Embedding of semigroups of Lipschitz maps into positive linear semigroups on ordered Banach spaces generated by measures

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== Report MI-2008-12 ==

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7 July 2008

Abstract

Interpretation, derivation and application of a variation of constants formula for measure-valued functions motivate our investigation of properties of particular Banach spaces of Lipschitz functions on a metric space and semigroups defined on their (pre)duals. Spaces of measures densely embed into these preduals. The metric space embeds continuously in these preduals, even isometrically in a specific case. Under mild conditions, a semigroup of Lipschitz transformations on the metric space then embeds into a strongly continuous semigroups of positive linear operators on these Banach spaces generated by measures.

1 Introduction

The concept of a continuous-time deterministic or causal dynamical system in a set S can be expressed by the existence of a family of maps $\Phi_t : S \to S$, parametrised by the nonnegative real numbers $t \in \mathbb{R}_+$, that satisfy the semigroup properties: $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and $\Phi_0 = \mathrm{Id}_S$. The evolution of the system in time from its initial state $x_0 \in S$ is described by the orbit $t \mapsto \Phi_t(x_0)$. (An interesting essay on the history of this concept can be found in [10]). If Σ is a σ-algebra of subsets of S and each Φ_t is (Σ, Σ)-measurable, then each Φ_t induces a linear operator $T_\Phi(t)$ on the space of signed measures $\mathcal{M}(S)$ on Σ by means of

$$T_\Phi(t)\mu := \mu \circ \Phi_t^{-1}. \tag{1}$$

The family of operators $(T_\Phi(t))_{t\geq 0}$ leaves the cone of positive measures $\mathcal{M}^+(S)$ invariant. It constitutes a positive linear semigroup in $\mathcal{M}(S)$ and Φ_t can be recovered from $T_\Phi(t)$ through the relation $T_\Phi(t)\delta_x = \delta_{\Phi_t(x)}$. In this sense, any semigroup of measurable maps on a measurable space (S, Σ) embeds into a positive linear semigroup on the space of signed measures on S.

This paper studies properties of this embedding in detail when S is a metric space with the Borel σ-algebra and the transformations Φ_t are Lipschitz maps. We are motivated by the study of long-term dynamics in structured population models where deterministic behaviour of an individual is 'perturbed' at random discrete time points by a deterministic or random (approximately) instantaneous change in state. Examples include branching random evolution [9], kinetic chemotaxis models concerning the run-and-tumble type of movement of flagellated bacteria like *E. coli*, *B. subtilis* or *V. cholerae* [18, 19, 2] and the extension of these to amoebae like *Dictyostelium discoideum* [3, 11], and cell cycle models in which a cell divides at random time points coupled to deterministic growth [5]. Our approach to these systems is to consider them as deterministic dynamical systems in the space (or cone of positive) finite Borel measures on the individual's state space S. The dynamics are then governed by a suitable variation of constants formula

$$\mu_t = T_\Phi(t)\mu_0 + \int_0^t T_\Phi(t-s)F(\mu_s)ds \tag{2}$$

in a space of measures on S. The interpretation, derivation and application of (2) require a detailed examination of topologies and functional analytic properties of spaces or sets of measures and operators thereon. There are some preliminary issues here, which are the primary concern of this paper.

First, the representation (1) of $T_\Phi(t)$ is practical in the context of (2) only when Φ_t is invertible, which is rarely the case in applications. For a functional analytic treatment we therefore need a 'better' representation of $T_\Phi(t)$. Second, what topology is 'natural' in this setting and allows the application of numerous results on perturbations of linear semigroups in the literature? The total variation norm in $\mathcal{M}(S)$ is of little use in our context. The embedding $x \mapsto \delta_x : S \to \mathcal{M}(S)$ is not continuous for $\|\cdot\|_{\mathrm{TV}}$, nor is $(T_\Phi(t))_{t\geq 0}$ strongly continuous, unless $(T_\Phi(t))_{t\geq 0}$ is constant. Our investigations continue along the line set out by Dudley [7, 8] mainly, based on [21]. Third, we need to have appropriate regularity of the map $t \mapsto T_\Phi(t)\mu$ for the existence of the integral in (2) in some sense (weak, Bochner, etc.).

Concerning the topologies on spaces of measures we would like to point out that clearly $\mathcal{M}(S)$ is a subspace of $C_b(S)^*$ and can therefore be endowed with the restriction of the weak-star topology on $C_b(S)^*$. This topology is often used in probability theory. There is an interesting result by Varadarajan, that the restriction to $\mathcal{M}^+(S)$ is metrisable (when S is separable, or when one restricts to separable positive measures), by a complete metric if S is complete ([21, Theorem 13 and Theorem 18]). Later Dudley showed ([7, Theorem 9 and Theorem 18]) that the metric given by

$$d_{\mathrm{BL}}(\mu, \nu) = \|\mu - \nu\|^*_{\mathrm{BL}} = \sup\left\{ \left|\int f d(\mu-\nu)\right| : \|f\|_{\mathrm{BL}} \leq 1 \right\},$$

may be used. It is this point that we pursue further.

Moreover, Peng and Xu [20] provide an embedding of a nonlinear semigroup of Lipschitz transformations into a linear semigroup as well. Their approach involves the use of quotient spaces and their duals however. These are rather inconvenient and the relationship of their results to the targeted semigroup $(T_\Phi(t))_{t\geq 0}$ on measures is not as clear and direct as the approach we advocate.

The outline of the paper is as follows: Section 2 and 3 introduce Banach spaces of Lipschitz functions on S, $\mathrm{BL}(S)$ and $\mathrm{Lip}_e(S)$, investigate their dual spaces and introduce preduals for both, $\mathcal{S}_{\mathrm{BL}}$ and $\mathcal{S}_e$ respectively. The latter are closed subspaces of $\mathrm{BL}(S)^*$ and $\mathrm{Lip}_e(S)^*$. While assuming for simplicity of this introductory exposition that S is separable, the space of finite measures $\mathcal{M}(S)$ and its subspace of measures with first moment, $\mathcal{M}_1(S)$, are densely embedded in $\mathcal{S}_{\mathrm{BL}}$ and $\mathcal{S}_e$ respectively (Theorem 3.9 and 3.14). The latter spaces equal these spaces of measures only in the case that S is uniformly discrete (Theorem 3.11). The embeddings yield through the map $x \mapsto \delta_x$, the Dirac measure at $x \in S$, an embedding of S into $\mathrm{BL}(S)^*$ and $\mathrm{Lip}_e(S)^*$ that is continuous in the first case (Lemma 3.5) and an *isometric embedding* in the latter (Lemma 3.4). Section 4 discusses the relationship between the natural pointwise ordering on Lipschitz functions, positive functionals on $\mathrm{BL}(S)$ and $\mathrm{Lip}_e(S)$ and cones of positive measures. Section 5 presents the main result on the embedding of a semigroup of Lipschitz transformations Φ_t on S into a positive linear semigroup on $\mathcal{S}_{\mathrm{BL}}$ and $\mathcal{S}_e$. We give a sufficient condition for strong continuity of these semigroups in terms of $(\Phi_t)_{t\geq 0}$. Section 6 concludes with a discussion of some issues concerning topologies on spaces or cones of measures.

2 Banach spaces of Lipschitz functions

Let (S, d) be a metric space, consisting of at least two points. $\mathrm{Lip}(S)$ denotes the vector space of *real-valued* Lipschitz functions on S. We only consider real-valued functions, because ordering will play a role. Moreover, it seems that real-valued functions are more 'natural' in the theory of spaces of Lipschitz functions (see [22, p. 13]). The Lipschitz seminorm $|\cdot|_{\mathrm{Lip}}$ is defined on $\mathrm{Lip}(S)$ by means of

$$|f|_{\mathrm{Lip}} := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in S, x \neq y\right\}$$

Clearly, $|f|_{\mathrm{Lip}} = 0$ if and only if f is constant.

We start with some basic facts on Lipschitz functions that we will use repeatedly. First, the distance function is a Lipschitz function:

Lemma 2.1. *Let E be a nonempty subset of S. Then $x \mapsto d(x, E)$ is in $\mathrm{Lip}(S)$. If $\overline{E} = S$, then $d(\cdot, E) \equiv 0$ and if $\overline{E}$ is a proper subset of S, then $|d(\cdot, E)|_{\mathrm{Lip}} = 1$.*

This follows from the triangle inequality and the fact that $d(x, E) = d(x, \overline{E})$. In particular Lemma 2.1 implies that $x \mapsto d(x, y) \in \mathrm{Lip}(S)$ for all $y \in S$.

The pointwise minima and maxima of a finite number of Lipschitz functions are again Lipschitz functions:

Lemma 2.2. *([7, Lemma 4]) Given $f_1, ..., f_n \in \mathrm{Lip}(S)$ we define*

$$g(x) := \min(f_1(x), ..., f_n(x)) \text{ and } h(x) := \max(f_1(x), ..., f_n(x)).$$

Then $g, h \in \mathrm{Lip}(S)$ *and*

$$\max(|g|_{\mathrm{Lip}}, |h|_{\mathrm{Lip}}) \leq \max(|f_1|_{\mathrm{Lip}}, ..., |f_n|_{\mathrm{Lip}}).$$

In the sequel two normed spaces of Lipschitz functions on S and their Banach space properties will be the central objects of study. First, for each $e \in S$ we introduce the norm $\|\cdot\|_e$ on $\mathrm{Lip}(S)$ by $\|f\|_e := |f(e)| + |f|_{\mathrm{Lip}}$, $f \in \mathrm{Lip}(S)$. If e' is another element in S, then

$$\begin{aligned} \|f\|_e &\leq |f(e')| + |f(e) - f(e')| + |f|_{\mathrm{Lip}} \leq |f(e')| + |f|_{\mathrm{Lip}}(d(e, e') + 1) \\ &\leq \|f\|_{e'}(d(e, e') + 1). \end{aligned}$$

Thus $\|\cdot\|_e$ and $\|\cdot\|_{e'}$ are equivalent norms on $\mathrm{Lip}(S)$.

For the rest of the paper, we fix an element $e \in S$ and write $\mathrm{Lip}_e(S)$ for the normed vector space $\mathrm{Lip}(S)$ with norm $\|\cdot\|_e$. The following property is straightforward:

Lemma 2.3. *If* $f \in \mathrm{Lip}_e(S)$ *and* $x \in S$, *then* $|f(x)| \leq \max(1, d(x, e))\|f\|_e$.

