Covers of surfaces with fixed branch locus

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Abstract

Given a connected smooth projective surface X over $\mathbb{C}$, together with a simple normal crossings divisor D on it, we study finite normal covers $Y \to X$ that are unramified outside D. Given moreover a fibration of X onto a curve C, we prove that the 'height' of Y over C is bounded quadratically in terms of the degree of $Y \to X$. We explain how an arithmetic analogue of this result can be auxiliary in proving the existence of a polynomial time algorithm that computes the mod-ℓ Galois representations associated to a given smooth projective geometrically connected surface over $\mathbb{Q}$.

1 Introduction

In this paper, we suppose given the following data:

- a connected smooth projective surface X over $\mathbb{C}$;
- a simple normal crossings divisor D on X (i.e., all components of D are smooth);
- a connected smooth projective curve C over $\mathbb{C}$;
- a flat morphism $h\colon X \to C$.

We denote by U the complement of D in X. We are interested in connected finite étale covers $V \to U$; these are considered to be the 'variable' in our set-up. Given a connected finite étale cover $V \to U$ denote by $\pi\colon Y \to X$ the normalisation of X in the function field of V. By 'finiteness of integral closure', the map π is finite. As the topological fundamental group of U is finitely generated (cf. [9, Exposé II, Théorème 2.3.1]), we have only finitely many $V \to U$ of a given degree. In particular the height over C of the associated covers $Y \to X$ is bounded. Our aim is to prove an effective version of this result. Let $\rho\colon Y' \to Y$ be a minimal resolution of singularities of Y, and denote by $f\colon Y' \to C$ the composed morphism $h\pi\rho$. Note that Y' and Y are projective and flat over C (C being a Dedekind scheme).

1.1 Theorem. *The height* $\deg\det \mathrm{R}^{\cdot} f_* O_{Y'}$ *of* Y' *over* C *is bounded from above and below by polynomials of degree at most* 2 *in the degree of* π*. The coefficients of these polynomials depend only on* D *and* h*.*

Here $\det \mathrm{R}^{\cdot} f_* O_{Y'}$ stands for the determinant of cohomology of $O_{Y'}$, cf. [2]. This is an invertible sheaf on C with $c_1(\det \mathrm{R}^{\cdot} f_* O_{Y'}) = c_1(\mathrm{R}^0 f_* O_{Y'}) - c_1(\mathrm{R}^1 f_* O_{Y'})$ in the Chow ring of C. According to [12, Theorem 3.6(v)] the degree $\deg\det \mathrm{R}^{\cdot} f_* O_{Y'}$ is non-negative if the fibres of f are connected and the arithmetic genus of the fibres of f is positive.

Our proof uses the Grothendieck-Riemann-Roch theorem, intersection theory on the normal surface Y and precise information about the minimal resolution of singularities of Y. In a concluding section we discuss an arithmetic analogue of our result and a possible application.

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2 Preliminaries

As above, let $V \to U = X - D$ be a connected finite étale cover, and let $\pi\colon Y \to X$ be the normalisation of X in the function field of V. Denote by d the degree of $V \to U$. Write $D = \sum_{i\in I} D_i$ for the decomposition of D into prime components, and write $\pi^{-1}(D_i) = \sum_{j\in J_i} D_{ij}$ for the decomposition into prime components of the inverse image with reduced structure under π of a D_i. For $i \in I$ and $j \in J_i$ denote by e_{ij} the ramification index of π along D_{ij} (i.e., the ramification index of π at the generic point of D_{ij}) and denote by f_{ij} the degree of the extension of function fields of D_{ij} over D_i. Note that for each $i \in I$ we have $\sum_{j\in J_i} e_{ij} f_{ij} = d$. We write D^{sing} for the singular locus of D.

2.1 Lemma. *(i) The singularities of* Y *occur in the inverse image under* π *of* D^{sing}*. Furthermore, the map* $\pi^{-1}(D - D^{\mathrm{sing}}) \to D - D^{\mathrm{sing}}$ *is étale.*
(ii) Let y *be a point of* Y *mapping to* D^{sing}*, say* $\pi(y) \in D_i \cap D_{i'}$ *(recall that the components* D_i *of* D *are smooth). Then there are unique* $j \in J_i, j' \in J_{i'}$ *such that* $y \in D_{ij} \cap D_{i'j'}$*.*
(iii) Let e *be an upper bound for both* e_{ij} *and* $e_{i'j'}$*, and assume* y *is a singular point of* Y*. Then* (Y, y) *is a cyclic quotient singularity with order of the cyclic group bounded by* e*.*

