PATHWISE LYAPUNOV EXPONENTS FOR LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the problem of estimating the pathwise Lyapunov exponent for linear stochastic systems with multiplicative noise. We present a Lyapunov type matrix inequality that is closely related to this problem, and show under what conditions we can solve the matrix inequality. From this we can deduce an upper bound for the Lyapunov exponent.

In the converse direction it is shown that a necessary condition for the stochastic system to be pathwise asymptotically stable can be formulated in terms of controllability properties of the matrices involved.

1. Introduction

Consider, in $\mathbb{R}^n$, the linear SDE

$$\begin{cases} dx(t) = Ax(t)\ dt + \sum_{i=1}^k B_i x(t)\ dW_i(t), & t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n \times n}$, $i = 1, \dots k$ and $(W_i)_{i=1}^k$ are independent standard Brownian motions in $\mathbb{R}$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. As a special case we will often consider the equation

$$dx(t) = Ax(t)\ dt + Bx(t)\ dW(t). \tag{2}$$

It is well known (see e.g. [Karatzas and Shreve, 1991]) that for any choice of A, $(B_i)_{i=1}^k$ and x_0 a unique solution to (1), denoted as $x(t; x_0)$, exists.

In this paper we are interested in the stability properties of the solution of (1). More specifically, we want to estimate the *pathwise Lyapunov exponent*, defined as

$$\lambda := \lim_{t\to\infty} \frac{1}{t} \log |x(t; x_0)|, \quad \text{a.s.} \quad \text{(see [Arnold et al., 1985])} \tag{3}$$

1.1. **Deterministic systems.** For square matrices A and B, let $\mathfrak{s}(A)$ denote the *spectral bound of* A, and $\mathfrak{r}(B)$ the *spectral radius of* B, i.e.

$$\mathfrak{s}(A) := \sup\{\mathrm{Re}\ \lambda : \lambda \in \sigma(A)\} \quad \text{and} \quad \mathfrak{r}(B) := \sup\{|\lambda| : \lambda \in \sigma(B)\}.$$

Furthermore let $\omega_0(A)$ denote the growth bound of A, i.e.

$$\omega_0(A) = \inf\{\omega \in \mathbb{R} : \exists_{M\geq 1} \text{ s.t. } ||\exp(At)|| \leq M e^{\omega t} \text{ for all } t \geq 0\}.$$

In finite dimensions the following equalities hold:

$$\mathfrak{s}(A) = \omega_0(A) = \frac{1}{t} \log\ \mathfrak{r}(\exp(At)), \quad \text{for all } t \geq 0. \tag{4}$$

See [Engel and Nagel, 2000], Proposition IV.2.2 and Theorem IV.3.11.

1.2. **Commutative case.** As an appetizer, consider the particular case of (1) where A and all B_i, $i = 1, \ldots, k$ commute. Then the solution is given by

$$x(t) = \exp\left[t\left(A - \tfrac{1}{2}\sum_{i=1}^{k} B_i^2\right) + \sum_{i=1}^{k} W_i(t) B_i\right] x_0 \quad \text{a.s.},$$

and

$$\frac{1}{t}\log|x(t)| \leq \frac{1}{t}\log|x_0| + \frac{1}{t}\log\left|\left|\left(\exp\left(A - \tfrac{1}{2}\sum_{i=1}^{k} B_i^2\right)t\right)\right|\right| + \frac{1}{t}\sum_{i=1}^{k} ||B_i||\ |W_i(t)|.$$

Now using the strong law of large numbers for martingales (see Theorem A.1)

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{k} ||B_i||\ |W_i(t)| = 0 \quad \text{a.s.},$$

and using the observations above,

$$\lim_{t\to\infty}\frac{1}{t}\log\left|\left|\left(\exp\left(A - \tfrac{1}{2}\sum_{i=1}^{k} B_i^2\right)t\right)\right|\right| = \mathfrak{s}\left(A - \tfrac{1}{2}\sum_{i=1}^{k} B_i^2\right).$$

Hence we find

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq \mathfrak{s}\left(A - \tfrac{1}{2}\sum_{i=1}^{k} B_i^2\right), \quad \text{a.s.} \tag{5}$$

1.3. **Interpretation of the pathwise Lyapunov exponent.** The expression for the Lyapunov exponent given in (3) implies that there exists a random variable M such that

$$|x(t)| \leq M\mathrm{e}^{\lambda t} \quad \text{for all } t \geq 0, \text{ a.s.}$$

It should be noted that if we can estimate $\inf_{t\geq 0}\mathbb{E}[|x(t)|^p] \geq k^p$ for some $p > 0$, then

$$k^p \leq \mathbb{E}[|x(t)|^p] \leq \mathbb{E}[M^p\mathrm{e}^{\lambda p t}],$$

so that

$$\mathbb{E}[M^p] \geq k^p\mathrm{e}^{-\lambda p t} \quad \text{for all } t \geq 0,$$

and in particular $\mathbb{E}[M^p] = \infty$ if $\lambda < 0$.

So in this sense a negative Lyapunov exponent does not guarantee a fast convergence to zero. Examples of this behaviour are easy to construct: in one dimension take $A = 0$ and $B_1 = b \in \mathbb{R}$; then $\mathbb{E}[|x(t)|^2] \geq \mathbb{E}[|x(0)|^2$, $t \geq 0$, but $\lambda = -\frac{1}{2}b^2$.

1.4. **Outline of this paper.** In Section 2 a short history of the Lyapunov exponent is presented. In Section 3 an upper bound for the Lyapunov exponent is obtained by studying a particular matrix inequality. In Section 4 we apply this result to the case of a control theoretic system with noisy linear feedback. In Section 5 we obtain a necessary condition for pathwise stability.