Proposition 2.4. $\mathrm{Lip}_e(S)$ *is a Banach space.*

Proof. Let $(f_n)_n$ be a Cauchy sequence in $\mathrm{Lip}_e(S)$. Let $x \in S$. Then Lemma 2.3 implies that $(f_n(x))_n$ is a Cauchy sequence for every $x \in S$. Put $f(x) := \lim_{n\to\infty} f_n(x)$. Let $\epsilon > 0$. There is an $N \in \mathbb{N}$, such that $|f_n - f_m|_{\mathrm{Lip}} \leq \epsilon$ for all $n, m \geq N$. Then for $x, y \in S$, $m \geq N$,

$$\begin{aligned} |(f - f_m)(x)) - (f - f_m)(y)| &= \lim_{n\to\infty} |(f_n - f_m)(x) - (f_n - f_m)(y)| \\ &\leq \epsilon d(x, y). \end{aligned}$$

Hence $|f - f_m|_{\mathrm{Lip}} \leq \epsilon$ for all $m \geq N$. This implies that $f \in \mathrm{Lip}_e(S)$ and $|f - f_n|_{\mathrm{Lip}} \to 0$ as $n \to \infty$. Thus $\|f - f_n\|_e \to 0$ as $n \to \infty$, and $\mathrm{Lip}_e(S)$ is complete. □

Second, let $\mathrm{BL}(S)$ be the vector space of bounded Lipschitz functions from S to $\mathbb{R}$. For $f \in \mathrm{BL}(S)$ we define: $\|f\|_{\mathrm{BL}} := \|f\|_\infty + |f|_{\mathrm{Lip}}$. Then $\|\cdot\|_{\mathrm{BL}}$ is a norm on $\mathrm{BL}(S)$.

Proposition 2.5. $\mathrm{BL}(S)$ *is complete with respect to* $\|\cdot\|_{\mathrm{BL}}$.

The proof of this proposition proceeds in a similar way to that of Proposition 2.4. See also [22, Proposition 1.6.2 (a)]. There, completeness is proved for the alternative (but equivalent) norm $\|f\|_{\mathrm{BL,max}} = \max(\|f\|_\infty, |f|_{\mathrm{Lip}})$.

If $f \in \mathrm{BL}(S)$, then $f \in \mathrm{Lip}_e(S)$, so there is a canonical embedding $j : \mathrm{BL}(S) \to \mathrm{Lip}_e(S)$, where $j(f) = f$. Clearly $\|j(f)\|_e \leq \|f\|_{\mathrm{BL}}$. Thus $\mathrm{BL}(S)$ embeds continuously into $\mathrm{Lip}_e(S)$. If S has finite diameter, then $\mathrm{BL}(S) = \mathrm{Lip}_e(S)$, and it is easy to see that in this case the norms $\|\cdot\|_{\mathrm{BL}}$ and $\|\cdot\|_e$ are equivalent. Otherwise we can consider the closure of $\mathrm{BL}(S)$ in $\mathrm{Lip}_e(S)$ with respect to $\|\cdot\|_e$:

Proposition 2.6. *Let S be a metric space with infinite diameter. Then*

$$\mathrm{BL}(S) \subsetneq \overline{\mathrm{BL}(S)}^{\|\cdot\|_e} \subsetneq \mathrm{Lip}_e(S).$$

Proof. Define $f(x) := \sqrt{d(x,e)+1}$. Then

$$|f(x) - f(y)| = \frac{|d(x,e) - d(y,e)|}{\sqrt{d(x,e)+1} + \sqrt{d(y,e)+1}} \leq \frac{d(x,y)}{\sqrt{d(x,e)+1} + \sqrt{d(y,e)+1}}.$$

So f is in $\mathrm{Lip}_e(S)$, but not in $\mathrm{BL}(S)$, since S has infinite diameter. We will show that $f \in \overline{\mathrm{BL}(S)}^{\|\cdot\|_e}$. Let $f_n(x) := \min(f(x), n)$. Then $f_n \in \mathrm{BL}(S)$ by Lemma 2.2.

Let $g_n := f - f_n$. Now let $x, y \in S$, $x \neq y$. Then if $f(x) \leq n$ and $f(y) \leq n$, $|g_n(x) - g_n(y)| = 0$. If $f(x) > n$ and $f(y) > n$, then $|g_n(x) - g_n(y)| = |f(x) - f(y)| \leq \frac{d(x,y)}{2n}$. If $f(x) > n$ and $f(y) \leq n$, then $|g_n(x) - g_n(y)| = |f(x) - n| \leq |f(x) - f(y)| \leq \frac{d(x,y)}{n+1}$. So $|f - f_n|_{\mathrm{Lip}} = |g_n|_{\mathrm{Lip}} \leq \frac{1}{n+1}$. Therefore $\|f - f_n\|_e \leq \frac{1}{n+1}$ and $f_n \to f$ in $\mathrm{Lip}_e(S)$.

Now define $g(x) = d(x,e)$. Then g is in $\mathrm{Lip}_e(S)$, but not in $\mathrm{BL}(S)$. Suppose that $g \in \overline{\mathrm{BL}(S)}^{\|\cdot\|_e}$, then there is a $h \in \mathrm{BL}(S)$, with $\|g - h\|_e < \frac{1}{2}$. Moreover, Lemma 2.3 yields

$$|g(x) - h(x)| \leq \frac{1}{2} \max(1, d(x,e)).$$

This implies that

$$|h(x)| \geq |g(x)| - |g(x) - h(x)| \geq \frac{1}{2} d(x,e) - \frac{1}{2}.$$

Because S has infinite diameter, this contradicts that h is bounded. □

Note that the adjoint map $j^* : \mathrm{Lip}_e(S)^* \to \mathrm{BL}(S)^*$, which restricts a $\varphi \in \mathrm{Lip}_e(S)^*$ to $\mathrm{BL}(S)$, is continuous, with $\|j^*(\varphi)\|_{BL}^* \leq \|\varphi\|_e^*$.

Whenever S has infinite diameter, $\overline{\mathrm{BL}(S)}^{\|\cdot\|_e} \subsetneq \mathrm{Lip}_e(S)$, by Proposition 2.6. From this and the Hahn-Banach Theorem it follows that there exists a non-zero $\phi \in \mathrm{Lip}_e(S)^*$ such that $\phi|_{\mathrm{BL}(S)} = 0$, hence j^* is not injective.

We will use the term *Lipschitz spaces* to refer to $\mathrm{BL}(S)$ and $\mathrm{Lip}_e(S)$.

Remark. Various authors consider other Banach spaces of Lipschitz functions, such as e.g. Weaver [22], looking at $\mathrm{Lip}_0(S)$ consisting of all Lipschitz functions on S that vanish at some distinct point $e \in S$. On this subspace of $\mathrm{Lip}(S)$, $|\cdot|_{\mathrm{Lip}}$ is a norm for which $\mathrm{Lip}_0(S)$ is complete. Peng and Xu [20] for example, perform the standard construction of dividing out the constant functions in $\mathrm{Lip}(S)$. Then this space of equivalence classes of Lipschitz functions $\mathrm{Lip}(S)/\mathbb{R}\mathbf{1}$ is complete with respect to the norm $|\cdot|_{\mathrm{Lip}}$ and it is isometrically isomorphic to $\mathrm{Lip}_0(S)$. Working with these spaces is somewhat cumbersome for our applications.

3 Dual and predual of Lipschitz spaces

Various spaces of Lipschitz functions have been shown to be isometrically isomorphic to the dual of a Banach space. For instance, $\mathrm{Lip}_0(S)$ is the dual of the so-called Arens-Eells space (see [1] and [22, Section 2.2]). It is also known that $\mathrm{BL}(S)$ endowed with the norm $\|f\|_{\mathrm{BL},\max} := \max(\|f\|_\infty, |f|_{\mathrm{Lip}})$ is isometrically isomorphic to the dual of a Banach space. For instance in [15, Theorem 4.1] the more general result is proven for $\mathrm{BL}(S, E^*)$, where E^* is the dual of a Banach space. Our aim in this section is to show that $\mathrm{BL}(S)$ with the norm $\|\cdot\|_{\mathrm{BL}}$ can also be viewed as the dual of a Banach space, $\mathcal{S}_{\mathrm{BL}}$, and that $\mathrm{Lip}_e(S)$ is the dual of a Banach space, $\mathcal{S}_e$, as well. Furthermore, we will show that natural spaces of measures are densely contained in $\mathcal{S}_{\mathrm{BL}}$ and $\mathcal{S}_e$.

3.1 Embedding of measures in dual of Lipschitz spaces

In this section we are concerned with embedding measures into $\mathrm{BL}(S)^*$ and $\mathrm{Lip}_e(S)^*$. We shall write $\|\cdot\|^*_{\mathrm{BL}}$ to denote the dual norm on $\mathrm{BL}(S)^*$ and $\|\cdot\|^*_e$ to denote the dual norm on $\mathrm{Lip}_e(S)^*$.

Let $\mathcal{M}(S)$ be the space of all signed finite Borel measures on S and $\mathcal{M}^+(S)$ the convex cone of positive measures in $\mathcal{M}(S)$. Let $\|\cdot\|_{\mathrm{TV}}$ denote the total variation norm on $\mathcal{M}(S)$. It is a standard result that $\mathcal{M}(S)$ endowed with $\|\cdot\|_{\mathrm{TV}}$ is a Banach space.

The Baire σ-algebra is the smallest σ-algebra on S for which all continuous real-valued functions on S are measurable. Since S is a metric space, the Baire and Borel σ-algebras coincide, because for any closed $C \subset S$, $f_C : x \mapsto d(x, C)$ is Lipschitz continuous by Lemma 2.1. Therefore we can apply some of the results from Dudley [7] on Baire measures.

