Collision local time of transient random walks and intermediate phases in interacting stochastic systems

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Abstract

In a companion paper, a quenched large deviation principle (LDP) has been established for the empirical process of words obtained by cutting an i.i.d. sequence of letters into words according to a renewal process. We apply this LDP to prove that the radius of convergence of the moment generating function of the collision local time of two strongly transient random walks on $\mathbb{Z}^d$, $d \geq 1$, strictly increases when we condition on one of the random walks, both in discrete time and in continuous time. We conjecture that the same holds for two transient but not strongly transient random walks. The presence of these gaps implies the existence of an *intermediate phase* for the long-time behaviour of a class of coupled branching processes, interacting diffusions, respectively, directed polymers in random environments.

Key words: Random walks, collision local time, annealed vs. quenched, large deviation principle, interacting stochastic systems, intermediate phase.
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1 Introduction and main results

In this note, we derive variational representations for the radius of convergence of the moment generating functions of the collision local time of two transient random walks in discrete and continuous time, respectively. These representations are used to establish the existence of an intermediate phase for the large time behaviour of a class of interacting stochastic systems.

1.1 Collision local time of random walks

1.1.1 Discrete time

Let $S = (S_k)_{k=0}^{\infty}$ and $S' = (S'_k)_{k=0}^{\infty}$ be two independent random walks on $\mathbb{Z}^d$, $d \geq 1$, both starting at the origin, with a *symmetric* transition kernel $p(\cdot,\cdot)$. We write p^n for the n-th convolution power of p and abbreviate $p^n(x) := p^n(0,x)$. Suppose that

$$\lim_{n\to\infty} \frac{\log p^{2n}(0)}{\log(2n)} =: -\alpha, \quad \alpha \in (1,\infty). \tag{1.1}$$

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Write $\mathbb{P}$ to denote the joint law of S, S'. Let

$$V := \sum_{k=0}^{\infty} 1_{\{S_k = S'_k\}} \tag{1.2}$$

be the *collision local time* of S, S', and define

$$z_1 := \sup\left\{z \geq 0\colon\ \mathbb{E}\left[\, z^V \mid S\,\right] < \infty \ \ S\text{-a.s.}\right\}, \quad z_2 := \sup\left\{z \geq 0\colon\ \mathbb{E}\left[\, z^V\,\right] < \infty\right\}. \tag{1.3}$$

(The lower indices indicate the number of random walks being averaged over.) Note that, by the tail triviality of S, the range of z's for which $\mathbb{E}[\, z^V \mid S\,]$ converges is S-a.s. constant. Also note that (1.1) implies that $p(\cdot,\cdot)$ is transient, so that $\mathbb{P}(V < \infty) = 1$.

Let $E := \mathrm{supp}(p) \subset \mathbb{Z}^d$, let $\widetilde{E} = \cup_{n\in\mathbb{N}} E^n$ be the set of finite *words* drawn from E, and let $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ denote the shift-invariant probability measures on $\widetilde{E}^{\mathbb{N}}$, the set of infinite *sentences* drawn from $\widetilde{E}$. Define $f : \widetilde{E} \to [0,\infty)$ via

$$f((x_1,\dots,x_n)) = \frac{p^n(x_1+\cdots+x_n)}{p^n(0)}\,[G(0)-1], \quad n\in\mathbb{N}:\ p^n(0)>0,\ x_1,\dots,x_n\in E, \tag{1.4}$$

where $G(0) = \sum_{n=0}^{\infty} p^n(0)$ is the Green function.

Theorem 1.1. *Assume* (1.1). *Then* $z_1 = 1+\exp[-r_1]$, $z_2 = 1+\exp[-r_2]$ *with*

$$r_1 = \sup_{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})} \left\{\int_{\widetilde{E}} \pi_1 Q(dy)\,\log f(y) - I^{\mathrm{que}}(Q)\right\}, \tag{1.5}$$

$$r_2 = \sup_{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})} \left\{\int_{\widetilde{E}} \pi_1 Q(dy)\,\log f(y) - I^{\mathrm{ann}}(Q)\right\}, \tag{1.6}$$

where the rate functions I^{que} *and* I^{ann} *are given in Theorems* 2.2 *and* 2.1 *below.*

Theorem 1.2. *Assume* (1.1). *If* $p(\cdot,\cdot)$ *is strongly transient, then* $1 < z_2 < z_1 < \infty$.

Theorems 1.1 and 1.2 will be proved in Section 3. Since $\mathbb{P}(V=k) = (1-F^{(2)})[F^{(2)}]^{k-1}$, $k\in\mathbb{N}$, with $F^{(2)} := \mathbb{P}\big(\exists\, k\in\mathbb{N}\colon\ S_k = S'_k\big)$, an easy computation gives $z_2 = 1/F^{(2)}$. Note that $F^{(2)} = 1-[1/G^{(2)}(0)]$ with $G^{(2)}(0) = \sum_{n=0}^{\infty} p^{2n}(0)$ (see Spitzer [17], Section 1).

Unlike for z_2, no closed form expression is known for z_1. By evaluating the function inside the supremum in (1.5) at well-chosen Q's, one can easily obtain an upper bound.

Corollary 1.3. *Under the assumptions of Theorem* 1.2,

$$z_1 \leq 1 + \left(\sum_{n\in\mathbb{N}} e^{-h(p^n)}\right)^{-1}, \tag{1.7}$$

where $h(p^n) = -\sum_{x\in\mathbb{Z}^d} p^n(x)\log p^n(x)$ *is the entropy of* $p^n(\cdot)$.

Proof. Note that for $q \in \mathcal{P}(\widetilde{E})$ of the form

$$q(x_1,\dots,x_n) = \rho_q(n)\nu(x_1)\cdots\nu(x_n), \quad n\in\mathbb{N},\ x_1,\dots,x_n\in E, \tag{1.8}$$

for some $\rho_q \in \mathcal{P}(\mathbb{N})$, we have $I^{\mathrm{que}}(q^{\otimes\mathbb{N}}) = H(q^{\otimes\mathbb{N}} \mid q_{\rho,\nu}) = h(\rho_q \mid \rho)$, since $\Psi_{[q^{\otimes\mathbb{N}}]_{\mathrm{tr}}} = \nu^{\otimes\mathbb{N}}$ for any $\mathrm{tr}\in\mathbb{N}$ (and $\Psi_{q^{\otimes\mathbb{N}}} = \nu^{\otimes\mathbb{N}}$ when ρ_q has a finite mean). The claim therefore follows from (1.5) by choosing $Q = q^{\otimes\mathbb{N}}$, $\nu(x) = p(x)$, $x\in\mathbb{Z}^d$, and

$$\rho_q(n) = \frac{\exp[-h(p^n)]}{\sum_{m\in\mathbb{N}}\exp[-h(p^m)]}, \quad n\in\mathbb{N}. \tag{1.9}$$

□

It is easy to see that the choice (1.9) is optimal in the class of q's of the form (1.8).

Theorem 1.4. *If $p(\cdot,\cdot)$ satisfies* (1.1) *with $\alpha = 1$, then $z_1 = z_2$.*

Proof. This follows from the representations (1.5–1.6) in Theorem 1.1 and the fact that $I^{\text{que}} = I^{\text{ann}}$ when $\alpha = 1$. □

1.1.2 Continuous time

Next we turn the discrete-time random walks S and S' into continuous-time random walks $\widetilde{S} = (S_t)_{t\geq 0}$ and $\widetilde{S}' = (\widetilde{S}'_t)_{t\geq 0}$ by allowing them to make steps at rate 1, keeping the same $p(\cdot,\cdot)$. Then the collision local time becomes

$$\widetilde{V} := \int_0^\infty 1_{\{\widetilde{S}_t = \widetilde{S}'_t\}}\, dt. \tag{1.10}$$

For the analogous quantities $\widetilde{z}_1$ and $\widetilde{z}_2$, we have the following.

