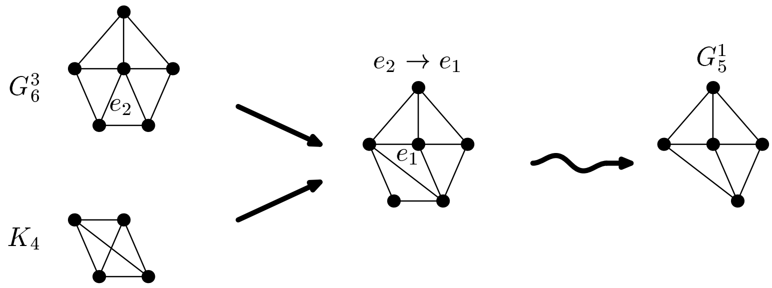


FIGURE 9. Using Proposition 4.1 we get that $\tau_2(G_6^3) = \tau_2(G_5^1) = 1$.

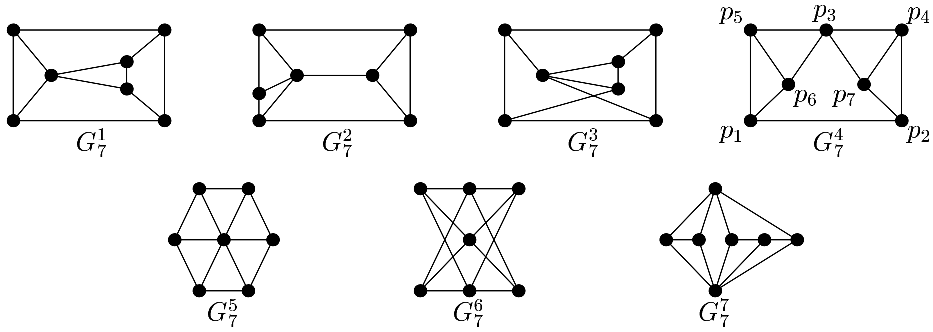


FIGURE 10. The seven graphs with seven vertices, $\kappa \geq 2$, $\lambda \geq 3$ and a minimal number of edges.

To prove that $\tau_2(G_7^1) = \tau_2(G_7^2) = 0$ we use Corollary 4.7 applied to G_6^1. Note that the geometric conditions for G_6^1 to underlie a nonzero tensegrity (see Subsection 5.1) allow to apply Corollary 4.7. Similarly, we apply Corollary 4.7 to G_6^2 to conclude that $\tau_2(G_7^3) = 0$. By computations analogous to [8, Section 4] we find that $\tau_2(G_7^4) = 0$. Indeed, one can show that this graph underlies a nonzero tensegrity if and only if at least one of the following codimension 1 conditions holds:

$$p_1 \nabla p_2 \nabla p_3 = 0, \quad p_1 \nabla p_5 \nabla p_6 = 0, \quad p_2 \nabla p_4 \nabla p_7 = 0, \quad p_3 \nabla p_4 \nabla p_7 = 0, \quad p_3 \nabla p_5 \nabla p_6 = 0.$$

So the first four graphs with 11 edges have zero 2-TC. The other three have 12 edges. We apply Corollary 4.7 to G_6^3 and G_6^4 to obtain that

$$\tau_2(G_7^5) = 1 \quad \text{and} \quad \tau_2(G_7^7) = 1.$$

To prove that the 2-TC of $G_7^6 = K_{3,4}$ is 1 we proceed as follows. First, $\tau_2(G_7^6) \geq 1$ by Corollary 3.10. Then we apply Proposition 4.1 as shown in Figure 11. The graph G has 6 vertices and 10 edges and thus we have $\tau_2(G) = 1$. It is easy to check that for a general position framework $G(P)$ with a nonzero self-stress, all edges of $G(P)$ have nonzero stress. On the middle picture we get a vertex of degree 2, so we reduce to the graph H on the right. Note that H is isomorphic to G, so $\tau_2(H) = 1$. Hence $\tau_2(G_7^6) = 1$ as well.