Proof. We follow the text of [1] on pp. 102–103. Let x be a point of X lying on D. Assume first of all that $x \notin D^{\mathrm{sing}}$. Locally for the analytic topology we identify a neighbourhood W of x in X with the bi-disk $Z = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < 1\}$, identifying x with the origin and D locally with the zero set of z_1. Let y be a point of Y mapping to x and consider the connected component B of $\pi^{-1}W$ that contains y. We have then that $B - \pi^{-1}(D) \to W - D$ is a connected finite degree topological covering. Thus $\Gamma = \pi_*(\pi_1(B - \pi^{-1}(D)))$ is a subgroup of finite index of $\pi_1(W - D)$. The latter is infinite cyclic; let e be the index of the subgroup Γ. Note that the map $W \to W$ given by $(z_1, z_2) \mapsto (z_1^e, z_2)$ is a connected cover of W, homeomorphic above $W - D$ to the covering $B - \pi^{-1}(D) \to W - D$. By a theorem of Grauert-Remmert [8, pp. 14–15] the holomorphic map $B \to W$ is determined by the topological covering $B - \pi^{-1}(D) \to W - D$ and hence we know $B \to W$ locally in coordinates. In particular we see that Y is regular above $D - D^{\mathrm{sing}}$ and that $\pi^{-1}(D - D^{\mathrm{sing}}) \to D - D^{\mathrm{sing}}$ is étale. This gives (i). Let us now assume that the point x lies on D^{sing}. Let $D_i, D_{i'}$ be the components of D passing through x. Again we identify a local neighbourhood W of x in X with the bi-disk Z, letting D_i correspond to the zero set of z_1 and $D_{i'}$ to the zero set of z_2. Let $Z^* = Z - \{z_1 z_2 = 0\}$ and W^* the corresponding open subset of W. If $\gamma_i \in \pi_1(Z^*)$ is the class of a positively oriented little loop around the z_i-axis then $\pi_1(Z^*) = \mathbb{Z} \times \mathbb{Z}$ with generators $\gamma_1 = (1, 0)$ and $\gamma_2 = (0, 1)$. Let y be a point of Y mapping to x and consider the connected component B of $\pi^{-1}W$ that contains y. Put $B^* = B - \pi^{-1}D$. We have then that $B^* \to W^*$ is a connected finite degree topological covering. As such it is determined up to homeomorphism by the image $\Gamma = \pi_*(\pi_1(B^*))$ of its topological fundamental group in $\pi_1(W^*) = \mathbb{Z} \times \mathbb{Z}$. Note that Γ is of finite index in $\pi_1(W^*)$. We pick generators of Γ as follows: $\Gamma \cap (\mathbb{Z} \times 0)$ is non-trivial, so there is some $(n', 0) \in \Gamma$ generating this intersection, with $n' > 0$. As the quotient $\Gamma/\mathbb{Z}(n', 0)$ is isomorphic to $\mathbb{Z}$ there is some $(q', m_2) \in \Gamma$ with $0 \le q' < n'$ and $m_2 > 0$ such that $(n', 0)$ and (q', m_2) generate Γ. If $q' = 0$ then $B^* \to W^*$ is homeomorphic to the covering $Z^* \to Z^*$ corresponding to $\gamma_1 \mapsto n'\gamma_1, \gamma_2 \mapsto m_2\gamma_2$ i.e. the one given in coordinates by $(z_1, z_2) \mapsto (z_1^{n'}, z_2^{m_2})$. Applying the result of Grauert-Remmert mentioned above we find that locally analytically $B \to W$ is isomorphic to $Z \to Z$ given by $(z_1, z_2) \mapsto (z_1^{n'}, z_2^{m_2})$. In particular $\pi^{-1}W$ is non-singular, and statement (ii) follows for this case. If $q' > 0$, let $m_1 = \gcd(n', q')$ and $n' = nm_1$, $q' = qm_1$. Then Γ is contained in the subgroup $\Gamma' = \mathbb{Z}(m_1, 0) + \mathbb{Z}(0, m_2)$. By the case just described, the subgroup Γ' corresponds to a covering $\pi' \colon W' \to W$ with W' non-singular, and with the original covering factoring as $\pi = \pi'\pi''$ for some $\pi'' \colon B \to W'$. If we identify $\pi_1(W'^*)$ again with $\mathbb{Z} \times \mathbb{Z}$ then $\pi'' \colon B \to W'$ corresponds to the subgroup with generators $(n, 0)$ and $(q, 1)$. An analysis of this case gives then that B is analytically equivalent to a neighbourhood of the Hirzebruch-Jung singularity of type $A_{n,q}$ (see [1, p. 101]), i.e. the singularity obtained by taking the quotient of $\mathbb{C}^2$ under the action $(w_1, w_2) \mapsto (\zeta w_1, \zeta^q w_2)$ where ζ is a root of unity of order n. Statement (ii) also follows for this case. Note finally that the order of the cyclic group giving the Hirzebruch-Jung singularity, namely n, is bounded by the ramification indices of D_{ij} and $D_{i'j'}$ over D_i and $D_{i'}$, which are nm_1 and nm_2, respectively. Statement (iii) follows therefore as well. $\square$