Theorem 1.5. *Assume* (1.1). *If $p(\cdot,\cdot)$ is strongly transient, then $0 < \widetilde{z}_2 < \widetilde{z}_1 < \infty$.*

Theorem 1.5 will be proved in Section 3.3. An easy computation gives $\log \widetilde{z}_2 = 2/G(0)$ with $G(0) = \sum_{n=0}^{\infty} p^n(0)$. There is again no simple expression for $\widetilde{z}_1$.

1.1.3 Discussion

As the reader will see in Section 3, our proof of Theorem 1.2 is based on the representations given in Theorem 1.1. Additional technical difficulties arise in the situation where the maximiser in (1.6) has infinite mean word length, which happens exactly when $p(\cdot,\cdot)$ is transient but not strongly transient. This will be pursued in future work, for the moment we close with the following conjecture.

Conjecture 1.6. *The gaps in Theorems* 1.2 and 1.5 *are present also when $p(\cdot,\cdot)$ is transient but not strongly transient.*

Random walks with zero mean and finite variance are transient for $d \geq 3$ and strongly transient for $d \geq 5$ (Spitzer [17], Section 1). In a paper by Birkner and Sun [4], the gap in Theorem 1.2 is proved for simple random walk on $\mathbb{Z}^d$, $d \geq 4$, and the proof is in principle extendable to a more general class of random walks (see the discussion in [4] after the proof of Theorem 1.3). The proof in [4] is an adaptation of the fractional moment technique developed by Derrida, Giacomin, Lacoin and Toninelli [10] in the context of random pinning models. Note that simple random walk on $\mathbb{Z}^4$ is just on the border of not being strongly transient.

1.2 The gaps settle three conjectures

In this section we use Theorems 1.2–1.5 to prove the existence of an *intermediate phase* for three classes of interacting particle systems.

1.2.1 Coupled branching processes

A Theorem 1.5 *proves* a conjecture put forward in Greven [12], [13]. Consider a spatial population model, defined as the Markov process $(\eta_t)_{t\geq 0}$ taking values in $(\mathbb{N} \cup \{0\})^{\mathbb{Z}^d}$ (counting the number of individuals at the different sites of $\mathbb{Z}^d$) evolving as follows:

(1) Individuals migrate at rate 1 according to $a(\cdot,\cdot)$.

(2) A new individual is born at site x at rate $b\eta(x)$.

(3) One individual at site x dies at rate $(1-p)b\eta(x)$.

(4) All individuals at site x die simultaneously at rate pb.

Here, $a(\cdot,\cdot)$ is an irreducible random walk transition kernel on $\mathbb{Z}^d \times \mathbb{Z}^d$, $b \in (0,\infty)$ is a birth-death rate, $p \in [0,1]$ is a coupling parameter, while (1)–(4) occur independently at every $x \in \mathbb{Z}^d$. The case $p = 0$ corresponds to a critical branching random walk, for which the average number of individuals per site is preserved. The case $p > 0$ is challenging because the individuals descending from different ancestors are no longer independent.

A critical branching random walk satisfies the following *dichotomy* (where for simplicity we restrict to the case where $a(\cdot,\cdot)$ is symmetric): if the initial configuration η_0 is drawn from a shift-invariant probability distribution with finite mean, then η_t as $t \to \infty$ locally dies out ("extinction") when $a(\cdot,\cdot)$ is recurrent, but converges to a non-trivial equilibrium ("survival") when $a(\cdot,\cdot)$ is transient, both irrespective of the value of b. In the latter case, the equilibrium has the same mean as the initial distribution and has all moments finite.

For the coupled branching process with $p > 0$ there is a dichotomy too, but it is controlled by a subtle interplay of $a(\cdot,\cdot)$, b and p: extinction holds when $a(\cdot,\cdot)$ is recurrent, but also when $a(\cdot,\cdot)$ is transient and p is sufficiently large. Indeed, it is shown in Greven [12] that if $a(\cdot,\cdot)$ is transient, then there is a unique $p_* \in (0,1)$ such that survival holds for $p < p_*$ and extinction holds for $p > p_*$.

Recall the critical values $\widetilde{z}_1, \widetilde{z}_2$ introduced in Section 1.1.2. Then survival holds if $\mathbb{E}(\exp[bp\widetilde{V}] \mid \widetilde{S}) < \infty$ $\widetilde{S}$-a.s., i.e., if $p < p_1$ with $p_1 = b^{-1}\log\widetilde{z}_1$. This can be shown by a size-biasing of the population in the spirit of Kallenberg [15]. On the other hand, survival with a *finite second moment* holds if and only if $\mathbb{E}(\exp[bp\widetilde{V}]) < \infty$, i.e., if and only if $p < p_2$ with $p_2 = b^{-1}\log\widetilde{z}_2$. Clearly, $p_* \geq p_1 \geq p_2$. Theorem 1.5 shows that if $a(\cdot,\cdot)$ satisfies (1.1) and is strongly transient, then $p_1 > p_2$, implying that there is an intermediate phase of survival with an *infinite second moment*.

B Theorem 1.2 *corrects* an error in Birkner [1], Theorem 6. Here, a system of individuals living on $\mathbb{Z}^d$ is considered subject to migration and branching. Each individual independently migrates at rate 1 according to a random walk transition kernel $a(\cdot,\cdot)$, and branches at a rate that depends on the number of individuals present at the same location. It is argued that this system has an intermediate phase in which the numbers of individuals at different sites tend to an equilibrium with a finite first moment but an infinite second moment. The proof was, however, based on a wrong rate function. The rate function claimed in Birkner [1], Theorem 6, must be replaced by that in [3], Corollary 1.5, after which the intermediate phase persists. This also affects [1], Theorem 5, which uses [1], Theorem 6, to compute z_1 in Section 1.1 and finds an incorrect formula. Corollary 1.3 shows that this formula actually is an upper bound for z_1.

1.2.2 Interacting diffusions

Theorem 1.5 *proves* a conjecture put forward in Greven and den Hollander [14]. Consider the system of interacting diffusions on $[0,\infty)$ defined by the following collection of coupled stochastic differential equations:

$$dX_x(t) = \sum_{y\in\mathbb{Z}^d} a(x,y)[X_y(t) - X_x(t)]\,dt + \sqrt{bX_x(t)^2}\,dW_x(t), \qquad x \in \mathbb{Z}^d, t \geq 0. \tag{1.11}$$

Here, $a(\cdot,\cdot)$ is an irreducible random walk transition kernel on $\mathbb{Z}^d \times \mathbb{Z}^d$, $b \in (0,\infty)$ is a diffusion parameter, and $(\{W_x(t)\}_{x\in\mathbb{Z}^d})_{t\geq 0}$ is a collection of independent standard Brownian motions on $\mathbb{R}$. The initial condition is chosen such that $\{X_x(0)\}_{x\in\mathbb{Z}^d}$ is a shift-invariant and shift-ergodic random field on $[0,\infty)$ with mean $\Theta \in (0,\infty)$ (the evolution preserves the mean).

It was shown in [14], Theorems 1.4–1.6, that if $a(\cdot,\cdot)$ is symmetric and transient, then there exist $0 < b_2 \leq b_*$ such that the system in (1.11) converges to an equilibrium when $0 < b < b_*$, and this equilibrium has a *finite second moment* when $0 < b < b_2$ and an *infinite second moment* when $b_2 \leq b < b_*$. It was conjectured in [14], Conjecture 1.8, that $b_* > b_2$. As explained in [14], Section 4.2, the gap in Theorem 1.5 settles this conjecture (at least when $a(\cdot,\cdot)$ is strongly transient), with $b_2 = \log \widetilde{z}_2$ and $b_* = \log \widetilde{z}_1$.