□

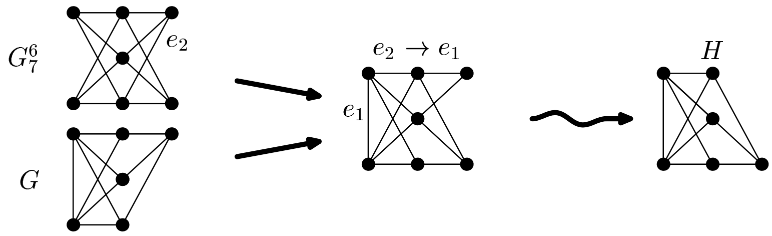


FIGURE 11. Using Proposition 4.1 one sees that $\tau_2(G_7^6) = 1$.

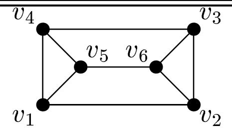
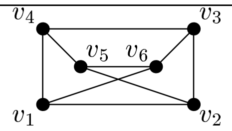
6.2. **Geometric conditions for realizability of plane tensegrities for graphs with zero tensegrity 2-characteristic.** Like in intersection theory of algebraic varieties, it often happens that strata for a graph with negative 2-TC are obtained as intersections of closures of some strata of graphs with zero 2-TC. So the conditions for realizability of plane tensegrities for graphs with zero 2-TC are the most important. In this subsection we give all the conditions for the zero 2-TC graphs with number of vertices not exceeding 8.

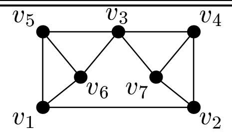
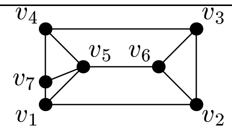
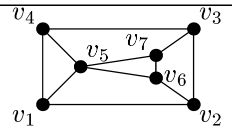
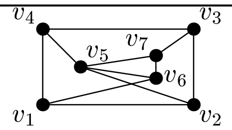
In practice one would like to construct a tensegrity without struts or cables with zero tension. So it is natural to give the following definition. We say that a graph G is *visible* at the framework P if there exists a self-stress that is nonzero at each edge of this framework.

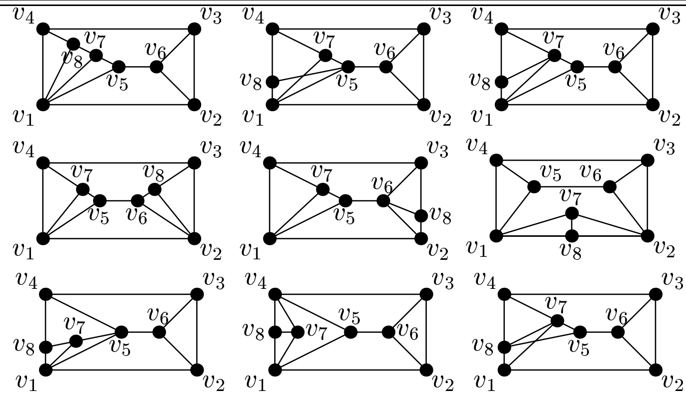
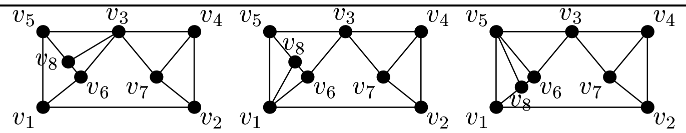
Remark 6.4. Visibility restrictions remove many degenerate strata. For instance if a zero 2-TC graph G has a complete subgraph on vertices v_1, v_2, and v_3, then the codimension 1 stratum defined by the condition: *the points* v_1*,* v_2*, and* v_3 *are on one line* does in many cases not contain visible frameworks.