In the appendices two proofs are given: one for the strong law of large numbers for martingales (Appendix A) and an alternative proof for Theorem 3.6 in Appendix (B).

2. History of the problem

In this section I present a short history of establishing the pathwise Lyapunov exponent for solutions of linear SDEs. This is for illustrative purposes only, and we do not claim any completeness. The methods described in this section will not be used later on.

An early reference to the problem is found in [Leibowitz, 1963], where it is postulated that if the solution of

$$\dot{x}(t) = Ax(t), \quad t \geq 0$$

is unstable, then the solution of the Stratonovich SDE

$$dx(t) = Ax(t) + \sum_{i=1}^{k} B_i x(t) \circ dW_i(t),$$

is never (pathwise) stable.

2.1. Khas'minskii's work.

Already in 1967, Khas'minskii [Khasminskii, 1967] shows that this postulate does not hold by studying (1) in spherical coordinates. His approach is as follows.

First we write (1) in Stratonovich form

$$dx(t) = \widetilde{A}x(t)\ dt + \sum_{i=1}^{k} B_i x(t) \circ dW_i(t), \tag{6}$$

with $\widetilde{A} = A - \frac{1}{2}\sum_{i=1}^{k} B_i^2$.

Write $y(t) := x(t)/|x(t)|$ and $\lambda(t) := \log|x(t)|$, $t \geq 0$. Then y is a process on S^{n-1}, the unit sphere of dimension $n-1$. Using Itô's formula it can be calculated that y satisfies

$$\begin{cases} dy(t) = h(\widetilde{A}, y(t))\ dt + \sum_{i=1}^{k} h(B_i, y(t)) \circ dW_i(t), \quad t \geq 0, \\ y(0) = y_0 := \frac{x_0}{|x_0|}, \end{cases} \tag{7}$$

where

$$h(C,z) := (C - q(C,z)I)z, \quad q(C,z) := z^T C z, \quad z \in S^{n-1}, C \in \mathbb{R}^{n\times n}.$$

Note that $h(C,z)^T z = 0$ for $z \in S^{n-1}$ so that indeed $h(C,z)$ is a vector in the tangent space $T_z S^{n-1}$ and that the process $(y(t))_{t\geq 0}$ is autonomous, as can be seen from (7). By compactness of S^{n-1} at least one invariant measure μ exists for y.

Khas'minskii then assumes a strong non-singularity condition on the (B_i), namely

$$\sum_{i=1}^{k} (B_i x)(B_i x)^T \text{ is positive definite for all } x \in \mathbb{R}^n,\ x \neq 0. \tag{8}$$

Due to this condition, $\mathbb{P}_{y_0}(y(t) \in U) > 0$ for all open $U \subset S^1$ and $(y(t))_{t\geq 0}$ is strong Feller. By an earlier theorem of Khas'minskii ([Khasminskii, 1960]), μ is therefore the unique invariant measure for $(y(t))_{t\geq 0}$ on S^{n-1}, and hence it is ergodic.

By Itô's formula, the process $(\lambda(t))_{t\geq 0}$ can be shown to satisfy

$$\lambda(t) = \lambda(0) + \int_0^t \Phi(y(s))\ ds + \sum_{i=1}^{k} \int_0^t \langle B_i y(s), y(s)\rangle\ dW_i(s) \quad \text{a.s.,} \tag{9}$$

with

$$\Phi(z) := \langle Az, z\rangle + \tfrac{1}{2}\sum_{i=1}^{k} ||B_i z||^2 - \sum_{i=1}^{k} \langle B_i z, z\rangle^2, \quad z \in S^{n-1}. \tag{10}$$

Now using the strong law of large numbers for martingales (Theorem A.1),

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^k \int_0^t \langle B_i y(s), y(s)\rangle \, dW_i(s) = 0 \quad \text{a.s.},$$

and by ergodicity of μ,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Phi(y(s)) \, ds = \int_{S^{n-1}} \Phi(z) \, d\mu(z) \quad \text{a.s.}$$

We may conclude that

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t)| = \int_{S^{n-1}} \Phi(z) \, d\mu(z) \quad \text{a.s.} \tag{11}$$

As stated above, (8) is unnecessarily strong for establishing the uniqueness of the invariant measure μ on S^{n-1}. A better understanding of the structure of ergodic invariant measures on manifolds (e.g. S^{n-1}) is provided by [Kliemann, 1987].

2.2. **A different approach by Mao.** In [Mao, 1994], Mao provides a new way of estimating the Lyapunov exponent. In the linear case this boils down to requiring that

$$\langle Az, z\rangle \leq \alpha \quad \text{and} \quad \tfrac{1}{2} \sum_{i=1}^k ||B_i z||^2 - \sum_{i=1}^k \langle B_i z, z\rangle^2 \leq \beta, \quad z \in S^{n-1} \tag{12}$$

so that $\Phi(z) \leq \alpha + \beta$, $z \in S^{n-1}$. Therefore (9) shows that

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq \alpha + \beta \quad \text{a.s.}$$

This approach can be extended using a Lyapunov function, see [Mao, 1997], Theorem 4.3.3.