1.2.3 Directed polymers in random environments

Theorem 1.2 *disproves* a conjecture put forward in Monthus and Garel [16]. Let $a(\cdot,\cdot)$ be a symmetric and irreducible random walk transition kernel on $\mathbb{Z}^d \times \mathbb{Z}^d$, let $S = (S_k)_{k=0}^\infty$ be the corresponding random walk, and let $\xi = \{\xi(x,n) \colon x \in \mathbb{Z}^d, n \in \mathbb{N}\}$ be i.i.d. $\mathbb{R}$-valued non-degenerate random variables satisfying

$$\lambda(\beta) := \log \mathbb{E}\big(\exp[\beta\xi(x,n)]\big) \in \mathbb{R} \qquad \forall\, \beta \in \mathbb{R}. \tag{1.12}$$

Put

$$e_n(\xi, S) := \exp\left[\sum_{k=1}^n \{\beta\xi(S_k,k) - \lambda(\beta)\}\right], \tag{1.13}$$

and set

$$Z_n(\xi) := \mathbb{E}[e_n(\xi,S)] = \sum_{s_1,\dots,s_n \in \mathbb{Z}^d} \left[\prod_{k=1}^n p(s_{k-1},s_k)\right] e_n(\xi,s), \qquad s = (s_k)_{k=0}^\infty,\ s_0 = 0, \tag{1.14}$$

i.e., $Z_n(\xi)$ is the normalising constant in the probability distribution of the random walk S whose paths are reweighted by $e_n(\xi,S)$, which is referred to as the "polymer measure". The $\xi(x,n)$'s describe a random space-time medium with which S is interacting, with β playing the role of the interaction strength or inverse temperature.

It is well known that $(Z_n)_{n\in\mathbb{N}}$ is a non-negative martingale with respect to the family of sigma-algebras $\mathcal{F}_n := \sigma(\xi(x,k), x \in \mathbb{Z}^d, 1 \leq k \leq n)$, $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} Z_n = Z_\infty \geq 0 \qquad \xi - a.s., \tag{1.15}$$

with the event $\{Z_\infty = 0\}$ being ξ-trivial. One speaks of *weak disorder* if $Z_\infty > 0$ ξ-a.s. and of *strong disorder* otherwise. As shown in Comets and Yoshida [8], there is a unique critical value $\beta_* \in [0,\infty]$ such that weak disorder holds for $0 \leq \beta < \beta_*$ and strong disorder holds for $\beta > \beta_*$. Moreover, in the weak disorder region the paths have a Gaussian scaling limit under the polymer measure, while this is not the case in the strong disorder region.

Recall the critical values z_1, z_2 defined in Section 1.1. Bolthausen [5] observed that

$$\mathbb{E}\left[Z_n^2\right] = \mathbb{E}\Big[\exp\left[\{\lambda(2\beta) - 2\lambda(\beta)\}\, V_n\right]\Big], \qquad \text{with } V_n := \sum_{k=1}^n \mathbb{1}_{\{S_k = S'_k\}}, \tag{1.16}$$

where S and S' are two independent random walks with transition kernel $p(\cdot,\cdot)$, and concluded that $(Z_n)_{n\in\mathbb{N}}$ is L^2-bounded if and only if $\beta < \beta_2$ with $\beta_2 \in (0,\infty]$ the unique solution of

$$\lambda(2\beta_2) - 2\lambda(\beta_2) = z_2. \tag{1.17}$$

Since $\mathbb{P}(Z_\infty > 0) \geq \mathbb{E}[Z_\infty]^2/\mathbb{E}[Z_\infty^2]$ and $\mathbb{E}[Z_\infty] = Z_0 = 1$ for an L^2-bounded martingale, it follows that $\beta < \beta_2$ implies weak disorder, i.e., $\beta_* \geq \beta_2$. By a stochastic representation of the size-biased law of Z_n, it was shown in Birkner [2], Proposition 1, that in fact weak disorder holds if $\beta < \beta_1$ with $\beta_1 \in (0,\infty]$ the unique solution of

$$\lambda(2\beta_1) - 2\lambda(\beta_1) = z_1, \tag{1.18}$$

i.e., $\beta_* \geq \beta_1$. Since $\beta \mapsto \lambda(2\beta) - 2\lambda(\beta)$ is strictly increasing for any non-trivial law for the disorder satisfying (1.12), it follows from (1.17–1.18) and Theorem 1.2 that $\beta_1 > \beta_2$ when $a(\cdot,\cdot)$ satisfies (1.1) and is strongly transient and when ξ is such that $\beta_2 < \infty$. In that case the weak disorder region contains a subregion for which $(Z_n)_{n\in\mathbb{N}}$ is *not* L^2-bounded. This disproves a conjecture of Monthus and Garel [16], who argued that $\beta_2 = \beta_*$. Camanes and Carmona [6] consider the same problem for simple random walk and specific choices of disorder. With the help of fractional moment estimates of Evans and Derrida [11] and numerical computation they prove $\beta_* > \beta_2$ for Gaussian disorder in $d \geq 5$, for Binomial disorder with small mean in $d \geq 4$ and for Poisson disorder with small mean in $d \geq 3$.

Outline

In Section 2 we recall the LDPs in [3] that are needed for Theorem 1.1 and its counterpart for continuous-time random walk. In Section 3 we use these LDPs to prove Theorems 1.2 and 1.5.

2 Word sequences and annealed and quenched LDP

We recall the problem setting in [3]. Let E be a finite or countable set of *letters*. Let $\widetilde{E} = \cup_{n\in\mathbb{N}} E^n$ be the set of finite *words* drawn from E. Both E and $\widetilde{E}$ are Polish spaces under the discrete topology. Let $\mathcal{P}(E^{\mathbb{N}})$ and $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$ denote the set of probability measures on sequences drawn from E, respectively, $\widetilde{E}$, equipped with the topology of weak convergence. Write θ and $\widetilde{\theta}$ for the left-shift acting on $E^{\mathbb{N}}$, respectively, $\widetilde{E}^{\mathbb{N}}$. Write $\mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}}), \mathcal{P}^{\mathrm{erg}}(E^{\mathbb{N}})$ and $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}), \mathcal{P}^{\mathrm{erg}}(\widetilde{E}^{\mathbb{N}})$ for the set of probability measures that are invariant and ergodic under θ, respectively, $\widetilde{\theta}$.

