

# BERNOULLI-EULER NUMBERS AND MULTIBOUNDARY SINGULARITIES OF TYPE $B_n^l$

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ABSTRACT. In this paper we study properties of numbers  $K_n^l$  of connected components of bifurcation diagrams for multiboundary singularities  $B_n^l$ . These numbers generalize classic Bernoulli-Euler numbers. We prove a recurrent relation on the numbers  $K_n^l$ . As it was known before,  $K_n^1$  is  $(n+1)$ -th Bernoulli-Euler number, this gives us a necessary boundary condition to calculate  $K_n^l$ . We also find the generating functions for  $K_n^l$  with small fixed  $l$  and write partial differential equations for the general case. The recurrent relations lead to numerous relations between Bernoulli-Euler numbers. We show them in the last section of the paper.

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## 1. INTRODUCTION

Like binomial coefficients and Fibonacci numbers Bernoulli-Euler numbers  $K_n$  are widely used in many different branches of mathematics. We define Bernoulli-Euler numbers as Taylor series coefficients for the function  $\sec + \tan$ , namely

$$\sec t + \tan t = \sum_{n=0}^{\infty} K_n \frac{t^n}{n!}.$$

There exists an equivalent definition of Bernoulli-Euler numbers by means of a “classical triangle”, one can construct  $K_n$  by analogy with finding binomial coefficients using Pascal triangle. We refer the reader to [1] for the detailed description of the triangle for Bernoulli-Euler numbers. For arithmetical properties of Bernoulli-Euler numbers we refer to papers [1], [4], and [5]).

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We study one of the geometrical aspects of the numbers  $K_n$  in singularity theory. As it was shown by V. I. Arnold in [2], the combinatorics of the components of the set of very nice M-morsifications for the boundary singularities of type  $B_n$  is closely related to the combinatorics of the corresponding Springer cones. In [1] V. I. Arnold also proved that the numbers of connected components coincides with Bernoulli-Euler numbers.

In this paper we give proofs of theorems announced by author in [3]. We deal with a generalization of boundary singularities  $B_n$  for the functions on the line to the case of multiboundary singularities  $B_n^l$  with boundaries consisting of  $l$  points. The numbers of connected components of the set of very nice M-morsifications for  $B_n^l$  (denoted by  $K_n^l$ ) is in its turn a natural generalization of Bernoulli-Euler numbers. In particular, we prove the recurrent relation on the numbers  $K_n^l$ :

$$K_{n-2}^{l+1} = K_n^l - nlK_n^{l-1}.$$

Note also that the numbers  $K_n^l$  also enumerate certain strata of singularities of  $A_{2l+n-1}$ , see Corollary 3.4.

This work is organized as follows. We start in Section 2 with necessary notions and definition, in particular we give definitions of singularities of type  $B_k^l$ . In Section 3 we formulate and prove the main theorem on recurrence relation for numbers  $K_n^l$ . Further in Section 4 we study the case of small number of boundary points  $l$ . We give explicit formulae for the numbers  $K_n^l$  in this case. Finally, in Section 5 we show general expressions for  $K_n^l$  in terms of Bernoulli-Euler numbers.

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## 2. DEFINITION OF $B_n^l$ SINGULARITIES

Let  $N^m$  be a smooth manifold,  $U^m$  be a smooth manifold with smooth boundary, and  $\pi : U^m \rightarrow N^m$  — an immersion, i.e. a smooth embedding with nondegenerate Jacobian matrix at any point. We also suppose that the number of preimages of any point of  $N^m$  is bounded from above by some constant. Let  $f$  be a smooth real function on the manifold  $N^m$ . Denote by  $\hat{f}$  the lifting  $f \circ \pi : U^m \rightarrow \mathbb{R}$ .

