Generalizations of some irreducibility results by Schur

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Section 1. Introduction

Let $a \geq 0$ and $a_0, a_1, \ldots, a_n$ be integers with

$$|a_0| = |a_n| = 1. \tag{1.1}$$

In this paper we study whether the polynomial

$$f(x) = a_n \frac{x^n}{(n+a)!} + a_{n-1} \frac{x^{n-1}}{(n-1+a)!} + \cdots + a_1 \frac{x}{(1+a)!} + a_0 \frac{1}{a!} \tag{1.2}$$

can have a factor of given degree over the rationals. In 1929 Schur [25], [26] proved that a polynomial of the form (1.2) satisfying (1.1) is irreducible if $a = 0$ and also if $a = 1$ unless $n+1$ is a power of 2 when it may have a linear factor or $n = 8$ when it may even have a quadratic factor. Also for $a = 2$ and many other values of a the polynomial f may have a linear factor. On the other hand, a factor of degree $> n/2$ of f has a cofactor of degree at most $n/2$. Therefore we consider the question whether f has a factor of degree k with

$$2 \leq k \leq \frac{n}{2}$$

which we always assume unless specified otherwise. One of our results reads as follows.

Theorem 1.1. *Let a and k be integers such that*

$$2 \leq k \leq \frac{n}{2}, \ 0 \leq a \leq \frac{3}{2}k.$$

Let $f(x)$ be given by (1.2) *where $a_0, a_1, \cdots, a_n$ are integers satisfying* (1.1). *Assume that $f(x)$ has a factor of degree k. Then*

$$(n,k,a) \in \{(6,2,3),(7,2,2),(7,2,3),(7,3,3),(8,2,1), (8,3,2),(12,3,4),(13,2,3),(22,2,3),(46,3,4),(78,2,3)\}. \tag{1.3}$$

We shall show that all the exceptions other than the last two are necessary. See after the proofs of Lemma 4.1 and Theorem 1. In Theorem 4.1 we give a similar result for $k \geq 5$ with the upper bound 30 for a. In Theorem 5.1 we show that the requirement that a is bounded by a constant times k can be relaxed to

$$a < ck \log k \frac{\log\log k}{\log\log\log k}$$

for some constant c.

We also relax the restriction (1.1). Filaseta [7] showed in 1996 that if $a = 0, |a_0| = 1$ and $0 < |a_n| < n$, then (1.2) is irreducible unless $(n, a_n) \in (6, \pm 5), (10, \pm 7)$. He also considered values of a_n less than $n^{3/2}/\sqrt{2}$. He further proved a theorem due to Lam that (1.2) is irreducible if $a = 0$ and $\gcd(a_0 a_n, n!) = 1$. Allen and Filaseta [1] computed for every n the smallest integer $M = M(n)$ such that the polynomial $f(x)$ given by (1.2) with $a = 1, |a_0| = 1, |a_n| = M$ may be reducible for suitable coefficients a_i.

Denoting by $\omega(r)$ the number of distinct prime factors of the integer r, we obtain the following result for the situation that $a_0 = \pm 1$ and a_n is a prime power, or conversely.

Theorem 1.2. *Let $f(x)$ be given by (1.2). Let $n \geq 1$, $2 \leq k \leq \frac{n}{2}$, $0 \leq a \leq .75k$ and $\omega(a_n a_0) = 1$. Assume that $f(x)$ has a factor of degree k. Then $(k, a) = (2, 1)$ or $(n, k, a) \in \{(6, 3, 0), (8, 3, 2), (10, 5, 0)\}$.*
The latter exceptions are necessary. See the end of Section 6. Theorem 7.1 provides a result in case a_n has more than one prime factor.

For real x and integer j we write $(x)_j = x(x+1)\cdots(x+j-1)$. We can express $f(x)$ from (1.2) by

$$a!f(x) = \sum_{j=0}^{n} a_j \frac{x^j}{(a+1)_j}. \tag{1.4}$$

Since every product of j consecutive integers is divisible by $j!$, the hypergeometric polynomial

$$g_{a,b,c}(x) := \sum_{j=0}^{n} \frac{(a)_j}{(b)_j (c)_j} x^j \quad (a, b, c \in \mathbb{Z}) \tag{1.5}$$

with $c = 1, b = a + 1$ is a special case of $f(x)$ in (1.4) by choosing $a_j = (a)_j/(c)_j$. Some orthogonal polynomials can be expressed in the form (1.5) such as the Laguerre polynomials

$$L_n^{(\alpha)}(x) := \sum_{j=0}^{n} \frac{(n+\alpha)\cdots(j+1+\alpha)}{j!(n-j)!}(-x)^j = \frac{(\alpha+1)_n}{(1)_n} g_{-n,\alpha+1,1}(x),$$

where n is a positive integer and α a complex number. Schur [25] proved in 1929 that $L_n^{(0)}(x)$ is irreducible for all n, and in 1931 that $L_n^{(1)}(x)$ is irreducible for all n too [27]. Filaseta and Lam [10] proved the irreduciblity for all but finitely many n of

$$\mathcal{L}_n^{(\alpha)}(x) := \sum_{j=0}^{n} a_j \frac{(n+\alpha)\cdots(j+1+\alpha)}{j!(n-j)!}(-x)^j$$

where α is a fixed rational number which is not a negative integer and $a_0, a_1, \ldots, a_n$ are any integers with $|a_0| = |a_n| = 1$. Filaseta, Kidd and Trifonov [9] proved that either $L_n^{(n)}(x)$ is irreducible or it is a linear polynomial times an irreducible polynomial of degree $n-1$ and the latter possibility is excluded when $n \equiv 2 \pmod 4$ with $n > 2$. This settled completely the inverse Galois problem that for every positive integer n, there exists an explicitly given polynomial of degree n whose Galois group is the alternating group A_n.

In 2008 Filaseta, Finch and Leidy [8] proved that $L_n^{(\alpha)}(x)$ is irreducible for $0 \le \alpha \le 10$ unless $(n, \alpha) \in \{(2,2), (4,5), (2,7)\}$ in which cases $L_n^{(\alpha)}(x)$ is divisible by $x - 6$. A consequence of Theorem 4.1 of the present paper is that for $5 \le k \le n/2, 0 \le \alpha \le 30$ the polynomial $L_n^{(\alpha)}(x)$ has no factor of degree k. We refer to [8] for a survey of irreducibility results connected with orthogonal polynomials and, in particular, Laguerre polynomials.

First we restrict our attention to integral values of α. We extend the range of a and c as follows.

Theorem 1.3. *Let $\epsilon > 0$ and $1 \le k \le \frac{n}{2}$. Assume*

$$a = -n - s \text{ with } s \ge 0, b = \alpha + 1, c \ge 1, s + c < \left(\frac{1}{3} - \epsilon\right) k \tag{1.6}$$

where c, n, s and α are integers with $0 \le \alpha \le k$. Suppose that $g_{a,b,c}(x)$ is divisible by a factor of degree k. Then k is bounded by an effectively computable *number depending only on* ϵ.

Consider the Bessel polynomials

$$y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{2^j (n-j)! j!} x^j.$$

A simple calculation gives

$$(-x)^n y_n\left(\frac{2}{x}\right) = n! L_n^{(-2n-1)}(x) = (-2n)_n \; g_{-n,-2n,1}(x).$$

In 2002 Filaseta and Trifonov [11] proved that $L^{(-2n-1)}(x)$, hence $y_n(x)$ is irreducible for all n. We also consider factors of functions $g_{-n,b,1}$ for negative b. The irreducibility of $g_{-n,-n-s,1}$ has been proved by Schur [25] for $s=0$, by Hajir [12] for $s=1$, by Sell [28] for $s=2$, and for $3 \le s \le 7$ and $n \ge n_0(s)$ by Hajir [13]. We consider s bounded in terms of k and settle the following result.

Theorem 1.4. *Let* $g_{a,b,c}(x)$ *be given by* (1.5) *such that* $a = -n, b = \alpha + 1$ *with* $\alpha = -n - s - 1, c = 1$ *where* $0 \le s \le .8k$ *is an integer. Let* $2 \le k \le \frac{n}{2}$. *Then* $g_{a,b,c}(x)$ *has no factor of degree* k.

A result for $s \le .95k$ is given in Theorem 9.1.

So far we were assuming that a is an integer. In Section 10 we study the case of rational a. Let $a > 0$ be a rational number such that

$$a = u + \frac{\alpha}{\beta}$$

where u, α, β are integers with $u \ge 0, 0 < \alpha < \beta$ with gcd $(\alpha, \beta) = 1$. Put

$$(\alpha)_{\beta,m} = \alpha(\alpha+\beta)\cdots(\alpha+(m-1)\beta).$$

Hence $(\alpha)_{1,m} = (\alpha)_m$. Further put

$$F(x) := F_a(x) := a_n \frac{\beta^n x^n}{(\alpha)_{\beta,n+u}} + a_{n-1}\frac{\beta^{n-1}x^{n-1}}{(\alpha)_{\beta,n-1+u}} + \cdots + a_1 \frac{\beta x}{(\alpha)_{\beta,1+u}} + a_0 \frac{1}{(\alpha)_{\beta,u}}$$

where $a_0, a_1, \cdots, a_n \in \mathbb{Z}$ with $|\, a_0 \,|=|\, a_n \,|= 1$.

The height $H(a)$ of a non-zero rational number a, in its reduced form, is defined as the maximum of the absolute values of its numerator and denominator. Then an analogue of Theorem 1.1 for the rational case is as follows.

