Gibbsian and non-Gibbsian states at Eurandom

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Abstract

We review some of the work on non-Gibbsian states of the last ten years, emphasizing the developments in which Eurandom played a role.

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1 Introduction

Thirty years ago some unexpected mathematical difficulties in rigorously implementing many of the generally used real-space Renormalization Group transformations as maps on a space of Hamiltonians were discovered in GRIFFITHS and PEARCE (1978, 1979). In van ENTER et al. (1993) these difficulties were explained by observing that under a renormalization group map, Gibbs measures were mapped on non-Gibbsian measures, making a map at the level of Hamiltonians (interactions, coupling constants) ill defined.

This point of view led to further papers covering the area of non-Gibbsianness and Renormalization–Group peculiarities (van ENTER et al., 1993, 1994; van ENTER, 1996; van ENTER and FERNÁNDEZ, 1999; van ENTER et al., 2002, 2004; FERNÁNDEZ, 1999; FERNÁNDEZ et al., 2003; FERNÁNDEZ, 2006; HALLER and KENNEDY, 1996; KENNEDY, 1997; KÜLSKE, 1999, 2001; KÜLSKE et al., 2004; LE NY, 2007; LÖRINCZI, 1995; VANDE VELDE, 1995)

Afterwards, other occasions where non-Gibbsian measures appear were found. In particular interacting particle systems (such as stochastic Ising models) in the transient regime were found to display a mathematically very similar behavior (van ENTER et al., 2002). The main observation here is that time evolution provides a continuum of stochastic maps, which are susceptible to a similar analysis. In fact, if the evolution is an infinite-temperature one (independent spin flips), then one can view it as a family of single-site stochastic renormalization maps; and these had been already considered, however, with a different interpretation, by Griffiths and Pearce. Interactions can be controlled by cluster expansion techniques, cf. also MAES and NETOČNÝ (2002), when either time or interaction strength is small enough.

This was probably the first contribution on the subject which started at Eurandom. Further developments were based on Dobrushin's programme: can one consider non-Gibbsian measures as Gibbs measures in some more generalized sense, and how much of the Gibbsian structure, e.g. the variational principle, survives under such more general notions?. Moreover, the occurrence of non-Gibbsianness in disordered systems has been addressed, as well as various other examples.

In 2003, the first meeting exclusively devoted to non-Gibbsian issues was organized, at Eurandom. One of the highlights of this meeting was the participation of Robert Israel, who probably was the first person to clearly identify non-Gibbsianness as the source of the Griffiths-Pearce problems (ISRAEL, 1981). In his contribution to the proceedings, he gave the proof (which he had announced earlier, although it was not published before) that in the topological sense Gibbs measures are exceptional, and that thus non-Gibbsianness is a generic property (ISRAEL, 2004).

The paper is organized as follows. We start with a brief description of Gibbs states. In following sections we review the recent work on preservation and loss of Gibbsianity under stochastic dynamics and variational characterization of generalized Gibbs states. We end the paper with a list of open problems.

2 Gibbs measures and quasilocality

In this section we will describe some definitions and facts we will need about the theory of Gibbs measures. For a more extensive treatment we refer to GEORGII (1988) or van ENTER et al. (1993).

We will consider spin systems on a lattice $\mathbb{Z}^d$, where in most cases we will take a single-spin space Ω_0 which is finite. The configuration space of the whole system is $\Omega = \Omega_0^{\mathbb{Z}^d}$. Configurations will be denoted by small Greek letters such as σ or ω, and their coordinates at lattice site i are denoted by σ_i or ω_i. A regular (absolutely summable) interaction Φ is a collection of functions $\Phi(\Lambda, \cdot)$ on Ω_0^{Λ}, indexed by finite sets $\Lambda \subset \mathbb{Z}^d$, which is translation invariant and satisfies:

$$\sum_{0 \in \Lambda} ||\Phi(\Lambda, \cdot)||_{\infty} < \infty,$$

where $||\Phi(\Lambda, \cdot)||_{\infty} = \sup_{\sigma} |\Phi(\Lambda, \sigma)|$. Formally Hamiltonians are given by

$$H^{\Phi} = \sum_{\Lambda \subset \mathbb{Z}^d} \Phi(\Lambda, \cdot)$$

Under the above regularity condition these type of expressions make mathematical sense if the sum is taken over all subsets having non-empty intersections with a finite volume Λ. For regular interactions one can define Gibbs measures as probability measures on Ω having conditional probabilities which are described in terms of appropriate Boltzmann-Gibbs factors:

