CHARACTERIZATIONS OF WORDS WITH MANY PERIODS

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ABSTRACT. In 2001 Droubay, Justin and Pirillo studied the so-called standard episturmian words. In 2006 and 2007 Fischler defined another type of words in the framework of simultaneous diophantine approximation. Both classes of words were described in terms of palindromic prefixes of infinite words. In 2003 the authors introduced words which they called extremal FW words after they had met so-called FW words later. Words from both classes appeared as unique words (up to word isomorphism) satisfying certain properties related to periods. In the present paper the connections between all these words are displayed in the form of four comparable characterizations of the mentioned four word classes. (Theorems 1-4)

1. INTRODUCTION

Let A be a finite set with at least two elements, the so-called alphabet. The free monoid A^* generated by A is the set of the finite words on A. The empty word is denoted by ϵ. For a word $u = u_1u_2 \dots u_m$ with $u_i \in A$ for $i = 1, \dots, m$ we denote the length m of u by $|u|$ and the number of distinct letters occurring in u by $\sharp u$.

A factor of a word $u = \{u_i\}_{i=1}^m$ is a word $u_hu_{h+1}\dots u_j$ with $1 \le h \le j \le m$. It is called a prefix of u if $h = 1$ and a suffix if $j = m$. The word u is called periodic with period p if $u_i = u_{i+p}$ for $1 \le i \le m - p$. The reversal of $u = u_1u_2 \dots u_m$ is $\overline{u} := u_mu_{m-1}\dots u_1$. The word u is called a palindrome if $u = \overline{u}$. Given u, its palindromic right-closure is the (unique) shortest palindrome $u^{(+)}$ which has prefix u. In this paper we study the palindromic prefixes of words with many periods.

A word is called standard episturmian whenever for every prefix v of u also $v^{(+)}$ is a prefix of u (cf. [1], [7]). Justin and Pirillo (cf. [7], Theorem 2.10) proved that a word u is standard episturmian if and only if there exists a word $\Delta(u) = x_1x_2\dots x_K \in A^{(+)}$ such that if $u[0] = \epsilon$ and $u[k] = (u[k-1]x_k)^{(+)}$ for $k = 1, \dots, K$, then $u[K] = u$. For comparison with other characterizations we shall prove the following variant.

Theorem 1. *A word u is standard episturmian if and only if there exists a K such that u can be generated as follows: $u[0] := \epsilon$; for $k = 1, \dots, K$ either*
$u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is a letter which does not occur in $u[k-1]$,
or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l]u'[k-1]$ and l is the largest integer $< k$ such that if $u[l+1] = (u[l]x)^{(+)}$, then $u[k] = (u[k-1]x)^{(+)}$.

The latter condition implies that if $\Delta(u) = x_1x_2\cdots x_K$, then l is the largest integer $< k$ such that $x_k = x_{l+1}$.

We call a word u an extremal FW word (i.e. extremal Fine and Wilf word) if there exist positive integers $p_1 < p_2 < \cdots < p_r = |u|$ such that u has periods $p_1, p_2, \dots, p_r$, but not period

Date: 15 September, 2008.
1991 *Mathematics Subject Classification.* 68R15.
Key words and phrases. Fine and Wilf theorem, palindromes, episturmian words, FW words.

$\gcd(p_1, p_2, \ldots, p_r)$, u has the maximal length for such a word, and $\sharp u \geq \sharp v$ for every word v with these properties. We say that u is an extremal FW word for period set $P := \{p_1, p_2, \ldots, p_r\}$. We proved in [8] that if u is an extremal FW word for the set P, then u is unique apart from word isomorphism and it is a palindrome. We denote the length of the word u which is extremal for the period set P by $L(P) - 1 = L(p_1, p_2, \ldots, p_r) - 1$. It follows from the following result that every standard episturmian word is an extremal FW word.