Theorem 1.5. *Let $2 \le k \le \frac{n}{2}$ and assume that $F(x)$ has a factor of degree k. Then there exist effectively computable absolute constants k_0 and $C_1 > 0$ such that for $k \ge k_0$, we have*

$$H(a) \ge C_1 \log\log k.$$

As applications of an explicit version of Theorem 1.5 we obtain in Theorems 10.2 and 10.3 some irreducibility results on the so-called Hermite polynomials.

In our proofs we follow the p-adic method of Coleman and Filaseta. See Lemma 2.12 and Corollary 2.1. This has been combined by a method arising out of a theorem of Sylvester that a product of k consecutive positive integers each exceeding k is divisible by a prime greater than k. See Lemmas 2.2-2.5, 2.7-2.9, 2.13 and Theorem 2.1. Further we need some results from Prime number theory and factors of composite numbers, see Lemmas 2.6, 2.10 and 2.11. For the results from Prime number theory we refer to [3] and [14]. We also use the theory of linear forms in logarithms in the proofs of Theorems

5.1 and 10.1. In many cases we show that the exceptions in our theorems are unavoidable by giving examples of factorizations.

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Section 2. Lemmas

Let x, k be positive integers with $x \geq 2k$. We put

$$ {}_k(x) = x(x-1)\cdots(x-k+1) $$

and

$$ (x)_k = x(x+1)\cdots(x+k-1). $$

Hence $(x)_k = {}_k(x+k-1)$.

Lemma 2.1. (a) *If* $k > 4, x > 42$ *and* $P({}_k(x)) \leq 41$, *then* $(x,k) = (290,6)$ *or* $k = 5$ *and* $x \in \{52, 58, 66, 78, 156, 289, 290, 495, 1521\}$.

(b) *For* $x > 100$ *the inequality* $P({}_3(x)) \leq 29$ *is possible only if* $x \in \{116, 117, 121, 145, 154, 162, 170, 171, 176, 209, 210, 232, 290, 324, 325, 352, 392, 442, 495, 552, 784, 1276, 2002, 2025, 2432, 3250, 9802, 13312\}$.

(c) *For* $x > 100$ *the inequality* $P({}_3(x)) \leq 19$ *is possible only if* $x \in \{121, 154, 170, 171, 210, 325, 352, 442, 2432\}$.

(d) *For* $x > 16$ *the inequality* $P({}_3(x)) \leq 13$ *is possible only if* $x \in \{22, 26, 27, 28, 50, 56, 65, 66, 100, 352\}$.

(e) *For* $x > 16$ *the inequality* $P({}_4(x)) \leq 13$ *is possible only if* $x \in \{27, 28, 66\}$.

(f) *For* $x > 100$ *the inequality* $P(x(x-1)) \leq 7$ *is possible only if* $x \in \{126, 225, 2401, 4375\}$.

(g) *For* $x > 100$ *the inequality* $P(x(x-2)) \leq 7$ *is possible only if* $x \in \{128, 162, 245\}$.

Proof. (a). We use Table IA of [21]. We collect all the numbers for which $x, x-1, x-2, x-3, x-4$ occur in the union of the tables for $t \leq 13$.

(b) Check whether x and $x-1$ are both in the tables for $t \leq 10$. (c) Use (b).

(d) Use (c). (e) Use (d). (f) Use Table IA of [21] for $t \leq 4$. (g) Use Table IIA of [21] for $t \leq 4$.

Lemma 2.2. *Let $x \geq 2k \geq 4$. Then*

$$P({}_k(x)) > 1.8k$$

unless $(x, k) \in \{(9, 2), (10, 3), (9, 4), (10, 4), (27, 13), (28, 13)\}$ *or*
$x = 2k$ *with* $k \in \{2, 3, 4, 5, 8, 11, 13, 14, 18, 63\}$.

Proof. For $k = 2$, the assertion follows from Lemma 2.1(f). For $k \geq 3$ see [18] Corollary 1 = [15] Corollary 1.3.2.

Lemma 2.3. *If $x \geq 2k \geq 4$, then*

$$P({}_k(x)) > 1.95k$$

unless $(x, k) \in \{(4, 2), (9, 2), (10, 3)\}$ *or* $x \in \{2k, \ldots, 2k + h - 1\}$ *for* $k \in E_h$ *with* $1 \leq h \leq 11$ *where* $E_{11} := E_{10} := \{58\}, E_9 := E_8 := \{58, 59\}$,
$E_7 := E_6 := \{58, 59, 60\}$,
$E_5 := E_4 := E_6 \cup \{12, 16, 46, 61, 72, 93, 103, 109, 151, 163\}$,
$E_3 := E_2 := E_4 \cup \{4, 7, 10, 13, 17, 19, 25, 28, 32, 38, 43, 47, 62, 73, 94, 104, 110, 124, 152, 164, 269\}$,
$E_1 := E_2 \cup \{3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 23, 26, 29, 33, 35, 39, 41, 44, 48, 50, 53, 56, 63, 68, 74, 78, 81, 86, 89, 95, 105, 111, 125, 146, 153, 165, 173, 270\}$.

Proof. See [18] Theorem 2 = [15] Theorem 1.3.1.

Lemma 2.4. *If $k \geq 2$ and $x \geq \max(2k + 13, \frac{541}{262}k)$, then*

$$P({}_k(x)) > 2k.$$

Proof. See [18] Theorem 1(a) = [15] Theorem 1.3.3(a).

Lemma 2.5. (a) *Between 60 and 7000 there is no block of 21 consecutive integers each of which is composed of primes* ≤ 85*.*
(b) *Let* $21 \le k \le 25$ *and* $x > k$*. Assume that*

$$P({}_k(x)) \le 2.5k.$$

Then $x \le 90$*.*

Proof. (a) By direct calculation.
(b) By deleting the terms divisible by $11 \le p \le 2.5k$, we check that there are at least five terms in ${}_k(x)$ composed of 2, 3, 5 and 7. For each prime we omit a term which contains the highest power of that prime. There remains a term x^* in ${}_k(x)$ divisible only by primes 2, 3, 5 and 7 such that

$$x \le x^* + 24 \le 16 \cdot 9 \cdot 5 \cdot 7 + 24 = 5064.$$

Note that $2.5k \le 85$. Now apply part (a). □

Lemma 2.6. *Let* $p_1 = 2 < p_2 < \dots$ *denote the sequence of primes. Then*

$$p_{i+1} - p_i \le \begin{cases} 14 \text{ if } p_i < 523 \\ 34 \text{ if } p_i < 9551 \\ 52 \text{ if } p_i < 31397 \end{cases}$$

Proof. By direct calculation (cf. Lander and Parkin [20]).

Lemma 2.7. *If* $20 \leq k < x \leq 12k$ *and* $x \geq 60$, *then*

$$P(_k(x)) > x - \frac{5k}{6}.$$

Proof. We put

$$x = 1.072X.$$

Then we observe that

$$x - \frac{5k}{6} < X$$

since $x \leq 12k$. By Lemma 2.6, for $60 \leq x \leq 128$ the gaps between consecutive primes are at most 14 which is $\leq 5k/6$. Further $X \geq 120$ when $x > 128$. It is a direct consequence of [1] Lemma 3 that for $X \geq 120$ there is a prime in $[X, 1.072X]$. Thus $(x - \frac{5k}{6}, x]$ contains also a prime for $x > 128$. $\square$

We write $\omega(x)$ for the number of distinct primes which divide x.

Lemma 2.8. (a) *For* $k \geq 3$ *and* $x \geq 2k$, *we have*

$$\omega(_k(x)) \geq \min(\pi(k) + [\frac{3}{4}\pi(k)] - 1 + \delta(k), \pi(2k) - 1)$$

where $\delta(k) = 0$ *for* $k \geq 17, \delta(k) = 1$ *for* $7 \leq k \leq 16$ *and* $\delta(k) = 2$ *for* $3 \leq k \leq 6$.

(b) *If* $x > \frac{29}{12}k - 1$ *and* $(x, k) \neq (9, 4)$, *then*

$$\omega(_k(x)) \geq \pi(2k).$$

Proof. (a) [16] Corollary 1 = [15] Corollary 1.2.2.

(b) [16] Theorem 2 = [15] Theorem 1.2.4.

Lemma 2.9. *Let $x \geq k^2$ and $k \geq 19$. Assume $f_1 < f_2 < \cdots < f_\mu$ are all integers in $[0, k)$ satisfying*

$$P((x - f_1) \ldots (x - k_\mu)) \leq k.$$

Then $\mu < k - \pi(2k) + \pi(k)$.

Proof. [18] Lemma 4 = [15] Lemma 6.1.6.

For s a non-negative integer we put

$${}_k(x)^{(s)}$$

to be some product obtained from ${}_k(x)$ by omitting any s terms.

Lemma 2.10. (a) *For any integer $x > 66$ we have*

$$\frac{x}{\log x - 0.5} < \pi(x).$$

(b) *For any integer $x > 1$ we have*

$$\pi(x) < \frac{x}{\log x}(1 + \frac{1.2762}{\log x}).$$

Proof. (a) [23] p. 69, (b) [5] p. 14.

Lemma 2.11. *For any integer $x > 1$ we have*

$$\sqrt{2\pi x}e^{-x}x^x e^{\frac{1}{12x+1}} < x! < \sqrt{2\pi x}e^{-x}x^x e^{\frac{1}{12x}}$$

Proof. [22].