Let the preimage of the point  $x_0$  in  $N^m$  consists of  $l$  boundary points of  $U^m$  and any number of nonboundary points of  $U^m$ . Let also the collection of  $l$  images of tangent hyperplanes to the boundary points projecting to  $x_0$  is a collection of hyperplanes in general position: the intersection of any  $s$  (for  $s \leq l$ ) hyperplanes is a  $(m-s)$ -dimensional plane.

**Definition 2.1.** We say that the function  $\hat{f}$  has a *multiboundary singularity of type  $B_n^l$*  at the preimage  $\pi^{-1}(x_0)$ , if  $f$  has a singularity  $A_{n-1}$  at point  $x_0$ . In addition it is required that the kernel of the Hessian matrix at  $x_0$  is transversal to the projection of any tangent hyperplane at the boundary point in the preimage  $\pi^{-1}(x_0)$ .

Let us remind a definition of a very nice M-morsification of the boundary singularity  $B_\mu$  of the paper [2].

Consider the space  $\mathbb{R}^{\mu-1}$  of real polynomials with zero constant term

$$x^\mu + \lambda_1 x^{\mu-1} + \dots + \lambda_{\mu-1} x$$

on the real line with fixed “boundary”  $x = 0$ .

**Definition 2.2.** A *very nice M-morsification of the boundary singularity*  $B_\mu$  is a polynomial of this family, whose all  $\mu-1$  critical points are real and the values at these points are distinct and nonzero (i.e. also distinct to the value at the boundary  $x = 0$ ).

Let us generalize Definition 2.2 to the case of multiboundary singularities of type  $B_n^l$ . Consider the Cartesian product of the spaces of real polynomials ( $\mathbb{R}^{n-1}$ )

$$x^n + \lambda_2 x^{n-2} + \dots + \lambda_{n-1} x$$

and the space  $\mathbb{R}^l$  of the space of “boundary values” at the points  $x = b_i$ , for  $i = 1, \dots, l$ . We call the points  $b_i$  the *boundary points*.

**Definition 2.3.** A *very nice M-morsification of the multiboundary singularity*  $B_n^l$  is a degree  $n$  polynomial with  $l$  marked boundary points, whose all  $n-1$  critical points are real, and the values at critical points and at  $l$  boundary values are all pairwise distinct.

Notice that we enumerate all boundary points, otherwise we should consider the factor of the space  $\mathbb{R}^l$  by the group of all permutations of the coordinates. We enumerate the boundary points since they correspond to distinct branches, whose permutations do not make sense here.

**Definition 2.4.** The *M-domain* is the closed subset of the space of all polynomials  $x^n + \lambda_2 x^{n-2} + \dots + \lambda_{n-1} x$ , consisting of polynomials which critical points are real.

The set of very nice M-morsifications is an open subset of  $\mathbb{R}^{n-1} \times \mathbb{R}^l$ , and its closure is  $(M\text{-domain}) \times \mathbb{R}^l$ . The set of very nice M-morsifications splits in connected components by the bifurcation diagram consisting of five hypersurfaces. Three of these hypersurfaces come from the case of boundary singularities of type  $B_n$  (see also in [2]):

- (a): the boundary caustic consisting of functions with a boundary critical point;
- (b): the ordinary Maxwell stratum consisting of functions with equal critical values at different points;
- (c): the boundary Maxwell stratum consisting of functions having some value at the boundary being equal to some critical value (the corresponding critical point is not at the boundary).

Notice that in our notation  $B_n = B_n^1$ , moreover the boundary point does not fixed at zero. So the definition of the very nice M-morsification of the boundary singularity  $B_n$  in the paper [2] is a special case of Definition 2.3.

In the case of multiboundary immersion singularities of type  $B_n^l$  for  $l \geq 2$  we have additional two hypersurfaces:

- (d): the double boundary caustic consisting of functions with some double boundary point;
- (e): the double boundary Maxwell stratum consisting of functions with equal values at different boundary points.