Theorem 2. *A word u is an extremal FW word if and only if $\sharp u > 1$ and for some K it can be generated as follows: $u[0] := \epsilon$; for $k = 1, \ldots, K$ either*
$u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is a letter which does not occur in $u[k-1]$,
or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l]u'[k-1]$ for some l with $0 \leq l < k-1$.

The numbers $L(p_1, p_2, \ldots, p_r)$ also appear in a seemingly different context. In [10] we defined a quantity denoted by $L'(p_1, p_2, \ldots, p_r)$ which is the minimal value n for the following property: Let $X_1, X_2, .., X_r$ be non-empty finite words in the alphabet A. For each i with $1 \leq i \leq r$, let $W_i \in X_i\{X_1, ..., X_r\}^\infty$ (in other words W_i is an infinite concatenation of $X_1, .., X_r$ beginning in X_i.) If $W_1, ..., W_r$ agree on a prefix of length n then $W_i = W_j$ for all i and j. We proved in [10] that $L'(p_1, p_2, \ldots, p_r) = L(p_1, p_2, \ldots, p_r)$ if $p_r \leq L(p_1, p_2, \ldots, p_{r-1})$ and $L'(p_1, p_2, \ldots, p_r) = p_r$ otherwise. Furthermore, examples of words realizing the maximal value $L' - 1$ for not satisfying the property are constructed by using extremal FW words. As was pointed out by T. Harju, the existence of L', but not the optimal bound, may be deduced from the details of the proof of the so-called *graph lemma* for infinite words (see Theorem 5.1 in Harju and Karhumäki [5]).

Extremal FW words are closely related to words u introduced by S. Fischler. Let $\{n_i\}_{i=1}^\infty$ denote the increasing sequence (assumed to be infinite) of all lengths of palindromic prefixes of u. Fischler [2] gave an explicit construction of all words u such that $n_{i+1} \leq 2n_i + 1$ for all i. He proved that among all such non-periodic words u the quantity limsup n_{i+1}/n_i is minimal for the Fibonacci word. We call the palindromic prefixes of the word u Fischler words. Later, in [3] Fischler applied his study to simultaneous approximation to a fixed real number and its square by rational numbers with the same denominator. The following result shows that every extremal FW word is a Fischler word, but not conversely.

Theorem 3. *A word u is a Fischler word if and only if for some K it can be generated as follows: $u[0] := \epsilon$; for $k = 1, \ldots, K$ either*
$u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is some letter,
or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l]u'[k-1]$ for some l with $0 \leq l < k-1$.

We call a word u a FW word (i.e. Fine and Wilf word) if there exist positive integers n and $p_1, p_2, \ldots, p_r$ such that u has length n and periods $p_1, p_2, \ldots, p_r$ and $\sharp u \geq \sharp v$ for any word v of length n and with periods $p_1, p_2, \ldots, p_r$. Note that for every positive integer n and periods $p_1, \ldots, p_r$ there exists a FW word. It is proved in [9] that the FW word for length n and periods $p_1, p_2, \ldots, p_r$ is unique apart from word isomorphism. We call u a FW word for period set $\{p_1, p_2, \ldots, p_r\}$ if it is the FW word for length $|u|$ and periods $p_1, p_2, \ldots, p_r$.

The word u is called a pseudo-palindrome if u is a fixed point of some involutary antimorphism θ of A^*; an involutary antimorphism is given by a map $\theta : A^* \to A^*$ such that $\theta \circ \theta = \mathrm{id}$ and $\theta(uv) = \theta(v)\theta(u)$ for all $u, v \in A^*$. The reversal operator $R : A^* \to A^*$ given by $R(u) = \overline{u}$ is a basic example, hence every palindrome is a pseudo-palindrome. It is proved in [9] that every FW word is a pseudo-palindrome.

Similar to the previous theorems we have the following characterization of FW words which implies that every extremal FW word is a FW word.

Theorem 4. *A word u is a FW word if and only if for some K it can be generated as follows: $u[0] := \epsilon$; for $k = 1, \dots, K$ either*
$u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is not ϵ and consists of distinct letters none of which occurs in $u[k-1]$,
or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l]u'[k-1]$ for some l with $0 \le l < k-1$.