Each $\mu \in \mathcal{M}(S)$ defines a linear functional I_μ on $\mathrm{BL}(S)$, by means of $I_\mu(f) := \int_S f d\mu$. Then

$$\begin{aligned}\|I_\mu\|^*_{\mathrm{BL}} &= \sup\left\{ \left|\int f d\mu\right| : \|f\|_{\mathrm{BL}} \le 1 \right\} \\ &\le \sup\left\{ \int |f| d|\mu| : \|f\|_{\mathrm{BL}} \le 1 \right\} \le |\mu|(S) = \|\mu\|_{\mathrm{TV}},\end{aligned} \tag{3}$$

thus $I_\mu \in \mathrm{BL}(S)^*$. Moreover, one has

Lemma 3.1. *Let $\mu \in \mathcal{M}^+(S)$. Then $\|I_\mu\|^*_{\mathrm{BL}} = \|\mu\|_{\mathrm{TV}}$.*

Proof. Suppose $\mu \in \mathcal{M}^+(S)$. From (3) it follows that $\|I_\mu\|^*_{\mathrm{BL}} \le \|\mu\|_{\mathrm{TV}}$. Clearly the constant function $\mathbf{1}$ is in $\mathrm{BL}(S)$, with $\|\mathbf{1}\|^*_{\mathrm{BL}} = 1$. Then $\|\mu\|_{\mathrm{TV}} = \mu(S) = \int \mathbf{1} d\mu \le \|I_\mu\|^*_{\mathrm{BL}}$. Hence $\|I_\mu\|^*_{\mathrm{BL}} = \|\mu\|_{\mathrm{TV}}$. □

Lemma 3.2. *([7, Lemma 6])*
The linear map $\mu \mapsto I_\mu : \mathcal{M}(S) \to \mathrm{BL}(S)^*$ is injective.

Thus we can continuously embed $\mathcal{M}(S)$ into $\mathrm{BL}(S)^*$ and identify $\mu \in \mathcal{M}(S)$ with $I_\mu \in \mathrm{BL}(S)^*$. When a functional $\varphi \in \mathrm{BL}(S)^*$ can be represented by a measure, we shall write $\varphi \in \mathcal{M}(S)$.

We define the subspace of $\mathcal{M}(S)$ of measures with finite first moment as follows:

$$\mathcal{M}_1(S) := \left\{ \mu \in \mathcal{M}(S) : \int d(x,e) d|\mu|(x) < \infty \right\}.$$

And we put $\mathcal{M}_1^+(S) := \mathcal{M}_1(S) \cap \mathcal{M}^+(S)$. For $\mu \in \mathcal{M}_1(S)$ we define $\|\mu\|_1 := \int \max(1, d(x,e)) d|\mu|(x)$. Then $\|\cdot\|_1$ is a norm on $\mathcal{M}_1(S)$. Let $\mu \in \mathcal{M}_1(S)$. Then $I_\mu(f) := \int f \, d\mu$ is well defined for every $f \in \mathrm{Lip}_e(S)$, and I_μ is a linear functional on $\mathrm{Lip}_e(S)$.

Lemma 3.3. *Let $\mu \in \mathcal{M}_1(S)$. Then $I_\mu \in \mathrm{Lip}_e(S)^*$ and*

$$\|I_\mu\|_e^* \le \|\mu\|_1.$$

Moreover, the linear map $\mu \mapsto I_\mu : \mathcal{M}_1(S) \to \mathrm{Lip}_e(S)^*$ is injective.

Proof. Let $\mu \in \mathcal{M}_1(S)$ and $f \in \mathrm{Lip}_e(S)$. Using Lemma 2.3 we obtain

$$|\int f d\mu| \le \int |f| d|\mu| \le \|f\|_e \int \max(1, d(x,e)) d|\mu| \le \|f\|_e \|\mu\|_1.$$

$\mathcal{M}_1(S)$ is a subspace of $\mathcal{M}(S)$ and thus embeds into $\mathrm{BL}(S)^*$. The image of $\mu \in \mathcal{M}_1(S)$ in $\mathrm{BL}(S)^*$ coincides with the one obtained by mapping $\mathcal{M}_1(S)$ into $\mathrm{Lip}_e(S)^*$ and then restricting to $\mathrm{BL}(S)$. Therefore $\mu \mapsto I_\mu$ is injective. $\square$

Thus we can identify $\mu \in \mathcal{M}_1(S)$ with $I_\mu \in \mathrm{Lip}_e(S)^*$, and embed $\mathcal{M}_1(S)$ into $\mathrm{Lip}_e(S)^*$. When a functional $\varphi \in \mathrm{Lip}_e(S)^*$ can be represented by a measure in $\mathcal{M}_1(S)$, we shall write $\varphi \in \mathcal{M}_1(S)$.

We can embed S into $\mathcal{M}(S)$ or $\mathcal{M}_1(S)$, by sending x to the Dirac measure δ_x. This embedding is not continuous in general with respect to the total variation norm, since $\|\delta_x - \delta_y\|_{\mathrm{TV}} = 2$ whenever $x \neq y$. However, we do have an isometric embedding into $\mathrm{Lip}_e(S)^*$:

Lemma 3.4. *Let $x \in S$, then δ_x is in $\mathrm{Lip}_e(S)^*$ with $\|\delta_x\|_e^* = \max(1, d(x,e))$. The map $x \mapsto \delta_x$ is an isometric embedding from S into $\mathrm{Lip}_e(S)^*$.*

Proof. Let $f \in \mathrm{Lip}_e(S)$ and $x \in S$. Then Lemma 2.3 implies that $\|\delta_x\|_e^* \le \max(1, d(x,e))$. For the reverse estimate, consider $f(x) := d(x,e)$. Then $f \in \mathrm{Lip}_e(S)$ and $|f|_{\mathrm{Lip}} = 1$, according to Lemma 2.1. Hence $\|f\|_e = 1$, and $|\delta_x(f)| = d(x,e)$ for every $x \in S$. Also, the constant function $\mathbf{1} \in \mathrm{Lip}_e(S)$ and $\|\mathbf{1}\|_e = 1$. Furthermore, $|\delta_x(\mathbf{1})| = 1$. Hence $\|\delta_x\|_e^* \ge \max(1, d(x,e))$ and thus $\|\delta_x\|_e^* = \max(1, d(x,e))$.

Now, let $x, y \in S, x \neq y$ and $f \in \mathrm{Lip}_e(S)$. Then

$$|(\delta_x - \delta_y)(f)| = |f(x) - f(y)| \le |f|_{\mathrm{Lip}} d(x,y) \le \|f\|_e d(x,y).$$

Let $f(z) := d(x,z) - d(x,e)$. Then $|f|_{\mathrm{Lip}} = |d(x,\cdot)|_{\mathrm{Lip}} = 1$, $\|f\|_e = 1$ and $|\delta_x(f) - \delta_y(f)| = d(x,y)$. Hence $\|\delta_x - \delta_y\|_e^* = d(x,y)$ and $x \mapsto \delta_x$ is an isometric embedding from S into $\mathrm{Lip}_e(S)^*$. $\square$

The situation for the embedding of S into $\mathrm{BL}(S)^*$ is similar, though slightly different: the embedding is not isometric in general.

Lemma 3.5. *For every $x \in S$, δ_x is in $\mathrm{BL}(S)^*$, and $\|\delta_x\|_{\mathrm{BL}}^* = 1$. Furthermore for every $x, y \in S$,*

$$\|\delta_x - \delta_y\|_{\mathrm{BL}}^* = \frac{2d(x,y)}{2 + d(x,y)} \leq \min(2, d(x,y)). \tag{4}$$

Proof. Let $x \in S$ and $f \in \mathrm{BL}(S)$. Then $|\delta_x(f)| = |f(x)| \leq \|f\|_{\mathrm{BL}}$, hence $\|\delta_x\|_{\mathrm{BL}}^* \leq 1$. The constant function $\mathbf{1}$ is in $\mathrm{BL}(S)$ and $|\delta_x(\mathbf{1})| = 1 = \|\mathbf{1}\|_{\mathrm{BL}}$, so $\|\delta_x\|_{\mathrm{BL}}^* = 1$.

If $x = y$, then (4) is satisfied. Suppose $x \neq y$. Let $f \in \mathrm{BL}(S)$. Then

$$|f(x) - f(y)| \leq \min(|f|_{\mathrm{Lip}} d(x,y), 2\|f\|_\infty).$$

Hence

$$(2 + d(x,y))|f(x) - f(y)| \leq 2d(x,y)\|f\|_{\mathrm{BL}},$$

so

$$\|\delta_x - \delta_y\|_{\mathrm{BL}}^* = \sup_{\|f\|_{\mathrm{BL}} \leq 1} |f(x) - f(y)| \leq \frac{2d(x,y)}{2 + d(x,y)}.$$

Define $f(z) := \frac{d(z,y) - d(z,x)}{2 + d(x,y)}$. Then

$$|f|_{\mathrm{Lip}} \leq \frac{1}{2 + d(x,y)} |d(\cdot, y) - d(\cdot, x)|_{\mathrm{Lip}} \leq \frac{2}{2 + d(x,y)},$$

where we use that $|d(\cdot, x)|_{\mathrm{Lip}} = 1$, by Lemma 2.1. Since $|d(z,y) - d(z,x)| \leq d(x,y)$ for all $z \in S$, we can conclude that $\|f\|_\infty \leq \frac{d(x,y)}{2+d(x,y)}$. Hence $\|f\|_{\mathrm{BL}} \leq 1$. Furthermore

$$|\delta_x(f) - \delta_y(f)| = |f(x) - f(y)| = \frac{2d(x,y)}{2 + d(x,y)}.$$

Hence $\|\delta_x - \delta_y\|_{\mathrm{BL}}^* = \frac{2d(x,y)}{2+d(x,y)}$. ☐

Remark. Instead of the norms $\|\cdot\|_{\mathrm{BL}}$ and $\|\cdot\|_e$, we could also consider the equivalent norms $\|\cdot\|_{\mathrm{BL,max}}$ and $\|f\|_{e,\max} := \max(|f(e)|, |f|_{\mathrm{Lip}})$. Then Lemma 3.4 holds with $\|\cdot\|_e^*$ replaced by $\|\cdot\|_{e,\max}^*$. The corresponding statement to (4) in Lemma 3.5 for $\|\cdot\|_{\mathrm{BL,max}}^*$ norm is that $\|\delta_x - \delta_y\|_{\mathrm{BL,max}}^* = \min(2, d(x,y))$, which can be shown using the function $f(z) := \min(-1 + d(x,z), 1)$ if $d(x,y) < 2$ and $f(z) := \min(-1 + \frac{2d(x,z)}{d(x,y)}, 1)$ if $d(x,y) \geq 2$.

3.2 Predual of $\mathrm{Lip}_e(S)$ and $\mathrm{BL}(S)$

Let

$$D := \mathrm{span}\{\delta_x | x \in S\} = \left\{ \sum_{k=1}^n \alpha_k \delta_{x_k} : n \in \mathbb{N}, \alpha_k \in \mathbb{R}, x_k \in S \right\}.$$

We define $\mathcal{S}_e$ to be the closure of the linear subspace D in $\mathrm{Lip}_e(S)^*$ with respect to $\|\cdot\|_e^*$, and $\mathcal{S}_{\mathrm{BL}}$ to be the closure of D in $\mathrm{BL}(S)^*$ with respect to $\|\cdot\|_{\mathrm{BL}}^*$.