Theorem 2.1. *Let $k \geq 10$ and $x \geq 2k$.*

Then

$$P({}_k(x)^{(1)}) > 1.5k. \tag{2.1}$$

Proof. Suppose (2.1) does not hold. Let $10 \leq k < 19$. Then $P({}_k(x)^{(1)}) \leq 23$. This is excluded by Lemma 2.1(a) if $x > 42$. By a further calculation we find that among the finitely many remaining possibilities no pairs (x, k) with $x \geq 2k \geq 20$ satisfy (2.1).

Thus $k \geq 19$. Then, by Lemmas 2.9 and 2.8(a), the number of the integers i such that $x - i$ is a term in ${}_k(x)^{(1)}$ satisfying $P(x-i) > k$ is at least $\pi(2k) - \pi(k)$ if $x \geq k^2$ and $[3\pi(k)/4] - 2$ if $x < k^2$. In either case ${}_k(x)^{(1)}$ has a term divisible by a prime greater than $1.5k$. Hence, by Lemma 2.10,

$$P({}_k(x)^{(1)}) > 1.5k.$$

□

For a prime p and integers a and b with $ab \neq 0$ we make use of the p-adic notation

$$\nu(a/b) = \nu_p(a/b) = e_1 - e_2 \quad \text{where} \quad p^{e_1} || a \quad \text{and} \quad p^{e_2} || b.$$

We define $\nu(0) = \infty$. Let $f(x) = \sum_{n=0}^{n} f_j x^j \in \mathbb{Z}[x]$ with $f_0 f_n \neq 0$. Let

$$S := \{(0, \nu(f_n)), (1, \nu(f_{n-1})), \ldots, (n-1, \nu(f_1)), (n, \nu(f_0))\}$$

a set of points in the extended plane. We consider the lower edges along the convex hull of the elements of S. The slopes of the edges are increasing when calculated from left to right. We define the Newton function with respect to the prime p as the real function $F_p(x)$ on the interval $[0, n]$ which has the polygonal path formed by these edges as its graph. Hence $F_p(i) = \nu(f_{n-i})$ for $i = 0, n$ and at all points i where the slopes of the edges change.

The next lemma plays a fundamental role in this paper. It is a refinement of Lemma 2 due to Filaseta [6] which is based on a result of Dumas [4].

Lemma 2.12. *Let k, n and r be integers with $n \geq 2k > 0$. Let $h(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and let p be a prime such that $p \not| b_n$. Denote the Newton function of $h(x)$ with respect to p by $H_p(x)$. Let $a_0, a_1, \ldots, a_n$ be integers with $p \not| a_0, p \not| a_n$. Put $f(x) = \sum_{j=0}^{n} a_j b_j x^j$. If $H_p(k) > r$ and $H_p(n) - H_p(n-k) < r + 1$, then $f(x)$ cannot have a factor of degree k.*

Proof. Suppose $f(x)$ has a factor $g(x)$ of degree k. Let $G_p(x)$ be the Newton function with respect to p of $g(x)$. Then $G_p(0) = 0$ and $G_p(k)$ is an integer m, hence $G_p(x)$ has average slope m/k on the interval $[0, k]$. It follows from Dumas's theorem ([4], cf. [6] Lemma 1) that the Newton function $F_p(x)$ with respect to p of $f(x)$ has segments corresponding to the irreducible factors of $g(x)$ which has average slope m/k. Since the slopes of the segments of $F_p(x)$ are increasing and $F_p(0) = 0$, we obtain that $F_p(k) \leq m$ and $F_p(n) - F_p(n-k) \geq m$. From the definition of f we see that $F_p(x) \geq H_p(x)$ for $0 \leq x \leq n$, $F_p(0) = H_p(0) = 0$ and $F_p(n) = H_p(n)$. It follows that $F_p(k) > r$ and $F_p(n) - F_p(n-k) < r + 1$. We conclude that $r < m$ and $m < r + 1$. Since m and r are integers, this yields a contradiction. □

Almost always we apply Lemma 2.12 in the following form which is equivalent with Lemma 2 of [6].

Corollary 2.1. *Let k and n be integers with $n \geq 2k > 0$. Let $h(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and let p be a prime such that $p \not| b_n$. Denote the Newton function of $h(x)$ with respect to p by $H_p(x)$. Let $a_0, a_1, \ldots, a_n$ be integers with $p \not| a_0, p \not| a_n$. Put $f(x) = \sum_{j=0}^{n} a_j b_j x^j$. If $p|b_j$ for $j = 0, 1, \ldots, n-k$ and the left derivative of H_p at n is $< 1/k$, then $f(x)$ cannot have a factor of degree k.*

Proof. Since $p|b_j$ for $j = 0, \ldots, n-k$, we have $H_p(k) > 0$. Since the left derivative of H_p at n is $< 1/k$ and the derivative of H_p can only change at integer points, we have $H_p(n) - H_p(n-k) < 1$. Apply Lemma 2.12 with $r = 0$. $\square$

Lemma 2.13. Let $n > k \geq 5$. Then $\omega((n)_{2,k}) \geq \pi(2k)$.

Proof. For $k \geq 9$ apply Theorem 1 of [17] = Theorem 1.4.1 of [15] with $d = 2$ and $\rho = 0$. For $5 \leq k \leq 8$ the assertion follows from the results given in [15] before the statement of Theorem 1.4.1.

Section 3. Some preliminary estimates

We introduce the following notation.

Let $N = P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r}$ where $P_1 > P_2. \cdots > P_r$ are primes and $a_1, \ldots, a_r$ positive integers. Then we set

$$P(N) = P_1(N) = P_1, P_2(N) = P_2, \ldots, P_r(N) = P_r.$$

Lemma 3.1. *Let $l \geq 1, s \geq 0, \beta > 1, \rho > 1, k \geq 200, m = k\rho$ and $x \geq k^\beta$. Assume that*

$$P_l(\ {}_k(x)^{(s)}\ ) \leq m. \tag{3.1}$$

Then

$$1 > e^{\frac{54}{55}} k^{\beta - 1 - \frac{\beta\rho}{\log m}(1 + \frac{1.2762}{\log m}) - \frac{l+s-1}{k}\beta} \left(1 - \frac{1}{k^{\beta-1}}\right). \tag{3.2}$$

Proof. Assume (3.1). For every prime p with $k < p \leq m$ dividing ${}_k(x)^{(s)}$, there is a unique term of ${}_k(x)^{(s)}$ divisible by p if it exists and we omit it. The remaining product is

$$(x - f_1)(x - f_2) \cdots (x - f_\mu)$$

where

$$0 \leq f_1 < f_2 < \cdots < f_\mu < k$$

and

$$\mu \geq k - s - (\pi(m) - \pi(k)) - (l - 1) \tag{3.3}$$

satisfying

$$P((x - f_1)(x - f_2) \cdots (x - f_\mu)) \leq k. \tag{3.4}$$

Notice that

$$(x-f_1)(x-f_2)\cdots(x-f_\mu) \mid k!\binom{x}{k}.$$

Thus

$$(x-f_1)\cdots(x-f_\mu) \le k!\prod_{p\le k} p^{\mathrm{ord}_p(\binom{x}{k})} \le k!\prod_{p\le k} p^{\log x/\log p} = k!x^{\pi(k)}.$$

On the other hand

$$(x-f_1)(x-f_2)\cdots(x-f_\mu) > (x-k)^\mu = x^\mu\left(1-\frac{k}{x}\right)^\mu.$$

By combining upper and lower estimates for $(x-f_1)(x-f_2)\cdots(x-f_\mu)$ we derive

$$k! > x^{\mu-\pi(k)}\left(1-\frac{k}{x}\right)^\mu.$$

Since $x \ge k^\beta$ and $\mu \le k$ we obtain

$$k! > k^{\beta(\mu-\pi(k))}\left(1-\frac{1}{k^{\beta-1}}\right)^k. \tag{3.5}$$

By (3.3) and Lemma 2.10(b),

$$\mu-\pi(k) \ge k-\pi(m)-l-s+1 \ge k-l-s+1-\frac{m}{\log m}\left(1+\frac{1.2762}{\log m}\right).$$

Thus, applying Lemma 2.11,

$$\sqrt{2\pi k}e^{-k}k^k e^{\frac{1}{12k}} > k^{\beta\left(k-\frac{m}{\log m}(1+\frac{1.2762}{\log m})-l-s+1\right)}\left(1-\frac{1}{k^{\beta-1}}\right)^k.$$

By taking kth roots on both the sides we have

$$(2\pi k)^{1/(2k)}e^{-1}ke^{\frac{1}{12k^2}} > k^{\beta\left(1-\frac{\rho}{\log m}(1+\frac{1.2762}{\log m})-\frac{l+s-1}{k}\right)}\left(1-\frac{1}{k^{\beta-1}}\right).$$

Since $k \ge 200$, we have

$$\frac{\log(2\pi k)}{2k}+\frac{1}{12k^2} < \frac{1}{55}.$$

Thus

$$1 > e^{\frac{54}{55}} k^{\beta - 1 - \frac{\beta\rho}{\log m}(1 + \frac{1.2762}{\log m}) - \beta\frac{l+s-1}{k}} \left(1 - \frac{1}{k^{\beta-1}}\right)$$

and the assertion follows from (3.2). □

Corollary 3.1. *Under the conditions of Lemma 3.1 with* $\beta > 1.1$ *inequality* (3.1) *implies*

$$\beta\left(1 - \frac{\rho}{\log m}\left(1 + \frac{1.2762}{\log m}\right) - \frac{l+s-1}{k}\right) < 1.$$

Proof. For $k \geq 200$ and $\beta > 1.1$ we have $e^{54/55}(1 - k^{-(\beta-1)}) > 1$ and the assertion follows from (3.2).