$$\mu(\sigma_{\Lambda} \,|\, \omega_{\Lambda^c}) = \frac{1}{Z_{\Lambda}^{\omega_{\Lambda^c}}} \exp\Big(- \sum_{V \cap \Lambda \neq \varnothing} \Phi(V, \sigma_{\Lambda}\omega_{\Lambda^c})\Big)$$

for each volume Λ, μ-almost every boundary condition ω_{Λ^c} outside Λ and each configuration σ_{Λ} in Λ. The expression on the right-hand side will be denoted by $\gamma_{\Lambda}(\sigma_{\Lambda} \,|\, \omega_{\Lambda^c})$; the collection of $\gamma = \{\gamma_{\Lambda}\}$ is the *Gibbsian specification* for the potential Φ.

As long as Ω_0 is compact, there always exists at least one Gibbs measure for every regular interaction; the existence of more than one Gibbs measure is one definition of the occurrence of a first-order phase transition of some sort. Thus the map from interactions to measures is one to at-least-one. Every Gibbs measure has the property that (for one of its versions) its conditional probabilities are continuous functions of the boundary condition ω_{Λ^c}, in the product topology. It is a non-trivial fact that this continuity, which goes by the name "quasilocality" or "almost Markovianness", in fact characterizes the Gibbs measures (KOZLOV, 1974; SULLIVAN, 1973), once one knows that all the conditional probabilities are bounded away from zero (that is, the measure is *nonnull* or has the *finite-energy* property). In some examples it turns out to be possible to check this continuity (quasilocality) property quite explicitly. If a measure is a Gibbs measure for a regular interaction, this interaction is essentially uniquely determined. Thus the map from measures to interactions is one to at-most-one.

A second characterization of Gibbs measures uses the variational principle expressing that in equilibrium a system minimizes its free energy. A probabilistic formulation of this fact naturally occurs in terms of the theory of large deviations. The (third level) large deviation rate function is up to a constant and a sign equal to a free energy density. To be precise, let μ be a translation invariant Gibbs measure, and let ν be an arbitrary translation invariant measure. Then the relative entropy density $h(\nu|\mu)$ can be defined as the limit:

$$h(\nu|\mu) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} H_\Lambda(\nu|\mu)$$

where

$$H_\Lambda(\nu|\mu) = \int \log\left(\frac{d\nu_\Lambda}{d\mu_\Lambda}\right) d\nu_\Lambda$$

and μ_Λ and ν_Λ are the restrictions of μ and ν to Ω_0^Λ. It has the property that $h(\nu|\mu) = 0$ if and only if the measure ν is a Gibbs measure for the same interaction as the base measure μ. We can use this result in applications if we know for example that a known measure ν cannot be a Gibbs measure for the same interaction as some measure μ we want to investigate. For example, if ν is a point measure, or if it is the case that ν is a product measure and μ is not, then we can conclude from the statement: $h(\nu|\mu) = 0$, that μ lacks the Gibbs property.

For another method of proving that a measure is non-Gibbsian because of having the "wrong" type of (in this case too small) large deviation probabilities, see SCHONMANN (1989).

3 Results on non-Gibbsian measures

As was mentioned before, non-equilibrium models, both in the steady state and in the transient regime have been considered. After the papers van ENTER et al. (2002) and MAES and NETOČNÝ (2002) on Glauber dynamics for discrete spins, extensions were developed in DEREUDRE and RŒLLY (2005); van ENTER and RUSZEL (2007); KÜLSKE and REDIG (2006); KÜLSKE and OPOKU (2007); LE NY and REDIG (2002) to more general spins and types of dynamics.

Also, joint quenched measures of disordered systems have been shown sometimes to be non-Gibbsian (van ENTER et al., 2000a,b; van ENTER and KÜLSKE, 2007; KÜLSKE, 1999, 2001), affecting the Morita approach to disordered systems (see KÜHN, 1996; MORITA, 1964). In this last case, the peculiarity can be so strong – and it actually is in the 3-dimensional random-field Ising model– as to violate the variational principle. This means in particular that the (weakly Gibbsian) interactions belonging to the plus state and the minus state are different, despite their relative entropy density being zero, see section 5 for further discussion. Non-Gibbsianness here means that the quenched measure cannot be written as an annealed measure, that is a Gibbs measure on the joint space of spins and disorder variables for some "grand potential", such as Morita proposed. The Eurandom contribution KÜLSKE et al. (2004) is especially relevant here.