Theorem 3.6. *$\mathcal{S}_e^*$ is isometrically isomorphic to* $\mathrm{Lip}_e(S)$ *under the map $\psi \mapsto T\psi$, where $T\psi(x) := \psi(\delta_x)$.*

Proof. Since $\mathcal{S}_e \subset \mathrm{Lip}_e(S)^*$, we can define $R : \mathrm{Lip}_e(S) \to \mathcal{S}_e^*$ such that $Rf(\varphi) := \varphi(f)$ for all $\varphi \in \mathcal{S}_e$. Clearly $|Rf(\varphi)| \leq \|\varphi\|_e^* \|f\|_e$, hence $\|Rf\|_{\mathcal{S}_e^*} \leq \|f\|_e$.

Now define $T : \mathcal{S}_e^* \to \mathrm{Lip}_e(S)$ such that $T\psi(x) := \psi(\delta_x)$ for all $x \in S$. It can easily be verified that $T\psi$ is indeed in $\mathrm{Lip}_e(S)$, and that T is linear. Now we want to show that $\|T\psi\|_e \leq \|\psi\|_{\mathcal{S}_e^*}$. Let $x, y \in S$, $x \neq y$. Note that for real numbers a and b, $|a| + |b| = \max(|a - b|, |a + b|)$. Therefore, using the fact that we only consider real-valued Lipschitz functions and hence real Banach spaces,

$$\begin{aligned} |\psi(\delta_e)| + \frac{|\psi(\delta_x - \delta_y)|}{d(x,y)} &= \max\left(|\psi(\delta_e) - \frac{\psi(\delta_x - \delta_y)}{d(x,y)}|, |\psi(\delta_e) + \frac{\psi(\delta_x - \delta_y)}{d(x,y)}|\right) \\ &= \max\left(|\psi(\delta_e - \frac{\delta_x - \delta_y}{d(x,y)})|, |\psi(\delta_e + \frac{\delta_x - \delta_y}{d(x,y)})|\right) \\ &\leq \|\psi\|_{\mathcal{S}_e^*} \max\left(\|\delta_e - \frac{\delta_x - \delta_y}{d(x,y)}\|_e^*, \|\delta_e + \frac{\delta_x - \delta_y}{d(x,y)}\|_e^*\right) \end{aligned}$$

Now for all $f \in \mathrm{Lip}_e(S)$, with $\|f\|_e \leq 1$, we have

$$\begin{aligned} |(\delta_e - \frac{\delta_x - \delta_y}{d(x,y)})(f)| &= |f(e) - \frac{f(x) - f(y)}{d(x,y)}| \\ &\leq |f(e)| + \frac{|f(x) - f(y)|}{d(x,y)} \leq 1. \end{aligned}$$

And since $|(\delta_e - \frac{\delta_x - \delta_y}{d(x,y)})(\mathbf{1})| = 1$, $\|\delta_e - \frac{\delta_x - \delta_y}{d(x,y)}\|_e^* = 1$. By interchanging x and y, we also get $\|\delta_e + \frac{\delta_x - \delta_y}{d(x,y)}\|_e^* = 1$ Thus for all $x, y \in S$, $x \neq y$,

$$|\psi(\delta_e)| + \sup_{\substack{x,y \in S \\ x \neq y}} \frac{|\psi(\delta_x - \delta_y)|}{d(x,y)} \leq \|\psi\|_{\mathcal{S}_e^*}.$$

Consequently,

$$\|T\psi\|_e \leq \|\psi\|_{\mathcal{S}_e^*}, \text{ for all } \psi \in \mathcal{S}_e^*.$$

Now we need to show R and T are each other's inverses. Let $f \in \mathrm{Lip}_e(S)$, then

$$T(Rf)(x) = Rf(\delta_x) = f(x), \text{ for all } x \in S.$$

Hence $T \circ R = \mathrm{Id}_{\mathrm{Lip}_e(S)}$. Now let $\psi \in \mathcal{S}_e^*$, and let $d \in D$, then $d = \sum_{k=1}^n \alpha_k \delta_{x_k}$, for certain $\alpha_k \in \mathbb{R}$ and $x_k \in S$. Then

$$R(T\psi)(d) = \sum_{k=1}^n \alpha_k T\psi(x_k) = \sum_{k=1}^n \alpha_k \psi(\delta_{x_k}) = \psi(d).$$

Hence $R(T\psi) = \psi$ on a dense subset of $\mathcal{S}_e$, so $R(T\psi) = \psi$ on $\mathcal{S}_e$. Hence $R \circ T = \mathrm{Id}_{\mathcal{S}_e^*}$. Consequently we get that for all $f \in \mathrm{Lip}_e(S) : \|Rf\|_e^* \leq \|f\|_e = \|T(Rf)\|_e \leq \|Rf\|_e^*$, hence R is an isometric isomorphism from $\mathrm{Lip}_e(S)$ to $\mathcal{S}_e^*$, with T as its inverse. $\square$

A similar result holds for $\mathrm{BL}(S)$.

Theorem 3.7. *$\mathcal{S}_{\mathrm{BL}}^*$ is isometrically isomorphic to $\mathrm{BL}(S)$ under the map $\psi \mapsto T\psi$, where $T\psi(x) := \psi(\delta_x)$.*

Proof. We define $R : \mathrm{BL}(S) \to \mathcal{S}_{\mathrm{BL}}^*$ such that $Rf(\varphi) := \varphi(f)$ for all $\varphi \in \mathcal{S}_{\mathrm{BL}} \subset \mathrm{BL}(S)^*$. And we define $T : \mathcal{S}_{\mathrm{BL}}^* \to \mathrm{BL}(S)$ such that $T\psi(x) := \psi(\delta_x)$ for all $x \in S$. Then analogous to the proof of Theorem 3.6 we can show that $\|Rf\|_{\mathcal{S}_{\mathrm{BL}}^*} \leq \|f\|_e$, that $\|T\psi\|_{\mathrm{BL}} \leq \|\psi\|_{\mathcal{S}_{\mathrm{BL}}^*}$ and that R and T are each other's inverses. Hence R is an isometric isomorphism from $\mathrm{BL}(S)$ to $\mathcal{S}_{\mathrm{BL}}^*$, with T as its inverse. $\square$

3.3 Identification of $\mathcal{S}_{\mathrm{BL}}$

A Borel measure $\mu \in \mathcal{M}(S)$ is called *separable* if there is a separable Borel measurable subset E of S, such that μ is concentrated on E, i.e. $|\mu|(S\backslash E) = 0$. Let $\mathcal{M}_s(S)$ be the separable Borel measures on S, and $\mathcal{M}_s^+(S)$ the set of positive, finite and separable Borel measures on S. If S is separable, $\mathcal{M}_s(S) = \mathcal{M}(S)$. It is easy to see that $\mathcal{M}_s(S)$ is a closed subspace of $\mathcal{M}(S)$ with respect to $\|\cdot\|_{\mathrm{TV}}$.

Let

$$D^+ := \left\{ \sum_{i=1}^n \alpha_i \delta_{x_i} : n \in \mathbb{N}, \alpha_i \in \mathbb{R}_+, x_i \in S \right\}.$$

We define $\mathcal{S}_{\mathrm{BL}}^+$ to be the closure of D^+ with respect to $\|\cdot\|_{\mathrm{BL}}^*$. Notice that $\mathcal{S}_{\mathrm{BL}}^+ \subset \mathcal{S}_{\mathrm{BL}}$ and all $\varphi \in \mathcal{S}_{\mathrm{BL}}^+$ are positive: $\varphi(f) \geq 0$ for all $0 \leq f \in \mathrm{BL}(S)$.

We will need the following theorem, which is based on a result from [7]:

Theorem 3.8. *$\mathcal{M}_s^+(S)$ is norm closed in $\mathrm{BL}(S)^*$ if and only if S is complete.*

Proof. If S is complete, then $\mathcal{M}_s^+(S)$ is norm closed in $\mathrm{BL}(S)^*$ by [7, Theorem 9]. Suppose S is not complete. Then there exists a Cauchy sequence $(x_n)_n$ in S that does not converge to an element in S. Then $(x_n)_n$ cannot have a convergent subsequence. This implies that for every $x \in S$ there must be an $\epsilon > 0$ and an $M \in \mathbb{N}$, such that $d(x, x_m) \geq \epsilon$ for all $m \in \mathbb{N}, m \geq M$, otherwise $(x_n)_{n\in\mathbb{N}}$ has a subsequence that converges to x.

We will show that $\mathcal{M}_s^+(S)$ cannot be norm closed in $\mathrm{BL}(S)^*$. By Lemma 3.5 δ_{x_n} is a Cauchy sequence in $\mathrm{BL}(S)^*$. Now assume there is a $\mu \in \mathcal{M}_s^+(S)$, such that $\|\delta_{x_n} - \mu\|_{\mathrm{BL}}^* \to 0$. Then

$$\|\mu\|_{\mathrm{BL}}^* = \lim_{n\to\infty} \|\delta_{x_n}\|_{\mathrm{BL}}^* = 1.$$

We will show that μ must be zero, which gives a contradiction. We can assume, by taking a subsequence, that $\|\delta_{x_n} - \mu\|_{\mathrm{BL}}^* < \frac{1}{n^2}$. Now define $f_n(x) := \min(nd(x, x_n), 1)$. Then $f_n \in \mathrm{BL}(S)$, with $|f_n|_{\mathrm{Lip}} \leq n$ and $\|f_n\|_\infty \leq 1$. Hence

$$|\int f_n \, d\mu| = |\delta_{x_n}(f_n) - \int f_n \, d\mu| < \frac{n+1}{n^2} \to 0 \text{ as } n \to \infty.$$

Now let $x \in S$. Then there exists an $\epsilon > 0$ and an $M \in \mathbb{N}$, such that $d(x, x_m) \geq \epsilon$ for all $m \in \mathbb{N}, m \geq M$. This implies that $f_n(x) \to 1$ as $n \to \infty$. Hence, by the Lebesgue Dominated Convergence Theorem,

$$|\int \mathbf{1}\, d\mu| = |\lim_{n\to\infty} \int f_n\, d\mu| = 0,$$

which implies that μ is zero. □

Our main result in this section is the following theorem:

Theorem 3.9. $\mathcal{M}_s^+(S) \subset \mathcal{S}_{\mathrm{BL}}^+$. *Furthermore,* $\mathcal{S}_{\mathrm{BL}}^+ = \mathcal{M}_s^+(S)$ *if and only if* S *is complete.*

Proof. First we show that $\mathcal{M}_s^+(S) \subset \mathcal{S}_{\mathrm{BL}}^+$. Let $\mu \in \mathcal{M}_s^+(S)$, and let E be a measurable separable subset of S on which μ is concentrated. We want to show that there is an element $\varphi \in \mathcal{S}_{\mathrm{BL}}^+$ such that $\varphi(f) = \int f\, d\mu$ for all $f \in \mathrm{BL}(S)$. If $\mu(S) = 0$ this is clear, so we assume $\mu(S) > 0$.