Let $\rho\colon Y' \to Y$ be a minimal resolution of singularities of Y and denote by $E_1, \ldots, E_s$ the exceptional components of $Y' \to Y$. Let K_Y be the Weil divisor obtained by taking the closure in Y of a canonical divisor on the non-singular locus of Y, and let $K_{Y'}$ be a canonical divisor on Y'. Since Y has only cyclic quotient singularities each Weil divisor on Y is $\mathbb{Q}$-Cartier, i.e., has the property that a certain integer multiple of it is a Cartier divisor on Y. Actually we have the following more precise local statement. Let y in Y be a singular point, hence a cyclic quotient singularity, say of type $A_{n,q}$. Then for any Weil divisor W on Y, $n{\cdot}W$ is Cartier at y (see [6, Prop. 5.15] and its proof).

2.2 Lemma. *Let y be a singular point of Y. Let $D_{ij}, D_{i'j'}$ be the unique prime divisors mapping to D and passing through y whose existence is guaranteed by Lemma 2.1, and let e be a bound for the ramification indices e_{ij} and $e_{i'j'}$. The reduced exceptional locus $\rho^{-1}(y)$ of y is a chain of $\mathbb{P}^1$'s with self-intersections at most -2 and bounded from below by $-e$. The number of exceptional components above y of $Y' \to Y$ is bounded from above by e. Finally, a linear equivalence of $\mathbb{Q}$-Cartier divisors*

$$K_{Y'} \equiv \rho^* K_Y + \sum_{i=1}^{s} a_i E_i$$

holds on Y' with $a_i \in \mathbb{Q} \cap (-1, 0]$.

Proof. We have seen in Lemma 2.1 that (Y, y) is a cyclic quotient singularity of Hirzebruch-Jung type $A_{n,q}$ with $0 < q < n \leq e$ and $\gcd(n, q) = 1$. According to the discussion in [1, pp. 100–101] we get if we write

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots}}$$

for the Hirzebruch-Jung continued fraction of n/q that the reduced exceptional locus $\rho^{-1}(y)$ is a chain of λ $\mathbb{P}^1$'s with self-intersections $-b_1, -b_2, \ldots, -b_\lambda$. We note that $b_i \geq 2$. The integers b_i are determined by the recursion (variant of the Euclidean algorithm) for integers c_i: $c_i = b_{i+2}c_{i+1} - c_{i+2}$, $0 \leq c_{i+2} < c_{i+1}$ for $i = -1, \ldots, \lambda - 2$ with initial conditions $c_{-1} := n, c_0 := q$ so that $b_i \leq c_{i-2} \leq c_{-1} = n < e$. The first statement follows. Next we have $e > n = c_{-1} > c_0 > c_1 > \ldots > c_\lambda = 0$ so that the number λ is bounded from above by e. Finally it is clear that we can write $K_{Y'} \equiv \rho^* K_Y + \sum_{i=1}^{s} a_i E_i$ at least for some $a_i \in \mathbb{Q}$. By the adjunction formula we have $(K_{Y'} + E_i, E_i) = -2$ so the a_i form the unique solution to the recursion $b_i a_i - a_{i-1} - a_{i+1} = 2 - b_i$ for $i = 1, \ldots, \lambda$ where we put $a_0 = 0 = a_{\lambda+1}$.

Suppose that there is an i in $\{1, \ldots, \lambda\}$ with $a_i \geq 0$. Let j with $1 \leq j \leq \lambda$ be an index with $a_j \geq 0$ and with $a_j = \max_i a_i$. We find

$$a_j = \frac{a_{j-1} + a_{j+1}}{b_j} + \frac{2 - b_j}{b_j} = \frac{2a_j}{b_j} + \frac{a_{j-1} - a_j + a_{j+1} - a_j}{b_j} + \frac{2 - b_j}{b_j} \leq a_j + \frac{a_{j-1} - a_j}{b_j} + \frac{a_{j+1} - a_j}{b_j}$$

whence $a_{j-1} = a_j = a_{j+1}$ and $b_j = 2$. Hence the maximum of the a_i is also attained at $j - 1$ and $j + 1$. Continuing with the same reasoning we find that all $b_i = 2$ and all $a_i = 0$. Hence all $a_i \leq 0$.