For $\nu \in \mathcal{P}(E)$, let $X = (X_i)_{i\in\mathbb{N}}$ be i.i.d. with law ν. For $\rho \in \mathcal{P}(\mathbb{N})$, let $\tau = (\tau_i)_{i\in\mathbb{N}}$ be i.i.d. with law ρ having infinite support and satisfying the *algebraic tail property*

$$\lim_{\substack{n\to\infty \\ \rho(n)>0}} \frac{\log \rho(n)}{\log n} =: -\alpha, \quad \alpha \in (1,\infty). \tag{2.1}$$

(No regularity assumption is imposed on $\mathrm{supp}(\rho)$.) Assume that X and τ are independent and write $\mathbb{P}$ to denote their joint law. Cut words out of X according to τ, i.e., put (see Figure 1)

$$T_0 := 0 \quad \text{and} \quad T_i := T_{i-1} + \tau_i, \quad i \in \mathbb{N}, \tag{2.2}$$

and let

$$Y^{(i)} := \big(X_{T_{i-1}+1}, X_{T_{i-1}+2}, \dots, X_{T_i}\big), \quad i \in \mathbb{N}. \tag{2.3}$$

Then, under the law $\mathbb{P}$, $Y = (Y^{(i)})_{i\in\mathbb{N}}$ is an i.i.d. sequence of words with marginal law $q_{\rho,\nu}$ on $\widetilde{E}$ given by

$$q_{\rho,\nu}\big((x_1,\dots,x_n)\big) := \mathbb{P}\big(Y^{(1)} = (x_1,\dots,x_n)\big) = \rho(n)\,\nu(x_1)\cdots\nu(x_n), \quad n \in \mathbb{N},\ x_1,\dots,x_n \in E. \tag{2.4}$$

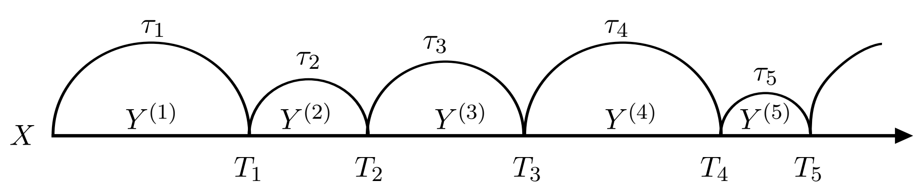

Figure 1: Cutting words from a letter sequence according to a renewal process.

For $N \in \mathbb{N}$, let $(Y^{(1)}, \dots, Y^{(N)})^{\mathrm{per}}$ stand for the periodic extension of $(Y^{(1)}, \dots, Y^{(N)})$ to an element of $\widetilde{E}^{\mathbb{N}}$, and define

$$R_N := \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\widetilde{\theta}^i (Y^{(1)}, \dots, Y^{(N)})^{\mathrm{per}}} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}), \tag{2.5}$$

the *empirical process of N-tuples of words.*

The following large deviation principle (LDP) is standard (see e.g. Dembo and Zeitouni [9], Corollaries 6.5.15 and 6.5.17). Let

$$H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) := \lim_{N\to\infty} \frac{1}{N}\, h\left(Q_{|\mathscr{F}_N} \mid (q_{\rho,\nu}^{\otimes\mathbb{N}})_{|\mathscr{F}_N} \right) \in [0,\infty] \tag{2.6}$$

be the *specific relative entropy of Q w.r.t. $q_{\rho,\nu}^{\otimes\mathbb{N}}$*, where $\mathscr{F}_N = \sigma(Y^{(1)}, \dots, Y^{(N)})$ is the sigma-algebra generated by the first N words, $Q_{|\mathscr{F}_N}$ is the restriction of Q to $\mathscr{F}_N$, and $h(\cdot \mid \cdot)$ denotes relative entropy.

Theorem 2.1. [Annealed LDP] *The family of probability distributions $\mathbb{P}(R_N \in \cdot)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with rate function $I^{\mathrm{ann}} \colon \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \to [0,\infty]$ given by*

$$I^{\mathrm{ann}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}). \tag{2.7}$$

This rate function is lower semi-continuous, has compact level sets, has a unique zero at $Q = q_{\rho,\nu}^{\otimes\mathbb{N}}$, and is affine.

Let $\kappa \colon \widetilde{E}^{\mathbb{N}} \to E^{\mathbb{N}}$ denote the *concatenation map* that glues a sequence of words into a sequence of letters. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ such that $m_Q := \mathbb{E}_Q[\tau_1] < \infty$, define $\Psi_Q \in \mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}})$ as

$$\Psi_Q(\cdot) := \frac{1}{m_Q} \mathbb{E}_Q \left[\sum_{k=0}^{\tau_1 - 1} \delta_{\theta^k \kappa(Y)}(\cdot) \right]. \tag{2.8}$$

Think of Ψ_Q as *the shift-invariant version of the concatenation of Y under the law Q obtained after randomising the location of the origin.*

For $\mathrm{tr} \in \mathbb{N}$, let $[\cdot]_{\mathrm{tr}} \colon \widetilde{E} \to [\widetilde{E}]_{\mathrm{tr}} := \cup_{n=1}^{\mathrm{tr}} E^n$ denote the *word length truncation* map defined by

$$y = (x_1, \dots, x_n) \mapsto [y]_{\mathrm{tr}} := (x_1, \dots, x_{n \wedge \mathrm{tr}}), \qquad n \in \mathbb{N},\ x_1, \dots, x_n \in E. \tag{2.9}$$

Extend this to a map from $\widetilde{E}^{\mathbb{N}}$ to $[\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}}$ via $\left[(y^{(1)}, y^{(2)}, \dots)\right]_{\mathrm{tr}} := \left([y^{(1)}]_{\mathrm{tr}}, [y^{(2)}]_{\mathrm{tr}}, \dots\right)$ and to a map from $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}^{\mathrm{inv}}([\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}})$ via $[Q]_{\mathrm{tr}}(A) := Q(\{z \in \widetilde{E}^{\mathbb{N}} \colon [z]_{\mathrm{tr}} \in A\})$ for $A \subset [\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}}$ measurable. Note that if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, then $[Q]_{\mathrm{tr}}$ is an element of the set

$$\mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) = \{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \colon\ m_Q < \infty\}. \tag{2.10}$$

The following theorem summarises the main results from [3].

Theorem 2.2. [Quenched LDP] *Assume* (2.1)*. Then, for $\nu^{\otimes\mathbb{N}}$–a.s. all X, the family of (regular) conditional probability distributions $\mathbb{P}(R_N \in \cdot \mid X)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with deterministic rate function $I^{\mathrm{que}} \colon \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \to [0,\infty]$ given by*

$$I^{\mathrm{que}}(Q) := \begin{cases} I^{\mathrm{fin}}(Q), & \text{if } Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \\ \lim_{\mathrm{tr}\to\infty} I^{\mathrm{fin}}\big([Q]_{\mathrm{tr}}\big), & \text{otherwise}, \end{cases} \tag{2.11}$$

where

$$I^{\mathrm{fin}}(Q) := H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) + (\alpha-1)\, m_Q\, H(\Psi_Q \mid \nu^{\otimes\mathbb{N}}). \tag{2.12}$$

The rate function I^{que} is lower semi-continuous, has compact level sets, has a unique zero at $Q = q_{\rho,\nu}^{\otimes\mathbb{N}}$, and is affine. Moreover, it is equal to the lower semi-continuous extension of I^{fin} from $\mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$.

If (2.1) *holds with $\alpha = 1$, then for $\nu^{\otimes\mathbb{N}}$–a.s. all X, the family $\mathbb{P}(R_N \in \cdot \mid X)$ satisfies the LDP with rate function I^{ann} given by* (2.7)*.*

Note that the quenched rate function (2.12) equals the annealed rate function (2.7) plus an additional term which quantifies the deviation of Ψ_Q from the reference law $\nu^{\otimes\mathbb{N}}$ on the letter sequence. The set

$$\mathscr{R}_\nu := \left\{ Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \colon \ \mathrm{w}-\lim_{L\to\infty} \frac{1}{L} \sum_{k=0}^{L-1} \delta_{\theta^k \kappa(Y)} = \nu^{\otimes\mathbb{N}} \ Q-a.s. \right\}. \tag{2.13}$$

is formed by those Q's for which *the concatenation of words has the same statistical properties as the letter sequence X*. For $Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}})$, we have (see [3], Equation (1.22))

$$\Psi_Q = \nu^{\otimes\mathbb{N}} \quad \Longleftrightarrow \quad I^{\mathrm{que}}(Q) = I^{\mathrm{ann}}(Q) \quad \Longleftrightarrow \quad Q \in \mathscr{R}_\nu. \tag{2.14}$$

3 Proof of Theorems 1.2 and 1.5

3.1 Proof of Theorem 1.2

Proof. The idea is to put the problem into the framework of (2.1–2.5) and then apply Theorem 2.2. To that end, we pick

$$E := \mathbb{Z}^d, \qquad \widetilde{E} := \cup_{n\in\mathbb{N}} (\mathbb{Z}^d)^n, \tag{3.1}$$

and choose

$$\nu(u) := p(u), \quad u \in E, \qquad \rho(n) := \frac{p^n(0)}{G(0)-1}, \quad n \in \mathbb{N}, \tag{3.2}$$

where

$$p(u) = p(0,u),\ u \in \mathbb{Z}^d, \quad p^n(u-v) = p^n(u,v),\ u,v \in \mathbb{Z}^d, \quad G(0) = \sum_{n=0}^{\infty} p^n(0), \tag{3.3}$$

the latter being the Green function at the origin.