3. RECURRENT RELATION ON NUMBERS  $K_n^l$ 

We use  $K_n$  to denote Bernoulli-Euler numbers (let us list some first numbers  $K_n$  starting from  $n = 0$ : 1, 1, 1, 2, 5, 16, 61, ...). Denote by  $K_n^l$  the number of connected components of the set of very nice M-morsifications of the singularity  $B_n^l$ . The Bernoulli-Euler numbers are boundary conditions for the numbers  $K_n^l$ , namely  $K_n^0 = K_{n-1}$  and  $K_n^1 = K_{n+1}$  (see in [1]).

**Theorem 3.1.** *Let  $n \geq 2$  and  $l \geq 1$ . Then the following identity holds.*

$$K_{n-2}^{l+1} = K_n^l - nlK_n^{l-1}.$$

Consider an example of the singularity of type  $B_2^3$ . We have

$$K_3^2 = K_5^1 - 5K_5^0 = K_6 - 5K_4 = 61 - 5 \cdot 5 = 36.$$

We can calculate  $K_3^2$  using recurrence relations in a different way:

$$K_3^2 = K_1^3 + 2 \cdot 3K_3^1 = 3! + 6 \cdot 5 = 36.$$

We start the proof with two lemmas. In these lemmas we use the following notation.

Denote by  $L_n^l$  the number of connected components of the boundary caustic in the complement to the union of the strata and caustics of codimension 2. Here we suppose that the function has one critical point at the projection of a boundary point, all the rest critical and boundary points are distinct and the values at them are also distinct.

**Lemma 3.2.** *Let  $n \geq 2$  and  $l \geq 1$ . Then the following identity holds.*

$$L_n^l = l(n-1)K_n^{l-1}.$$

*Proof.* We take any very nice morsification with  $l-1$  boundary points and  $n-1$  critical points. Consider the action of the permutation group on the boundary points. This action naturally defines  $(l-1)!$  very nice M-morsifications in different connected components but with the same set of boundary points and the same polynomial. We put a new  $l$ -th boundary point to one of  $n-1$  critical points. The action of the permutation group on the boundary points now defines  $l!$  distinct M-morsifications with the same set of boundary points and the same polynomial. Therefore, we uniquely associate to the collection of  $(l-1)!$  old connected components the collection of  $(l-1)!$  new connected components. This implies the statement of Lemma 3.2.  $\square$

Further we find the number of connected components of very nice M-morsifications for which one of the boundary values is either greater than all the critical values or less than all the critical values, we denote this number by  $\hat{K}_n^l$ . Note that we count twice connected components of all M-morsifications for which all the critical values are contained in the interval with endpoints at two boundary values.

**Lemma 3.3.** *Let  $n \geq 2$  and  $l \geq 1$ . Then the following identity holds.*

$$\hat{K}_n^l = 2lK_n^{l-1}$$

*Proof.* Let  $n$  be even and  $l \geq 1$ . Then any M-morsification has two branches tending to plus infinity. Consider the maximal boundary value. If it is greater than all the critical values than it can be attained only at a point corresponding to two branches described above. Hence, for any very nice M-morsification with  $l-1$  boundary point we have exactly  $l$  distinct M-morsifications for which one of the boundary points  $b_i$  where  $1 \leq i \leq l$  is on the right branch, and the value at it is the maximal among all the boundary and critical values (here we keep the order of the rest points). The same reasoning is valid for the boundary point with maximal value at the left branch. Hence, we obtain  $\hat{K}_{2n}^l = 2lK_{2n}^{l-1}$  for even  $n$ .