Finally let j with $1 \leq j \leq \lambda$ be an index with $a_j = \min_i a_i$. Our recursion can be written as $(b_i - 2)(a_i + 1) = (a_{i-1} - a_i) + (a_{i+1} - a_i)$, so we see that if $a_j \leq -1$, then $a_{j-1} = a_j = a_{j+1}$ and a_{j-1} and a_{j+1} are also minimal and ≤ -1, and we get the contradiction $0 = a_0 \leq -1$. Hence for all i we have $a_i > -1$. $\square$

We will need to compare the topological Euler characteristics of X and Y. The following general lemma is useful for this. We denote by $\mathrm{H}_c^{\cdot}(-, \mathbb{Q})$ cohomology with compact supports and with rational coefficients on the category of paracompact Hausdorff spaces. We use the notation $e_c(-)$ for the compactly supported Euler characteristic $e_c(-) = \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{H}_c^i(-, \mathbb{Q})$; this is a well-defined integer for separated algebraic varieties over $\mathbb{C}$.

2.3 Lemma. *Let M, N be separated algebraic varieties over $\mathbb{C}$.*
(i) If Z is a closed subvariety of M, then $e_c(M) = e_c(Z) + e_c(M - Z)$.
(ii) If $M \to N$ is a finite étale cover of degree n then $e_c(M) = n \cdot e_c(N)$.

Proof. The first statement follows from the long exact sequence of compactly supported cohomology

$$\cdots \to \mathrm{H}_c^i(M - Z) \to \mathrm{H}_c^i(M) \to \mathrm{H}_c^i(Z) \to \mathrm{H}_c^{i+1}(M - Z) \to \cdots$$

As to the second statement, we may assume first of all that M and N are connected. Second, we may reduce to the case that $M \to N$ is Galois. Indeed, let $P \to N$ be a Galois closure of $M \to N$, and denote by G the group of automorphisms of P such that $N = P/G$. Let H be the subgroup of G such that $M = P/H$. If the result is true for Galois covers, we find:

$$e_c(N) = \frac{1}{\#G} e_c(P) = \frac{\#H}{\#G} e_c(M) = \frac{1}{n} e_c(M)$$

and the result also follows in the general case. So let's assume that $M \to N$ is Galois, with group G. If V is a $\mathbb{Q}[G]$-module of finite type, let $[V]$ be the class of V in the Grothendieck group of such modules. More generally, if V is a $\mathbb{Z}$-graded $\mathbb{Q}[G]$-module of finite type, like $\mathrm{H}_c(M)$ for example, then we denote by $[V]$ the class of $\sum_i (-1)^i V^i$. Now remark that G acts freely on M, hence by the Lefschetz trace formula for compactly supported cohomology (see [3, Theorem 3.2]) we have for all non-trivial $g \in G$ that $\sum_i (-1)^i \mathrm{trace}(g, \mathrm{H}_c^i(M)) = 0$. By character theory it follows that $[\mathrm{H}_c(M)]$ is a multiple of $[\mathbb{Q}[G]]$, the class of the regular representation of G, say $[\mathrm{H}_c(M)] = m \cdot [\mathbb{Q}[G]]$ with $m \in \mathbb{Z}$. Since we also have that $\mathrm{H}_c(N) = \mathrm{H}_c(M)^G$ we get $e_c(N) = \dim_{\mathbb{Q}} \mathrm{H}_c(M)^G = m$. As $e_c(M) = \dim_{\mathbb{Q}} \mathrm{H}_c(M) = m \cdot \#G$ the result follows. $\square$

Finally we want to work with the Grothendieck-Riemann-Roch theorem. We recall the statement and all notions that go into it. Let M, N be smooth quasi-projective varieties over $\mathbb{C}$. One has a Grothendieck group $K_0(M)$ for coherent sheaves on M. This group is isomorphic to its analogue for locally free O_M-modules of finite rank, and therefore, it has a natural ring structure. There is also a Chow ring $\mathrm{CH}(M)$, coming with a natural grading. For $p\colon M \to N$ a projective morphism one has a map $p_!\colon K_0(M) \to K_0(N)$ given by $p_!([\mathcal{F}]) = \sum_{i=0}^{\infty} (-1)^i [\mathrm{R}^i p_* \mathcal{F}]$. Also one has a map $p_*\colon \mathrm{CH}(M) \to \mathrm{CH}(N)$ given by proper pushforward of cycles. The Chern character ch gives a ring homomorphism $\mathrm{ch}\colon K_0(M)_{\mathbb{Q}} \to \mathrm{CH}(M)_{\mathbb{Q}}$. Each coherent sheaf $\mathcal{F}$ on M has a Todd class $\mathrm{td}(\mathcal{F})$ in $\mathrm{CH}(M)_{\mathbb{Q}}$. The Todd class $\mathrm{td}(M)$ of M is by definition the Todd class of the tangent bundle T_M of M.