Recalling (1.2), and writing

$$z^V = \big((z-1)+1\big)^V = 1 + \sum_{N=1}^{V} (z-1)^N \, \frac{V(V-1)\cdots(V-N+1)}{N!} \tag{3.4}$$

with

$$\frac{V(V-1)\cdots(V-N+1)}{N!} = \sum_{0<j_1<\cdots<j_N<\infty} \mathbb{1}_{\{S_{j_1}=S'_{j_1},\dots,S_{j_N}=S'_{j_N}\}}, \tag{3.5}$$

we have

$$\mathbb{E}\left[\, z^V \mid S \,\right] = 1 + \sum_{N=1}^{\infty} (z-1)^N F_N^{(1)}(X), \qquad \mathbb{E}\left[\, z^V \,\right] \ = 1 + \sum_{N=1}^{\infty} (z-1)^N F_N^{(2)}, \tag{3.6}$$

with

$$F_N^{(1)}(X) := \sum_{0<j_1<\cdots<j_N<\infty} \mathbb{P}(S_{j_1} = S'_{j_1}, \dots, S_{j_N} = S'_{j_N} \mid X), \qquad F_N^{(2)} \ := \mathbb{E}\big[F_N^{(1)}(X)\big], \tag{3.7}$$

where $X = (X_k)_{k\in\mathbb{N}}$ denotes the sequence of increments of S. (The upper indices 1 and 2 indicate the number of random walks being averaged over.)

The notation in (3.1–3.2) allows us to rewrite the first line of (3.7) as

$$\begin{aligned} F_N^{(1)}(X) &= \sum_{0<j_1<\cdots<j_N<\infty} \prod_{i=1}^N p^{j_i-j_{i-1}}\left(\sum_{k=j_{i-1}+1}^{j_i} X_k\right) \\ &= \sum_{0<j_1<\cdots<j_N<\infty} \prod_{i=1}^N \rho(j_i-j_{i-1})\ \exp\left[\sum_{i=1}^N \log\left(\frac{p^{j_i-j_{i-1}}(\sum_{k=j_{i-1}+1}^{j_i} X_k)}{\rho(j_i-j_{i-1})}\right)\right] \end{aligned} \tag{3.8}$$

Let $Y^{(i)} = (X_{j_{i-1}+1}, \cdots, X_{j_i})$. Recall the definition (1.4) of $f:\widetilde{E}\to[0,\infty)$ as

$$f((x_1,\dots,x_n)) = \frac{p^n(x_1+\cdots+x_n)}{p^n(0)}\,[G(0)-1], \quad n\in\Lambda,\ x_1,\dots,x_n\in E, \tag{3.9}$$

with

$$\Lambda := \{n\in\mathbb{N} : \rho(n) = p^n(0) > 0\} \supset 2\mathbb{Z}, \tag{3.10}$$

let $R_N \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ be the empirical process of words defined in (2.5), and $\pi_1 R_N \in \mathcal{P}(\widetilde{E})$ the projection of R_N onto the first coordinate. Then we have

$$F_N^{(1)}(X) = \mathbb{E}\left[\exp\left(\sum_{i=1}^N \log f(Y^{(i)})\right)\Bigg|\, X\right] = \mathbb{E}\left[\exp\left(N\int_{\widetilde{E}} (\pi_1 R_N)(dy)\ \log f(y)\right)\Bigg|\, X\right]. \tag{3.11}$$

The second line of (3.7) is obtained by averaging (3.11) over X:

$$F_N^{(2)} = \mathbb{E}\left[\exp\left(N\int_{\widetilde{E}} (\pi_1 R_N)(dy)\ \log f(y)\right)\right]. \tag{3.12}$$

Without conditioning on X, the sequence $(Y^{(i)})_{i\in\mathbb{N}}$ is i.i.d. with law (recall (2.4))

$$q_{\rho,\nu}^{\otimes\mathbb{N}} \quad \text{with} \quad q_{\rho,\nu}(x_1,\dots,x_n) = \frac{p^n(0)}{G(0)-1}\prod_{k=1}^n p(x_k). \tag{3.13}$$

Next we note that f as in (3.9) is bounded from above. Indeed, the Fourier representation of $p^n(x,y)$ reads

$$p^n(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi)^d} dk\, e^{-i(k\cdot x)}\,\widehat{p}(k)^n \tag{3.14}$$

with $\widehat{p}(k) = \sum_{x\in\mathbb{Z}^d} e^{i(k\cdot x)} p(0,x)$. Because $p(\cdot,\cdot)$ is symmetric, it follows that

$$\max_{x\in\mathbb{Z}^d} p^{2n}(x) = p^{2n}(0), \qquad \max_{x\in\mathbb{Z}^d} p^{2n+1}(x) \le p^{2n}(0), \qquad \forall\, n\in\mathbb{N}. \tag{3.15}$$

Consequently, $f((x_1,\dots,x_n)) \le [p^{n-1}(0)/p^n(0)][G(0)-1]$, $n\in\Lambda$, which is bounded from above because of (1.1). The *annealed* LDP in Theorem 2.1, together with Varadhan's lemma applied to (3.12), therefore gives $z_2 = 1+\exp[-r_2]$ with

$$\begin{aligned} r_2 := \lim_{N\to\infty}\frac{1}{N}\log F_N^{(2)} &= \sup_{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})}\left\{\int_{\widetilde{E}} \pi_1 Q(dy)\ \log f(y) - I^{\mathrm{ann}}(Q)\right\} \\ &= \sup_{q\in\mathcal{P}(\widetilde{E})}\left\{\int_{\widetilde{E}} q(dy)\ \log f(y) - h(q\mid q_{\rho,\nu})\right\} \end{aligned} \tag{3.16}$$

(recall (1.3) and (3.6)). The last equality stems from the fact that, on the set of Q's with a given marginal $\pi_1 Q = q$, the function $Q\mapsto I^{\mathrm{ann}}(Q) = H(Q\mid q_{\rho,\nu}^{\otimes\mathbb{N}})$ has a unique minimiser $Q = q^{\otimes\mathbb{N}}$.

In order to carry out the second supremum in (3.16), we use the following.