If  $n$  is odd, then any M-morsification has two branches, one of them tends to plus infinity, and the other tends to minus infinity. By the same reason the statement of Lemma 3.3 hold for odd  $n$ . We remind that we count connected component corresponding to the odd case twice.  $\square$

*Proof of Theorem 3.1.* Consider a connected component of the set of very nice M-morsifications with  $l+1$  boundary points and  $n-3$  critical points. Take any M-morsification of this component. Consider the last boundary point  $b_{l+1}$  and add to it a  $\delta$ -shaped function concentrated in a small neighborhood of this point such that the new M-morsification has a critical point at  $b_{l+1}$  with old boundary value. The value of the second critical point that occurs while adding the  $\delta$ -shaped function is the maximal among all critical and boundary values. New M-morsification has now  $l$  boundary and  $n-1$  critical points. In the same way we can subtract  $\delta$ -shaped function. This provides us the decomposition of the set of M-morsifications with  $l$  boundary and  $n-1$  critical points into couples (corresponding to addition and subtraction of  $\delta$ -shaped function). For any connected component of the set of very nice M-morsifications with  $l+1$  boundary and  $n-3$  critical points by the above procedure we bijectively associate a couple of connected components of very nice M-morsifications with  $l$  boundary and  $n-1$  critical points.

Consider an arbitrary connected component of the set of M-morsifications of the boundary caustic with  $n-1$  critical point and  $l$  boundary points. So, one of the boundary points is critical, and in other boundary and critical points the values are pairwise distinct. Consider the critical boundary point. Let us move the boundary point a little to the right or to the left. If in the critical point we have local maximum than we add  $\delta$ -shaped function in this critical point, otherwise we subtract  $\delta$ -shaped function. The critical value becomes then maximal (or minimal respectively). For the boundary point there are two positions in correspondence with the direction we have moved it. This gives us to two very nice M-morsifications with  $n-1$  critical points and  $l$  boundary points. Hence, we associate to each of the components of the boundary caustics a couple of components of very nice M-morsifications with maximal or minimal critical values.

Finally, consider a component of very nice M-morsifications with  $n-1$  critical points and  $l$  boundary points with one critical value greater than all boundary values. Consider an M-morsification of this component. Let  $x_i$  be the point with maximal critical value. Let us take critical or boundary points that are the closest to  $x_i$  from the left and from the right respectively. Choose one of this two points with maximal value. Let us make an

operation inverse to the adding  $\delta$ -shaped function that “pulls down” the maximal critical value to the level of the neighbor chosen point. The result can be of two types. The first possibility is if we obtain a function with double critical point. In this case we replace a double point by a new boundary point  $b_{l+1}$ . The second possibility is if we obtain a function having one critical point coinciding with one boundary point.

We act in the same way for the case of a critical value being less than  $a$ ; boundary values.

The above observations prove the following identity:

$$2K_n^l - \hat{K}_n^l = 2L_n^l + 2K_n^{l+1}.$$

Now we apply Lemmas 3.2 and 3.3:

$$2K_n^l - 2lK_n^{l-1} = 2l(n-1)K_n^{l-1} + 2K_{n-2}^{l+1}.$$

Therefore,

$$K_{n-2}^{l+1} = K_n^l - nlK_n^{l-1}.$$

This concludes the proof of Theorem 3.1.  $\square$

In conclusion of this section we indicate the relation between  $K_n^l$  and the numbers of connected components of special strata of bifurcation diagram of critical points and critical values for degree  $2l+n$  polynomials (i.e. of  $A_{2l+n-1}$ ).

**Corollary 3.4.** *Consider open strata of caustic of the singularity  $A_{2l+n-1}$  corresponding to polynomials of  $M$ -domain with  $l$ -couples of double critical points with all distinct critical values at distinct critical points. The number of connected components of such strata is  $K_n^l/l!$ .*

#### 4. COROLLARIES OF THEOREM 3.1. CASES OF SMALL NUMBERS OF BOUNDARY POINTS

Let us give explicit formulae for  $K_n^l$  for  $l \leq 5$ . In [1] V. I. Arnold proved that the number of connected components of very nice  $M$ -morsifications for singularity  $A_{n-1}$  equals to the  $n$ -th Bernoulli-Euler number, namely  $K(A_{n-1}) = K_{n-1}$ . It [1] there is also a formula for numbers  $K(B_n)$  for multiboundary singularities of type  $B_n$ :  $K(B_n) = K_{n+1}$ . So, we have:  $K_n^0 = K_{n-1}$ , and  $K_n^1 = K_{n+1}$ .