Observe that standard episturmian words and Fischler words are defined in terms of palindromes whereas extremal FW words and FW words are defined in terms of periods. The relation between periods and (pseudo-)palindromic prefixes of (pseudo-)palindromic words is exhibited in the following equivalence.

Lemma 1. *Let w be a (pseudo-)palindrome and u a prefix of w. Then u is a (pseudo-)palindrome if and only if $|w| - |u|$ is a period of w.*

2. Proof of Lemma 1

Proof of Lemma 1. Denote the involutary antimorphism by θ. Thus $\theta(w) = w$. In case of proper palindromes θ is the reversal operator.

Suppose $|w| - |u|$ is a period of w. Then there is a word $\tilde{v}$ such that $w = uv = \tilde{v}u$. Hence

$$\theta(v)\theta(u) = \theta(uv) = \theta(w) = w = uv = \tilde{v}u.$$

Since $|\theta(v)| = |v| = |\tilde{v}|$, we have $\theta(u) = u$. Thus u is a (pseudo-)palindrome.

Suppose u is a (pseudo-)palindrome. Then

$$uv = w = \theta(w) = \theta(v)\theta(u) = \theta(v)u.$$

Hence $|w| - |u| = |v| = |\theta(v)|$ is a period of u. □

3. Proof of Theorems 1 and 3

Both in Theorem 1 and in Theorem 3 we have either $u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is some letter or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l]u'[k-1]$ and $l < k-1$. Observe that in the former case $u[k] \le 2|u[k-1]|+1$ and in the latter case $|u[k]| = 2|u[k-1]| - |u[l]| \le 2|u[k-1]|$.

Proof of Theorem 3. Let u be a Fischler word. Then $n_k \le 2n_{k-1} + 1$ for all k, where $\{n_k\}_{k=1}^K$ denotes the sequence of the lengths of all palindromic prefixes of u. We define a directive sequence $\{x_k\}_{k=1}^K$ by $u[0] = \epsilon, u[k] = (u[k-1]x_k)^{(+)}$ for $k = 1, \dots, K$. If $n_k = 2n_{k-1} + 1$, then $u[k] = u[k-1]x_k u[k-1]$ and we choose $v[k] = x_k$. If $n_k \le 2n_{k-1}$, then write $u[k] = u[k-1]u'[k-1]$ where x_k is the initial letter of $u'[k-1]$. By Lemma 1, $|u'[k-1]|$ is a period of $u[k]$, hence of $u[k-1]$. Write $u[k-1] = wu'[k-1]$. This is possible by $n_k \le 2n_{k-1}$. Since

$$\overline{u'[k-1]}wu'[k-1] = \overline{u'[k-1]}u[k-1] = \overline{u'[k-1]}\;\overline{u[k-1]} = \overline{u[k-1]u'[k-1]} = \overline{u[k]}$$

$$= u[k] = u[k-1]u'[k-1] = \overline{u[k-1]}u'[k-1] = \overline{wu'[k-1]}u'[k-1] = \overline{u'[k-1]}\;\overline{w}u'[k-1]$$

we see that w is a palindrome. Hence $w = u[l]$ for some $l < k-1$.

Let u be constructed as in Theorem 3. Let $\{n_k\}_{k=1}^K$ be the increasing sequence of the lengths of the palindromic prefixes of u. Then $n_k \le 2n_{k-1} + 1$ by the remark at the beginning of this section. □

Proof of Theorem 1. Let u be a standard episturmian word. Then, by Theorem 2.10 of Justin and Pirillo [7], $u = u[K]$ can be constructed by $u[0] = \epsilon$ and $u[k] = (u[k-1]x_k)^{(+)}$ for $k = 1, \dots, K$. If x_k does not occur in $u[k-1]$, then obviously $u[k] = u[k-1]x_k u[k-1]$ and we choose $v[k] = x_k$.