Corollary 3.2. *Let* $l = 1, s = 0, \beta \geq 3/2, \rho = 5/2, m = k\rho$ *and* $x \geq k^{\beta}$. *Then* (3.1) *implies*

(3.6)
$$k \leq \begin{cases} 310 \text{ if } \beta = 3/2 \\ 85 \text{ if } \beta = 7/4 \\ 48 \text{ if } \beta = 2 \\ 34 \text{ if } \beta = 9/4 \\ 25 \text{ if } \beta = 5/2 \end{cases}$$

Proof. Suppose the assumptions of Corollary 3.2 are satisfied. Then (3.2) and (3.5) are valid. Corollary 3.1 implies that $k \leq 3000$. By using (3.2, (3.5) and the exact values of the function $\pi(x)$, we obtain the claimed bounds.

Let $k \leq 310$. Now we conclude (3.6) from (3.5) with $\beta \in \{7/4, 2, 9/4, 5/2\}$ and using the exact values of the function $\pi(x)$. □

Corollary 3.3. *Let $\epsilon > 0, \beta = 3/2, \rho = 2$ and*

$$l + s < (\frac{1}{3} - \epsilon)k. \tag{3.7}$$

Then (3.1) *implies that k is bounded by an effectively computable number depending only on ϵ.*

Lemma 3.2. *Let $l = 1, s = 0, \rho = 5/2,\ m = k\rho$ and $x \geq k^{3/2}$. Then* (3.1) *implies that $k \leq 20$.*

Proof. Let $l, s, \rho,\ m$ and x be as in Lemma 3.2 and assume (3.1) with $k > 20$. Then (3.6) is valid. Further we may suppose that none of the terms in $_k(x)$ is prime otherwise

$$P(_k(x)) > x - k \geq k^{3/2} - k > \frac{5}{2}k$$

contradicting (3.1). If $k > 310$, then the assertion of the lemma follows from Corollary 3.2.

Let $85 < k \leq 310$. Then, by Corollary 3.2, $x < k^{7/4} \leq 310^{7/4} < 22903$ and we observe that $p_{i+1} - p_i \geq 85$, since none of the terms in $_k(x)$ is prime. This is not possible since $p_{i+1} - p_i \leq 52$ for $p_i < 31397$ by Lemma 2.6. Similarly, if $48 < k \leq 85$, then $x < 85^2 \leq 7225$ and if $34 < k \leq 48$, then $x < 48^{9/4} < 6065$. But $p_{i+1} - p_i \leq 34$ for $p_i < 9551$ by Lemma 2.6.

Let $25 < k \leq 34$. Then $132 < 26\sqrt{26} < x < (34)^{5/2} < 6741$. By (3.1) we have $P(_k(x)) \leq 5k/2 \leq 85$. Now we apply Lemma 2.5(a).

Let $21 \leq k \leq 25$. Then $96 < 21\sqrt{21} < x \leq 90$ by Lemma 2.5(b). $\square$

For proceeding further, we need Lemma 3 of [24] which equals Corollary 4.2.4 of [15].

Lemma 3.3. *Let* $2k \le x < k^{3/2}$ *and assume* (3.4)*. Then*

$$\binom{x}{k} \le (2.83)^{k+\sqrt{x}} x^{k-\mu}.$$

We observe that the inequality (3) of [24] is not used in the proof of Lemma 3 of [24]. We apply the above result in the proof of the following lemma.

Lemma 3.4. *Let* $x < k^{3/2}$ *and* $\rho = 2$*. Assume* (3.1) *and* $l+s < 1.05k/\log k$*. Then there exists an effectively computable absolute constant* k_0 *such that for* $k > k_0$ *we have* $x < 25k$*.*

Proof. Put $\gamma = 25$ and assume $x \ge \gamma k$. Then, by Lemma 3.3,

$$\binom{\gamma k}{k} \le \binom{x}{k} \le (2.83)^{k+\sqrt{x}} x^{k-\mu}. \tag{3.8}$$

Now, by Lemma 2.11,

$$\binom{\gamma k}{k} = \frac{(\gamma k)!}{k!((\gamma-1)k)!} > \frac{1}{\sqrt{2\pi k}} \left(\frac{\gamma^\gamma}{(\gamma-1)^{\gamma-1}}\right)^k e^{-\frac{\gamma}{\gamma-1}\frac{1}{12k}}. \tag{3.9}$$

Let $k \ge k_1$ where k_1 is a sufficiently large effectively computable number depending only on γ. Then

$$e^{-\frac{\gamma}{\gamma-1}\frac{1}{12k}} > \frac{999}{1000}$$

and

$$\frac{\gamma^\gamma}{(\gamma-1)^{\gamma-1}} = (\gamma-1)\left(1+\frac{1}{\gamma-1}\right)^\gamma \ge e(\gamma-1). \tag{3.10}$$

Thus

$$\binom{\gamma k}{k} > \frac{999}{1000}\frac{1}{\sqrt{2\pi k}}(e(\gamma-1))^k.$$

On the other hand, by $m = 2k$, (3.8), (3.3) and Lemma 2.10,

$$\binom{\gamma k}{k} \le (2.83)^{k+k^{3/4}} k^{\frac{3}{2}(\pi(2k)-\pi(k)+l+s-1)}$$

$$\le (2.83)^{k+k^{3/4}} k^{\frac{3k}{\log(2k)}\left(1+\frac{1.2762}{\log(2k)}\right)-\frac{3k}{2\log k-1}+\frac{3}{2}(l+s-1)} < e^{4.116k}.$$

By combining upper and lower estimates for $\binom{\gamma k}{k}$, we have

$$(e(\gamma-1))^k < \frac{1000}{999}\sqrt{2\pi k}\, e^{4.116k} < e^{4.12k}.$$

Thus $24 = \gamma - 1 < e^{3.12} < 24$ which yields a contradiction. $\square$

Lemma 3.5. *Let $l = 1, s = 0, \rho = 5/2, m = k\rho$ and $x < k^{3/2}$. Then* (3.1) *implies that $x < 12k$.*

Proof. Let $k > 1000$ and $x \ge 16k$. Put $\gamma = 16$. Then (3.8) and (3.9) are valid. Further we observe that

$$e^{-\frac{\gamma}{\gamma-1}\frac{1}{12k}} > .999.$$

Thus, by (3.9) and (3.10),

$$\binom{\gamma k}{k} \ge .999\frac{1}{\sqrt{2\pi k}}(e(\gamma-1))^k. \tag{3.11}$$

On the other hand, as in the proof of Lemma 3.4, we have

$$\binom{\gamma k}{k} \le (2.83)^{k+k^{0.75}} k^{\frac{3.75k}{\log(2.5k)}\left(1+\frac{1.2762}{\log(2.5k)}\right)-\frac{3k}{2\log k}}.$$

This combined with (3.11) implies $k \le 1000$.

Let $12k \le x < 16k$. Put $\gamma = 12$. Then (3.8), (3.9) and (3.11) are valid. On the other hand, as in the proof of Lemma 3.4, we have

$$\binom{\gamma k}{k} \le (2.83)^{k+4\sqrt{k}} (16k)^{\frac{2.5k}{\log(2.5k)}\left(1+\frac{1.2762}{\log(2.5k)}\right)-\frac{k}{\log k}}$$

This combined with (3.11) implies $k < 1000$. The lemma is trivial for $k \leq 144$, since $k^{3/2} \leq 12k$ for such k. We use exact values of $\binom{\gamma k}{k}$ and the π function in

$$\binom{\gamma k}{k} \leq (2.83)^{k+k^{0.75}} k^{1.5(\pi(2.5k)-\pi(k))}$$

to show that this inequality is wrong for $144 < k < 1000$. $\square$

Section 4. The basic results.

First we prove Theorem 1.1 for $a \le k$ which extends Schur's results for $a = 0, 1$. In fact his result for $a = 0$ is used in the proof. We tacitly assume in Sections 4-7 that $a \ge 0$.

Lemma 4.1. *Let a and k be integers such that*

$$2 \le k \le \frac{n}{2},\ 0 \le a \le k.$$

Let $f(x)$ be given by (1.2) *where $a_0, a_1, \ldots, a_n$ are integers satisfying* (1.1). *Assume that $f(x)$ has a factor of degree k. Then*

$$(n, k, a) \in \{(7, 2, 2), (7, 3, 3), (8, 2, 1), (8, 3, 2)\}.$$

In this section we assume that the conditions of Theorem 1.1 hold. The proof of Lemma 4.1 depends on the following criterion for irreducibility.