We define the map $\delta : S \to \mathcal{S}_{\mathrm{BL}}$, sending x to δ_x. Then δ is Lipschitz continuous by Lemma 3.5. Also, since E is separable and δ is continuous, $\delta(E)$ is a separable subset of $\mathcal{S}_{\mathrm{BL}}$. Because $\mu(S\backslash E) = 0$, δ is μ-essentially separably valued. For any $f \in \mathrm{BL}(S) \cong \mathcal{S}_{\mathrm{BL}}^*$ the function $x \mapsto \langle \delta_x, f\rangle = f(x)$ is measurable, so $x \mapsto \delta_x$ is weakly measurable. By the Pettis Measurability Theorem (e.g. [6, Theorem 2]), δ is strongly μ-measurable. Furthermore,

$$\int \|\delta_x\|_{\mathrm{BL}}^* d\mu(x) = \int d\mu < \infty,$$

therefore $\delta : S \to \mathcal{S}_{\mathrm{BL}}$ is μ-Bochner integrable and $\int \delta_x d\mu(x)$ defines an element in $\mathcal{S}_{\mathrm{BL}}$. By [6, Corollary 8] we get that

$$\frac{1}{\mu(S)} \int \delta_x d\mu(x) \in \overline{\mathrm{conv}}\{\delta_x : x \in E\} \subset \mathcal{S}_{\mathrm{BL}}^+.$$

Hence $\int \delta_x d\mu(x) \in \mathcal{S}_{\mathrm{BL}}^+$. Furthermore, by [6, Theorem 6] we obtain for all $f \in \mathrm{BL}(S)$ that $\langle \int \delta_x d\mu(x), f\rangle = \int \langle \delta_x, f\rangle d\mu(x) = \int f d\mu$. This implies $\int \delta_x d\mu(x)$ is a functional in $\mathcal{S}_{\mathrm{BL}}^+$ represented by μ. Thus $\mathcal{M}_s^+(S) \subset \mathcal{S}_{\mathrm{BL}}^+$.

Now assume S is complete. It is clear that for all $x \in S$, $\delta_x \in \mathcal{M}_s^+(S)$. Hence $D^+ \subset \mathcal{M}_s^+(S)$. From Theorem 3.8 we obtain that $\mathcal{M}_s^+(S)$ is norm closed in $\mathrm{BL}(S)^*$, hence $\mathcal{S}_{\mathrm{BL}}^+ \subset \mathcal{M}_s^+(S)$. If S is not complete, then by Theorem 3.8, $\mathcal{M}_s^+(S)$ is not norm closed in $\mathrm{BL}(S)^*$, which implies that $\mathcal{M}_s^+(S) \subsetneq \mathcal{S}_{\mathrm{BL}}^+$. □

The crucial observation towards identification of $\mathcal{S}_{\mathrm{BL}}$ is the following:

Corollary 3.10. $\mathcal{M}_s(S)$ *is a* $\|\cdot\|_{\mathrm{BL}}^*$*-dense subspace of* $\mathcal{S}_{\mathrm{BL}}$.

One might ask when $\mathcal{S}_{\mathrm{BL}} = \mathcal{M}_s(S)$. To answer this question we need the notion of a uniformly discrete metric space. S is *uniformly discrete* if there is an $\epsilon > 0$ such that $d(x, y) > \epsilon$ for all $x, y \in S$, $x \neq y$. The following theorem settles our question:

Theorem 3.11. *$\mathcal{M}_s(S)$ is norm closed in $\mathrm{BL}(S)^*$ if and only if S is uniformly discrete.*

Proof. Suppose $\overline{\mathcal{M}_s(S)}^{\|\cdot\|_{\mathrm{BL}}^*} = \mathcal{M}_s(S)$. Then $(\mathcal{M}_s(S), \|\cdot\|_{\mathrm{BL}}^*)$ is a Banach space. Let I be the identity map from $(\mathcal{M}_s(S), \|\cdot\|_{\mathrm{TV}})$ to $(\mathcal{M}_s(S), \|\cdot\|_{\mathrm{BL}}^*)$. Then, since $\|\mu\|_{\mathrm{BL}}^* \leq \|\mu\|_{\mathrm{TV}}$, I is a bounded linear map. Clearly, I is bijective, hence by the Inverse Mapping Theorem the inverse of I is a bounded linear map.

Assume S is not uniformly discrete, then there are $x_n, y_n \in S$, such that $0 < d(x_n, y_n) < \frac{1}{n}$. Let $\mu_n = \delta_{x_n} - \delta_{y_n}$. Then $\|\mu_n\|_{\mathrm{TV}} = 2$, while $\|\mu_n\|_{\mathrm{BL}}^* \leq d(x_n, y_n) < \frac{1}{n}$, for all $n \in \mathbb{N}$. This implies I^{-1} cannot be bounded, which gives us a contradiction. Hence S must be uniformly discrete.

Now suppose S is uniformly discrete. Then there is an $\epsilon > 0$ such that $d(x,y) > \epsilon$ for all $x, y \in S$, $x \neq y$. Let $\mu \in \mathcal{M}_s(S)$. Let $S = P \cup N$ be the Hahn decomposition of S corresponding to μ, then $\mu^+ = \mu|_P$ and $\mu^- = \mu|_N$. Define

$$f(x) := \begin{cases} \min(\epsilon/4, 1/2) & \text{if } x \in P; \\ -\min(\epsilon/4, 1/2) & \text{if } x \in N. \end{cases}$$

Then $\|f\|_\infty \leq 1/2$ and

$$|f|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} \leq \frac{\epsilon/2}{\epsilon} = \frac{1}{2}.$$

Hence $\|f\|_{\mathrm{BL}} \leq 1$. Furthermore,

$$\begin{aligned} |\int f\, d\mu| &= |\int_P \min(\epsilon/4, 1/2) d\,\mu - \int_N \min(\epsilon/4, 1/2) d\mu| \\ &= |\mu^+(S) + \mu^-(S)| \min(\epsilon/4, 1/2) = \|\mu\|_{\mathrm{TV}} \min(\epsilon/4, 1/2). \end{aligned}$$

Hence

$$\|\mu\|_{\mathrm{TV}} \leq \|\mu\|_{\mathrm{BL}}^* \frac{1}{\min(\epsilon/4, 1/2)},$$

for all $\mu \in \mathcal{M}_s(S)$. Also, $\|\mu\|_{\mathrm{BL}}^* \leq \|\mu\|_{\mathrm{TV}}$ for all $\mu \in \mathcal{M}_s(S)$, hence the norms $\|\cdot\|_{\mathrm{BL}}^*$ and $\|\cdot\|_{\mathrm{TV}}$ are equivalent on $\mathcal{M}_s(S)$. This implies that

$$\overline{\mathcal{M}_s(S)}^{\|\cdot\|_{\mathrm{BL}}^*} = \overline{\mathcal{M}_s(S)}^{\|\cdot\|_{TV}} = \mathcal{M}_s(S).$$

□

Remark. Note that all the arguments in the proof of Theorem 3.11 hold when we replace $\mathcal{M}_s(S)$ by $\mathcal{M}(S)$. Hence $\overline{\mathcal{M}(S)}^{\|\cdot\|_{\mathrm{BL}}^*} = \mathcal{M}(S)$ if and only if S is uniformly discrete.

Corollary 3.12. *If S is not uniformly discrete, there are elements in $\mathcal{S}_{\mathrm{BL}}$, hence in $\mathrm{BL}(S)^*$, that cannot be represented by a measure in $\mathcal{M}(S)$.*

3.4 Identification of $\mathcal{S}_e$

We start with the observation that each $\varphi \in \mathcal{S}_e$ is completely determined by its restriction to $\mathrm{BL}(S)$; more precise:

Lemma 3.13. *Let $\varphi \in \mathcal{S}_e$, $f \in \mathrm{Lip}_e(S)$. Define $f_n(x) := \max(\min(f(x), n), -n)$. Then $\lim_{n\to\infty} \varphi(f_n) = \varphi(f)$.*

Proof. Obviously, $\|f_n\|_e \leq \|f\|_e$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Then there is a $d \in D$ such that $\|\varphi - d\|_e^* < \frac{\epsilon}{2(\|f\|_e+1)}$. Let N_d be such that $d(f - f_n) = 0$ for all $n \geq N_d$. Then for $n \geq N_d$ we have

$$\begin{aligned} |\varphi(f) - \varphi(f_n)| &\leq |\varphi(f) - d(f)| + |d(f) - d(f_n)| + |\varphi(f_n) - d(f_n)| \\ &\leq 2\|\varphi - d\|_e^* \|f\|_e < \epsilon. \end{aligned}$$

Hence $\lim_{n\to\infty} \varphi(f_n) = \varphi(f)$. □

Just as before, we restrict to the separable Borel measures: Let $\mathcal{M}_{s,1}(S) := \mathcal{M}_1(S) \cap \mathcal{M}_s(S)$, and $\mathcal{M}_{s,1}^+(S) := \mathcal{M}_s(S) \cap \mathcal{M}_+^1(S)$. Similar to $\mathcal{S}_{\mathrm{BL}}^+$, we define $\mathcal{S}_e^+$ to be the closure of D^+ with respect to $\|\cdot\|_e^*$.

Now we can prove the analogue to Theorem 3.9:

Theorem 3.14. *$\mathcal{M}_{s,1}^+(S) \subset \mathcal{S}_e^+$. Furthermore, $\mathcal{S}_e^+ = \mathcal{M}_{s,1}^+(S)$ if and only if S is complete.*

Proof. First we will show that $\mathcal{M}_{s,1}^+(S) \subset \mathcal{S}_e^+$. Let $\mu \in \mathcal{M}_{s,1}^+(S)$ and define $\delta : S \to \mathrm{Lip}_e(S)^*$, $x \mapsto \delta_x$. Then we can prove, using similar techniques as in the proof of Theorem 3.9, that δ is μ-Bochner integrable, that $\int \delta_x \, d\mu(x) \in \mathcal{S}_e^+$ and that $\langle \int \delta_x \, d\mu(x), f\rangle = \int f d\mu$ for all $f \in \mathrm{Lip}_e(S)$. This implies that $\mathcal{M}_{s,1}^+(S) \subset \mathcal{S}_e^+$.