Suppose on the contrary that x_k occurs in $u[k-1]$. Then $x_k = x_l$ for some $l < k$. Choose l maximal and write $u[k-1] = u[l]u'[k-1]$. Then x_k is the initial letter of $u'[k-1]$. Furthermore,

$$\overline{u[k-1]u'[k-1]} = \overline{u'[k-1]}\ \overline{u[k-1]} = \overline{u'[k-1]}\ u[k-1] = \overline{u'[k-1]}u[l]u'[k-1] \tag{3.1}$$

$$\overline{u'[k-1]}\ \overline{u[l]}u'[k-1] = \overline{u[l]u'[k-1]}u'[k-1] = \overline{u[k-1]}u'[k-1] = u[k-1]u'[k-1].$$

Therefore $u[k] := u[k-1]u'[k-1]$ is a palindrome. Thus $u[k-1]x_k$ is a prefix of the palindrome $u[k-1]u'[k-1]$.

In order to show that $(u[k-1]x_k)^{(+)} = u[k] = u[k-1]u'[k-1]$ we observe that, by Lemma 1, $|u'[k-1]|$ is a period of $u[k]$, whence of $u[k-1]$, and, again by Lemma 1, the prefix of length $|u[k-1]| - |u'[k-1]|$ has to be a palindromic prefix of u. Hence, by the definition of episturmian word, it has to be equal to $u[l]$ for some l. Since the next letter has to be x_k, the maximal such l leads to the shortest word $u'[k-1]$, is the therefore the only choice.

Suppose a word u is constructed as described in Theorem 1. If $u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ does not occur in $u[k-1]$, then obviously $(u[k-1]v[k])^{(+)} = u[k]$. In the other case we have $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l]u'[k-1]$, $l < k$, and $u[l]$ and $u[k]$ are palindromes. Formula (3.1) implies that $u[k]$ is a palindrome. Let x_k be the first letter of $u'[k-1]$. The proof that $u[k] = (u[k-1]x_k)^{(+)}$ is the same as given above. □

4. Some lemmas

Lemma 2. a) *If $p_1, p_2, \dots, p_r$ are coprime integers and $p \in \{p_1, p_2, \dots, p_r\}$, then $p, p_1 + p, p_2 + p, \dots, p_r + p$ are coprime integers too.*
b) *If $p_1 < p_2 < \dots < p_r$ are coprime integers, then $p_1, p_2 - p_1, \dots, p_r - p_1$ are coprime integers.*

Proof. a) Suppose $d|p, d|p + p_i$ for all i. Then $d|p_i$ for all i. b) Similar. □

Lemma 3. ([9], Lemma 1) *Let $u = u_1 \cdots u_m$ be a word with $\sharp u = s$ and periods $q_1 < \dots < q_r$. Put $u' := u_1 \cdots u_{m-q_1}$. If $m \geq 2q_1 - y$, with $0 \leq y < q_1$, then u' is a word with $\sharp u' \geq s - y$ and periods $q_1, q_2 - q_1, ..., q_r - q_1$.*

Proof. Because of the period q_1, every letter of u occurs in u' with the possible exception of some of the y letters $u_{q_1-y+1}, u_{q_1-y+2}, \dots, u_{q_1}$. Hence $\sharp u' \geq s - y$. For $t \leq m - q_j$ we have $u_t = u_{t+q_j} = u_{t+q_j-q_1}$. So u' has period $q_j - q_1$ for $j = 2, ..., r$. □

Lemma 4. ([9], Lemma 2) *Suppose $u = u_1 \cdots u_m$ has periods $q_1, ..., q_r$. Let $u' := u_1u_2 \cdots u_{m+q_1}$ have period q_1. Then the word u' has periods $q_1, q_2 + q_1, ..., q_r + q_1$.*

Proof. Note that if $q_1 \leq m$, then $u_{m+1} \cdots u_{m+q_1}$ is the suffix of u of length q_1. If $q_1 > m$, then $u_{q_1+1} \cdots u_{m+q_1}$ is a suffix of u. For $t \leq m - q_j$ we have $u_t = u_{t+q_j} = u_{t+q_j+q_1}$ for $j = 2, ..., r$. □

Lemma 5. *Every extremal FW word is a palindrome.*

Proof. [8] Theorem 4.