Lemma 4.2. *Assume that $f(x)$ has a factor of degree k. Suppose that there exists a prime $p > k + a$ such that p divides $_k(n + a)$. Then p divides $a_0 a_n$.*

Corollary 4.1. *Let $|a_0| = |a_n| = 1$. Assume that $f(x)$ has a factor of degree k. Then $P(_k(n + a)) \le k + a$.*

Proof of Lemma 4.2. Suppose that $_k(n+a)$ is divisible by a prime p with $p > k+a$ and $p \not| a_0 a_n$. Consider the Newton function $G_p(x)$ with respect to p of

$$G(x) := \sum_{j=0}^{n} b_j\ x^j := (n + a)! \sum_{j=0}^{n} \frac{x^j}{(j + a)!}$$

with

$$b_j = \frac{(n+a)!}{(j+a)!}.$$

On the one hand $p| \ _k(n+a)$, hence $p|b_j = \frac{(n+a)!}{(j+a)!}$ for $j \leq n-k$. So, by Corollary 2.1, it remains to show that the slope ϕ of the right most edge of the Newton function of G_p is $< \frac{1}{k}$. Denoting ν_p by ν we have

$$\phi = \max_{1\leq j\leq n} \frac{\nu(b_0)-\nu(b_j)}{j} \leq \max_{1\leq j\leq n} \frac{\nu((j+a)!)}{j}.$$

We may assume $p \leq j+a$, otherwise $\nu((j+a)!) = 0$. Thus $j \geq p-a$ and

$$\phi < \max_{j\geq p-a} \frac{j+a}{j(p-1)} = \frac{p}{p-1}\cdot\frac{1}{p-a} \leq \frac{k+1}{k}\cdot\frac{1}{k+1} = \frac{1}{k}.$$

By Corollary 2.1 the polynomial $G(x)$ cannot have a factor of degree k. □

Proof of Lemma 4.1. Let $2 \leq k \leq n/2$ and (n,k,a) not equal to $(8,2,1)$. We assume that $f(x)$ has a factor of degree k. Then $a \geq 1$ by the result of Schur [25]. If $(n,k) = (8,2)$, then $a = 2$, which is not possible by Corollary 2.1 with respect to $p = 5$. From now on $(n,k) \neq (8,2)$.

First, we suppose that $a \leq .5k$. By Corollary 4.1, we see that

$$P(_k(n+a)) \leq k+a \leq 1.5k$$

which, by Lemma 2.2 implies that $(n+a,k) = (4,2), (9,2), (10,5)$. Further we observe that $a = 1$ if $k = 2$ and $a \in \{1,2\}$ if $k = 5$. Then $(n,k) \in \{(3,2),(8,5),(9,5)\}$ which is ruled out since $n \geq 2k$. Thus $a > .5k$ implying $a \geq 2$.

Let $.5k < a \leq .8k$. Then $k \geq 3$. Further we see from Corollary 4.1 that $P(_k(n+a)) \leq k+a = 1.8k$. Now we apply again Lemma 2.2 with $x = n+a$

to conclude that

$$(n+a,k) \in \{(10,3),(10,4),(28,13)\} \tag{4.1}$$

since $x = n+a \geq 2k+2$. In all cases but (10,3) we have $k \geq 4$ and $n+a \leq 2k+2$ and therefore

$$k \leq \frac{n}{2} \leq k+1-\frac{a}{2} < \frac{3}{4}k+1,$$

which yields a contradiction. If $(n+a,k) = (10,3)$, then $a = 2$, hence $(n,k,a) = (8,3,2)$. Thus we conclude that $a > .8k$.

Assume that $.8k \geq 12$. Then

$$n+a \geq \max(2k+13, \frac{541}{262}k).$$

Now we derive from Lemma 2.4 that $P(_k(n+a)) > 2k \geq k+a$ contradicting Corollary 4.1. Thus $.8k < 12$, i.e. $k < 15$.

Let $.8k < a \leq .95k$. By Corollary 4.1, we see that

$$P(_k(n+a)) \leq k+a \leq 1.95k.$$

Now we apply Lemma 2.3 to conclude that

$$n+a \in \{2k, \cdots, 2k+h-1\}$$

with $k \in E_h$ and $1 \leq h \leq 5$. Let $h \in \{4,5\}$. Then $k \geq 12$ and $n+a = 2k+j$ with $0 \leq j \leq 4$. On the other hand, $2k \leq n = 2k+j-a < 2k+4-.8k < 2k$ since $k \geq 12$. Let $h \in \{2,3\}$. Then $n+a = 2k+j$ with $0 \leq j \leq 2$ and $2k \leq n \leq 2k+2-.8k < 2k$ since $k \geq 4$. Let $h = 1$. Then $n+a = 2k+j$ with $j = 0$ and $2k \leq n \leq 2k-.8k < 2k$. Hence $.95k < a \leq k$. Since $k < 15$, we have $a = k$. Now we see from Corollary 4.1 that

$$P(_k(n+k)) \leq 2k. \tag{4.2}$$

Let $7 \leq k \leq 14$. Then we see that there are three consecutive terms in ${}_k(n+k)$ divisible by primes ≤ 13 and we derive from Lemma 2.1(d) a contradiction with (4.2). Let $4 \leq k \leq 6$. Then, for $n+k > 16$ we refer to Lemma 2.1(e) and use that $P({}_4(n+k)) \leq 12$. Hence we may assume $12 \leq n+k \leq 16$ implying $k \in \{4, 5\}$ and we check that $P({}_k(n+k)) > 2k$. Let $k = 3$. Then $n = 7$ by $n \geq 6$ and Lemma 2.1(d). Thus $(n, k, a) = (7, 3, 3)$. Let $k = 2$. Since $a = k$, we see from (4.2) that $P((n+1)(n+2)) \leq k + a = 4$ implying $n = 7$. Thus $(n, k, a) = (7, 2, 2)$. $\square$

The following examples show that the exceptions in Theorem 4.1 are necessary. In fact, it is possible to derive congruence conditions on the a_i's to guarantee that there is a factor of the indicated degree. In this way one can give infinitely examples for each triple.

$$\frac{x^7}{9!} + 23\frac{x^6}{8!} - 73\frac{x^5}{7!} + 16\frac{x^2}{4!} + 8\frac{x}{3!} + \frac{1}{2!} =$$
$$\frac{1}{9!}(x^2 - 3x - 6)(x^5 + 210x^4 - 4620x^3 - 12600x^2 - 65520x - 30240),$$

$$\frac{x^7}{10!} + 2\frac{x^6}{9!} + 4\frac{x^5}{8!} + 15\frac{x^4}{7!} + 4\frac{x^3}{6!} - 10\frac{x^2}{5!} + \frac{1}{3!} =$$
$$\frac{1}{10!}(x^3 + 20x^2 - 60x - 120)(x^4 + 420x^2 + 2520x - 5040),$$

$$\frac{x^8}{9!} + 3\frac{x^7}{8!} - \frac{x^6}{7!} - 2\frac{x^5}{6!} + \frac{x^3}{4!} + \frac{x^2}{3!} + \frac{x}{2!} + \frac{1}{1!} =$$
$$\frac{1}{9!}(x^2 + 6x + 12)(x^6 + 21x^5 - 210x^4 + 2520x^2 + 30240),$$

$$\frac{x^8}{10!} - 104\frac{x^7}{9!} - 121\frac{x^6}{8!} - 56\frac{x^5}{7!} - 15\frac{x^4}{6!} + 6\frac{x^3}{5!} + 6\frac{x^2}{4!} + 4\frac{x}{3!} + \frac{1}{2!} =$$
$$\frac{1}{10!}(x^3 + 10x^2 + 30x + 60)(x^5 - 1050x^4 - 420x^3 - 2520x^2 + 25200x + 30240).$$

We use Lemma 4.1 to derive an explicit result in case $0 \le a \le 30$. As already stated in Section 1 the cases $a = 0$ and $a = 1$ are due to Schur [25], [26]. Precise results for $2 \le a \le 10$ have been given by Filaseta, Finch and Leidy [8].

Theorem 4.1. *Let $f(x)$ be given by (1.2) and $3 \le k \le \frac{n}{2}$. Then*
(a) *If $0 \le a \le 30$ and $k \ge 5$, then $f(x)$ has no factor of degree k, unless*

$$(n, k, a) \in \{(17, 5, 11), (19, 5, 9), (40, 5, 12)\}.$$

(b) *If $0 \le a \le 10$ and $3 \le k \le 4$, then $f(x)$ has no factor of degree k unless*

$$(n, k, a) \in \{(7, 3, 3), (8, 3, 2), (12, 3, 4), (18, 4, 9), (18, 4, 10), (46, 3, 4), (56, 4, 10)\}.$$

Proof. Assume that $f(x)$ has a factor of degree k. By Lemma 4.1 and Corollary 4.1 we may assume $k < a$ and $P({}_k(n + a)) \le a + k$.
(a) Let $5 \le k \le 6$. Then $n \ge 10, n+a > 15$ and $P({}_k(n+a)) \le 31$. By Lemma 2.1(a) this is only possible if $n + a \le 36$ or $k = 5, n + a \in \{52, 58, 66, 156\}$.

Let $23 \le n + a < 27$. Then $23 |\ {}_k(n + a)$, hence $a + k \ge 23$ and $a \ge 17$ contradicting that $n \ge 10, n + a < 27$. Similarly contradictions are obtained for $19 \le n + a < 23, 17 \le n + a < 19, n + a = 16$.

For $34 \le n + a \le 36$ we find $17|_k(n + a)$, hence $a + k \ge 17, 10 \le n \le 25$ and eliminations by Corollary 2.1 with $p = 17$ for $a \ge 17$, with $p = 31$ if $k = 6, a = 16$ and with $p = 11$ for the other values of (n, k, a). For $29 \le n + a < 34$ we find $a = 23, n = 10$ and elimination by Corollary 2.1 with $p = 29$. For $n + a = 28$ we find $a + k \ge 13, n \le 21$ and eliminations by Corollary 2.1 with $p = 23$ for $k = 6$, and for $k = 5$ by Corollary 2.1 with $p = 13$ for $10 \leq n \leq 15$ and with $p = 7$ for $n = 20, 21$ and by Lemma 2.12 with $p = 3, r = 2$ for $n = 16$ and $n = 18$. For $n + a = 27$ we find $a + k \geq 23, n = 10, a = 17, k = 6$. This is eliminated by Corollary 2.1 with $p = 13$.