3. Estimating the Lyapunov exponent by means of a matrix inequality

In this section we use the particular Lyapunov function $V(x) := \langle Qx, x\rangle$ with Q positive definite, to obtain an estimate for the pathwise Lyapunov exponent for the solution x of (1). It is shown that a Lyapunov type inequality for Q can be formulated which gives a sufficient condition for x to have a particular Lyapunov exponent. In Theorem 3.6 general conditions are formulated such that a positive definite solution to the mentioned matrix inequality exists. As a corollary we formulate conditions on A and B such that the solution of (2) has a particular Lyapunov exponent in Theorem 3.7.

3.1. **Proposition.** Suppose there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ such that

$$\langle Qx, x\rangle \left[2\langle QAx, x\rangle + \sum_{i=1}^k \langle QB_i x, B_i x\rangle - 2\lambda \langle Qx, x\rangle \right] \leq 2 \sum_{i=1}^k \langle Qx, B_i x\rangle^2 \quad \text{for all } x \in \mathbb{R}^n. \tag{13}$$

Then

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq \lambda \quad \text{a.s.},$$

with x the solution of (1).

Proof: If $x_0 = 0$, then $\mathbb{P}(x(t) = 0) = 1$, $t \geq 0$, and the required estimate holds trivially.

Suppose $x_0 \neq 0$. Let $y(t) := \langle Qx(t), x(t)\rangle$, $t \geq 0$. Then by unicity of the solution of SDEs and positiveness of Q, $\mathbb{P}(y(t) = 0) = 0$ for all $t \geq 0$.

By Itô's formula,

$$\begin{aligned} d \log y(t) &= \frac{1}{y(t)} \left[2\langle Qx(t), Ax(t)\rangle \ dt + 2 \sum_{i=1}^k \langle Qx(t), B_i x(t)\rangle \ dW_i(t) \right] \\ &\quad - \sum_{i=1}^k \frac{2}{y(t)^2} \langle Qx(t), B_i x(t)\rangle^2 \ dt + \sum_{i=1}^k \frac{1}{y(t)} \langle QBx(t), B_i x(t)\rangle \ dt \\ &= \left\{ \frac{1}{y(t)} \left[2\langle QAx(t), x(t)\rangle + \sum_{i=1}^k \langle QB_i x(t), B_i x(t)\rangle \right] - \sum_{i=1}^k \frac{2}{y(t)^2} \langle Qx(t), B_i x(t)\rangle^2 \right\} \ dt \\ &\quad + \sum_{i=1}^k \frac{2}{y(t)} \langle Qx(t), B_i x(t)\rangle \ dW_i(t) \\ &\leq 2\lambda \ dt + \sum_{i=1}^k \frac{2}{y(t)} \langle Qx(t), B_i x(t)\rangle \ dW_i(t) \quad \text{a.s.} \end{aligned}$$

Now by boundedness of $\frac{\langle Qx(t), B_i x(t)\rangle}{\langle Qx(t), x(t)\rangle}$, $i = 1, \ldots, k$ and the law of large numbers for martingales (Theorem A.1)

$$\frac{1}{t} \int_0^t \frac{2\langle Qx(s), B_i x(s)\rangle}{y(s)} \ dW_i(s) \to 0 \quad (t \to \infty), \quad \text{a.s.}, \quad i = 1, \ldots, k,$$

so

$$\limsup_{t\to\infty} \frac{1}{t} \log y(t) \leq 2\lambda \quad \text{a.s.},$$

and by positiveness of Q, this implies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq \lambda \quad \text{a.s.}$$

□

3.2. **Remark.** Recall the definition of Φ in (10). If the conditions of Lemma 3.1 hold and we equip $\mathbb{R}^n$ with the inner product

$$\langle x, y\rangle_Q := \langle Qx, y\rangle, \quad x, y \in \mathbb{R}^n,$$

then we see that (13) is equivalent to stating that

$$\Phi_Q(z) := \langle Az, z\rangle_Q + \tfrac{1}{2} \sum_{i=1}^k \langle B_i z, B_i z\rangle_Q - \sum_{i=1}^k \langle B_i z, z\rangle_Q^2 \leq \lambda$$

for $z \in S_Q^{n-1} := \{x \in \mathbb{R}^n : \langle x, x\rangle_Q = 1\}$, the unit sphere corresponding to the inner product $\langle \cdot, \cdot\rangle_Q$. So contrary to the setting of Section 2, we do not require a unique invariant measure on S_Q^{n-1}, since $\Phi_Q(\cdot) \leq \lambda$ on the entire Q-unit sphere anyway.

3.3. **Example.** One might ask whether it is at all possible that $\Phi_Q \leq \lambda$ on S_Q^{n-1} for some but not all positive definite matrices Q. Already in the deterministic case this can be seen in the next example, phrased as a proposition:

3.4. **Proposition.** Let $A = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix}$ with $\alpha_1, \alpha_2 \in \mathbb{R}$ eigenvalues of A, and let $Q := \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$ with $q > 0$. Then for any $\lambda > \max(\alpha_1, \alpha_2)$ there exists a q such that

$$\langle QAx, x\rangle \leq \lambda \langle Qx, x\rangle, \quad x \in \mathbb{R}^2,$$

and consequently

$$A^* Q + QA - 2\lambda Q \leq 0.$$

Here, and in the following, I write "$\leq$" for the partial order induced by the positive cone consisting of positive semidefinite matrices.