The non-Gibbsian character of the various measures considered comes often as an unwelcome surprise. A description in terms of an effective interaction is often convenient, and even seems essential for some applications. Thus, the fact that such a description is not available can be a severe drawback.

The fact that the constraints which act as points of discontinuity often involve configurations which are very untypical for the measure under consideration, suggested a notion of *almost* Gibbsian or *weakly* Gibbsian measures. These are measures whose conditional probabilities are either continuous only on a set of full measure or can be written in terms of an interaction which is summable only on a set of full measure. Intuitively, the difference is that in one case the "good" configurations can shield off *all* influences from infinitely far away, and in the other case only *almost all* influences. The weakly Gibbsian approach was first suggested by Dobrushin to various people; his own version was published only later DOBRUSHIN (1995); DOBRUSHIN and SHLOSMAN (1997, 1999). An early definition of almost Gibbsianness appeared in print in LÖRINCZI and WINNINK (1993), see also van ENTER et al. (2004); van ENTER and VERBITSKIY (2004); FERNÁNDEZ and PFISTER (1997); KÜLSKE et al. (2004); MAES and VANDE VELDE (1995, 1997); MAES et al. (1999) for further developments. Some examples of measures which are at the worst almost Gibbsian measures in this sense are decimated or projected Gibbs measures in an external field, random-cluster measures on regular lattices, and low-temperature fuzzy Potts measures.

Another source of non-Gibbsian examples, which was developed at Eurandom, is Random Walk in Random Scenery (den HOLLANDER et al., 2005).

4 Preservation, loss, and recovery of Gibbsianity under stochastic dynamics

Consider a lattice spin system, initially in a Gibbs state μ^Φ corresponding to a translation invariant interaction Φ. This initial state is chosen to be the starting measure of a Markovian dynamics which has as a reversible measure a Gibbs measure μ^Ψ with interaction $\Psi \neq \Phi$. The dynamics considered in van ENTER et al. (2002) is high-temperature Glauber dynamics, which is informally described as follows: at each lattice site $x \in \mathbb{Z}^d$ the spin σ_x flips at rate

$$c(x, \sigma) = \exp\left(-\frac{1}{2}\left(H_\Psi(\sigma^x) - H_\Psi(\sigma)\right)\right)$$

where σ^x denotes the spin configuration $\in \{-1, 1\}^{\mathbb{Z}^d}$ obtained by changing the sign of the spin at x and leaving all other spins unchanged, and where H_Ψ denotes the (formal) Hamiltonian corresponding to Ψ, i.e.,

$$H_\Psi(\sigma^x) - H_\Psi(\sigma) = \sum_{A \ni x} \left(\Psi(A, \sigma^x) - \Psi(A, \sigma)\right)$$

By high-temperature we mean that we choose the interaction Ψ to be small and (for technical reasons) of finite range. Small is in the sense of the norm

$$||\Psi||_\alpha = \sum_{A\ni 0} e^{\alpha|A|}||\Psi(A,\cdot)||_\infty$$

for some $\alpha > 0$. This implies in particular that the reversible Gibbs measure μ^Ψ is unique, and that from any initial measure ν, the distribution ν_t at time $t > 0$ converges exponentially fast to μ^Ψ.

A good and intuition-guiding example to keep in mind is when $\Phi = \Phi_{\text{Ising}}$ is the potential of the Ising model with magnetic field h, i.e.,

$$\Phi(\{x\},\sigma) = h\sigma_x,$$
$$\Phi(\{x,y\},\sigma) = \beta\sigma_x\sigma_y$$

for x, y nearest neighbors in $\mathbb{Z}^d$, and $\Phi(A,\sigma) = 0$ for all other subsets $A \subset \mathbb{Z}^d$.