Let $n + a = 156$. Because 31 divides 155, we see that
$a + k \geq P(_k(n + a)) \geq 31$, and thus we need only consider a's with $a \geq 26$. The resulting cases $(n, a) = (126, 30), (127, 29), (128, 28), (129, 27), (130, 26)$ can each be eliminated by Corollary 2.1 with $p = 13$. For $n + a = 66$ a similar reasoning gives $a + k \geq 31, 36 \leq n \leq 40$ and eliminations by Corollary 2.1 with $p = 13$. For $n + a = 58$ we obtain $a + k \geq 29, 28 \leq n \leq 34$ and eliminations by Corollary 2.1 with $p = 19$. For $n + a = 52$ we find $a + k \geq 17, 22 \leq n \leq 40$. These cases are eliminated by Corollary 2.1 with $p = 17$ for $24 \leq n \leq 35$, and with $p = 13$ for the remaining values of n except for $n = 40$.

Let $7 \leq k \leq 10$. Then $n \geq 14, n+a > 21$ and $P(_k(n+a)) \leq 37$. By Lemma 2.1(a) this is only possible if $n+a \leq 40$. Let $37 \leq n+a < 41$. Then $a+k \geq 37$, hence $a \geq 27$ contradicting that $n \geq 14$. Similarly contradictions are obtained for $29 \leq n + a < 33, 23 \leq n + a < 27, n + a = 22$. Let $33 \leq n + a < 37$. Then $a + k \geq 31$, hence $a \geq 21$. This implies $n + a \in \{35, 36\}, a \in \{21, 22\}, k \geq 9$. These cases are eliminated by Corollary 2.1 with $p = 17$. Let $27 \leq n+a < 29$. Then $a + k \geq 23$, hence $a \geq 13$. This implies $a \in \{13, 14\}, k \geq 9$. These cases are eliminated by Corollary 2.1 with $p = 13$.

Let $k \in \{11, 12\}$. Then $n + a > 33$ and $P(_k(n + a)) \leq 41$. By Lemma 2.1(a) this implies $n + a \leq 42$. If $37 \leq n + a \leq 42$, then $a + k \geq 37$, hence $a \geq 25$ in contradiction with $n \geq 22$. A similar reasoning excludes $34 \leq n+a \leq 36$.

Let $13 \leq k \leq 16$. Then $n+a > 39$ and $P({}_k(n+a)) \leq 43$ and we see that there are seven consecutive terms in ${}_k(n+a)$ composed of primes ≤ 41 or two pairs of six separated by an integer x with $P(x) = 43$ in between. By Lemma 2.1(a) the latter possibility is excluded and the former implies $n + a \leq 46$. The remaining possibilities lead to contradictions by using Corollary 2.1 with $p = 37$.

Let $17 \leq k \leq 22$. Then $n + a > 51$ and $P({}_k(n + a)) \leq 47$ and there are six consecutive terms in ${}_k(n + a)$ with prime factors ≤ 41 or three blocks of five prime factors ≤ 41 such that consecutive blocks are separated by one integer. This is excluded by Lemma 2.1(a).

Let $23 \leq k \leq 28$. Then $n+a > 69$ and $P({}_k(n+a)) \leq 53$ and the previous argument applies.

Let $29 \leq k \leq 30$. Then $n+a > 87$ and $P({}_k(n+a)) \leq 59$ and the previous argument applies again.

(b) Let $k \in \{3, 4\}$ and $(n, k, a) \neq (7, 3, 3), (8, 3, 2)$. Then we see from Lemma 4.1 that $a \geq 4, n \geq 6$ if $k = 3$ and $a \geq 5, n \geq 8$ if $k = 4$.

Let $a \in \{4, 5, 6, 7\}$ if $k = 3$ and $a \in \{4, 5, 6\}$ if $k = 4$. Then, by Corollary 4.1, $P({}_k(n + a)) \leq 7$ which implies $k = 3$ by Lemma 2.1(e). Further we see from Lemma 2.1(d) that

$$(n, a) \in \{(6, 4), (9, 7), (10, 6), (11, 5), (12, 4), (43, 7), (44, 6), (45, 5), (46, 4)\}.$$

We observe that (6,4) is excluded by Lemma 2.12 with $p = 3, r = 1, (n, a) = (10, 6), (11, 5), (44, 6), (45, 5)$ by Corollary 2.1 with $p = 5$ and $(n, a) = (9, 7), (43, 7)$

with $p = 7$.

Let $a \in \{8, 9\}$ if $k = 3$ and $a \in \{7, 8\}$ if $k = 4$. Then $P(_k(n + a)) \le 11$. Then we derive from Lemma 2.1(d),(e) that $(n, a) \in$

$$\{(7, 9), (8, 8), (13, 9), (14, 8), (41, 9), (42, 8), (47, 9), (48, 8), (91, 9), (92, 8)\}$$

with $k = 3$. All these cases are excluded by Corollary 2.1 with $p = 7$.

Let $k = 4$ and $a \in \{9, 10\}$. Then $P(_k(n + a)) \le 13$. We derive from Lemma 2.1(e) that

$$(n, a) \in \{(17, 10), (18, 9), (18, 10), (19, 9), (56, 10), (57, 9)\}.$$

The cases $(n, a) = (19, 9), (57, 9)$ are excluded by Corollary 2.1 with $p = 7$, the case $(n, a) = (17, 10)$ by Lemma 2.12 with $p = 3, r = 2$.

Let $k = 3$ and $a = 10$. Then we derive from Lemma 2.1(d) that $n \in \{6, 12, 16, 17, 18, 40, 46, 55, 56, 90, 342\}$. The case $n = 18$ is excluded by Corollary 2.1 with $p = 7$ and all others with $p = 5$. □

Proof of Theorem 1.1. Let the assumptions of Theorem 1.1 be satisfied and (n, k, a) not in the set given in (1.3). Then

$$k < a \le \frac{3k}{2}$$

by Lemma 4.1.

Let $k > 2$. Then $k \ge 5$ by Theorem 4.1(b). Now we see from Theorem 4.1(a) that $a > 30$ implying $k > 20$. We put $x = n + a$. Thus $x > 2k + k = 3k > 60$. Let $l = 1, s = 0, \rho = 5/2, m = k\rho$. By Corollary 4.1 we have

$$P(_k(x)) \le \frac{5k}{2}. \tag{4.3}$$

Consequently $x < k^{3/2}$ by Lemma 3.2. We derive from Lemma 3.5 that $x < 12k$. Now we derive from Lemma 2.7 that

$$P(_k(x)) > x - \frac{5k}{6} \geq 3k - \frac{5k}{6} = \frac{13k}{6}$$

Now, by Corollary 4.1, we see that $a > \frac{7}{6}k$ implying that $x > \frac{19}{6}k$. Another application of Lemma 2.7 gives $P(_k(x)) > \frac{7}{3}k$ whence $a > \frac{4}{3}k, x > \frac{10}{3}k, P(_k(x)) > 2.5k$ contradicting (4.3).

Let $k = 2$. Then $a = 3, n + a \geq 7$ and $P(_2(n + a)) \leq 5$ by Corollary 4.1. Now we apply Lemma 2.1(f) to obtain $n \in \{6, 7, 13, 22, 78\}$. □

Remark. Examples of factorizations for $(6, 2, 3), (7, 2, 3), (12, 3, 4), (13, 2, 3)$, $(22, 2, 3)$ are given by:

$$\frac{x^6}{9!} + 4\frac{x^5}{8!} - 6\frac{x^4}{7!} + \frac{x^3}{6!} - \frac{x^2}{5!} + 2\frac{x}{4!} + \frac{1}{3!}$$

is divisible by $x^2 - 6x + 12$,

$$\frac{x^7}{10!} + 10\frac{x^6}{9!} + 24\frac{x^5}{8!} + 7\frac{x^4}{7!} + 5\frac{x^3}{6!} + 2\frac{x^2}{5!} + 2\frac{x}{4!} + \frac{1}{3!}$$

is divisible by $x^2 + 30x + 60$,

$$\frac{x^{12}}{16!} - 107\frac{x^9}{13!} - 45\frac{x^6}{10!} + 9\frac{x^3}{7!} + \frac{1}{4!}$$

is divisible by $x^3 + 840$,

$$\frac{x^{13}}{16!} - \frac{x^{12}}{15!} + 44\frac{x^{11}}{14!} - 94\frac{x^{10}}{13!} + 2\frac{x^9}{12!} + 15\frac{x^8}{11!} - 2\frac{x^7}{10!} + \frac{x^6}{9!} - 8\frac{x^5}{8!} + 5\frac{x^4}{7!} - 12\frac{x^3}{6!} + 2\frac{x^2}{5!} - \frac{x}{4!} + \frac{1}{3!}$$

is divisible by $x^2 - 30x + 60$.

$$\frac{x^{22}}{25!} + 76\frac{x^{20}}{23!} + 9\frac{x^{18}}{21!} + \frac{x^6}{9!} + 2\frac{x^4}{7!} + 2\frac{x^2}{5!} + \frac{1}{3!}$$

is divisible by $x^2 + 60$.

Factorizations for (7, 2, 2), (7, 3, 3), (8, 2, 1), (8, 3, 2) have been given after the proof of Lemma 4.1.