Lemma 6. *If w is a FW word for the period set $Q = \{q_1, \dots, q_s\}$ and $u := wvw$ where $v \neq \epsilon$ and v consists of distinct letters none of which occurs in w, then u is a FW word for the period set $P := \{|w| + |v|, |w| + |v| + q_1, |w| + |v| + q_2, \dots, |w| + |v| + q_s\}$.*

Proof. By Lemma 4, P is a set of periods of u. Suppose u is not a FW word for P. Then there exists a word u' with period set P such that $|u'| = |u|$ and $\sharp u' > \sharp u$. Let w' be the prefix of u' of length $|w|$. Then, by Lemma 3, w' has all periods from Q and $\sharp w' \geq \sharp u' - |v| > \sharp u - |v| = \sharp w$. This shows that w is not a FW word for Q, which contradicts our assumption. □

Lemma 7. *If w is a FW word for the period set $Q := \{q_1, \dots, q_s\}$ and $u = ww'$ where $w = vw'$ for some v with $|v| \in Q$, then u is a FW word for the period set $P := \{|w'|, |w'| + q_1, |w'| + q_2, \dots, |w'| + q_s\}$.*

Proof. By Lemma 4, P is a set of periods of u. Suppose u is not a FW word for period set P. Then there exists a word u' with period set P such that $|u'| = |u|$ and $\sharp u' > \sharp u$. Let w' be the prefix of u' of length $|w|$. Then, by Lemma 3, w' has all periods from Q and $\sharp w' = \sharp u' > \sharp u = \sharp w$. This shows that w is not a FW word for Q, contradicting our assumption. $\square$

Lemma 8. *If w is a FW word for the period set $Q := \{q_1, \dots, q_s\}$ with $q_1 < q_2 < \dots < q_s$ and $w = uv$ with $|v| = q_1$, then u is a FW word for the period set $Q' := \{q_1, q_2 - q_1, \dots, q_s - q_1\}$ where we omit the q_1 if $|u| < q_1$ or $q_i = 2q_1$ for some i.*

Proof. Suppose u is not a FW word for Q'. Then there exists a word u' with $|u'| = |u|$, $\sharp u' > \sharp u$ and period set Q'.

If $|v| \le |u|$, then consider $w' := u'v'$ where v' is the suffix of u' of length $|v|$. By Lemma 4, w' has all periods from Q. Moreover, $\sharp w' = \sharp u' > \sharp u = \sharp w$ and $|w'| = |w|$. This gives a contradiction with the assumption that w is a FW word.

If $|v| > |u|$, then v is of the form $v''u$. Consider $w' := u'v'u'$ where v' consists of $|v''|$ distinct letters none of which appears in u'. Then w' has all periods from Q by Lemma 4. Moreover, $\sharp w' = \sharp u' + |v'| > \sharp u + \sharp v'' \ge \sharp w$, again in contradiction with the assumption. $\square$

5. Proofs of Theorems 2 and 4

Proof of Theorem 4. We proved in Lemma 6 of [9] that every FW word can be generated in the way stated in the theorem. It remains to prove that every word u which can be generated by the given inductive procedure is a FW word. We use induction on k.

It is clear that $u[1] = v[1]$. Since $v[1]$ consists of $|v[1]|$ distinct letters, it is a FW word with period set $\{|v[1]|\} = \{|u[1]| - |u[0]|\}$. Suppose $u[k-1]$ is a FW word with period set

$$\{|u[k-1]| - |u[k-2]|, |u[k-1]| - |u[k-3]|, \dots, |u[k-1]| - |u[0]|\}.$$

If $u[k] = u[k-1]v[k]u[k-1]$ with $v[k]$ as in the statement of the theorem, then, by Lemma 6, $u[k]$ is a FW word for periods

$$\{|u[k-1]|+|v[k]|, |u[k-1]|+|v[k]|+|u[k-1]|-|u[k-2]|, \dots, |u[k-1]|+|v[k]|+|u[k-1]|-|u[0]|\} =$$
$$\{|u[k]| - |u[k-1]|, |u[k]| - |u[k-2]|, \dots, |u[k]| - |u[0]|\}.$$