PROOF: For fixed $\lambda \in \mathbb{R}$ and $q > 0$ we calculate

$$\langle QAx, x\rangle - \lambda\langle Qx, x\rangle = (\alpha_1 - \lambda)x_1^2 + q(\alpha_2 - \lambda)x_2^2 + x_1x_2.$$

Now since for any $\gamma \in \mathbb{R}$

$$|x_1x_2| = \left|\frac{1}{\gamma}x_1\gamma x_2\right| \leq \left|\frac{1}{\gamma}x_1\right| \; |\gamma x_2| \leq \frac{1}{2\gamma^2}x_1^2 + \frac{\gamma^2}{2}x_2^2$$

holds, we have

$$\langle QAx, x\rangle - \lambda\langle Qx, x\rangle \leq \left(\alpha_1 - \lambda + \frac{1}{2\gamma^2}\right)x_1^2 + (q(\alpha_2 - \lambda) + \frac{\gamma^2}{2})x_2^2,$$

which is equal to zero for any x_1, x_2 if

$$\alpha_1 - \lambda + \frac{1}{2\gamma^2} = 0 \quad \text{and} \quad q(\alpha_2 - \lambda) + \frac{\gamma^2}{2} = 0.$$

This is satisfied for

$$\gamma^2 = 2q(\lambda - \alpha_2) \quad \text{and} \quad \lambda = \frac{\alpha_1 + \alpha_2}{2} + \frac{1}{2}\sqrt{(\alpha_1 - \alpha_2)^2 + \frac{1}{q}}.$$

When $q \to \infty$ then

$$\lambda \to \frac{\alpha_1 + \alpha_2}{2} + \frac{|\alpha_1 - \alpha_2|}{2} = \max(\alpha_1, \alpha_2).$$

□

In the remainder of this section we will establish conditions such that Q and λ exist as required in Proposition 3.1.

3.5. **Lemma.** Suppose there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$ and constants $b_i \in \mathbb{R}$, $i = 1, \ldots, k$ such that

$$\left(A + \sum_{i=1}^k b_i B_i\right)^* Q + Q\left(A + \sum_{i=1}^k b_i B_i\right) + \sum_{i=1}^k B_i^* Q B_i + \left(\tfrac{1}{2}\sum_{i=1}^k b_i^2 - 2\lambda\right)Q \leq 0. \tag{14}$$

Then (13) holds, and hence

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq \lambda, \quad \text{a.s.} \tag{15}$$

PROOF: Note that for $i = 1, \ldots, k$, by the *abc*-formula,

$$\frac{\langle Qx, B_ix\rangle^2}{\langle Qx, x\rangle^2} + b_i\frac{\langle Qx, B_ix\rangle}{\langle Qx, x\rangle} + \frac{1}{4}b_i^2 \geq 0, \quad \text{for all } x \in \mathbb{R}^n. \tag{16}$$

So, by Proposition 3.1, if

$$2\langle QAx, x\rangle + \sum_{i=1}^k \langle QB_ix, B_ix\rangle - 2\lambda\langle Qx, x\rangle \leq -\sum_{i=1}^k 2b_i\langle Qx, B_ix\rangle - \frac{1}{2}b_i^2\langle Qx, x\rangle, \quad \text{for all } x \in \mathbb{R}^n,$$

then the claimed result holds.

But this is equivalent to the stated condition. □

The following theorem gives sufficient conditions in order for a solution to (14) to exist.

3.6. **Theorem.** Suppose $L, D_i \in \mathbb{R}^{n\times n}$, $i = 1, \dots, k$ such that

$$\left|\left|\mathrm{e}^{Lt}\right|\right| \leq m\mathrm{e}^{\omega t} \quad \text{for all } t \geq 0,$$

with $m \geq 1$, $\omega \in \mathbb{R}$, and

$$m^2 \sum_{i=1}^{k} ||D_i||^2 + 2\omega < 0. \tag{17}$$

Then for any $M \in \mathbb{R}^{n\times n}$ there exists a unique solution $Q \in \mathbb{R}^{n\times n}$ to

$$L^*Q + QL + \sum_{i=1}^{k} D_i^* Q D_i + 2\nu Q = M. \tag{18}$$

This Q also satisfies

$$Q = \int_0^\infty \mathrm{e}^{L^*t} \left(\sum_{i=1}^{k} D_i^* Q D_i - M \right) \mathrm{e}^{Lt} \, dt. \tag{19}$$

Furthermore,

(i) if $M = 0$, then $Q = 0$,
(ii) if $M \leq 0$, then $Q \geq 0$, and
(iii) if $M < 0$ then $Q > 0$.

Proof: Define a recursion by

$$Q_0 := 0, \quad Q_{j+1} := \int_0^\infty \mathrm{e}^{L^*t} \left(\sum_{i=1}^{k} D_i^* Q_j D_i - M \right) \mathrm{e}^{Lt} \, dt.$$

The recursion is actually a contraction, since

$$\begin{aligned} ||Q_{j+1} - Q_j|| &= \left|\left| \int_0^\infty \mathrm{e}^{L^*t} \left(\sum_{i=1}^{k} D_i^* (Q_j - Q_{j-1}) D_i \right) \mathrm{e}^{Lt} \, dt \right|\right| \\ &\leq m^2 \sum_{i=1}^{k} ||D_i||^2 \int_0^\infty \mathrm{e}^{2\omega t} \, dt \, ||Q_j - Q_{j-1}|| \\ &= -\frac{m^2 \sum_{i=1}^{k} ||D_i||^2}{2\omega} ||Q_j - Q_{j-1}||. \end{aligned}$$

Note that the recursion is defined such that Q_{j+1} satisfies

$$L^*Q_{j+1} + Q_{j+1}L = M - \sum_{i=1}^{k} D_i^* Q_j D_i,$$

a basic result from Lyapunov theory (see e.g. [Lancaster and Tismenetsky, 1985]).