Section 5. A better estimate for a when k is sufficiently large

We prove

Theorem 5.1. *Let $f(x)$ be given by* (1.2). *Let* $n \geq 1$, $1 \leq k \leq \frac{n}{2}$ and $|a_0| = |a_n| = 1$. *Assume that* $f(x)$ *has a factor of degree* k. *Then there exist effectively computable absolute constants* k_0 *and* $C > 0$ *such that for* $k \geq k_0$, *we have*

$$a \geq Ck \log k \frac{\log\ \log k}{\log\ \log\ \log k}$$

Proof. Let k be sufficiently large. We show that there exists a prime $p > k + a$ dividing ${}_k(n + a)$. Let $n \leq k^{3/2}$. Then the interval $[n + a - k + 1, n + a]$ contains primes and each prime is at least $n + a - k + 1 > k + a$. The existence of the primes follows from a well-known result on differences between consecutive primes [14]. Thus we may suppose that $n > k^{3/2}$. Then we see from [29] that $P({}_k(n + a))$ is at least constant times $k \log k \frac{\log \log k}{\log \log \log k}$. Hence there is a positive constant C such that $P({}_k(n+a)) > k+a$ whenever $a < Ck \log k \frac{\log \log k}{\log \log \log k}$. Then, by Corollary 4.1, $f(x)$ has no factor of degree k, a contradiction. Hence we conclude that

$$a \geq Ck \log k \frac{\log \log k}{\log \log \log k}.$$

□

Section 6. The leading coefficient is a prime power

In the next two sections we shall relax the condition that both a_0 and a_n are ± 1. In this section we prove Theorem 1.2.

Lemma 6.1. *Let $f(x)$ be given by* (1.2)*. Let $n \geq 1,\ 2 \leq k \leq n/2,\ 0 \leq a \leq .5k$ and $\omega(a_n a_0) = 1$. Assume that $f(x)$ has a factor of degree k. Then $(k, a) = (2, 1)$ or $(n, k, a) \in \{(6, 3, 0), (10, 5, 0)\}$.*

Proof of Lemma 6.1. Let the assumptions of Lemma 6.1 be satisfied. We may suppose $\omega(a_n) = 1, |a_0| = 1$ since the proof of the other case $\omega(a_0) = 1, |a_n| = 1$ is similar. Assume that $f(x)$ has a factor of degree k. We omit a term in ${}_k(n+a)$ divisible by $P(a_n)$ and denote the remaining product as ${}_k(n+a)^{(1)}$. Then, by the proof of Lemma 4.2 applied to any prime different from $P(a_n)$,

$$P({}_k(n+a)^{(1)}) \leq k + a \leq 1.5k. \tag{6.1}$$

Now we derive from Theorem 2.1 that $k \leq 9$.

Let $k = 9$. Then $a \leq 4$. We may suppose by (6.1) that there is at most one term in ${}_k(n+a)$ divisible by a prime greater than 13 and we omit this term. There is at most one term divisible by each of 13 and 11 and there are at most 2 terms divisible by 7 and we omit all of them. Thus we are left with a term n^* composed of primes ≤ 5 and for each of these primes, there is another term in which it appears to a power which is at least the power in n^*. Thus $n + a - 8 \leq 8 \cdot 3 \cdot 5 = 120$. Thus $n + a \leq 128$ if $k = 9$. Similarly $n + a \leq 67$ if $k = 8, n + a \leq 18$ if $k = 7$ and $n + a \leq 65$ if $k = 6$. Further we check that for each of the above cases, ${}_k(n+a)$ is divisible by at least two primes exceeding $k+a$ contradicting (6.1). This follows by direct computation except for $(n,k,a) = (24,9,4)$ in which case we apply Corollary 2.1 with respect to prime 23 to derive that (1.2) has no factor of degree 9. Thus $k \leq 5$.

Let $3 \leq k \leq 5$ and $(n,k,a) \notin \{(6,3,0),(10,5,0)\}$. As above, we see that there are two terms in ${}_k(n+a)$ composed of 2 and 3 in each of the cases $3 \leq k \leq 5$. On using that all powers of 2 and 3 which differ by 1 are ≤ 9 (cf. [21] Table 1A) we derive that $n+a \leq 36$. By checking the possibilities we find for $k=5$ that (6.1) is not satisfied unless

$$(n,a) \in \{(10.1),(10,2),(11,1),(12,0),(14,2),(16,2),(26,2)\}$$

which is excluded by Corollary 2.1 with respect to $p = 11,11,11,11,13,17,13$, respectively, for $k=4$ that $(n,a) \in$

$$\{(8,1),(8,2),(9,0),(9,1),(9,2),(10,1),(10,2),(11,1),(16,2),(17,1),(25,2),(26,1)\}$$

which is excluded by Corollary 2.1 with

$$p = 7,7,7,7,11,11,11,11,17,17,13,13,$$

respectively, and for $k=3$ that

$$(n,a) \in \{(7,1),(8,0),(8,1),(9,0),(9,1),(10,0),(17,1),(18,0)\}$$

which is excluded by Corollary 2.1 with respect to

$$p = 7,7,7,7,5,5,17,17,$$

respectively. Consequently $k=2$. If $a=0$, then either n or $n-1$ is a power of 2 and 2 does not divide a_n. These cases have been excluded in [7]. □

For integers $N > 1$ and $k \geq 1$, we write $\omega_k(N)$ for the number of distinct prime divisors $> k$ of N. Further we put $\omega_k(1) = 0$ and $\omega_k(N) = \omega_k(|N|)$ for any non-zero integer N.

Proof of Theorem 1.2. By Lemma 6.1, we may suppose that $a > .5k$. Also

$$P({}_k(n+a)^{(1)}) \leq k + a \tag{6.2}$$

by Lemma 4.2. Let $k \geq 19$. We observe that $n+a > 2.5k$. Further we derive from Lemma 2.9 when $n+a > k^2$ and Lemma 2.8(b) when $n+a \leq k^2$ that

$$\omega_k(\ {}_k(n+a)^{(1)}\ ) \geq \pi(2k) - \pi(k) - 1.$$

On the other hand, in view of (6.2) and $a < .75k$ we may suppose that

$$\omega_k(\ {}_k(n+a)^{(1)}\ ) \leq \pi(1.75k) - \pi(k).$$

Thus

$$\pi(2k) - \pi(1.75k) \leq 1,$$

whence, by Lemma 2.10,

$$k \leq 18. \tag{6.3}$$

Let $13 \leq k \leq 18$. Then $n + a \geq 32, k + a \leq 31$ and there are six consecutive terms with prime factors ≤ 31 which possibility is excluded by Lemma 2.1(a).

Let $10 \leq k \leq 12$. Then $n+a > 25, k+a \leq 21$ and there are six consecutive terms composed of primes ≤ 19 or one block of four and one block of five such terms separated by one integer. This is also excluded by Lemma 2.1(a) unless $n+a=57$ in which case (6.2) is not satisfied.

Let $k=8$ or 9. Then $n+a>20$ and we see that there are four consecutive terms in $_k(n+a)$ divisible by primes at most 11 or 13 according as $k=8$ or 9, respectively. Now we derive from Lemma 2.1(e) that $k=9$ and that they are given by $_4(27)$, $_4(28)$ and $_4(66)$ and these are excluded by (6.2) unless $(n+a,k)=(28,9)$. However this is excluded by Corollary 2.1 with $p=23$.

Let $k=6$ or 7. Then $n+a>15$ and there are three consecutive terms in $_k(n+a)$ composed of primes not exceeding 7 and 11 according as $k=6$ or 7. Next we see from Lemma 2.1(d) that they are given by $_3(16)$, $_3(50)$ if $k\in\{6,7\}$ and moreover $_3(22)$, $_3(56)$, $_3(100)$ if $k=7$. By (6.2), $a\le .75k$ and $n+a>2.5k$ all are excluded.

Let $k=5$. Then $a=3, n\ge 10$. Further there is at most one term divisible by a prime $p_0>7$. Suppose that $p_0 \nmid n+a-2$. Then there are three consecutive terms divisible by primes ≤ 7 and we derive from Lemma 2.1(d) that $n\le 49$. By (6.2) we need only consider $n\in\{13,15,25\}$ which is excluded by Corollary 2.1 with $p=13,17,13$ respectively. Assume that $p_0 \mid n+a-2$. In that case, we see that

$$P((n+a)(n+a-1))\le 7, P((n+a-3)(n+a-4))\le 7.$$

Now we apply Lemma 2.1(f) and we see that the above inequalities are possible only if $n=25$ which is already excluded.

Let $k=4$. Then $a=3$ and $n\ge 8$. In this case, there are two consecutive terms divisible by primes ≤ 7 and another such number at distance at most

2. By Lemma 2.1(f),(g) this is possible only if

$$n \in \{8, 9, 12, 13, 14, 15, 18, 24, 25, 27, 47, 48, 125\}$$

which is excluded by Corollary 2.1 with respect to

$$p = 11, 11, 13, 13, 17, 17, 19, 13, 13, 29, 47, 17, 127,$$

respectively.

Let $k = 3$. Then $a = 2$ and $n \geq 6$. Now there are either two consecutive terms or two terms differing by 2 divisible by primes ≤ 5. We again apply Lemma 2.1(f)(g) to conclude that $n \in \{7, 8, 9, 10, 14, 15, 23, 24, 79, 80\}$ in the first possibility and $n \in \{6, 8, 10, 16, 18, 25, 30, 48, 160\}$ in the second. The cases

$$n = 6, 7, 9, 10, 14, 15, 16, 18, 23, 24, 25, 30, 48, 79, 80, 160$$

are excluded by Corollary 2.1 with respect to

$$p = 7, 7, 11, 11, 7, 17, 17, 19, 23, 13, 13, 31, 7, 79, 41, 161,$$

respectively. The case $n = 8$ remains.