Now suppose that S is complete. It is clear that $D^+ \subset \mathcal{M}_{s,1}^+(S)$. Let $\varphi \in \mathcal{S}_e^+$, then there are $d_n \in D^+$ such that $\|\varphi - d_n\|_e^* \to 0$. Because

$$\|j^*(\varphi) - d_n\|_{\mathrm{BL}}^* = \|j^*(\varphi - d_n)\|_{\mathrm{BL}}^* \leq \|\varphi - d_n\|_e^* \to 0,$$

there is a $\mu \in \mathcal{M}_s^+(S)$, according to Theorem 3.9, such that $j^*(\varphi(f)) = \varphi(f) = \int f d\,\mu$ for all $f \in \mathrm{BL}(S)$. We need to show that $\mu \in \mathcal{M}_{s,1}^+(S)$ and $\varphi(f) = \int f d\,\mu$ for all $f \in \mathrm{Lip}_e(S)$.

Let $f \in \mathrm{Lip}_e(S)$, $f \geq 0$. Then

$$\varphi(f) = \lim_{n\to\infty} \varphi(f_n) = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu < \infty$$

by Lemma 3.13 and the Monotone Convergence Theorem. In particular, $\int d(x,e) d\mu = \varphi(d(\cdot, e)) < \infty$, hence $\mu \in \mathcal{M}_{s,1}^+(S)$.

Using $f = f^+ - f^-$ for general $f \in \mathrm{Lip}_e(S)$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, we find that $f \in L^1(\mu)$ and $\varphi(f) = \int f d\mu$ for every $f \in \mathrm{Lip}_e(S)$. Hence $\mathcal{S}_e^+ \subset \mathcal{M}_{s,1}^+(S)$.

Now suppose S is not complete. Then there is a Cauchy sequence $(x_n)_n$ in S that does not converge to an element in S. This implies by Lemma 3.4 that $(\delta_{x_n})_n$ is a Cauchy sequence in $\mathrm{Lip}_e(S)^*$. Suppose that $\mu \in \mathcal{M}^+_{s,1}(S)$ is such that $\|\delta_{x_n} - \mu\|^*_e \to 0$. Then $\|\delta_{x_n} - \mu\|^*_{\mathrm{BL}} \to 0$, but from the proof of Theorem 3.8 it follows that this is not possible. Hence $\mathcal{M}^+_{s,1}(S)$ is not norm closed in $\mathrm{Lip}_e(S)$, and since $\mathcal{M}^+_{s,1}(S) \subset \mathcal{S}^+_e$, this implies that $\mathcal{M}^+_{s,1}(S) \subsetneq \mathcal{S}^+_e$. □

The following corollaries follows easily from Theorem 3.14:

Corollary 3.15. *$\mathcal{M}_{s,1}(S)$ is a $\|\cdot\|^*_e$-dense subspace of $\mathcal{S}_e$.*

Corollary 3.16. *$\mathcal{M}^+_{s,1}(S)$ is norm closed in $\mathcal{S}_e$ if and only if S is complete.*

Remark. In [16, Theorem 4.2] it is shown that the metric space S is complete if and only if the set of separable probability measures of finite first moment, $\mathcal{P}_{s,1}(S)$, is complete with respect to the metric H, where

$$H(\mu,\nu) = \sup_{\substack{f\in\mathrm{Lip}(S)\\ |f|_{\mathrm{Lip}}\le 1}} \left|\int f d\mu - \int f d\nu\right|.$$

From Corollary 3.16 we can also conclude this theorem: it follows that when S is complete, the subset of separable probability measures of finite first moment $\mathcal{P}^{1,s}(S)$ is also a closed set of $\mathcal{S}_e$, hence complete with respect to $\|\cdot\|^*_e$. Let $\mu,\nu \in \mathcal{P}^{1,s}(S)$, then $\|\mu-\nu\|^*_e$ is equal to $H(\mu,\nu)$, since for $f \in \mathrm{Lip}(S)$ with $|f|_{\mathrm{Lip}} \le 1$ we have

$$\left|\int f\, d\mu - \int f\, d\nu\right| = \left|\int f - f(e) d\,\mu - \int f - f(e) d\,\mu\right|,$$

and $g(x) := f(x) - f(e)$ satisfies: $\|g\|_e = |f|_{\mathrm{Lip}} \le 1$. Furthermore, when S is not complete, $\mathcal{M}^+_{s,1}(S)$ is not complete with respect to $\|\cdot\|^*_e$. Then it is not difficult to see that then $\mathcal{P}^{1,s}(S)$ also cannot be complete with respect to $\|\cdot\|^*_e$, hence it is not complete with respect to H.

Recall the natural embedding $j : \mathrm{BL}(S) \to \mathrm{Lip}_e(S)$, and the adjoint $j^* : \mathrm{Lip}_e(S)^* \to \mathrm{BL}(S)^*$. Then, as a consequence of Proposition 2.6, j^* is not injective whenever S has infinite diameter. Consider however the restriction j^*_e of j^* to $\mathcal{S}_e$.

Lemma 3.17. *j^*_e maps $\mathcal{S}_e$ injectively and densely into $\mathcal{S}_{\mathrm{BL}}$.*

Proof. Let $\phi \in \mathcal{S}_e$ be such that $j^*_e(\phi) = 0$. Then $\phi(f) = 0$ for all $f \in \mathrm{BL}(S)$. Hence Lemma 3.13 implies that $\phi(f) = 0$ for all $f \in \mathrm{Lip}_e(S)$, hence $\phi = 0$. So j^*_e is injective. By continuity of j^*_e,

$$j^*_e(\mathcal{S}_e) = j^*_e(\overline{D}^{\|\cdot\|^*_e}) \subset \overline{j^*_e(D)}^{\|\cdot\|^*_{\mathrm{BL}}} = \overline{D}^{\|\cdot\|^*_{\mathrm{BL}}} = \mathcal{S}_{\mathrm{BL}}.$$

So we can continuously embed $\mathcal{S}_e$ into $\mathcal{S}_{\mathrm{BL}}$ and $j^*_e(\mathcal{S}_e)$ is dense in $\mathcal{S}_{\mathrm{BL}}$, since $j^*_e(D) = D$ is dense in $\mathcal{S}_{\mathrm{BL}}$. □

4 Positivity

We can endow $\mathrm{BL}(S)$ and $\mathrm{Lip}_e(S)$ with pointwise ordering, so $f \geq g$ if $f(x) \geq g(x)$ for all $x \in S$. From Lemma 2.2 it follows that $\mathrm{BL}(S)$ and $\mathrm{Lip}_e(S)$ are Riesz spaces with respect to this ordering. However, $\|\cdot\|_{\mathrm{BL}}$ and $\|\cdot\|_e$ are not Riesz norms, since $|f| \leq |g|$ need not imply that $|f|_{\mathrm{Lip}} \leq |g|_{\mathrm{Lip}}$. We are interested in the question whether all the positive functionals

$$\mathrm{BL}(S)^*_+ := \{\phi \in \mathrm{BL}(S)^* : \phi(f) \geq 0 \text{ for all } f \in \mathrm{BL}(S), f \geq 0\}$$

can be represented by measures on S.

Let $C_{ub}(S)$ denote the Banach space of bounded uniformly continuous real-valued functions on S, with the supremum norm $\|\cdot\|_\infty$. Then $\mathrm{BL}(S) \subset C_{ub}(S)$ is dense [7, Lemma 8]. Let $\phi \in \mathrm{BL}(S)^*_+$. Then

$$|\phi(f)| = |\phi(f^+) - \phi(f^-)| \leq \phi(|f|) \leq \phi(\|f\|_\infty \cdot \mathbf{1}) = \|f\|_\infty \phi(\mathbf{1}) \tag{5}$$

by positivity of ϕ. This ϕ can be uniquely extended to a positive continuous linear functional on $C_{ub}(S)$.

Let S be complete. If S is compact, then $C_{ub}(S) = C(S)$, and by the Riesz representation theorem, every $\phi \in C_{ub}(S)^*$ can be represented by a measure. If S is not compact, then $C_0(S) \subsetneq C_{ub}(S)$, and it is possible to show the existence of a non-zero functional $\phi \in C_{ub}(S)^*_+$, such that $\phi|_{C_0(S)} = 0$, which implies that ϕ cannot be represented by a measure. However, when we also demand that ϕ is in $\mathcal{S}_{\mathrm{BL}}$, it *can* be represented by a measure, by a corollary of the following theorem:

Theorem 4.1. $\mathcal{S}_{\mathrm{BL}} \cap \mathrm{BL}(S)^*_+ = \mathcal{S}^+_{\mathrm{BL}}$.