Hence there exists a unique fixed point $Q \in \mathbb{R}^{n\times n}$ that satisfies both (18) and

$$Q = \int_0^\infty \mathrm{e}^{L^*t} \left(\sum_{i=1}^{k} D_i^* Q D_i - M \right) \mathrm{e}^{Lt} \, dt.$$

If $M = 0$ then $Q = 0$ by unicity of the solution.

Now suppose $M \leq 0$. Then we can check that the recursion for (Q_j) has the property that $Q_j \geq 0$ for all i. So $Q \geq 0$, and (19) shows that

$$Q \geq -\int_0^\infty \mathrm{e}^{L^*t} M \mathrm{e}^{Lt} \, dt.$$

If $M < 0$, then there exists a unique $P \in \mathbb{R}^{n\times n}$, $P > 0$ such that $M = L^*P + PL$. Then

$$Q \geq -\int_0^\infty \mathrm{e}^{L^*t} M \mathrm{e}^{Lt}\, dt = P > 0.$$

□

In Appendix B we prove the same result under slightly stronger conditions but in a different way.

Consider now the case of only one noise term, that is equation (2). The following theorem is the main result on estimation of the Lyapunov exponent. It shows that under conditions on $A+\sigma B$ and $B+\tau I$ for some constants $\sigma, \tau \in \mathbb{R}$, we have pathwise asymptotic stability of the corresponding solution of (1).

3.7. **Theorem.** Suppose $h : \mathbb{R} \to [1,\infty)$, $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to [0,\infty)$ are functions such that

$$||\exp\left((A+\sigma B)t\right)|| \leq h(\sigma)\exp(g(\sigma)t) \quad \text{and}$$
$$||B+\sigma I|| \leq f(\sigma), \quad t \in \mathbb{R}.$$

Suppose for some $\sigma, \tau, \lambda \in \mathbb{R}$ we have

$$\lambda > \tfrac{1}{2}(h(\sigma-\tau)f(\tau))^2 + g(\sigma-\tau) + \tfrac{1}{4}\sigma^2 - \tfrac{1}{2}\tau^2. \tag{20}$$

Then

$$\lim_{t\to\infty} \frac{1}{t}\log|x(t)| \leq \lambda \quad \text{a.s.},$$

where x is the solution to (2).

Proof: For the given combination of σ,τ and λ put

$$D := B + \tau I, \quad L := A + (\sigma-\tau)B + (\tfrac{1}{4}\sigma^2 - \tfrac{1}{2}\tau^2 - \lambda)I.$$

Note that

$$||D|| \leq f(\tau) \quad \text{and} \quad ||\exp(Lt)|| \leq m\mathrm{e}^{\omega t},$$

with $m = h(\sigma-\tau)$ and $\omega = g(\sigma-\tau) + \frac{1}{4}\sigma^2 - \frac{1}{2}\tau^2 - \lambda$. Then by (20), condition (17) of Theorem 3.6 is satisfied and hence we can find a Q such that (18) holds for $M = -I$. By the choice of D and L, we have

$$0 \geq M = L^*Q + QL + D^*QD = (A+\sigma B)^*Q + Q(A+\sigma B) + B^*QB + (\tfrac{1}{2}\sigma^2 - 2\lambda)I,$$

and it follows that Q is a solution to (14), with $b = \sigma$.

Therefore by Lemma 3.5, we may conclude that (15) holds. □

3.8. **Example.** Let

$$A = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}. \tag{21}$$

Equip $\mathbb{R}^2$ by the inner product $\langle x, y\rangle = x_1y_1 + qx_2y_2$, with $q > 0$.

By Proposition 3.4, for all $\sigma \in \mathbb{R}$ and $\varepsilon > 0$ there exists a $q > 0$ such that

$$\langle (A+\sigma B)x, x\rangle \leq (\max(a_1+\sigma b_1, a_2+\sigma b_2) + \varepsilon)\langle x, x\rangle,$$

and therefore

$$\exp((A+\sigma B)t) \leq \exp([\max(a_1+\sigma b_1, a_2+\sigma b_2)+\varepsilon]t), \quad t \geq 0.$$

Furthermore $||B+\sigma I|| = \max(|b_1+\sigma|, |b_2+\sigma|)$, irrespective of q.

So, by Theorem 3.7, if we pick

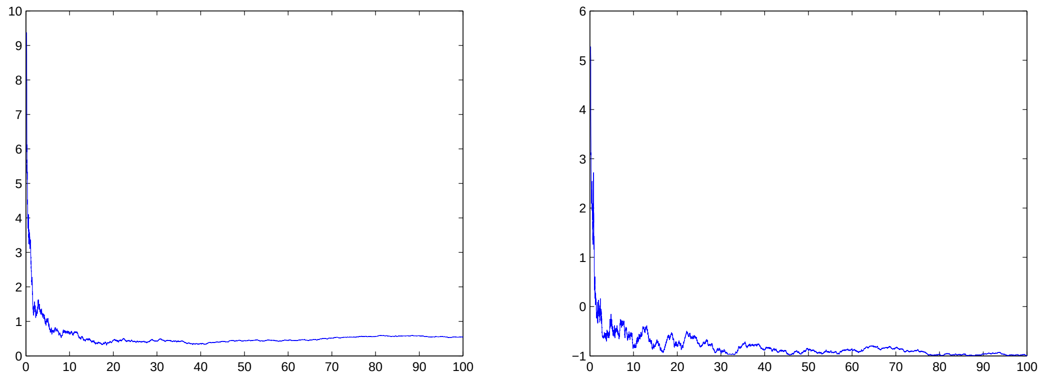
$$\begin{aligned} \lambda &> \tfrac{1}{2}\max((b_1+\tau)^2, (b_2+\tau)^2) + \max(a_1+(\sigma-\tau)b_1, a_2+(\sigma-\tau)b_2) + \tfrac{1}{4}\sigma^2 - \tfrac{1}{2}\tau^2 \\ &= \tfrac{1}{2}\max(2b_1\tau + b_1^2, 2b_2\tau + b_2^2) + \max(a_1+(\sigma-\tau)b_1, a_2+(\sigma-\tau)b_2) + \tfrac{1}{4}\sigma^2 \end{aligned}$$

for some optimal σ and τ, then

$$\lim_{t\to\infty} \frac{1}{t} \log|x(t)| \le \lambda,$$

when x is the solution of (2).