Direct calculations lead to the results of the following corollary of Theorem 3.1.

**Corollary 4.1.** *The following expressions of  $K_n^l$  in terms of Bernoulli-Euler numbers are correct.*

$$\begin{aligned} K_n^2 &= K_{n+3} - (n+2)K_{n+1}; \\ K_n^3 &= K_{n+5} - (3n+8)K_{n+3}; \\ K_n^4 &= K_{n+7} - (6n+20)K_{n+5} + 3(n+2)(n+4)K_{n+3}; \\ K_n^5 &= K_{n+9} - (10n+40)K_{n+7} + (15n^2 + 110n + 184)K_{n+5} \\ &\dots \end{aligned}$$

The exponential generating function for Bernoulli-Euler numbers is the function

$$K(t) = \tan(t) + \sec(t).$$

Let us show exponential generating functions for the numbers  $B_n^l$  for  $l \leq 4$ . Note that in the cases of  $l = 0$  and  $l = 1$  one can propose the exponential generating functions to be  $K(t)$ , still in our notation we have

$$\begin{aligned} K_0(t) &= \int K(t) dt = -\ln(\cos(t)) + \ln(\tan(\frac{t}{2} + \frac{\pi}{4})) + C \quad \text{and} \\ K_1(t) &= K'(t) = \frac{1+K^2}{2} = \frac{1+\sin(t)}{\cos^2 t} = \frac{1}{1-\sin(t)} \end{aligned}$$

respectively.

**Corollary 4.2.** *The exponential generating functions for the cases  $l = 2, 3, 4$  are as follows:*

$$\begin{aligned} K_2(t) &= K'''(t) - (tK(t))'' = \frac{3\sin(t) - t\cos(t)}{(1-\sin(t))^2}; \\ K_3(t) &= (K'' - 3tK' + K)''' = \frac{3}{(1-\sin(t))^3} \left( \sin(t)(3\sin(t) + 7) - 3t\cos(t)(5 + \sin(t)) \right); \\ K_4(t) &= (K''' - 6tK'' + (3t^2 + 4)K' - 3tK)^{(4)} = \\ &= \left( \frac{3t^2}{1-\sin(t)} - \frac{3t\cos(t)}{(1-\sin(t))^2} (3 - \sin(t)) + \frac{3(2-\sin(t))}{(1-\sin(t))^2} \right)^{(4)}. \end{aligned}$$

Let

$$K(x, y) = \sum \frac{K_n^l}{l!n!} x^l y^n$$

be an exponential generation function in two variables.

**Corollary 4.3.** *The function  $K(x, y)$  satisfies the following differential equation*

$$K_x = (1 - 2x)K_{yy} - xyK_{yyy}.$$

B. Z. Shapiro proposed to consider two exponential generating functions in two variables  $R(x, y)$  and  $S(x, y)$  separately for  $R_n^l = K_{2n}^l$  and for  $S_n^l = K_{2n-1}^l$ . This decreases the order of the differential equation.

**Corollary 4.4.** *The functions  $R(x, y)$  and  $S(x, y)$  satisfy the following differential equations:*

$$\begin{aligned} R_x &= (1 - 2x)R_y - 2xyR_{yy}, \\ S_x &= (1 - 2x)S_y - x(2y - 1)S_{yy}. \end{aligned}$$

Finally, let us describe geometrical structure of  $B_n^2$  with schematic “pictures” of fibers in the bundle

$$\pi : (M\text{-domain}) \times \mathbb{R}^2 \rightarrow M\text{-domain}.$$

For any polynomial  $f(x)$  we associate a skew-symmetric polynomial  $f(b_1) - f(b_2)$  in two variables  $b_1$  and  $b_2$ . This polynomial defines the double boundary Maxwell stratum and boundary caustics in the fiber. Boundary caustics and boundary Maxwell stratum are defined by vertical and horizontal straight lines and form a rectangular net, in which the curve  $f(b_1) = f(b_2)$  is inscribed. It is easy to draw combinatoric pictures of such curves. For example, we draw the pictures for the singularities  $B_3^2$  and  $B_4^2$  (see Figures 1 and 2).