Proof. Clearly, $\mathcal{S}^+_{\mathrm{BL}} \subset \mathcal{S}_{\mathrm{BL}} \cap \mathrm{BL}(S)^*_+$. Suppose that there exists a $\phi \in \mathcal{S}_{\mathrm{BL}} \cap \mathrm{BL}(S)^*_+$ such that $\phi \notin \mathcal{S}^+_{\mathrm{BL}}$. If $\phi(\mathbf{1}) = 0$, then $\phi(f) = 0$ for every $f \in \mathrm{BL}(S)$, by positivity of ϕ and (5), hence $\phi \in \mathcal{S}^+_{\mathrm{BL}}$. So $\phi(\mathbf{1}) > 0$. Let

$$M := \left\{ \sum_{i=1}^n \alpha_i \delta_{x_i} : n \in \mathbb{N}, 0 \leq \alpha_i \leq \phi(\mathbf{1}), x_i \in S, \text{ for } i = 1, ..., n \right\},$$

then $M \subset \mathcal{S}^+_{\mathrm{BL}}$. Let $\overline{M}$ be the closure of M in $\mathcal{S}^+_{\mathrm{BL}}$ with respect to $\|\cdot\|^*_{\mathrm{BL}}$. By assumption, ϕ is not in $\overline{M}$. Since M is convex, $\overline{M}$ is a closed convex subset of $\mathcal{S}_{\mathrm{BL}}$. Thus ϕ is *strictly separated* from $\overline{M}$ by [4, Corollary 3.10]: there is an $f \in \mathcal{S}^*_{\mathrm{BL}} = \mathrm{BL}(S)$ and an $\alpha \in \mathbb{R}$, such that $\langle m, f \rangle < \alpha$ for all $m \in \overline{M}$, and $\langle \phi, f \rangle = \phi(f) > \alpha$. Clearly $\phi(\mathbf{1})\delta_x \in \overline{M}$ for all $x \in S$, hence

$$\langle \phi(\mathbf{1})\delta_x, f \rangle = \phi(\mathbf{1}) f(x) < \alpha \text{ for all } x \in S.$$

So $f < \frac{\alpha}{\phi(\mathbf{1})}$ and by positivity of ϕ,

$$\phi(f) < \phi(\frac{\alpha \mathbf{1}}{\phi(\mathbf{1})}) = \alpha,$$

which is a contradiction. So $\mathcal{S}_{\mathrm{BL}} \cap \mathrm{BL}(S)^*_+ = \mathcal{S}^+_{\mathrm{BL}}$. $\square$

From Theorem 3.9 and Theorem 4.1 we get the following result:

Corollary 4.2. *$\mathcal{M}_s^+(S) \subset \mathcal{S}_{\mathrm{BL}} \cap \mathrm{BL}(S)^*_+$, and $\mathcal{S}_{\mathrm{BL}} \cap \mathrm{BL}(S)^*_+ = \mathcal{M}_s^+(S)$ if and only if S is complete.*

The following theorem can be proved similarly to Theorem 4.1:

Theorem 4.3. *$\mathcal{S}_e \cap \mathrm{Lip}_e(S)^*_+ = \mathcal{S}_e^+$.*

And the following corollary follows from Theorem 3.14 and Theorem 4.3:

Corollary 4.4. *$\mathcal{M}_{s,1}^+(S) \subset \mathcal{S}_e \cap \mathrm{Lip}_e(S)^*_+$, and $\mathcal{S}_e \cap \mathrm{Lip}_e(S)^*_+ = \mathcal{M}_{s,1}^+(S)$ if and only if S is complete.*

We have seen in Lemma 3.17 that $\mathcal{S}_e$ can be considered as a dense subspace of $\mathcal{S}_{\mathrm{BL}}$. The closed convex cones $\mathcal{S}_e^+$ and $\mathcal{S}_{\mathrm{BL}}^+$ in both spaces relate as follows:

Proposition 4.5. *$\mathcal{S}_{\mathrm{BL}}^+ \cap \mathcal{S}_e = \mathcal{S}_e^+$.*

Proof. Using Theorem 4.1, we obtain

$$\begin{aligned}\mathcal{S}_{\mathrm{BL}}^+ \cap \mathcal{S}_e &= \mathrm{BL}(S)^*_+ \cap \mathcal{S}_e \\ &= \{\phi \in \mathcal{S}_e : \phi(f) \geq 0, \text{ for all } 0 \leq f \in \mathrm{BL}(S)\} =: P.\end{aligned}$$

Now, if $\phi \in \mathcal{S}_e$ is such that $\phi(f) \geq 0$ for all positive $f \in \mathrm{BL}(S)$, then, by Lemma 3.13, $\phi(g) \geq 0$ for all positive $g \in \mathrm{Lip}_e(S)$. Hence $\phi \in \mathcal{S}_e^+$ and $P \subset \mathcal{S}_e^+$. Clearly $\mathcal{S}_e^+ \subset P$. □

The closed convex cone $\mathcal{S}_{\mathrm{BL}}^+$ defines a partial ordering '$\geq$' on $\mathcal{S}_{\mathrm{BL}}$ by means of $\phi \geq \psi$ if and only if $\phi - \psi \in \mathcal{S}_{\mathrm{BL}}^*$. Then $(\mathcal{S}_{\mathrm{BL}}, \geq)$ is an ordered Banach space. In a similar fashion, $\mathcal{S}_e^+$ introduces a partial ordering in $\mathcal{S}_e$. Proposition 4.5 implies that both orderings are compatible and obtained from the ordering in $\mathrm{BL}(S)^*_+$ and $\mathrm{Lip}_e(S)^*_+$ according to Theorem 4.1 and Theorem 4.3 respectively.

Note that $\mathcal{S}_{\mathrm{BL}}^+$ is not a generating cone in $\mathcal{S}_{\mathrm{BL}}$, unless S is uniformly discrete (Theorem 3.11).

5 Embedding into positive linear semigroups on dual Lipschitz spaces

Let $\mathrm{Lip}(S,S)$ be the space of Lipschitz maps on S. For $T \in \mathrm{Lip}(S,S)$, we define

$$|T|_{\mathrm{Lip}} := \sup\left\{\frac{d(T(x),T(y))}{d(x,y)} : x,y \in S, x \neq y\right\}.$$

Lemma 5.1. *Let $T \in \mathrm{Lip}(S,S)$. For any $f \in \mathrm{Lip}_e(S)$,*

$$\|f \circ T\|_e \leq \max(1, d(e,T(e)) + |T|_{\mathrm{Lip}})\|f\|_e,$$

and for $g \in \mathrm{BL}(S)$,

$$\|g \circ T\|_{\mathrm{BL}} \leq \max(1, |T|_{\mathrm{Lip}})\|g\|_{\mathrm{BL}}.$$

Proof. It is easy to check that for $f \in \mathrm{Lip}_e(S)$, $|f \circ T|_{\mathrm{Lip}} \leq |f|_{\mathrm{Lip}}|T|_{\mathrm{Lip}}$, hence we have

$$\begin{aligned} \|f \circ T\|_e &\leq |f(T(e))| + |f|_{\mathrm{Lip}}|T|_{\mathrm{Lip}} \\ &\leq |f(e)| + |f|_{\mathrm{Lip}} d(e, T(e)) + |f|_{\mathrm{Lip}}|T|_{\mathrm{Lip}} \\ &\leq \max(1, d(e, T(e)) + |T|_{\mathrm{Lip}})\|f\|_e. \end{aligned}$$

And for $g \in \mathrm{BL}(S)$, we have

$$\begin{aligned} \|g \circ T\|_{\mathrm{BL}} &\leq \|g \circ T\|_\infty + |g|_{\mathrm{Lip}}|T|_{\mathrm{Lip}} \\ &\leq \|g\|_\infty + |g|_{\mathrm{Lip}}|T|_{\mathrm{Lip}} \leq \max(1, |T|_{\mathrm{Lip}})\|g\|_{\mathrm{BL}}. \end{aligned}$$

. □

Definition 5.2. *A family of maps $(\Phi_t)_{t\geq 0}$ from S into S is a* Lipschitz semigroup *on S if*

(i) for all $t \geq 0$, $\Phi_t \in \mathrm{Lip}(S, S)$,

(ii) for all $s, t \geq 0$, $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and $\Phi_0 = \mathrm{Id}_S$.

A Lipschitz semigroup $(\Phi_t)_{t\geq 0}$ on S is called *strongly continuous* if $t \mapsto \Phi_t(x)$ is continuous at $t = 0$ for all $x \in S$. From property *(ii)* it then follows that $t \mapsto \Phi_t(x)$ is continuous on $\mathbb{R}_+$ for all $x \in S$.

Let $(\Phi_t)_{t\geq 0}$ be a Lipschitz semigroup on S. Then we define a semigroup of operators on $\mathrm{Lip}_e(S)$. Let $f \in \mathrm{Lip}_e(S)$ and $t \geq 0$, and let $S_\Phi(t)f := f \circ \Phi_t$. Then $S_\Phi(t)$ is a bounded linear operator on $\mathrm{Lip}_e(S)$, by Lemma 5.1, and $\|S_\Phi(t)\|_{\mathcal{L}(\mathrm{Lip}_e(S))} \leq \max(1, d(e, \Phi_t(e)) + |\Phi_t|_{\mathrm{Lip}})$. Hence $(S_\Phi(t))_{t\geq 0}$ is a semigroup of bounded linear operators on $\mathrm{Lip}_e(S)$.

So the dual operators $(S^*_\Phi(t))_{t\geq 0}$ form a semigroup of bounded linear operators on $\mathrm{Lip}_e(S)^*$.

Lemma 5.3. $S^*_\Phi(t)(\mathcal{S}_e) \subset \mathcal{S}_e$.

Proof. Let $f \in \mathrm{Lip}_e(S)$. Then

$$(S^*_\Phi(t)\delta_x)(f) = \delta_x(S_\Phi(t)f) = \delta_x(f \circ \Phi_t) = f(\Phi_t(x)) = \delta_{\Phi_t}(f), \tag{6}$$

for all $x \in S, t \geq 0$. Thus $S^*_\Phi(t)(D) \subset D$. Hence, by continuity of $S^*_\Phi(t)$, $S^*_\Phi(t)(\mathcal{S}_e) \subset \mathcal{S}_e$. □

Thus we can define a semigroup $(\hat{T}_\Phi(t))_{t\geq 0}$ of bounded linear operators on $\mathcal{S}_e$ by setting

$$\hat{T}_\Phi(t)\varphi := S^*_\Phi(t)\varphi, \text{ for all } \varphi \in \mathcal{S}_e, t \geq 0.$$

Theorem 5.4. *For all $x, y \in S$ and $s, t \geq 0$,*

$$d(\Phi_s(x), \Phi_t(y)) = \|\hat{T}_\Phi(s)\delta_x - \hat{T}_\Phi(t)\delta_y\|^*_e. \tag{7}$$

Furthermore, the following are equivalent:

(i) $(\hat{T}_\Phi(t))_{t\geq 0}$ *is a strongly continuous semigroup on* $\mathcal{S}_e$.

(ii) $(\Phi_t)_{t\geq 0}$ *is strongly continuous and* $\limsup_{t\downarrow 0} |\Phi_t|_{\mathrm{Lip}} < \infty$.

(iii) $(\Phi_t)_{t\geq 0}$ *is strongly continuous and there exist* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that* $|\Phi_t|_{\mathrm{Lip}} \leq Me^{\omega t}$ *for all* $t \geq 0$.