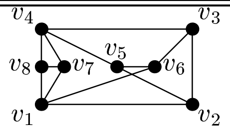
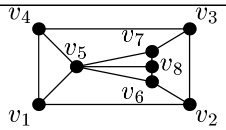
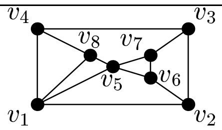
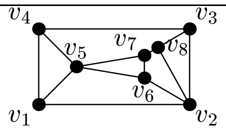
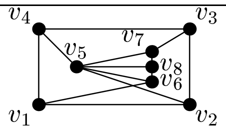
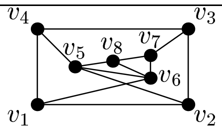
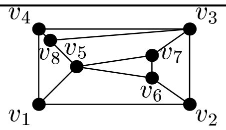
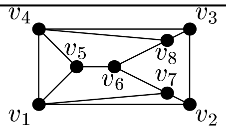
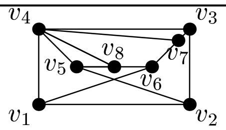
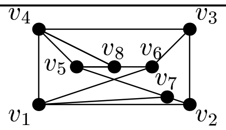
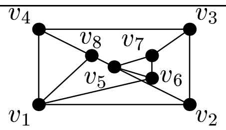
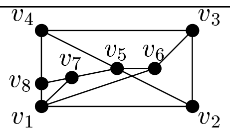
Let us list the geometric conditions for the vertices of all visible 2-vertex and 3-edge connected graphs with n vertices and zero 2-TC for $n \leq 8$. To find the geometric conditions we essentially use the surgeries of Section 4, see Propositions 4.1, 4.6 and 4.8.

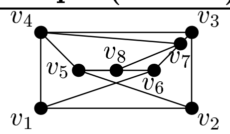
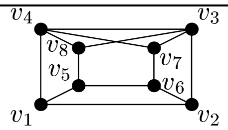
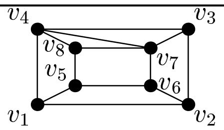
In the next table we use besides the elementary also the following two additional geometric conditions:
— six points are on a conic;
— for six points $p_1, \ldots, p_6$ the lines p_1p_2, p_3p_4, and p_5p_6 have a common nonempty intersection.
As we have seen in Examples 5.2 and 5.3 these conditions are equivalent to geometric conditional systems.

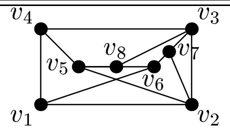
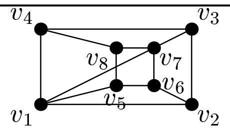
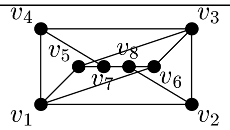
Graph (6 vert.)	**Sufficient geometric conditions**
	the lines v_1v_2, v_3v_4, and v_5v_6 have a common nonempty intersection
	the six points v_1, v_2, v_3, v_4, v_5, and v_6 are on a conic

Graph (7 vert.)	**Sufficient geometric conditions**
	$v_1 \nabla v_2 \nabla v_3 = 0$
	the lines v_1v_2, v_3v_4, and v_5v_6 have a common nonempty intersection
	the lines v_1v_2, v_3v_4, and v_5p where $p = [v_2,v_6; v_3,v_7]$ have a common nonempty intersection
	the six points v_1, v_2, v_3, v_4, v_5, and p, where $p = [v_1,v_6; v_3,v_7]$ are on a conic

Graph (8 vert.)	**Geometric conditions**
	the lines v_1v_2, v_3v_4, and v_5v_6 have a common nonempty intersection
	$v_1 \nabla v_2 \nabla v_3 = 0$