The Grothendieck-Riemann-Roch theorem reads as follows.

2.4 Theorem. *Let M, N be smooth quasi-projective varieties over $\mathbb{C}$. Let $p\colon M \to N$ be a projective morphism and let $\mathcal{F}$ be a coherent sheaf on M. Then the equality*

$$\mathrm{ch}(p_!\mathcal{F}) \cdot \mathrm{td}(N) = p_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(M))$$

holds in $\mathrm{CH}(N)_{\mathbb{Q}}$.

We recall the formulas

$$\mathrm{ch}(\mathcal{F}) = c_0(\mathcal{F}) + c_1(\mathcal{F}) + \frac{1}{2}(c_1^2(\mathcal{F}) - 2c_2(\mathcal{F})) + \mathrm{h.o.t.}$$

and

$$\mathrm{td}(\mathcal{F}) = 1 + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}(c_1^2(\mathcal{F}) + c_2(\mathcal{F})) + \mathrm{h.o.t.}$$

We have $c_0(\mathcal{F}) = \mathrm{rank}(\mathcal{F})$ if $\mathcal{F}$ is locally free. Finally $c_1(\mathcal{F}) = c_1(\det \mathcal{F})$. In particular $c_1(\det \mathrm{R}^{\cdot} f_* O_{Y'}) = c_1(f_! O_{Y'})$.

3 Proof of Theorem 1.1

We start by deriving a useful expression for $c_1(f_! O_{Y'})$. We recall from Lemma 2.1 that the singular points of Y are cyclic quotient singularities. According to [1, Proposition III.3.1] such singularities are rational, i.e. we have

$$\rho_* O_{Y'} = O_Y \,, \quad \mathrm{R}^i \rho_* O_{Y'} = 0 \quad \text{for}\, i > 0 \,.$$

Using the Leray spectral sequence we find, writing $\bar{\pi} = \pi\rho$, that $\mathrm{R}^i \bar{\pi}_* O_{Y'} = \mathrm{R}^i \pi_* O_Y$ for all i. As π is finite we obtain

$$\bar{\pi}_* O_{Y'} = \pi_* O_Y \,, \quad \mathrm{R}^i \bar{\pi}_* O_{Y'} = 0 \quad \text{for}\, i > 0 \,.$$

Applying then the Leray spectral sequence to the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{\pi}} & X \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle h} \\ & & C \end{array}$$

we obtain $\mathrm{R}^i f_* O_{Y'} = (\mathrm{R}^i h_*)(\bar{\pi}_* O_{Y'}) = (\mathrm{R}^i h_*)(\pi_* O_Y)$ for all i and hence

$$f_! O_{Y'} = h_!(\pi_* O_Y) \,.$$

The Grothendieck-Riemann-Roch theorem then gives

$$\mathrm{ch}(f_! O_{Y'}) \cdot \mathrm{td}(C) = h_*(\mathrm{ch}(\pi_* O_Y) \cdot \mathrm{td}(X)) \,.$$

We recall that we write d for the degree of π. By [11, Lemma 2] the sheaf $\pi_* O_Y$ is locally free of rank d. Comparing terms in degree 0 therefore yields

$$c_0(f_! O_{Y'}) = h_*(d \cdot \mathrm{td}(X)_{(1)} + c_1(\pi_* O_Y)) \,.$$

On the other hand the Grothendieck-Riemann-Roch theorem applied directly to f gives

$$\mathrm{ch}(f_! O_{Y'}) \cdot \mathrm{td}(C) = f_*(\mathrm{ch}(O_{Y'}) \cdot \mathrm{td}(Y')) = f_*(\mathrm{td}(Y')) \,.$$

Comparing terms in degree 1 we find

$$c_1(f_! O_{Y'}) + c_0(f_! O_{Y'}) \cdot \mathrm{td}(C)_{(1)} = f_*(\mathrm{td}(Y')_{(2)}) \,.$$

Combining with our previous expression for $c_0(f_! O_{Y'})$ we get

$$c_1(f_! O_{Y'}) = f_*(\mathrm{td}(Y')_{(2)}) - h_*(d \cdot \mathrm{td}(X)_{(1)} + c_1(\pi_* O_Y)) \cdot \mathrm{td}(C)_{(1)}$$

hence

$$\deg \det \mathrm{R}^{\cdot} f_* O_{Y'} = \deg \left\{ \frac{1}{12} f_* \left(c_1^2(T_{Y'}) + c_2(T_{Y'}) \right) - h_*(d \cdot \mathrm{td}(X)_{(1)} + c_1(\pi_* O_Y)) \cdot \mathrm{td}(C)_{(1)} \right\} \,.$$