Notice that the curve  $f(b_1) = f(b_2)$  is a union of a straight line  $b_1 = b_2$  with some curve of degree  $n-1$ . For instance, in the case of  $B_3^2$  it is the union of a straight line  $b_1 = b_2$  and an ellipse (see Figure 1). Here a natural problem arises. *Describe all combinatoric types of such pictures for general  $n$ .* At present moment we do not know the answer to this problem.

## 5. COROLLARIES OF THEOREM 3.1. CONNECTION WITH BERNOULLI-EULER NUMBERS

In this part we introduce some expressions for the numbers  $K_n^l$  similar to the expressions of Corollary 4.1. Further we calculate  $K_n^l$  for negative  $n$  satisfying  $n \leq -l$ . These numbers lead to nice relations on Bernoulli-Euler numbers.

**Corollary 5.1.** *The following expression for  $K_n^l$  is true:*

$$\begin{aligned} K_n^l &= K_{n+2l-1} - \\ &\left(\frac{l(l-1)}{2}n + \frac{(l+1)l(l-1)}{3}\right)K_{n+2l-3} + \\ &l(l-1)(l-2)(l-3)\left(\frac{1}{8}n^2 + \frac{2l+1}{12}n + \frac{(l+1)(5l-2)}{90}\right)K_{n+2l-5} + \\ &\sum_{k=4}^{\lfloor \frac{l}{2} \rfloor + 1} \left( \left( \frac{(-1)^{k-1}}{2^{k-1}(k-1)!} \frac{l!}{(l-2k+2)!} \sum_{d=0}^{k-1} (p_{k,d}(l)n^d) \right) K_{n+2(l-k)+1} \right), \end{aligned}$$

where  $p_{k,d}(x)$  is a polynomial of degree  $k-d-1$  with constant coefficients depending on  $k$  and  $d$ .

The idea of the proof is based on the induction on  $l$ .

Still we do not know the explicit formulae for the polynomials  $p_{k,d}(x)$ , nevertheless there exists a recursive method to find the coefficients of such polynomials. Let us show the polynomials for  $d = k-1$ ,  $d = k-2$ , and  $d = k-3$ .