As a numerical example, let $a_1 = 1, a_2 = -1, b_1 = 1$ and $b_2 = 0$. By picking $\tau = 0$ and $\sigma = -2$, we see that any $\lambda > \frac{1}{2}$ is an upper bound for the Lyapunov exponent. If $a_1 = 1, a_2 = -1, b_1 = 2$ and $b_2 = 0$, then by picking $\tau = -1$ and $\sigma = -2$ we obtain the estimate $\lambda > 0$. We see that the noise has a stabilizing effect, even though it is degenerate. Now compare these theoretical results to a simulation (see Figure 3.8). We see that in the case $b_1 = 1$ the estimate is sharp, whereas in the case $b_1 = 2$ the graph suggest a Lyapunov exponent of -1, which gives room for even further improvements.



(a) $b_1 = 1$ (b) $b_1 = 2$

Figure 1: Graph of $\frac{1}{t}\log|x(t)|$ for a sample path of the solution x of (2) with A and B as given in (21), with $a_1 = 1$, $a_2 = -1$, $b_2 = 0$.

4. Lyapunov exponent for noisy feedback

In this small section we consider a control system

$$\dot{x}(t) = Ax\;dt + Bu(t), \quad t \ge 0,$$

where we choose a noisy feedback as control u:

$$du(t) = Fx(t)\;dW(t).$$

4.1. **Proposition.** Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Suppose for some $K \in \mathbb{R}^{n\times n}$ we have that

$$\mathrm{Re}\,\langle (A+BK)x, x\rangle \le \nu|x|^2, \quad \text{for all } x \in \mathbb{R}^n,$$

with $\nu \in \mathbb{R}$, and suppose $F := \frac{1}{c}K$ for some $c \in \mathbb{R}, c \neq 0$ such that $2||BK|| \le c^2$.

Then for

$$\lambda > \nu + ||BK|| - \frac{1}{2c^2}||BK||^2$$

we have

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \le \lambda \quad \text{a.s.},$$

where x is the solution of

$$dx = Ax(t)\;dt + BFx(t)\;dW(t), \quad x(0) = x_0 \in \mathbb{R}^n.$$

PROOF: We can write

$$(A + bBF)^*Q + Q(A + bBF) + B^*QBF + (\tfrac{1}{2}b^2 - 2\lambda)Q = L^*Q + QL + D^*Q$$

with

$$L := A + BK + \tfrac{1}{4}b^2 - \lambda - \tfrac{1}{2}(b-c)^2,$$
$$D := BF + (b-c)I.$$

It can be calculated that for

$$\lambda > \nu + ||BK|| - \frac{1}{2c^2}||BK||^2, \quad 2||BK|| \leq c^2 \quad \text{and} \quad b = \frac{2}{c}||BK||$$

the conditions of Theorem 3.6 hold. □

In particular we may choose c optimal, i.e. $c^2 = 2||BK||$, to obtain the estimate

$$\lambda > \nu + \frac{3}{4}||BK||.$$

This proves the following corollary.

4.2. **Corollary.** If we are given (A, B) which are stabilizable in such a way that there exists a $K \in \mathbb{R}^{m\times n}$ such that

$$\text{Re } \langle (A + BK)x, x\rangle \leq \nu|x|^2,$$

with $\nu < -\frac{3}{4}||BK||$, then the solution of

$$dx(t) = Ax(t) + BFx(t)\ dW(t)$$

is almost surely asymptotically stable, for $F = \frac{1}{\sqrt{2||BK||}}K$.

In the next section we will prove a converse result, namely that a necessary condition for the solution (1) to be asymptotically stable is that (A, B) are stabilizable.

5. A necessary condition for pathwise stability

In this section we show that, in order for the solutions of the linear SDE (2) to be pathwise asymptotically stable, an assumption on the controllability properties of the pair (A, B) is necessary.

5.1. **Definition.** The stochastic differential equation

$$\begin{cases} dx(t) = f(x(t), t)\ dt + g(x(t), t)\ dW(t), & t \geq 0, \\ x(0) = x_0 \end{cases}$$

is *almost surely asymptotically stable* if

$$\lim_{t\to\infty} |x(t; x_0)| = 0 \quad \text{almost surely,}$$

and *almost surely exponentially stable* if

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t; x_0)| < 0 \quad \text{almost surely}$$

for all initial conditions $x_0 \in \mathbb{R}^n$.

5.2. **Definition.** Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The pair (A, B) is called *stochastically stabilizable* if there exists an $F \in \mathbb{R}^{m\times n}$ such that

$$dx(t) = Ax(t)\ dt + BFx(t)\ dW(t) \tag{22}$$

is almost surely asymptotically stable.

We would like to establish conditions on (A, B) such that (A, B) is stochastically stabilizable.