We are done once we show that the degrees of $c_1^2(T_{Y'})$, $c_2(T_{Y'})$ and of $h_*(c_1(\pi_* O_Y))\cdot \mathrm{td}(C)_{(1)}$ are bounded from above and below by quadratic polynomials in d with coefficients depending only on D and h. We start by considering the term involving $c_1(\pi_* O_Y)$. Again write $D = \sum_{i\in I} D_i$ for the decomposition of D into its prime components. Define R to be the Weil divisor, supported on $\pi^{-1}(D)$, given as follows: let D_{ij} be a component of $\pi^{-1}(D)$ mapping onto D_i, then the multiplicity of D_{ij} in R is $(e_{ij} - 1)$. Put $B := \pi_* R$. Note that we have a trace pairing $\pi_* O_Y \otimes_{O_X} \pi_* O_Y \to O_X$. This induces a monomorphism $(\det \pi_* O_Y)^{\otimes 2} \rightarrowtail O_X$, identifying $(\det \pi_* O_Y)^{\otimes 2}$ with the ideal sheaf $O_X(-B)$ of B, as a local computation (see e.g. [10, III, §6, Proposition 13]) shows. We obtain $c_1(\pi_* O_Y) = -\frac{1}{2}[B]$ in $\mathrm{CH}(X)_{\mathbb{Q}}$ so we are done for this term if we could show that the multiplicity of each D_i in B is bounded linearly in d. But this multiplicity is $\sum_{j\in J_i}(e_{ij} - 1)f_{ij}$ and this is bounded by d.

Next we consider the term $c_2(T_{Y'})$. We recall that by a version of the Gauss-Bonnet formula (see e.g. [5, p. 416]) we have $\deg c_2(T_{Y'}) = e_c(Y')$, the topological Euler characteristic of Y'. For each $i \in I$ write $d_i := \sum_{j\in J_i} f_{ij}$. By Lemmas 2.1 and 2.3 we have

$$\begin{aligned} e_c(Y) &= e_c(\pi^{-1}U) + e_c(\pi^{-1}D) \\ &= d e_c(U) + \sum_{i\in I} d_i e_c(D_i - D^{\mathrm{sing}}) + e_c(\pi^{-1}D^{\mathrm{sing}}) \\ &= d e_c(U) + \sum_{i\in I} d_i e_c(D_i - D^{\mathrm{sing}}) + \#\pi^{-1}D^{\mathrm{sing}} \end{aligned}$$

with D^{sing} the singular locus of D. This shows that $e_c(Y)$ is bounded from above and below by linear polynomials in d with coefficients depending only on D. Now

$$e_c(Y') = e_c(Y) + s$$

where s is the total number of exceptional components of $Y' \to Y$. If y is a singular point of Y let e_y be the sum of the ramification indices of the two unique components of the preimage of D in Y passing through y as in Lemma 2.1. By Lemma 2.2, above y the number of exceptional components is bounded by e_y. Since for any x on X we have $\sum_{y\colon y\mapsto x} e_y$ bounded by $2d$ we obtain that $e_c(Y')$ is bounded from above and below by linear polynomials in d with coefficients depending only on D as well.

Finally we consider $c_1^2(T_{Y'})$. Note that we can write $\deg c_1^2(T_{Y'}) = (K_{Y'}, K_{Y'})$, the self-intersection number of the divisor $K_{Y'}$ on Y'. We compute this self-intersection number. By [13, Theorem 4.1] the normal surface Y is an Alexander scheme, implying (cf. op. cit., Note 2.4) among other things that for the proper maps $\rho\colon Y' \to Y$ and $\pi\colon Y \to X$ one has a projection formula for Weil divisors, provided that one works on Y with the intersection theory with $\mathbb{Q}$-

coefficients as in [7, Section IIb]. Thus we compute

$$\begin{aligned}(K_{Y'}, K_{Y'}) &= (\rho^* K_Y + \sum_i a_i E_i, K_{Y'}) \\ &= (\rho^* K_Y, K_{Y'}) + \sum_i a_i(b_i - 2) \\ &= (K_Y, K_Y) + \sum_i a_i(b_i - 2)\,.\end{aligned}$$

But $K_Y = \pi^* K_X + R$ so

$$\begin{aligned}(K_Y, K_Y) &= d \cdot (K_X, K_X) + 2(\pi^* K_X, R) + (R, R) \\ &= d \cdot (K_X, K_X) + 2(K_X, B) + (R, R)\,.\end{aligned}$$