**Corollary 5.2.** *The following hold:*

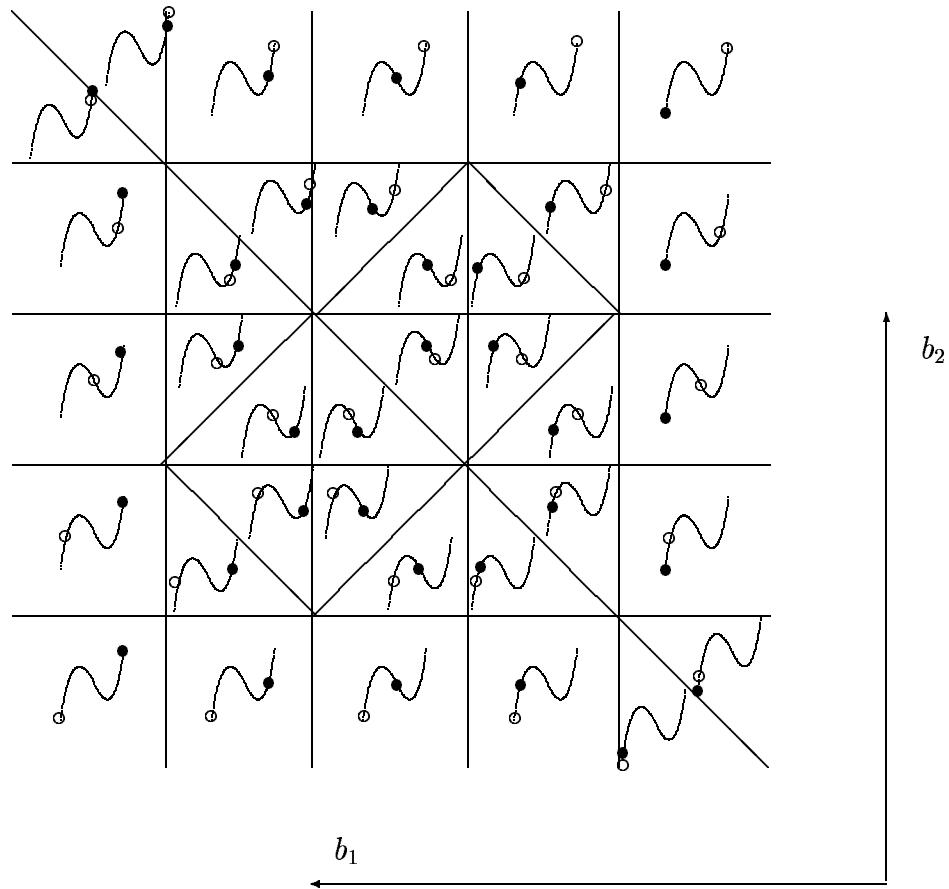
$$\begin{aligned} p_{k,k-1}(x) &= 1; \\ p_{k,k-2}(x) &= \frac{(k-1)}{3}(2x + 4 - k); \\ p_{k,k-3}(x) &= \frac{(k-1)(k-2)}{90} \left( 20x^2 + (72 - 20k)x + (5k^2 - 39k + 64) \right). \end{aligned}$$

Applying formulae of Corollary 5.1 one can obtain many relations for Bernoulli-Euler numbers. Let us consider some examples of such relations. Substitute  $n = 1$  and  $n = 2$  in the formulae of Corollary 5.1, we get the equalities of the following theorem.

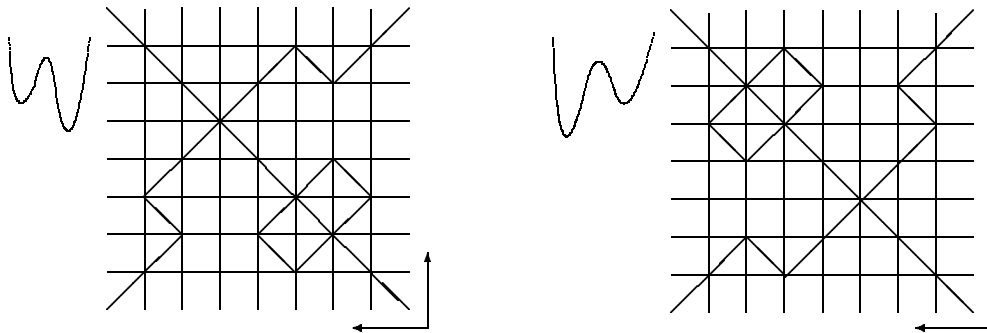
**Theorem 5.3.** *The following relations on Bernoulli-Euler numbers hold:*

$$\begin{aligned} K_{2l} - l! &= K_{2l} - K_1^l = \sum_{k=2}^{\lfloor \frac{l}{2} \rfloor + 1} \left( \frac{(-1)^k}{2^{k-1}(k-1)!} \frac{l!}{(l-2k+2)!} \sum_{d=0}^{k-1} p_{k,d}(l) \right) K_{2(l-k)+1}; \\ K_{2l+1} - 2^l l! &= K_{2l+1} - K_2^l = \sum_{k=2}^{\lfloor \frac{l}{2} \rfloor + 1} \left( \frac{(-1)^k}{2^{k-1}(k-1)!} \frac{l!}{(l-2k+2)!} \sum_{d=0}^{k-1} (p_{k,d}(l)2^d) \right) K_{2(l-k)+1}. \end{aligned}$$





**Figure 1.** A fiber of general position of the M-domain for the singularity  $B_3^2$ .



**Figure 2.** A fiber of general position of the M-domain for the singularity  $B_4^2$ .

Let us use a expressions of Corollary 4.1 to obtain the numbers  $K_n^l$ , for  $n \leq 0$ , and  $n \leq -l$ .