The basic question addressed in van ENTER et al. (2002) is: **"is the measure at time $t > 0$, μ_t^Φ, a Gibbs measure?"** In other words, is $\mu_t^\Phi = \mu^{\Phi_t}$ for some absolutely summable interaction Φ_t. The $\mathcal{B}$1-norm of Φ_t

$$||\Phi_t|| = \sum_{A\ni 0} ||\Phi_t(A,\cdot)||_\infty$$

can then be considered as a time-dependent inverse temperature. In the case $\Psi = 0$, i.e., "infinite-temperature" dynamics, the limiting Gibbs measure μ^Ψ is a product measure, and the dynamics then simply consists of spins that independently (for different lattice sites) flip at the event-times of a mean-one Poisson process. Intuitively speaking, this corresponds to "heating up" a system which starts at a finite temperature. The question of Gibbsianness then corresponds to the question whether we can still associate an intermediate time-dependent "effective temperature" to the non-equilibrium transient states, and how this temperature evolves. In this language, loss of Gibbsianness then corresponds to "loss of temperature".

To study this basic question, one considers the distribution of the so-called double-layer system consisting of the starting configurations, together with the configurations at time t. This is a Gibbs measure with formal Hamiltonian

$$H_t(\sigma,\eta) = H_\Phi(\sigma) - \log p_t(\sigma,\eta) \tag{1}$$

In particular, the term $\log p_t(\sigma,\eta)$ is formal, and, except at infinite temperature, a cluster expansion (requiring Ψ to be small in a strong norm) is used (MAES and NETOČNÝ, 2002) in order to see that it has the required structure of a sum of local terms. The Hamiltonian of the double-layer system plays a fundamental role in the detection of essential points of discontinuity of the conditional probabilities of the measure μ_t^Φ. Roughly speaking if η is such that the "random field" system with η the realization of the random field, thus having Hamiltonian $H_t(\cdot,\eta)$, has a phase transition, then η is a good candidate point of discontinuity (a so-called bad configuration), while if there is no phase transition, then η is a point of continuity (a so-called good configuration). Natural candidates for a bad configuration in the context of the Ising model (as a starting measure) are configurations η which give rise to a "neutral" field in (1), such as the alternating configuration, or a "typical" random configuration, chosen from a symmetric product measure.

The results of van ENTER et al. (2002) can then be summarized as follows.

1. **High-temperature region: Gibbsianness.** For Ψ and Φ finite range and small, the measure μ_t^Φ is Gibbs for all $t \geq 0$.

2. **Low-temperature unbiased region: loss of Gibbsianness.** Ψ is small, has zero single-site part, and Φ is the interaction of the Ising model with zero magnetic field at inverse temperature β. We can choose any Gibbs measure for Φ to be the starting measure. Then there exists β_0 such that for all $\beta > \beta_0$ there exist $t_0 \leq t_1$ such that for $t < t_0$, μ_t^Φ is Gibbs and for $t > t_1$, μ_t^Φ is not Gibbs.

3. **Low-temperature biased region: loss and recovery of Gibbsianness.** Ψ is small, has zero single-site part, and Φ is the interaction of the Ising model with small magnetic field $h > 0$ at inverse temperature β. Then there exists β_0 such that for all $\beta > \beta_0$ there exist $t_0 \leq t_1 < t_2 \leq t_3$ such that for $t < t_0$, μ_t^Φ is Gibbs, for $t_1 < t < t_2$, μ_t^Φ is not Gibbs (loss of Gibbsianness), and for $t > t_3$ is Gibbs again (recovery).

It is believed that the transitions in item 2 and 3 are sharp (i.e., $t_0 = t_1$ and $t_2 = t_3$) but this has not been proved, except in the context of mean-field models (cf. below). After van ENTER et al. (2002) there have been several further and new developments, of which we mention the following.

1. **Universality of short-time conservation of Gibbsianness.** In LE NY and REDIG (2002) it is proved that for arbitrary local dynamics (including e.g. Kawasaki dynamics or mixtures of Glauber and Kawasaki) and arbitrary initial Gibbs measure corresponding to a finite-range potential, for short times the measure remains Gibbs. The reason is that for short times, the system consists of a "sea" (in the percolation sense) of unflipped spins and isolated islands of spins where one or more flips happened. Technically speaking, this intuition can be made into a proof of Gibbsianness via a combination of cluster expansion with the Girsanov formula.

2. **Interacting diffusion processes at high temperature.** In DEREUDRE and RŒLLY (2005) weakly interacting diffusions are considered, and a starting measure that is high-temperature Gibbs. In that context, via a cluster expansion technique, Gibbsianness at all times is proved.

3. **Independent diffusions starting at high and low temperatures.** In KÜLSKE and REDIG (2006) independent diffusions are considered starting from a particular Gaussian model which can be mapped to a discrete spin system. Here loss without recovery of Gibbsianness is proved.