Graph (8 vert.)	**Geometric conditions**
	the six points v_1, v_2, v_3, v_4, v_5, and v_6 are on a conic
 	the lines v_1v_2, v_3v_4, and v_5p, where $p = [v_2, v_6; v_3, v_7]$ have a common nonempty intersection
	the lines v_1v_2, v_3v_4, and v_5p, where $p = [v_2, v_6; v_7, v_8]$ have a common nonempty intersection
 	the six points v_1, v_2, v_3, v_4, v_5, and p, where $p = [v_1, v_6; v_3, v_7]$, are on a conic
	the lines v_1v_2, v_3p, and v_5q, where $p = [v_1, v_4; v_5, v_8]$ and $q = [v_2, v_6; v_3, v_7]$ have a common nonempty intersection
	the lines v_5v_6, v_1p, and v_4q, where $p = [v_2, v_3; v_6, v_7]$ and $q = [v_2, v_3; v_6, v_8]$ have a common nonempty intersection
	the six points v_1, v_2, v_4, v_6, p, and q, where $p = [v_2, v_3; v_6, v_7]$ and $q = [v_2, v_5; v_6, v_8]$, are on a conic
	the six points v_1, v_3, v_4, v_6, p, and q, where $p = [v_2, v_3; v_5, v_7]$ and $q = [v_5, v_7; v_6, v_8]$, are on a conic
	the six points v_1, v_2, v_3, v_5, p, and q, where $p = [v_1, v_6; v_3, v_7]$ and $q = [v_3, v_4; v_5, v_8]$, are on a conic
	the six points v_1, v_2, v_3, v_5, v_6, and q, where $p = [v_1, q; v_3, v_4]$ and $q = [v_5, v_7; v_4, v_8]$, are on a conic

Graph (8 vert.)	**Geometric conditions**
	the six points v_1, v_2, v_4, v_5, p, and q, where $p = [v_1, v_6; v_5, v_8]$ and $q = [p, v_7; v_2, v_3]$, are on a conic
 	the three points $[v_1, v_4; v_2, v_3]$, $[v_1, v_5; v_2, v_6]$, and $[v_5, v_8; v_6, v_7]$ are on one line

Graph (8 vert.)	**Sufficient geometric conditions**
	the three points $[v_1, v_2; v_6, v_7]$, $[v_1, p; v_6, v_8]$, and $[p, q; v_3, v_8]$, where $p = [v_2, v_4; v_5, v_8]$ and $q = [v_1, v_5; v_3, v_4]$, are on one line, AND the lines $p'v_2$, $q'v_3$, and v_6v_7 have a common nonempty intersection, where $p' = [r', s'; v_1, v_6]$, $q' = [r', s'; v_6, v_8]$, $r' = [v_1, v_4; v_2, v_5]$, and $s' = [v_3, v_4; v_5, v_8]$
	the six points v_1, v_4, v_7, v_8, p, and q, where $p = [r, s; v_3, v_4]$, $q = [r, s; v_5, v_8]$, $r = [v_1, v_2; v_5, v_6]$, and $s = [v_2, v_3; v_6, v_7]$, are on a conic, AND the six points v_1, v_2, v_6, v_7, p', and q', where $p' = [r', s'; v_2, v_3]$, $q' = [r', s'; v_5, v_6]$, $r' = [v_1, v_4; v_5, v_8]$, and $s' = [v_3, v_4; v_7, v_8]$, are on a conic
	the six points v_1, v_3, v_4, v_5, v_7, and p, where $p = [v_1, q; v_7, v_8]$, $q = [r, s; v_2, v_3]$, $r = [v_3, v_6; v_7, v_8]$, and $s = [v_1, v_6; v_2, v_8]$, are on a conic, AND the six points v_1, v_2, v_3, v_6, v_8, and p', where $p' = [v_3, q'; v_7, v_8]$, $q' = [r', s'; v_1, v_4]$, $r' = [v_1, v_5; v_7, v_8]$, and $s' = [v_3, v_5; v_4, v_7]$, are on a conic

Remark 6.5. For the last three graphs in the table we have two distinct equations. Nevertheless, the 2-TC of the graphs are zero. This is similar to the case of non-complete intersections in algebraic geometry.

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E-mail address, Franck Doray: `doray@math.leidenuniv.nl`
E-mail address, Oleg Karpenkov: `karpenk@mccme.ru`
E-mail address, Jan Schepers: `janschepers1@gmail.com`

(Franck Doray, Oleg Karpenkov, Jan Schepers) Mathematisch Instituut, Universiteit Leiden, P.O. Box 9512, 2300 RA Leiden, The Netherlands