We are done once we show that $\sum_i a_i(b_i - 2)$, (K_X, B) and (R, R) are bounded from above and below by quadratic polynomials in d with coefficients depending only on D. We start with the term $\sum_i a_i(b_i - 2)$. We deal with the contribution from one singularity y of Y first. From the recursion in the proof of Lemma 2.2 it follows that

$$\sum_i a_i(b_i - 2) = \sum_i (-b_i + 2) - (a_1 + a_\lambda)\,.$$

But by Lemma 2.2 the a_i are in $(-1, 0]$, and $2 \leq b_i \leq e_y$. Second, the number of summands on the right hand side is bounded by e_y. Thus, for a given y, the contribution to the term $\sum_i a_i(b_i - 2)$ is bounded from above by 2 and from below by $e_y(-e_y + 2)$. Since $\sum_{y:\, y \mapsto x} e_y$ is bounded by $2d$ for each x on X we get in total that the sum $\sum_i a_i(b_i - 2)$ is bounded by at most quadratic polynomials in d with coefficients depending only on D.

The intersection number (K_X, B) is bounded linearly in d by our description of B given earlier in this proof.

As for (R, R), we obtain from Lemma 2.1(i) and (ii) that the irreducible components of the inverse image under π of a D_i are disjoint and hence we can write:

$$(R, R) = \sum_{i,j} (e_{ij} - 1)^2 (D_{ij}, D_{ij}) + \sum_{\substack{(i,i'),j,j' \\ i \neq i'}} (e_{ij} - 1)(e_{i'j'} - 1)(D_{ij}, D_{i'j'})\,.$$

Now we have, for each $i \in I$, that $\pi^* D_i = \sum_j e_{ij} D_{ij}$ so on the one hand for a given j_0,

$$(D_{ij_0}, \pi^* D_i) = \sum_j e_{ij} (D_{ij}, D_{ij_0}) = e_{ij_0} (D_{ij_0}, D_{ij_0})$$

by the disjointness of the D_{ij} and on the other hand

$$(D_{ij_0}, \pi^* D_i) = (\pi_* D_{ij_0}, D_i) = f_{ij_0} (D_i, D_i)$$

by the projection formula. Thus

$$(D_{ij_0}, D_{ij_0}) = \frac{f_{ij_0}}{e_{ij_0}}(D_i, D_i)$$

and hence for a given i,

$$\sum_j (e_{ij}-1)^2 (D_{ij}, D_{ij}) = \sum_j (e_{ij}-1)^2 \frac{f_{ij}}{e_{ij}} (D_i, D_i)\,.$$

Remark that $0 \leq \sum_j (e_{ij}-1)^2 \frac{f_{ij}}{e_{ij}} < d$ and we are done for the first term $\sum_{i,j}(e_{ij}-1)^2(D_{ij}, D_{ij})$. Finally we can write

$$\sum_{\substack{(i,i'),j,j' \\ i \neq i'}} (e_{ij}-1)(e_{i'j'}-1)(D_{ij}, D_{i'j'}) = \sum_{\substack{(i,i') \\ i\neq i'}} \sum_{x \in D_i \cap D_{i'}} \sum_{y \mapsto x} \sum_{j,j'} (e_{ij}-1)(e_{i'j'}-1)(D_{ij}, D_{i'j'})_y\,.$$

But as we saw in Lemma 2.1 for each $y \mapsto x$ with $x \in D_i \cap D_{i'}$ there is exactly one pair (j, j') such that $(D_{ij}, D_{i'j'})_y \neq 0$. So, the summation over j and j' can be replaced by a single term with indices $j(y)$ and $j'(y)$. Moreover, if $(D_{ij}, D_{i'j'})_y \neq 0$ then in fact $(D_{ij}, D_{i'j'})_y \leq 1$. This can be seen by the following local calculation. We may assume that y is a singular point of Y, with a neighbourhood that is analytically isomorphic to a cyclic quotient singularity $\mathbb{C}^2/\mu_n$ for some integer n. By the paragraph before Lemma 2.2, $n{\cdot}D_{ij}$ is a Cartier divisor at y. But then as $D_{i'j'}$ is smooth the intersection number $(D_{ij}, D_{i'j'})_y$ is just given as $1/n$ times the valuation in $O_{D_{i'j'},y}$ of a local function defining $n \cdot D_{ij}$ around y on Y. Since this function is a local coordinate around y on $D_{i'j'}$ we find $(D_{ij}, D_{i'j'})_y$ to be equal to $1/n$. All in all, keeping in mind that for fixed i we have $\sum_{j \in J_i} e_{ij} \leq d$ we find that

$$\sum_{\substack{(i,i') \\ i\neq i'}} \sum_{x \in D_i \cap D_{i'}} \sum_{y \mapsto x} (e_{ij(y)}-1)(e_{i'j'(y)}-1)(D_{ij(y)}, D_{i'j'(y)})_y$$

is bounded by a quadratic polynomial in d with coefficients depending only on D. This finishes the proof.