4. n**-vector models with interacting spin-diffusions.** In KÜLSKE and OPOKU (2007) and van ENTER and RUSZEL (2007) it was shown that for n-vector models under a diffusive single-spin evolution the measure remains Gibbs for short time. If, moreover, the starting measure is at high temperature, then the time-evolved measure will remain Gibbs forever. These conclusions remain true, if one adds a small interaction to the dynamics. For the zero-field plane rotor model in two or more dimensions, started at low temperature, and with infinite-temperature evolution, it is shown in van ENTER and RUSZEL (2007) that the Gibbs property is lost after some finite time, but possibly recovered after some larger time. In $d = 3$ no recovery takes place.

5. **Mean-field systems with infinite-temperature dynamics.** In KÜLSKE and LE NY (2007) the Curie-Weiss model evolved via independent spin flips is considered. In particular it is shown there that the transitions Gibbs-non-Gibbs are sharp, and that there is a region of parameters where the "bad" configurations are typical (of measure one) for the time-evolved measure and a region where they are untypical (i.e., of measure zero).

5 Variational principle

The second part of the so-called Dobrushin's restoration programme asks for the extension of classical results for Gibbs states (e.g., the variational principle) to the classes of generalized Gibbsian states. Similarly, stochastic dynamics (see preceding section) or deterministic and random transformations of Gibbs states might produce non-Gibbsian states. Is it possible to recover a variational principle for the transformations of Gibbs states?

5.1 Generalized Gibbs states.

The classical variational principle for Gibbs measures states that if μ is a translation invariant Gibbs measure on $\Omega_0^{\mathbb{Z}^d}$ for a potential Φ, and ν is another translation-invariant measure with $h(\nu|\mu) = 0$, then

(a) **Specification-dependent formulation:** ν is consistent with the Gibbs specification γ^Φ;

(b) **Specification-independent formulation:** ν is a Gibbs measure for Φ.

In general, $h(\nu|\mu) = 0$, then, according to FÖLLMER (1973), μ and ν have the same local characteristics:

$$\nu(\sigma_0|\sigma_{\mathbb{Z}^d\setminus\{0\}}) = \mu(\sigma_0|\sigma_{\mathbb{Z}^d\setminus\{0\}}) \tag{2}$$

for ν-almost all σ. One has to take into account, that the left-hand side is defined ν-a.s., and the right hand side is defined μ-a.s. Hence, if ν and μ are two ergodic measures (and hence singular), the interpretation of (2) without further assumptions is problematic. For example, there are measures μ such that $h(\nu|\mu) = 0$ for all ν. A natural assumption is that ν is concentrated on a set of continuity points for the conditional probabilities of μ. The notion of concentration must be made explicit.

Any measure admits infinitely many (consistent) specifications. For a Gibbs measure there is a unique **quasi-local** or **continuous** specification, hence the specification which is uniquely defined everywhere, and which is the specification of choice. For a non-Gibbsian measure we cannot construct a quasi-local specification, and simply must choose some specification as a reference. Naturally, a good specification is the one close to quasi-local specifications, i.e., the specification with a large set of continuity points.

A measure μ is called **almost Gibbs**, if there exists a specification γ such that μ is consistent with γ (denoted by $\mu \in \mathcal{G}(\gamma)$, and the set Ω_γ of continuity points of γ has μ-measure 1. For an almost Gibbs measure μ this specification γ is the natural reference specification.

In FERNÁNDEZ et al. (2003) it was shown that if μ is an almost Gibbs measure for specification γ, and γ is *monotonicity preserving*, then $h(\nu|\mu) = 0$ implies that $\nu \in \mathcal{G}(\gamma)$. In KÜLSKE et al. (2004), the strong monotonicity assumption was substituted by the requirement that ν is concentrated on a set of continuity points of γ: $\nu(\Omega_\gamma) = 1$.

If γ is a specification and $\mu \in \mathcal{G}(\gamma)$, define a set

$$\tilde{\Omega}_\gamma = \left\{ \omega : \mu(\omega_\Lambda | \omega_{[-n,n]^d \setminus \Lambda}) \to \gamma_\Lambda(\omega_\Lambda | \omega_{\mathbb{Z}^d \setminus \Lambda}) \text{ as } n \to \infty \text{ for all finite } \Lambda \right\}.$$

Since μ is consistent with γ, one has $\mu(\tilde{\Omega}_\gamma) = 1$. Moreover, if $h(\nu|\mu) = 0$ and $\nu(\tilde{\Omega}_\gamma) = 1$, then $\nu \in \mathcal{G}(\gamma)$ as well (van ENTER and VERBITSKIY, 2004). If μ is an almost Gibbs measure for specification γ and $\nu(\Omega_\gamma) = 1$, then $\nu(\tilde{\Omega}_\gamma) = 1$ as well, and hence the result of van ENTER and VERBITSKIY (2004) can be viewed as an extension of the corresponding result in KÜLSKE et al. (2004).