Lemma 3.1. *Let* $Z := \sum_{y\in E} f(y)q_{\rho,\nu}(y)$*. Then*

$$\int_{\widetilde{E}} q(dy)\,\log f(y) - h(q \mid q_{\rho,\nu}) = \log Z - h(q \mid q^*) \qquad \forall\, q \in \mathcal{P}(\widetilde{E}), \tag{3.17}$$

where $q^*(y) = f(y)q_{\rho,\nu}(y)/Z$*,* $y \in E$*.*

Proof. This follows from a straightforward computation. □

Inserting (3.17) into (3.16), we see that the suprema are uniquely attained at $q = q^*$ and $Q = (q^*)^{\otimes\mathbb{N}}$, and that $r_2 = \log Z$. From (3.9) and (3.13), we have

$$Z = \sum_{n\in\mathbb{N}} \sum_{x_1,\dots,x_n\in\mathbb{Z}^d} p^n(x_1+\cdots+x_n) \prod_{k=1}^n p(x_k) = \sum_{n\in\mathbb{N}} p^{2n}(0) = G^{(2)}(0) - 1, \tag{3.18}$$

where we use that $\sum_{v\in\mathbb{Z}^d} p^m(u+v)p(v) = p^{m+1}(u)$, $u \in \mathbb{Z}^d$, $m \in \mathbb{N}$, and $G^{(2)}(0)$ is the Green function at the origin associated with $p^2(\cdot,\cdot)$. Hence the maximizer in (3.16) is

$$q^*(x_1,\dots,x_n) = \frac{p^n(x_1+\cdots+x_n)}{G^{(2)}(0)-1} \prod_{k=1}^n p(x_k). \tag{3.19}$$

Note that $z_2 = 1 + \exp[-\log Z] = G^{(2)}(0)/[G^{(2)}(0)-1]$.

The *quenched* LDP in Theorem 2.2, together with Varadhan's lemma applied to (3.8), gives $z_1 = 1 + \exp[-r_1]$ with

$$r_1 := \lim_{N\to\infty} \frac{1}{N} \log F_N^{(1)}(X) = \sup_{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})} \left\{ \int_{\widetilde{E}} \pi_1 Q(dy)\,\log f(y) - I^{\mathrm{que}}(Q) \right\} \quad X - a.s., \tag{3.20}$$

where $I^{\mathrm{que}}(Q)$ is given by (2.11–2.12).

To compare (3.20) with (3.16), we need the following lemma, the proof of which is deferred to Section 3.2.

Lemma 3.2. *Assume* (1.1)*. Let* $Q^* = (q^*)^{\otimes\mathbb{N}}$ *with* q^* *as in* (3.19)*. If* $m_{Q^*} < \infty$*, then* $I^{\mathrm{que}}(Q^*) > I^{\mathrm{ann}}(Q^*)$*.*

With the help of Lemma 3.2 we complete the proof of the existence of the gap as follows. Since $\log f$ is bounded from above, the function

$$Q \mapsto \int \log f(y)\, \pi_1 Q(dy) - I^{\mathrm{que}}(Q) \tag{3.21}$$

is upper semicontinuous. By compactness of the level sets of $I^{\mathrm{que}}(Q)$, the function in (3.21) therefore achieves its maximum at some $\widetilde{Q}$ that satisfies

$$r_1 = \int_{\widetilde{E}} \pi_1\widetilde{Q}(dy)\,\log f(y) - I^{\mathrm{que}}(\widetilde{Q}) \le \int_{\widetilde{E}} \pi_1\widetilde{Q}(dy)\,\log f(y) - I^{\mathrm{ann}}(\widetilde{Q}) \le r_2. \tag{3.22}$$

If $r_1 = r_2$, then $\widetilde{Q} = Q^*$, because the unconditional variational problem (3.16) has Q^* as its unique maximiser. But $I^{\mathrm{que}}(Q^*) > I^{\mathrm{ann}}(Q^*)$ by Lemma 3.2, so this is a contradiction, and we arrive at $r_1 < r_2$ as required. □

3.2 Proof of Lemma 3.2

Proof. Note that

$$q^*(E^n) = \sum_{x_1,\dots,x_n\in\mathbb{Z}^d} \frac{p^n(x_1+\cdots+x_n)}{G^{(2)}(0)-1}\prod_{k=1}^n p(x_k) = \frac{p^{2n}(0)}{G^{(2)}(0)-1}, \quad n\in\mathbb{N}, \tag{3.23}$$

and hence, by assumption (1.2),

$$\lim_{n\to\infty}\frac{\log q^*(E^n)}{\log n} = -\alpha \tag{3.24}$$

and

$$m_{Q^*} = \sum_{n=1}^\infty n q^*(E^n) = \sum_{n=1}^\infty \frac{np^{2n}(0)}{G^{(2)}(0)-1}. \tag{3.25}$$

We will show that

$$m_{Q^*}<\infty \quad\Longrightarrow\quad Q^* = (q^*)^{\otimes\mathbb{N}} \notin \mathcal{R}_\nu, \tag{3.26}$$

the set defined in (2.13). This implies $\Psi_{Q^*}\neq\nu^{\otimes\mathbb{N}}$ (recall (2.14)), and hence $H(\Psi_{Q^*}|\nu^{\otimes\mathbb{N}})>0$, implying the claim.

In order to verify (3.26), we compute the first two marginals of Ψ_{Q^*}. Using the symmetry of $p(\cdot,\cdot)$, we have

$$\Psi_{Q^*}(a) = \frac{1}{m_{Q^*}}\sum_{n=1}^\infty\sum_{j=1}^n \sum_{\substack{x_1,\dots,x_n\in\mathbb{Z}^d\\ x_j=a}} \frac{p^n(x_1+\cdots+x_n)}{G^{(2)}(0)-1}\prod_{k=1}^n p(x_k) = p(a)\,\frac{\sum_{n=1}^\infty np^{2n-1}(a)}{\sum_{n=1}^\infty np^{2n}(0)}. \tag{3.27}$$

Hence, $\Psi_{Q^*}(a)=p(a)$ for all $a\in\mathbb{Z}^d$ with $p(a)>0$ if and only if

$$a\mapsto \sum_{n=1}^\infty n\,p^{2n-1}(a) \text{ is constant on the support of } p(\cdot). \tag{3.28}$$

There are many $p(\cdot,\cdot)$'s for which (3.28) fails, and for these (3.26) holds. However, for simple random walk (3.28) does not fail, because $a\mapsto p^{2n-1}(a)$ is constant on the $2d$ neighbours of the origin, and so we have to look at the two-dimensional marginal.

Observe that $q^*(x_1,\dots,x_n) = q^*(x_{\sigma(1)},\dots x_{\sigma(n)})$ for any permutation σ of $\{1,\dots,n\}$. For $a,b\in\mathbb{Z}^d$, we have

$$\begin{aligned} m_{Q^*}\Psi_{Q^*}(a,b) &= \mathbb{E}_{Q^*}\left[\sum_{k=1}^{\tau_1}\mathbb{1}\left(\kappa(Y)_k=a,\kappa(Y)_{k+1}=b\right)\right]\\ &= \sum_{n=1}^\infty\sum_{n'=1}^\infty\sum_{x_1,\dots,x_{n+n'}} q^*(x_1,\dots,x_n)\; q^*(x_{n+1},\dots,x_{n+n'})\sum_{k=1}^n \mathbb{1}_{(a,b)}(x_k,x_{k+1}) \\ &= q^*(x_1=a)\,q^*(x_1=b) + \sum_{n=2}^\infty (n-1)q^*\big(x_1=a,x_2=b\big). \end{aligned} \tag{3.29}$$