3.1 Remark. We can remove from the theorem the assumption that V is connected by replacing Y in the proof with the disjoint union of the normalisations of X in the function fields of the various components of V.

3.2 Remark. It follows from our proof that most contributions to the height of Y' over C are no worse than linear in d. In fact only the terms

$$\sum_i a_i(b_i - 2) = \sum_i (-b_i + 2) - (a_1 + a_\lambda) \quad \text{and} \quad \sum_{\substack{(i,i'),j,j' \\ i\neq i'}} (e_{ij}-1)(e_{i'j'}-1)(D_{ij}, D_{i'j'})$$

can give rise to quadratic terms. The first contribution is absolutely bounded from above by 2 and is usually non-positive, whereas the second contribution is always non-negative.

4 An arithmetic analogue

In [4] an algorithm is given that computes the $\mathrm{GL}_2(\mathbb{F}_\lambda)$ Galois representations associated to a given normalised Hecke eigenform f of level 1, in time polynomial in $\#(\mathbb{F}_\lambda)$, if one admits the Generalised Riemann Hypothesis for number fields. Here λ runs through the finite degree 1 places of the field of coefficients of the form. By a famous argument due to R. Schoof, this leads to an algorithm that on input a prime number p computes the p-th coefficient of the Fourier development of f, in time polynomial in $\log p$. In particular, the number of vectors with half length-squared equal to p in a fixed even unimodular lattice can be computed in time polynomial in $\log p$.

Generalisations of the above results seem possible in various different directions. For example, one could look at the case of mod-ℓ Galois representations occurring in the étale cohomology of a given smooth, projective and geometrically connected surface S over $\mathbb{Q}$. Letting ℓ be a prime number, one has the cohomology groups $\mathrm{H}^i(S_{\overline{\mathbb{Q}},\mathrm{et}}, \mathbb{F}_\ell)$ for $0 \leq i \leq 4$, being finite dimensional $\mathbb{F}_\ell$-vector spaces with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. It seems reasonable to suspect that, again, there is an algorithm that on input a prime ℓ computes these cohomology groups, with their $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, in time polynomial in ℓ. Once such an algorithm is known, one also has an algorithm that, on input a prime p of good reduction of S, gives the number of points $\#S(\mathbb{F}_p)$ of S over $\mathbb{F}_p$ in time polynomial in $\log p$. This result would be of interest because the known p-adic algorithms for finding such numbers have running time exponential in $\log p$.

The idea in [4] to compute mod-ℓ étale cohomology is to trivialise the sheaves involved, using suitable covers of (modular) curves, and to reduce to computing in the ℓ-torsion of their Jacobians. In our case, using a Lefschetz fibration, one can first reduce to computing cohomology groups $\mathrm{H}^1(U_{\overline{\mathbb{Q}}}, \mathcal{F}_l)$ where U is a non-empty open subset of $\mathbb{P}^1_{\mathbb{Q}}$ determined by S and the chosen Lefschetz fibration and where $\mathcal{F}_l$ are certain étale locally constant sheaves of $\mathbb{F}_\ell$-vector spaces of fixed dimension, say r. For each ℓ let $V_\ell := \underline{\mathrm{Isom}}_U(\mathbb{F}_\ell^r, \mathcal{F}_l)$. Then each cover $V_\ell \to U$ is finite Galois with group $G \cong \mathrm{GL}_r(\mathbb{F}_\ell)$, and the group $\mathrm{H}^1(U_{\overline{\mathbb{Q}}}, \mathcal{F}_l)$ can be related to $\mathrm{H}^1(V_{\ell,\overline{\mathbb{Q}}}, \mathcal{F}_l = \mathbb{F}_\ell^r)$ which sits in the ℓ-torsion of the Jacobian of the smooth projective model $\overline{V_\ell}$ of V_ℓ. It is our hope that methods as in [4] can show that we have a polynomial algorithm for computing these cohomology groups once we have a bound for the Faltings height of $\overline{V_\ell}$ that is polynomial in ℓ.

Now the point is that in fact the finite étale morphism $V_\ell \to U$ extends to a finite étale morphism $V'_\ell \to U'$ where U' is the intersection of a fixed (independent of ℓ) non-empty open subset of $\mathbb{P}^1_{\mathbb{Z}}$ with $\mathbb{P}^1_{\mathbb{Z}[1/\ell]}$. One may assume that the complement of U' in $\mathbb{P}^1_{\mathbb{Z}}$ is a normal crossings divisor. We would therefore be helped a lot if we could prove the following arithmetic analogue of the main theorem of this note: let $\pi\colon V' \to U'$ be a finite étale cover, and let $\overline{V'}$ be the smooth projective model of the generic fibre of V'. Then the Faltings height of $\overline{V'}$ is bounded by a polynomial in the degree of π.

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