If $u[k] = u[k-1]u'[k-1]$ with $u'[k-1]$ as in the statement of the theorem, then $|u'[k-1]| = |u[k-1]| - |u[l]|$ is in the period set of $u[k-1]$. From Lemma 7 it follows that $u[k]$ is a FW word for periods

$$\{|u'[k-1]|, |u'[k-1]| + |u[k-1]| - |u[k-2]|, \dots, |u'[k-1]| + |u[k-1]| - |u[0]|\} =$$
$$\{|u[k]| - |u[k-1]|, |u[k]| - |u[k-2]|, \dots, |u[k]| - |u[0]|\}.$$

This completes the induction step and shows that $u = u[K]$ is a FW word for period set $\{|u[K]| - |u[K-1]|, |u[K]| - |u[K-2]|, \dots, |u[K]| - |u[0]|\}$. $\square$

Proof of Theorem 2. Suppose u is an extremal FW word for some period set P. Then u is a FW word and can, by Theorem 4, be constructed in the indicated way. Suppose there is a k such that $u[k] = u[k-1]v[k]u[k-1]$ where $|v[k]| > 1$. Put $v[k] = x_1 \dots x_m$. Then in u every x_1 is followed by x_2 and every x_2 is preceded by x_1. Hence u is not a palindrome. This contradicts our assumption in view of Lemma 5. Thus u can be constructed by the given inductive procedure.

Let u be constructed according to the inductive procedure from the statement of Theorem 2. For $k = 0, 1, \ldots$ we define $P[k] := \{|u[k]| - |u[k-1]|, |u[k]| - |u[k-2]|, \ldots, |u[k]| - |u[0]|\}$.

It is clear that $u[1] = v[1]$ is a FW word for the period set $\{1\}$. Since $\sharp u > 1$, there is some minimal $j > 1$ with $v[j] \neq v[1]$. Then $u[j]$ is of the form $u_1 u_2 \ldots u_n u_{n+1} \ldots u_{2n-1}$ with $u_n = v[j]$ and $u_i = v[1]$ for $i \neq n$. Furthermore $u[j]$ has the period set $P[j]$ which is coprime, since we started with the coprime period set $P[1] := \{1\}$ and during the inductive process the period set remains coprime by Lemma 2a). By Lemmas 6 and 7 it follows inductively that $u[j]$ is a FW word for period set $P[j]$. Suppose $u[j]$ is not extremal for $P[j]$. Then there exists an extremal FW word w with period set $P[j]$ with either $|w| > |u[j]|, \sharp w > 1$ or $|w| = |u[j]|, \sharp w > 2$. We consider the prefix w' of w of length $|w| - n$. In the former case we have $|w| \geq 2n$. Since w has period n, we obtain $\sharp w' = \sharp w > 1$. By Lemma 8, w' is a FW word for period set $P[j-1]$. Furthermore $|w'| > |u[j-1]|$ and $\sharp w' > 1 = \sharp u[j-1]$. This contradicts that $u[j-1]$ is a FW word for period set $P[j-1]$. In the latter case we have $|w'| = |u[j-1]|$ and $\sharp w' \geq \sharp w - 1 > 1 = \sharp u[j-1]$. By Lemma 8, w' has period set $P[j-1]$. This again contradicts that $u[j-1]$ is a FW word for period set $P[j-1]$. Thus $u[j]$ is an extremal FW word for period set $P[j]$.

Next we apply induction on k, starting from $k = j+1$. Let $u[k-1]$ be an extremal FW word with period set $P[k-1]$. By Lemmas 6 and 7, $u[k]$ is a FW word for period set $P[k]$. Suppose it is not extremal. Let u' be an extremal FW word for period set $P[k]$.