Since

$$\begin{aligned} q^*(x_1=a) &= \frac{p(a)^2}{G^{(2)}(0)-1} + \sum_{n=2}^\infty\sum_{x_2,\dots,x_n\in\mathbb{Z}^d}\frac{p^n(a+x_2+\cdots+x_n)}{G^{(2)}(0)-1}\,p(a)\prod_{k=2}^n p(x_k)\\ &= \frac{p(a)}{G^{(2)}(0)-1}\sum_{n=1}^\infty p^{2n-1}(a) \end{aligned} \tag{3.30}$$

and

$$
\begin{aligned}
q^*\big(x_1 = a, x_2 = b\big) &= \mathbb{1}_{n=2}\,\frac{p(a)p(b)}{G^{(2)}(0)-1}\,p^2(a+b) \\
&\qquad + \mathbb{1}_{n\geq 3}\,\frac{p(a)p(b)}{G^{(2)}(0)-1}\sum_{x_3,\dots,x_n\in\mathbb{Z}^d} p^n(a+b+x_3+\cdots+x_n)\prod_{k=3}^n p(x_k) \\
&= \frac{p(a)p(b)}{G^{(2)}(0)-1}\,p^{2n-2}(a+b),
\end{aligned}
\tag{3.31}
$$

we find

$$
\Psi_{Q^*}(a,b) = \frac{p(a)p(b)}{\sum_{n=1}^\infty np^{2n}(0)}\left(\left[\sum_{n=1}^\infty p^{2n-1}(a)\right]\left[\sum_{n=1}^\infty p^{2n-1}(b)\right] + \sum_{n=2}^\infty (n-1)p^{2n-2}(a+b)\right). \tag{3.32}
$$

Pick $b=-a$ with $p(a)>0$. Then, shifting n to $n-1$ in the last sum, we get

$$
\Psi_{Q^*}(a,-a) - p(a)^2 = \frac{\left[\sum_{n=1}^\infty p^{2n-1}(a)\right]^2}{\sum_{n=1}^\infty np^{2n}(0)} > 0. \tag{3.33}
$$

This shows that consecutive letters are not uncorrelated under Ψ_{Q^*}, and implies that (3.26) holds as claimed. □

3.3 Proof of Theorem 1.5

The proof is a relatively minor extension of that of Theorem 1.2 in Sections 3.1–3.2.

Proof. The analogues of (3.4–3.7) are

$$
z^{\widetilde{V}} = \sum_{N=0}^\infty (\log z)^N\,\frac{\widetilde{V}^N}{N!}, \tag{3.34}
$$

with

$$
\frac{\widetilde{V}^N}{N!} = \int_0^\infty dt_1 \cdots \int_{t_{N-1}}^\infty dt_N\;\mathbb{1}_{\{\widetilde{S}_{t_1}=\widetilde{S}'_{t_1},\dots,\widetilde{S}_{t_N}=\widetilde{S}'_{t_N}\}}, \tag{3.35}
$$

and

$$
\mathbb{E}\left[z^{\widetilde{V}}\mid\widetilde{S}\right] = \sum_{N=0}^\infty (\log z)^N\,F_N^{(1)}(\widetilde{S}), \qquad \mathbb{E}\left[z^{\widetilde{V}}\right] = \sum_{N=0}^\infty (\log z)^N\,F_N^{(2)}, \tag{3.36}
$$

with

$$
F_N^{(1)}(\widetilde{S}) := \int_0^\infty dt_1\cdots\int_{t_{N-1}}^\infty dt_N\;\mathbb{P}\left(\widetilde{S}_{t_1}=\widetilde{S}'_{t_1},\dots,\widetilde{S}_{t_N}=\widetilde{S}'_{t_N}\mid\widetilde{S}\right), \qquad F_N^{(2)} := \mathbb{E}\big[F_N^{(1)}(\widetilde{S})\big], \tag{3.37}
$$

where the conditioning in the first expression in (3.36) is on the full continuous-time path $\widetilde{S} = (\widetilde{S}_t)_{t\geq 0}$. Our task is to compute

$$
\widetilde{r}_1 := \lim_{N\to\infty}\frac{1}{N}\log F_N^{(1)}(\widetilde{S}) \quad (\widetilde{S}-a.s.), \qquad \widetilde{r}_2 := \lim_{N\to\infty}\frac{1}{N}\log F_N^{(2)}, \tag{3.38}
$$

and show that $\widetilde{r}_1 < \widetilde{r}_2$.

The idea is to average over the jump times of $\widetilde{S}$ while keeping its jumps fixed, thereby reducing the problem to the one for the discrete-time random walk treated in the proof of Theorem 1.5.

For the first line in (3.37) this *partial annealing* gives an upper bound, while for the second line it is simply part of the averaging over $\widetilde{S}$. To that end, put $\sigma_0 := 0$, for $k \in \mathbb{N}$ put $\sigma_k := \inf\{t > \sigma_{k-1}\colon\ \widetilde{S}_t \neq \widetilde{S}_{\sigma_{k-1}}\}$, let

$$X^\natural = (X_k^\natural)_{k\in\mathbb{N}} \quad \text{with} \quad X_k^\natural := \widetilde{S}_{\sigma_k}, \tag{3.39}$$

and define

$$F_N^{(1)}(X^\natural) := \int_0^\infty dt_1 \cdots \int_{t_{N-1}}^\infty dt_N\ \mathbb{P}(\widetilde{S}_{t_1} = \widetilde{S}'_{t_1}, \dots, \widetilde{S}_{t_N} = \widetilde{S}'_{t_N} \mid X^\natural), \qquad F_N^{(2)} := \mathbb{E}\big[F_N^{(1)}(X^\natural)\big], \tag{3.40}$$

together with the critical values

$$\widetilde{r}_1^\natural := \lim_{N\to\infty} \frac{1}{N} \log F_N^{(1)}(X^\natural) \quad (X^\natural - a.s.), \qquad \widetilde{r}_2^\natural := \lim_{N\to\infty} \frac{1}{N} \log F_N^{(2)}. \tag{3.41}$$

Clearly,

$$\widetilde{r}_1 \le \widetilde{r}_1^\natural \text{ and } \widetilde{r}_2 = \widetilde{r}_2^\natural, \tag{3.42}$$

which can be viewed as a result of "partial annealing", and so it suffices to show that $\widetilde{r}_1{}^\natural < \widetilde{r}_2^\natural$.

To this end write out

$$\begin{aligned} &\mathbb{P}(\widetilde{S}_{t_1} = \widetilde{S}'_{t_1}, \dots, \widetilde{S}_{t_N} = \widetilde{S}'_{t_N} \mid X^\natural) \\ &\quad = \sum_{0\le j_1\le\cdots\le j_N<\infty} \left(\prod_{i=1}^N e^{-(t_i-t_{i-1})} \frac{(t_i-t_{i-1})^{j_i-j_{i-1}}}{(j_i-j_{i-1})!}\right) \\ &\qquad \sum_{0\le j'_1\le\cdots\le j'_N<\infty} \left(\prod_{i=1}^N e^{-(t_i-t_{i-1})} \frac{(t_i-t_{i-1})^{j'_i-j'_{i-1}}}{(j'_i-j'_{i-1})!}\right) \left[\prod_{i=1}^N p_{j'_i-j'_{i-1}}\left(\sum_{k=j_{i-1}+1}^{j_i} X_k^\natural\right)\right]. \end{aligned} \tag{3.43}$$

Integrating over $0 \le t_1 \le \cdots \le t_N < \infty$, we obtain

$$\begin{aligned} F_N^{(1)}(X^\natural) = &\sum_{0\le j_1\le\cdots\le j_N<\infty}\ \sum_{0\le j'_1\le\cdots\le j'_N<\infty} \\ &\prod_{i=1}^N \left[2^{-(j_i-j_{i-1})-(j'_i-j'_{i-1})-1} \frac{[(j_i-j_{i-1})+(j'_i-j'_{i-1})]!}{(j_i-j_{i-1})!(j'_i-j'_{i-1})!}\ p_{j'_i-j'_{i-1}}\left(\sum_{k=j_{i-1}+1}^{j_i} X_k^\natural\right)\right]. \end{aligned} \tag{3.44}$$

Abbreviating

$$\Theta_n(u) = \sum_{m=0}^\infty p_m(u)\, 2^{-n-m-1} \binom{n+m}{m}, \qquad n \in \mathbb{N}\cup\{0\},\ u \in \mathbb{Z}^d, \tag{3.45}$$

we may rewrite (3.44) as

$$F_N^{(1)}(X^\natural) = \sum_{0\le j_1\le\cdots\le j_N<\infty} \prod_{i=1}^N \Theta_{j_i-j_{i-1}}\left(\sum_{k=j_{i-1}+1}^{j_i} X_k^\natural\right). \tag{3.46}$$