Nevertheless, despite the positive results mentioned above, a specification-dependent formulation of the variational principle has its limitations, which were identified in KÜLSKE et al. (2004). Relying on a previous work on disordered systems KÜLSKE (1999, 2001), KÜLSKE et al. (2004) provides an example of two weakly Gibbs measures μ^+ and μ^- with natural specifications γ^+ and γ^-, respectively, such that $h(\mu^+|\mu^-) = h(\mu^-|\mu^+) = 0$, but

$$\mu^+ \notin \mathcal{G}(\gamma^-), \quad \mu^- \notin \mathcal{G}(\gamma^+).$$

For a recent analysis of how far one can set up the formalism, based on specifications, see MAHÉ (2007).

5.2 Transformations of Gibbs states.

Suppose μ is a Gibbs measure on $\Omega_0^{\mathbb{Z}^d}$ for potential Φ. There is a number of ways the state space $\Omega_0^{\mathbb{Z}^d}$ and hence the measure μ can be transformed:

1. **Decimation:** For $\ell \in \mathbb{N}$, let $T : \Omega_0^{\mathbb{Z}^d} \to \Omega_0^{\mathbb{Z}^d}$ be defined by $(T\omega)_{\mathbf{n}} = \omega_{\ell\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^d$, let $\nu = T^*\mu$ be the image of μ under T.

2. **Single-site projections:** Suppose Ω_1 is a finite set such that $|\Omega_1| < |\Omega_0|$ and $T : \Omega_0 \to \Omega_1$ is onto. Let $T : \Omega_0^{\mathbb{Z}^d} \to \Omega_1^{\mathbb{Z}^d}$ be defined by $(T\omega)_{\mathbf{n}} = T(\omega_{\mathbf{n}})$ for all $\mathbf{n} \in \mathbb{Z}^d$; again, let $\nu = T^*\mu$ be the image of μ under T.

3. **Random transformations or Hidden Gibbs fields:** For each $\mathbf{n} \in \mathbb{Z}^d$, $\sigma_{\mathbf{n}} \in \Omega_1$ chosen independently according to $T(\cdot\,|\omega_{\mathbf{n}})$. It is assumed that $T(\sigma_{\mathbf{n}}|\omega_{\mathbf{n}}) > 0$ for all $\sigma_{\mathbf{n}} \in \Omega_1$, $\omega_{\mathbf{n}} \in \Omega_0$, again $\nu = T^*\mu$.

It is known that these transformations, as well as the stochastic transformations introduced in the previous section can produce non-Gibbsian states. The general results on the non-Gibbsian nature (classification of possible pathologies) of measures $\nu = T^*\mu$ or $\nu = \mu_t$ in terms of the potential Φ of the source measure μ and the properties of T (Ψ) remains sketchy. Nevertheless, it is expected that the transformed measures will admit variational principles in some form.

In LE NY and REDIG (2004), it was shown that a Gibbs measure μ will remain asymptotically decoupled under Glauber dynamics for all $t > 0$. Hence, for $\nu = \mu_t$, $h(\rho|\nu)$ is well-defined for all ρ. This type of results is a prerequisite for a successful variational description.

In KÜLSKE et al. (2004), the authors considered transformations T of type (1)-(3) and Gibbs measures μ with specification γ under the condition that the specification $\gamma \otimes T$ is monotonicity-preserving. In this case, the image states $\nu = T^*\mu$ are almost Gibbs for some specification $\tilde{\gamma}$, and $h(\rho|\nu) = 0$ implies that $\rho \in \mathcal{G}(\tilde{\gamma})$.