5.3. **Definition.** The pair (A, B) is called *(deterministically) stabilizable* if there exists an $F \in \mathbb{R}^{m\times n}$ such that $\mathfrak{s}(A + BF) < 0$. It is well known that, through pole placement, controllability implies stabilizability. Stabilizability can be understood as controllability of the unstable part of A.

We can now state a necessary condition for a system (A, B) to be stochastically stabilizable:

5.4. **Theorem.** Suppose (A, B) is stochastically stabilizable. Then (A, B) is stabilizable.

This leads directly to the following corollary which is the main result of this section.

5.5. **Corollary.** Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ such that the solution of (2) is asymptotically stable. Then (A, B) is stabilizable.

Proof (of corollary): By taking $F = I$, we see that (A, B) is stochastically stabilizable. Now apply the proposition. □

In order to prove the theorem, we need some other notions and results from systems theory. See [Polderman and Willems, 1998] for details.

5.6. **Controllability, isomorphic systems.** Recall that the pair (A, B) is called *controllable* if

$$\operatorname{rank}\left[B, AB, \ldots, A^{n-1}B\right] = n.$$

Here $[T_1, T_2, \ldots, T_n]$ denotes the concatenation of all the columns of matrices $T_1, \ldots, T_n$.

$\lambda \in \mathbb{C}$ is called (A, B)-*controllable* if

$$\operatorname{rank}\left[A - \lambda I, B\right] = n,$$

Note that if $\lambda \notin \sigma(A)$, then λ is always (A, B)-controllable.

The system $(\overline{A}, \overline{B})$ is said to be *isomorphic* to (A, B) if there exists an invertible matrix S such that

$$\overline{A} = S^{-1}AS, \quad \overline{B} = S^{-1}B.$$

5.7. **Lemma.** If $\lambda \in \sigma(A)$ is not (A, B)-controllable, then $\lambda \in \sigma(A + BF)$ for all $F \in \mathbb{R}^{m\times n}$.

5.8. **Lemma.** Suppose (A, B) not controllable and $B \neq 0$. Then there exist $(\overline{A}, \overline{B})$ isomorphic to (A, B) such that

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

and such that (A_{11}, B_1) is controllable.

5.9. **Lemma.** Suppose (A, B) is not controllable and $(\overline{A}, \overline{B})$ is isomorphic to (A, B) such that

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with (A_{11}, B_1) controllable. Then $\lambda \in \mathbb{C}$ is (A, B)-controllable if and only if $\lambda \notin \sigma(A_{22})$.

Proof of Theorem 5.4: Suppose (A, B) not stabilizable. Then by Lemma 5.7 there exists $\lambda \in \sigma(A)$, $\lambda > 0$, such that λ is not (A, B)-controllable. By Lemma 5.8 and Lemma 5.9 there exist

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

isomorphic to (A, B) such that $\lambda \in \sigma(A_{22})$. Now x satisfies

$$dx(t) = Ax(t)\ dt + BFx(t)\ dW(t), \tag{23}$$

if and only if $y(t) = Sx(t)$ satisfies

$$dy(t) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} y(t)\, dt + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \overline{F} y(t)\, dW(t), \tag{24}$$

where $\overline{F} = FS^{-1}$.

Let $y_2 \neq 0$ such that $A_{22}y_2 = \lambda y_2$ and let $y_1 = 0$. Let y be the solution of (24) with initial condition $y(0) = [y_1\ y_2]^T$, and $x = S^{-1}y$ the corresponding solution of (23). Then

$$y_2(t) = \exp(A_{22}t)y_2 = \exp(\lambda t)y_2, \quad \text{a.s.}$$

Hence

$$||S||\ |x(t)| \geq |Sx(t)| = |y(t)| \geq |y_2(t)| = \exp(\lambda t)|y_2| \quad \text{a.s.}$$

and therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \geq \lambda, \quad \text{a.s.}$$

Hence (A, B) is not stochastically stabilizable. □

Appendix A. Strong law of large numbers for martingales

Some of the proofs in this paper rely on the strong law of large numbers for martingales, which can be found in [Mao, 1997], Theorem 1.3.4, where it appears without proof. To make the exposition here self-contained we provide the reader with a proof here.

A.1. **Theorem (Strong law of large numbers for martingales).** Let $(M(t))_{t\geq 0}$ be a continuous local martingale with $M(0) = 0$. If

$$\limsup_{t\to\infty} \frac{[M](t)}{t} < \infty, \quad \text{a.s.,} \tag{25}$$

then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0, \quad \text{a.s.}$$

Proof: Let $k, m \in \mathbb{N}$. For $n \in \mathbb{N}$, let E_n denote the event

$$E_n := \left\{ \sup_{t\geq 0} |M^{n+1}(t)| \geq \frac{n}{m} \text{ and } [M^{n+1}](\infty) \leq 2kn \right\}.$$

Here M^n denotes the martingale M stopped at time n.

Then by the exponential martingale inequality ([Revuz and Yor, 1999], Exercise IV.3.16)

$$\mathbb{P}(E_n) \leq 2\mathrm{e}^{-\frac{n}{4km^2}}, \quad n \in \mathbb{N}.$$

Hence by Borel-Cantelli, $\mathbb{P}(E_n^c,\ \text{eventually}) = 1$. Let $\widetilde{\Omega}_{k,m} := (E_n^c,\ \text{eventually})$, and $\Omega_k := \{[M](t) \leq kt \text{ for all } t \geq 0\}$, for $k \in \mathbb{N}$.