If $u[k] = u[k-1]v[k]u[k-1]$, then $|u'| \geq |u[k]| = 2|u[k-1]| + 1$. Let v' be the suffix of u' of length $|u[k]| - |u[k-1]|$. Then $u' = u''v'$ with $|u''| = |u'| - |u[k]| + |u[k-1]| \geq |u[k-1]|$. By Lemma 8, u'' is a FW word for $P[k-1]$. By our assumption that $u[k-1]$ is an extremal FW word for $P[k-1]$ we have that either $|u''| \leq |u[k-1]|$ or $|u''| > |u[k-1]|$ and u'' has $\gcd(|u[k-1]| - |u[k-2]|, |u[k-1]| - |u[k-3]|, \ldots, |u[k-1]| - |u[0]|)$ as a period. In the former case we obtain $|u''| = |u[k-1]|, \sharp u'' \leq \sharp u[k-1]$, hence $|u'| = |u[k-1]| + |v'| = |u[k]|$ and $u' = u''v''u''$ where $|v''| = 1$. Then $\sharp u[k] - 1 = \sharp u[k-1] \geq \sharp u'' \geq \sharp u' - 1$. Thus $u[k]$ is an extremal word for $P[k]$ itself. In the latter case u'' is constant, since the gcd of the period set is a period of u'' and it equals 1 by Lemma 2. Moreover, $|u''| \geq |u[k-1]| + 1 = |u[k]| - |u[k-1]| = |v'|$. Since $u' = u''v'$ and u' has period $|v'|$, we have $\sharp u' = \sharp u'' = 1$. This contradicts that u' is an extremal FW word for $P[k]$.

If $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l]u'[k-1]$ for some l with $0 \leq l < k-1$, then $|u'| \geq |u[k]| = 2|u[k-1]| - |u[l]|$. Let u'' be the prefix of u' of length $|u'| - (|u[k]| - u[k-1]|) = |u'| - |u'[k-1]|$. By Lemma 8, u'' is a FW word for $P[k-1]$. As above, it follows that $|u''| \leq |u[k-1]|$ or $|u''| > |u[k-1]|$ and u'' is the constant word. In the former case $|u''| = |u[k-1]|$ and $\sharp u'' = \sharp u' > \sharp u[k] = \sharp u[k-1]$ which contradicts that $u[k-1]$ is a FW word for $P[k-1]$. In the latter case $\sharp u' = \sharp u'' = 1$ contradicting that u' is an extremal FW word. This completes the induction step. We conclude that $u = u[K]$ is an extremal FW word for period set $P[K]$. $\square$

6. Further properties of FW words

Corollary 1. *If w is an extremal FW word, then every palindromic prefix u of w with $\sharp u > 1$ is an extremal FW word.*

Corollary 2. *If w is a FW word, then every pseudo-palindromic prefix u of w is a FW word.*

The truth of the analogous statements for standard episturmian words and Fischler words follow immediately from their definitions.

Proof of Corollary 1. It suffices to prove it for the largest palindromic prefix of w, since thereafter we can apply induction. Suppose w is an extremal FW word for period set $q_1, q_2, \ldots, q_r$. Let u be the largest palindromic prefix of w. Then, by Lemma 1, $|w| - |u|$ is the shortest period of w. Hence, by Lemma 3, u has periods $q_1, q_2 - q_1, \ldots, q_r - q_1$. Suppose u is not an extremal FW word. Then there exists a non-constant word v with periods $q_1, q_2 - q_1, \ldots, q_r - q_1$ and $|v| > |u|$. Let v' be the suffix of v of length q_1 and consider the word vv'. By Lemma 4 the word has periods $q_1, q_2, \ldots, q_r$. Furthermore it is non-constant and has length $> |u| + q_1 = |w|$. This contradicts that w is an extremal FW word. □

Proof of Corollary 2. This proof is similar to the proof of Corollary 1.

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