In VERBITSKIY (2006), for a transformation T of type (1)-(3) and any Gibbs state μ for potential Φ, it was shown that for the image measure $\nu = T^*\mu$ one has $h(\rho|\nu) = 0$ if and only if there exists a measure λ such that $h(\lambda|\mu) = 0$ and $\rho = T^*\lambda$. Equality $h(\lambda|\mu) = 0$ by the classical variational principle means that λ is Gibbs for the same potential Φ, and hence, $h(\rho|\nu) = 0$ implies that ρ, ν are transformations of Gibbs states with the same potential.

Yet another class of transformations is formed by restrictions to a layer – the so-called Schonmann projections: let μ be a Gibbs measure on $\Omega_0^{\mathbb{Z}^d}$, and ν be a restriction of μ to a lower dimensional hyperplane $\mathbb{L}$, say $\mathbb{L} = \mathbb{Z}^{d-1} \times \{0\} \subset \mathbb{Z}^d$. Such measures ν are often non-Gibbsian. Nevertheless, in some cases, for example, if μ is a plus phase of a low-temperature two-dimensional Ising model, the corresponding measure ν can be shown (KÜLSKE et al., 2004) to be consistent with a monotonicity-preserving specification γ, and hence the specification-independent variational principle for ν is valid.

6 Conclusions and some further open problems

In non-equilibrium statistical mechanics there are still many open questions about the occurrence of non-Gibbsian measures. Whether one can ascribe an effective temperature in a non-equilibrium situation is a topic of considerable interest, (also in the physics literature, see e.g. de OLIVEIRA and PETRI (2005)). The term non-Gibbsian or non-reversible is often used for invariant measures in systems in which there is no detailed balance ERNST and BUSSEMAKER (1995); EYINK et al. (1996); LIGGETT (1985). It is an open question to what extent such measures are non-Gibbsian in the sense we have described here. It has been conjectured that such measures for which there is no detailed balance are quite generally non-Gibbsian in systems with a stochastic dynamics, see for example LEBOWITZ and SCHONMANN (1988) or (EYINK et al., 1996, Appendix 1); on the other hand it has been predicted that non-Gibbsian measures are rather exceptional (LIGGETT, 1985, Open problem IV.7.5, p.224), at least for non-reversible spin-flip processes under the assumptions of rates which are bounded away from zero. The examples we have are for the moment too few to develop a good intuition on this point, but see LEFÉVERE and TASAKI (2005).

We add the remark that sometimes a dynamical description is possible in terms of a Gibbs measure on the space-time histories (in $d+1$ dimensions). In such cases, looking at the steady states is considering the d-dimensional projection of such Gibbs measures.

In another direction, a study of non-Gibbsianness in a mean-field setting has been developing. In this case, the characterization of Gibbs measures as having continuity properties in the product topology breaks down. For these developments, see HÄGGSTRÖM and KÜLSKE (2004); KÜLSKE (2003); KÜLSKE and LE NY (2007).

Several problems remain open in connection to dynamics of Gibbsian measure.

1. **Kawasaki dynamics.** Is the transition Gibbs-non Gibbs present if we start from a low-temperature model and consider infinite-temperature Kawasaki dynamics (the so-called simple symmetric exclusion process)? Since this dynamics conserves the density of plus spins, we should look for an initial measure that has two phases having the same density. A possible candidate is the Ising antiferromagnet $\Phi(\{x,y\},\sigma) = -\beta\sigma_x\sigma_y$ for x, y neighbors and zero elsewhere. This model has the two checkerboard configurations η_1, η_2 as ground states, which have the same density 1/2 of plus spins. The candidate bad configuration would then be a checkerboard configuration of two by two squares, which has also density 1/2, and is neutral with respect to the configurations η_1, η_2.

   For the Ising model, we do not expect Gibbs-non Gibbs transitions in the course of the evolution (because the groundstates have different density of plus spins), but this has also not been proved.

2. **Nature versus nurture transition.** In van ENTER et al. (2002) is suggested that the transition Gibbs-non Gibbs is related to a so-called nature versus nurture transition. This is informally described as follows. Consider the Ising model plus phase as starting measure and condition that at time t a neutral configuration (such as the alternating configuration) is observed. The question is then whether this configuration is produced by typical path of the dynamics starting from an atypical configuration (nature) or by an atypical path of the dynamics starting from a typical configuration (nurture). The second scenario is related to the "badness" of the configuration.

3. **Low-temperature dynamics.** If the norm of the potential describing the dynamics

is not small, then one cannot make sense of $-\log p_t(\sigma,\eta)$ in (1) as a sum of local terms via a cluster expansion. Therefore, this regime is still completely open.

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