On Ω_k, we have that

$$\frac{[M]^{n+1}(\infty)}{n} = \frac{[M](n+1)}{n} \leq \frac{(n+1)k}{n} \leq 2k, \quad n \in \mathbb{N},$$

so on $\Omega_k \cap \widetilde{\Omega}_{k,m}$ we have that

$$\sup_{t\geq 0} |M^{n+1}(t)| < \frac{n}{m}, \quad \text{eventually as } n \to \infty.$$

In particular, on $\Omega_k \cap \widetilde{\Omega}_{k,m}$, for $N \in \mathbb{N}$ large enough and $t \in [n, n+1]$, for $n > N$, $n \in \mathbb{N}$,

$$\frac{|M(t)|}{t} = \frac{|M^{n+1}(t)|}{t} \leq \frac{|M^{n+1}(t)|}{n} < \frac{1}{m},$$

that is

$$\frac{|M(t)|}{t} < \frac{1}{m} \quad \text{for } t \text{ large enough.}$$

Note that $\widetilde{\Omega}_k := \cap_m \widetilde{\Omega}_{k,m}$ has full measure and on $\Omega_k \cap \widetilde{\Omega}_k$, for all $m \in \mathbb{N}$ we have

$$\frac{|M(t)|}{t} < \frac{1}{m} \quad \text{for } t \text{ large enough.}$$

Note that, by (25), for all $\gamma > 0$ there exists an $k \in \mathbb{N}$ such that $\mathbb{P}(\Omega_k) \geq 1 - \gamma$. Therefore $\widetilde{\Omega} := \cup_k \widetilde{\Omega}_k$ has full measure and on $\widetilde{\Omega}$, for all $m \in \mathbb{N}$,

$$\frac{|M(t)|}{t} < \frac{1}{m} \quad \text{for } t \text{ large enough.}$$

□

Appendix B. Alternative solution to the matrix equation

The next result is a variant of Theorem 3.6. The result is obtained using a different proof but is actually weaker. It is provided here because the technique used in this proof might be useful to obtain sharper results for specific examples or perhaps even in the general case. The idea is the same as in Theorem 3.6: we solve the matrix equality (18) keeping some terms fixed, and iterate these solutions to a fixed point.

B.1. **Proposition.** Suppose $L, D \in \mathbb{R}^{n\times n}$ and $\nu < 0$ with

$$2\nu + ||D||^2 + 2||L|| < 0.$$

Then for any $M \in \mathbb{R}^{n\times n}$, there exists a unique solution to (18) (with $k = 1$ and $D_1 = D$).

Furthermore,

(i) if $M = 0$ then $Q = 0$,
(ii) if $M \leq 0$ then $Q \geq 0$, and
(iii) if $M < 0$ then $Q > 0$.

Proof: Rewrite the equation into

$$Q = \frac{D^* Q D}{-2\nu} + \frac{L^* Q + QL - M}{-2\nu}.$$

Define a recursion by $Q_0 := 0$, and Q_{i+1} the unique solution of

$$Q_{i+1} = \frac{D^* Q_{i+1} D}{-2\nu} + \frac{L^* Q_i + Q_i L - M}{-2\nu}.$$

This equation is of the form discussed in [Lancaster and Tismenetsky, 1985], Exercise 12.3.2, and we can write for the solution

$$Q_{i+1} := \sum_{j=0}^{\infty} \left(\frac{D^*}{\sqrt{-2\nu}}\right)^j \left(\frac{L^* Q_i + Q_i L - M}{-2\nu}\right) \left(\frac{D}{\sqrt{-2\nu}}\right)^j.$$

Here we used that the spectral radius of D, i.e.

$$\mathfrak{r}(D) := \sup\{|\lambda| \; : \; \lambda \in \sigma(D)\},$$

satisfies

$$\mathfrak{r}(D) \leq ||D|| < \sqrt{-2||L|| - 2\nu} \leq \sqrt{-2\nu}.$$

We may estimate

$$\begin{aligned} ||Q_{i+1} - Q_i|| &\leq \sum_{j=0}^{\infty} \left(\frac{||D||^2}{-2\nu}\right)^j \frac{||L||}{-\nu} ||Q_i - Q_{i-1}|| \\ &= \frac{2||L||}{-(2\nu + ||D||^2)} ||Q_i - Q_{i-1}||, \end{aligned}$$

to find that the recursion is contractive.

Therefore there exists a unique solution Q to (18), that also satisfies

$$Q = \sum_{j=0}^{\infty} \left(\frac{D^*}{\sqrt{-2\nu}} \right)^j \left(\frac{M - L^*Q - QL}{2\nu} \right) \left(\frac{D}{\sqrt{-2\nu}} \right)^j .$$

If $M = 0$ then $Q = 0$ by unicity of the solution.

By [Lancaster and Tismenetsky, 1985], Exercise 12.3.4, we have

$$Q \geq \frac{L^*Q + QL - M}{-2\nu},$$

that is

$$N := (L + \nu I)^* Q + Q(L + \nu I) \leq M.$$

Note that $L + \nu I$ is stable. Therefore, if $M \leq 0$, then $N \leq 0$ and it follows by Lyapunov's stability theorem that $Q \geq 0$. Similarly, if $M < 0$ then $Q > 0$. □

B.2. **Remark.** Note that under the conditions of Proposition B.1, we have for $\widetilde{L} := L + \nu I$ that

$$||e^{\widetilde{L}t}|| \leq m e^{\omega t},$$

with $m = 1$, $\omega = ||L|| + \nu$, and hence the conditions

$$2\omega + ||D||^2 < 0$$

of Theorem 3.6 holds. Hence we might have deduced the same conclusion from that result.

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