Proof. From Lemma 3.4 and (6) we get that for every $x, y \in S$ and $t, s \geq 0$

$$\begin{aligned} \|\hat{T}_\Phi(s)\delta_x - \hat{T}_\Phi(t)\delta_y\|_e^* &= \|\delta_{\Phi_s(x)} - \delta_{\Phi_t(y)}\|_e^* \\ &= d(\Phi_s(x), \Phi_t(y)). \end{aligned}$$

(i) $\Rightarrow$ *(iii)*: There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|\hat{T}_\Phi(t))\|_{\mathcal{L}(\mathcal{S}_e)} \leq Me^{\omega t}$ for all $t \geq 0$. Hence it follows from (7) that for $x, y \in S$ and $t \geq 0$,

$$\begin{aligned} d(\Phi_t(x), \Phi_t(y)) &= \|\hat{T}_\Phi(t)\delta_x - \hat{T}_\Phi(t)\delta_y\|_e^* \\ &\leq Me^{\omega t}\|\delta_x - \delta_y\|_e^* = Me^{\omega t} d(x, y). \end{aligned}$$

Hence $|\Phi_t|_{\mathrm{Lip}} \leq Me^{\omega t}$ for all $t \geq 0$. From (7) and strong continuity of $(\hat{T}_\Phi(t))_{t\geq 0}$ it follows that $(\Phi_t)_{t\geq 0}$ is strongly continuous.

(iii) $\Rightarrow$ *(ii)*: This is trivial.

(ii) $\Rightarrow$ *(i)*: We want to show that there is a $\delta > 0$ and an $M \geq 1$ such that $\sup_{0\leq t\leq\delta} \|\hat{T}_\Phi(t)\|_{\mathcal{L}(\mathcal{S}_e)} \leq M$, and that $(\hat{T}_\Phi(t))_{t\geq 0}$ is strongly continuous on D. Then we can conclude by [10, Proposition 5.3] that $(\hat{T}_\Phi(t))_{t\geq 0}$ is strongly continuous on $\mathcal{S}_e$, since D is dense in $\mathcal{S}_e$ by definition. Since $\limsup_{t\downarrow 0} |\Phi_t|_{\mathrm{Lip}} < \infty$, there exist $M_1, \delta > 0$ such that $|\Phi_t|_{\mathrm{Lip}} \leq M_1$ for all $0 \leq t \leq \delta$. We know that

$$\begin{aligned} \|\hat{T}_\Phi(t)\|_{\mathcal{L}(\mathcal{S}_e)} &\leq \|S_\Phi^*(t)\|_{\mathcal{L}(\mathrm{Lip}_e(S)^*)} \\ &= \|S_\Phi(t)\|_{\mathcal{L}(\mathrm{Lip}_e(S))} \leq \max(1, d(e, \Phi_t(e)) + |\Phi_t|_{\mathrm{Lip}}). \end{aligned}$$

Now, since $[0, \delta]$ is compact, $\Phi_{[0,\delta]}(e)$ is compact, hence bounded, in S, so there is an $M_2 > 0$ such that $d(e, \Phi_t(e)) \leq M_2$ for all $0 \leq t \leq \delta$. Hence $\sup_{0\leq t\leq\delta} \|\hat{T}_\Phi(t)\|_{\mathcal{L}(\mathcal{S}_e)} \leq \max(1, M_1 + M_2) =: M < \infty$.

By (7) and strong continuity of $(\Phi_t)_{t\geq 0}$ we have for every $x \in S$ that

$$\|\hat{T}_\Phi(t)\delta_x - \delta_x\|_e^* = d(\Phi_t(x), x) \to 0$$

as $t \downarrow 0$. Hence by linearity $\lim_{t\downarrow 0} \|\hat{T}_\Phi(t)d - d\|_e^* = 0$ for all $d \in D$. $\square$

Remarks. 1) Notice that for all $\varphi \in \mathcal{S}_e$, $f \in \mathrm{Lip}_e(S)$ and $t \geq 0$, we have

$$f(\hat{T}_\Phi(t)\varphi) = (\hat{T}_\Phi(t)\varphi)(f) = (S_\Phi^*(t)\varphi)(f) = \varphi(S_\Phi(t)(f)) = (S_\Phi(t)f)(\varphi).$$

Therefore $\hat{T}_\Phi^*(t)f = S_\Phi(t)f$ for all $f \in \mathrm{Lip}_e(S)$ and under the equivalent conditions of Theorem 5.4, $(S_\Phi(t))_{t\geq 0}$ is the dual semigroup of a strongly continuous semigroup. As $\mathcal{S}_e$ is not reflexive in general, $(S_\Phi(t))_{t\geq 0}$ cannot be expected to be strongly continuous. It *is* on the smaller space $\mathcal{S}_e^\odot$ by definition. It would be interesting to be able to identify the latter space.

2) In [20, Corollary 3 and Remark 4] a result similar to Theorem 5.4 is proven, but in less generality, since there S is taken to be a closed subset of a Banach space. In [20] the duality of spaces of Lipschitz functions is also exploited to show this result, but there the Banach space $\mathrm{Lip}_0(S)$ is used, consisting of the Lipschitz functions vanishing at some distinct point e in S. Since the semigroup $T_\Phi(t)$ will in general not map $\mathrm{Lip}_0(S)$ into itself, unless e is a fixed point of $(\Phi_t)_{t\geq 0}$, the proof in [20] needs to make use of the Banach space $\mathrm{Lip}(S)/\mathbb{R}\mathbf{1}$. By making use of the space $\mathrm{Lip}_e(S)$, we have no such difficulties.

3) Notice that the semigroup $(S_\Phi(t))_{t\geq 0}$ defined above is also a semigroup of bounded linear operators on $\mathrm{BL}(S)$, by Lemma 5.1. Then $(S_\Phi^*(t))_{t\geq 0}$ is a semigroup of bounded linear operators on $\mathrm{BL}(S)^*$. Using very similar techniques as above, we can show that $S_\Phi^*(t)(\mathcal{S}_{\mathrm{BL}}) \subset \mathcal{S}_{\mathrm{BL}}$ for all $t \geq 0$. Hence we can define a semigroup $(T_\Phi(t))_{t\geq 0}$ on $\mathcal{S}_{\mathrm{BL}}$ by restricting $S_\Phi^*(t)$ to $\mathcal{S}_{\mathrm{BL}}$. Under the equivalent conditions of Theorem 5.4 this semigroup is strongly continuous:

Theorem 5.5. *For all $x, y \in S$,$s, t \geq 0$,*

$$\begin{aligned} \|T_\Phi(s)(\delta_x) - T_\Phi(t)(\delta_y)\|^*_{\mathrm{BL}} &= \frac{2d(\Phi_s(x), \Phi_t(y))}{2 + d(\Phi_s(x), \Phi_t(y))} \\ &\leq \min(2, d(\Phi_s(x), \Phi_t(y))). \end{aligned}$$

If $\limsup_{t\downarrow 0} |\Phi_t|_{\mathrm{Lip}} < \infty$ and $(\Phi_t)_{t\geq 0}$ is strongly continuous, then $(T_\Phi(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{S}_{\mathrm{BL}}$.

The proof is similar to the proof of Theorem 5.4, but here the equality follows from Lemma 3.5.

Let $t \geq 0$. Then $\hat{T}_\Phi(t)(D^+) \subset D^+$ and $T_\Phi(t)(D^+) \subset D^+$, hence by the continuity of $\hat{T}_\Phi(t)$ and $T_\Phi(t)$ we can conclude that $\hat{T}_\Phi(t)(\mathcal{S}_e^+) \subset \mathcal{S}_e^+$ and $T_\Phi(t)(\mathcal{S}_{\mathrm{BL}}^+) \subset \mathcal{S}_{\mathrm{BL}}^+$. Thus $(\hat{T}_\Phi(t))_{t\geq 0}$ and $(T_\Phi(t))_{t\geq 0}$ are positive semigroups.

Thus, if S is complete,

$$\hat{T}_\Phi(t)(\mathcal{M}_{s,1}^+(S)) \subset \mathcal{M}_{s,1}^+(S)$$

and

$$T_\Phi(t)(\mathcal{M}_s^+(S)) \subset \mathcal{M}_s^+(S).$$

In the following proposition we will show that this also holds if S is not complete.

Proposition 5.6. *Let $t \geq 0$. Then $T_\Phi(t)$ and $\hat{T}_\Phi(t)$ leave $\mathcal{M}_s(S)$ and $\mathcal{M}_{s,1}(S)$ invariant, respectively. Moreover, they are given by* (1).

Proof. Let $\mu \in \mathcal{M}_s(S)$. Then for all $f \in \mathrm{BL}(S)$ and $t \geq 0$ we have:

$$T_\Phi(t)(\mu)(f) = \mu(S_\Phi(t)f) = \int f \circ \Phi_t d\mu = \int f d(\mu \circ \Phi_t^{-1}),$$

where $\mu \circ \Phi_t^{-1}$ is again a Borel measure, since Φ_t is continuous on S. Hence $T_\Phi(t)(\mu)$ is represented by the measure $\mu \circ \Phi_t^{-1}$. We now want to show that $\mu \circ \Phi_t^{-1}$ is a separable measure. Since μ is separable, there is a separable Borel measurable subset E of S, such that $|\mu|(S\backslash E) = 0$. By continuity of Φ_t, $\Phi_t(E)$ is separable, and so is $\overline{\Phi_t(E)}$. For any Borel measurable $A \subset S\backslash\overline{\Phi_t(E)}$, $\mu \circ \Phi_t^{-1}(A) = 0$. Therefore $|\mu \circ \Phi_t^{-1}|(S\backslash\overline{\Phi_t(E)}) = 0$, so $\mu \circ \Phi_t^{-1}$ is separable.

Similarly we get that for $\mu \in \mathcal{M}_{s,1}(S)$ and $t \geq 0$, $\hat{T}_\Phi(t)(\mu)$ is represented by the separable Borel measure $\mu \circ \Phi_t^{-1}$. Then, by Lemma 3.3, $\mu \circ \Phi_t^{-1} \in \mathcal{M}_1(S)$, hence in $\mathcal{M}_{s,1}(S)$. So $\hat{T}_\Phi(t)(\mathcal{M}_{s,1}(S)) \subset \mathcal{M}_{s,1}(S)$. $\square$

Corollary 5.7. *Let $t \geq 0$. Then $T_\Phi(t)$ and $\hat{T}_\Phi(t)$ leave $\mathcal{M}_s^+(S)$ and $\mathcal{M}_s^+(S)$ invariant, respectively.*

So we see that the strongly continuous semigroup $(T_\Phi(t))_{t\geq 0}$ on $\mathcal{S}_{\mathrm{BL}}$, when restricted to $\mathcal{M}_s(S)$, is the semigroup defined by (1). This gives us the proper functional analytic framework that will enable us to study (2).

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