$K_n^l$	l=1	l=2	l=3	l=4	l=5
n=0	1	0	0	0	0
n=-1	1	0	0	0	0
n=-2	?	1	0	0	0
n=-3	?	?	2	0	0
n=-4	?	?	?	6	0
n=-5	?	?	?	?	24

The numbers  $K_0^l$  can be interpreted as numbers of connected components of the space of degree 0 polynomials with  $l$  boundary points with distinct boundary values. Here we have only one non-zero entry corresponding to  $l = 1$ . The author does not know any reasonable interpretation for the numbers for negative  $n$ .

Notice that there are many zeroes in the table. Let us formulate this statement more precisely.

**Proposition 5.4.** *Let  $n \leq -1$ , then  $K_n^l = 0$  for  $l > -n$  and  $K_n^{-n} = (-n-1)!$ . The numbers  $K_0^l = 0$  for  $l > 1$ .*

*Proof.* The proof is based on Theorem 3.1. Let us use the induction on  $-n$ .

$$\begin{aligned} K_0^l &= K_2^{l-1} - 2(l-1)K_2^{l-2} = (2l-2)!! - (2l-2)(2l-4)!! = 0 \text{ for } l-2 \geq 0. \\ K_{-1}^l &= K_1^{l-1} - (l-1)K_2^{l-2} = (l-1)! - (l-1)(l-2)! = 0 \text{ for } l-2 \geq 0. \\ K_{-2}^l &= K_0^{l-1} = 0 \text{ for } l-1 \geq 2. \end{aligned}$$

In general case we have

$$\begin{aligned} K_n^l &= K_{n+2}^{l-1} - (n+2)(l-1)K_{n+2}^{l-2} \text{ for } n < -2, l-2 \geq -2-n. \\ K_n^{-n} &= K_{n+2}^{-n-1} - (n+2)(-n-1)K_{n+2}^{-n-2} = 0 + (n+2)(n+1)(-n-3)! = (-n-1)!. \end{aligned}$$

This proves the statement of the proposition.  $\square$

We conclude the paper with relations for the Bernoulli-Euler numbers that follows from the results of Corollary 5.1 and Proposition 5.4.

**Corollary 5.5.** *Let  $n \leq 0$ ,  $l > \max(1, -n)$ , then*

$$0 = K_n^l = K_{n+2l-1} + \sum_{k=2}^{\lfloor \frac{l}{2} \rfloor + 1} \left( \left( \frac{(-1)^{k-1}}{2^{k-1}(k-1)!} \frac{l!}{(l-2k+2)!} \sum_{d=0}^{k-1} (p_{k,d}(l)n^d) \right) K_{n+2(l-k)+1} \right).$$

If  $l = -n$ , then

$$(l-1)! = K_n^{-n} = K_{l-1} + \sum_{k=2}^{\lfloor \frac{l}{2} \rfloor + 1} \left( \left( \frac{(-1)^{k-1}}{2^{k-1}(k-1)!} \frac{l!}{(l-2k+2)!} \sum_{d=0}^{k-1} (p_{k,d}(l)(-l)^d) \right) K_{l-2k+1} \right).$$

In particular for  $l > 1$  we have:

$$0 = K_0^l = K_{n+2l-1} + \sum_{k=2}^{\lfloor \frac{l}{2} \rfloor + 1} \left( \left( \frac{(-1)^{k-1}}{2^{k-1}(k-1)!} \frac{l!}{(l-2k+2)!} p_{k,0}(l) \right) K_{2(l-k)+1} \right).$$

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