This expression is similar in form as the first line of (3.8), except that the order of the j_i's is not strict. However, defining

$$\widehat{F}_N^{(1)}(X^\natural) = \sum_{0<j_1<\cdots<j_N<\infty} \prod_{i=1}^N \Theta_{j_i-j_{i-1}}\left(\sum_{k=j_{i-1}+1}^{j_i} X_k^\natural\right), \tag{3.47}$$

we have

$$F_N^{(1)}(X^\natural) = \sum_{M=0}^{N} \binom{N}{M} \theta_0(0)^M \, \widehat{F}_{N-M}^{(1)}(X^\natural), \tag{3.48}$$

with the convention $\widehat{F}_0^{(1)}(X^\natural) \equiv 1$. Letting

$$\widehat{r}_1^\natural = \lim_{N\to\infty} \frac{1}{N} \log \widehat{F}_N^{(1)}(X^\natural), \qquad X^\natural - a.s., \tag{3.49}$$

and recalling (3.41), we therefore have the relation

$$\widetilde{r}_1^\natural = \log\left[\theta_0(0) + e^{\widehat{r}_1^\natural}\right], \tag{3.50}$$

and so it suffices to compute $\widehat{r}_1^\natural$.

Write

$$F_N^{(1)}(X^\natural) = \mathbb{E}\left[\exp\left(N \int_{\widetilde{E}} \log f^\natural(y)\,(\pi_1 R_N)(dy)\right)\middle| X^\natural\right], \tag{3.51}$$

where $f^\natural \colon \widetilde{E} \to [0,\infty)$ is defined by

$$f^\natural((x_1,\dots,x_n)) = \frac{\Theta_n(x_1+\cdots+x_n)}{p^n(0)}\,[G(0)-1], \quad n\in\mathbb{N},\, x_1,\dots,x_n\in E. \tag{3.52}$$

Equations (3.51–3.52) replace (3.8–3.9). We can now repeat the same argument as in (3.16–3.22), with the sole difference that f in (3.9) is replaced by $f^\natural$ in (3.52), and this, combined with Lemma 3.3 below, yields the gap $\widetilde{r}_1^\natural < \widetilde{r}_2^\natural$.

We first check that $f^\natural$ is bounded from above, which is necessary for the application of Varadhan's lemma. To that end, we insert the Fourier representation (3.14) into (3.45) to obtain

$$\theta_n(u) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} dk\, e^{-i(k\cdot u)}\,[2-\widehat{p}(k)]^{-n-1}, \qquad u\in\mathbb{Z}^d, \tag{3.53}$$

from which we see that $\theta_n(u) \le \theta_n(0)$, $u\in\mathbb{Z}^d$. Consequently,

$$f_n^\natural((x_1,\cdots,x_n)) \le \frac{\theta_n(0)}{p^n(0)}\,[G(0)-1], \qquad n\in\Lambda. \tag{3.54}$$

Next we note that

$$\lim_{n\to\infty}\frac{1}{n}\log\left[2^{-(a+b)n-1}\binom{(a+b)n}{an}\right] \begin{cases} =0, & \text{if } a=b,\\ <0, & \text{if } a\neq b.\end{cases} \tag{3.55}$$

From (1.1), (3.45) and (3.55) it follows that $\theta_n(0)/p^n(0) \le C < \infty$ for all $n\in\Lambda$, so that $f^\natural$ indeed is bounded from above.

Note that $X^\natural$ is the discrete-time random walk with transition kernel $p(\cdot,\cdot)$. The key ingredient behind $\widehat{r}_1^\natural < \widehat{r}_2^\natural$ is the analogue of Lemma 3.2, this time with $Q^* = (q^*)^{\otimes\mathbb{N}}$ and q^* given by

$$q^*(x_1,\dots,x_n) = \frac{\Theta_n(x_1+\cdots+x_n)}{G(0)-1}\prod_{k=1}^{n} p(x_k), \tag{3.56}$$

replacing (3.19).

Lemma 3.3. *Assume* (1.1). *Let* $Q^* = (q^*)^{\otimes\mathbb{N}}$ *with* q^* *as in* (3.56). *If* $m_{Q^*} < \infty$, *then* $I^{\rm que}(Q^*) > I^{\rm ann}(Q^*)$.

The analogue of (3.18) reads

$$
\begin{aligned}
Z^\natural &= \sum_{n\in\mathbb{N}} \sum_{x_1,\dots,x_n\in\mathbb{Z}^d} [\Theta_n(x_1+\dots+x_n)] \prod_{k=1}^n p(x_k) \\
&= \sum_{n\in\mathbb{N}} \sum_{m=0}^\infty \left\{ \sum_{x_1,\dots,x_n\in\mathbb{Z}^d} p^m(x_1+\dots+x_n) \prod_{k=1}^n p(x_k) \right\} 2^{-n-m-1} \binom{n+m}{m} \Bigg] \\
&= -\theta_0(0) + \sum_{n,m=0}^\infty p^{n+m}(0)\, 2^{-n-m-1} \binom{n+m}{m} \\
&= -\theta_0(0) + \frac{1}{2}\sum_{k=0}^\infty p^k(0) = -\theta_0 + \frac{G(0)}{2}.
\end{aligned}
\tag{3.57}
$$

Consequently,

$$
\log \widetilde{z}_2 = e^{-\widetilde{r}_2} = e^{-\widetilde{r}_2^\natural} = \frac{1}{\theta_0 + e^{\widetilde{r}_2^\natural}} = \frac{1}{\theta_0 + Z^\natural} = \frac{2}{G(0)}, \tag{3.58}
$$

where we use (3.36), (3.38), (3.42), (3.50) and (3.57). □

3.4 Proof of Lemma 3.3

Proof. We must adapt the proof in Section 3.2 to the fact that q^* has a slightly different form, namely, $p^n(x_1+\dots+x_n)$ is replaced by $\Theta_n(x_1+\dots+x_n)$, which averages transition kernels. The computations are straightforward and are left to the reader. The analogues of (3.23) and (3.25) are

$$
\begin{aligned}
q^*(E^n) &= \frac{1}{G(0)-1} \sum_{m=0}^\infty p^{n+m}(0)\, 2^{-n-m-1} \binom{n+m}{m}, \\
m_{Q^*} &= \sum_{n\in\mathbb{N}} n q^*(E^n) = \tfrac{1}{4} \sum_{k=0}^\infty k p^k(0),
\end{aligned}
\tag{3.59}
$$

while the analogues of (3.30–3.31) are

$$
\begin{aligned}
q^*(x_1=a) &= \frac{p(a)}{G(0)-1}\, \tfrac{1}{2} \sum_{k=0}^\infty p^k(a)[1-2^{-k-1}], \\
q^*(x_1=a, x_2=b) &= \frac{p(a)p(b)}{G(0)-1} \left[\tfrac{1}{4} \sum_{k=0}^\infty k p^k(a+b) + \sum_{k=0}^\infty p^k(a+b)\, 2^{-k-3} \right].
\end{aligned}
\tag{3.60}
$$

Recalling (3.29), we find

$$
\Psi_{Q^*}(a,-a) - p(a)^2 > 0, \tag{3.61}
$$

implying that $\Psi_{Q^*} \neq \nu^{\otimes\mathbb{N}}$ (recall (3.2)), and hence $H(\Psi_{Q^*} \mid \nu^{\otimes\mathbb{N}}) > 0$, implying the claim. □

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