Let $k = 2$. Then there is no integer in $(k/2, 3k/4]$. □

The exceptions (6,3,0), (8,3,2), (10,5,0) in Theorem 1.2 are necessary. An example of a factorization for (8,3,2) is given in Section 4. For (6,3,0) and (10,5,0) we have the well known examples

$$5\frac{x^6}{6!} - 1 = \frac{1}{6!}(x^3 - 12)(x^3 + 12),$$

$$7\frac{x^{10}}{10!} - 1 = \frac{1}{10!}(x^5 - 720)(x^5 + 720).$$

Section 7. The leading coefficient divisible by more than one prime

Theorem 7.1. *Let $\epsilon > 0, 1 \leq k \leq \frac{n}{2}, 0 \leq a \leq (1-\epsilon)k$ and $f(x)$ be given by*

$$f(x) = a_n \frac{x^n}{(n+a)!} + a_{n-1} \frac{x^{n-1}}{(n-1+a)!} + \cdots + a_1 \frac{x}{(1+a)!} + a_0 \frac{1}{a!} \tag{1.2}$$

where $a_0, a_1, \ldots, a_n$ are integers with $a_n \neq 0$ and $|a_0| = 1$. Then there exist effectively computable numbers $c_1 > 0$ and k_0 depending only on ϵ such that for $k \geq k_0$, the polynomial $f(x)$ has no factor of degree k whenever

$$\omega_k(a_n) < c_1 \frac{k}{\log k}.$$

Proof. Let $\epsilon > 0$ and $\omega_k(a_n) < \frac{k}{\log k}$. We may suppose that $k \geq k_0$ with k_0 sufficiently large. By Lemma 4.2 it suffices to find a prime $p > k + a$ satisfying

$$p \mid \ {}_k(n+a), \ p \not| a_n. \tag{7.1}$$

Let $n \leq 25k$. We consider the interval $(n + a - \epsilon k, n + a]$. By the prime number theorem, the above interval contains $c_2 \frac{k}{\log k}$ primes where $c_2 > 0$ is an effectively computable number depending only on ϵ. Each of the primes divides ${}_k(n + a)$ and is greater than $n + a - \epsilon k > (2 - \epsilon)k \geq k + a$, since $a \leq (1 - \epsilon)k$. If we choose $c_1 = c_2/2$, then there exists a prime $p > k + a$ satisfying (7.1).

Next we consider $n > 25k$. For each prime $p > k$ dividing a_n we take the unique term from ${}_k(n + a)$ divisible by p if it exists and we omit it. The number of these omitted terms is less than $1.05k/\log\ k$. Now we apply Lemma 3.4 and Corollary 3.3 with $l = 0, m = 2k, s < 1.05k/\log k, x = n + a$

if $n + a < k^{3/2}$ and if $n + a \geq k^{3/2}$, respectively, to conclude that (3.1) does not hold. Therefore there exists a prime $p > 2k \geq k + a$ satisfying (7.1). $\square$

Thus we see from Theorem 7.1 that there exists an effectively computable number $c_3 > 0$ depending only on ϵ such that $|a_n| > c_3^k$ whenever $f(x)$ has a factor of degree k.

Section 8. Generalised Laguerre polynomials with α non-negative

We consider polynomials

$$g_{a,b,c}(x) = \sum_{j=0}^{n} \frac{(a)_j}{(b)_j (c)_j} x^j \tag{1.5}$$

where

$$a = -n - s, b = \alpha + 1, c \geq 1.$$

For $s = 0, c = 1$ this is the classical Laguerre polynomial. We prove Theorem 1.3 stated in the introduction.

Proof of Theorem 1.3. We may assume that $k \geq k_1 = k_1(\epsilon)$ with k_1 sufficiently large. Put

$$h(x) = (n+s+c-1)! \sum_{j=0}^{n} \frac{(n+\alpha)\cdots(j+1+\alpha)}{(n+s-j)!(c+j-1)!}\ x^j.$$

$$= \sum_{j=0}^{n} \binom{n+s+c-1}{c+j-1}\ {}_{n-j}(n+\alpha)\ x^j.$$

Since

$$g_{-n-s,\alpha+1,c}(x) = \frac{(c-1)!(n+s)!}{{}_{n}(n+\alpha)} \sum_{j=0}^{n} \frac{(n+\alpha)\dots(j+1+\alpha)}{(n+s-j)!(c+j-1)!}(-x)^j,$$

we observe that $h(x)$ has a factor of degree k if and only if $g(x)$ has a factor of degree k. Therefore it suffices to prove Theorem 1.3 with $g(x)$ replaced by $h(x)$. For $0 \leq j \leq n$, we put

$$a_j = \binom{n+s+c-1}{c+j-1}.$$

Then

$$a_n = \binom{n+s+c-1}{n+c-1} = \frac{(n+c)\cdots(n+c+s-1)}{s!}$$

and

$$a_0 = \binom{n+s+c-1}{c-1} = \frac{(n+s+1)\cdots(n+s+c-1)}{(c-1)!}.$$

Because of Lemma 4.2 it suffices to show that there exists a prime $p > k+\alpha$ with $p \mid \ {}_k(n+\alpha)$ such that

$$(8.1) \qquad p \nmid (n+c)\cdots(n+c+s-1), p \nmid (n+s+1)\cdots(n+s+c-1).$$

Let $n < k^{3/2}$ and $\alpha < .5k$. We consider the interval $(n-(.6k-\alpha-1), n]$. This contains a prime $p > n+\alpha-.6k+1 \geq 1.4k+\alpha$. Assume that p divides $(n+c)\cdots(n+c+s-1)$. Then

$$n-j \equiv 0 \pmod{p}, n+i \equiv 0 \pmod{p}$$

for some i, j with $0 < j \leq .6k, c \leq i < c+s$. Thus $i+j \equiv 0 \pmod{p}$ implying $1.4k+\alpha < i+j < .6k+c+s$ contradicting (1.6). Consequently $p \nmid (n+c)_s$. Similarly $p \nmid (n+s+1)_{c-1}$. Next we consider $n < k^{3/2}, \alpha \geq .5k$. Then we take the interval $(n+c+s, n+\alpha]$. By (1.6) and [14] we see that it contains a prime $p > n+c+s \geq 2k \geq k+\alpha$. By a similar reasoning as above we obtain that p satisfies (8.1).

Let $n \geq k^{3/2}$. We omit all the terms $n+c, \cdots, n+c+s-1$ and $n+s+1, \cdots, n+s+c-1$ from $n+\alpha-k+1, \cdots, n+\alpha$. The number of omitted terms is at most $v \leq s+c$. Now we apply (1.6) and Corollary 3.3 with $x = n+\alpha, s = v$ and $\ell = 1$ to conclude that (3.1) does not hold. Therefore we find a prime $p > 2k \geq k+\alpha$ dividing ${}_k(n+\alpha)^{(v)}$. Assume that (8.1) is not satisfied. Then

$$n-j \equiv 0 \pmod{p}, n+i \equiv 0 \pmod{p}$$

with $-\alpha \leq j \leq k$ and $0 < i \leq s + c$. It follows from the construction that $i + j \neq 0$. This implies that

$$2k < p \leq |i + j| \leq s + c + k < \frac{4}{3}k,$$

which yields a contradiction. □

Section 9. Generalised Laguerre polynomials with α negative

In the following result the conditions as well as the conclusion are weaker than in Theorem 1.4.

Theorem 9.1. *Let $g_{a,b,c}(x)$ be given by (1.5) such that $a = -n, b = \alpha + 1, c = 1$, where $\alpha = -n - s - 1$ and s is an integer with $0 \leq s \leq .95k$. Let $1 \leq k \leq \frac{n}{2}$. Suppose that $g_{-n,\alpha+1,1}(x)$ has a factor of degree k. Then $k \leq 270$ and $n \leq 2k + 10$.*

Proof. It suffices to show that the following polynomial has no factor of degree k,

$$g(x) := \sum_{j=0}^{n} \frac{(n+s-j)!n!}{(n-j)!j!s!} x^j = n! \sum_{j=0}^{n} \binom{n+s-j}{n-j} \frac{x^j}{j!}.$$

For $0 \leq j \leq n$, we put

$$a_j = \binom{n+s-j}{n-j}.$$

Then

$$a_n = 1, \quad a_0 = \binom{n+s}{n}.$$

By Lemma 4.2 it suffices to show that there exists a prime $p > k$ such that $p \mid \ _k(n)$ and $p \nmid \ _s(n+s)$. We derive from Lemma 2.3 that, unless $k \leq 270$ and $n \leq 2k + 10$, there exists a prime $p > 1.95k$ dividing $_k(n)$. Assume that this prime divides $_s(n+s)$. Then $n - j \equiv 0 \pmod{p}, n + i \equiv 0 \pmod{p}$ with $0 \leq j < k, 0 < i \leq s$ implying $1.95k < p \leq i + j < s + k$. Thus $s > .95k$ which gives a contradiction. $\square$

Proof of Theorem 1.4. Suppose $f(x)$ has a factor of degree k with $1 < k \leq n/2$. By proceeding as in the proof of Theorem 9.1 we see that it suffices to disprove that there is a prime $p > k$ such that

$$p \mid \ {}_k(n) \quad \text{and} \quad p \nmid \ {}_s(n+s). \tag{9.1}$$

By Lemma 2.2 we derive that $p \mid {}_k(n)$ implies

$$(n, k) \in \{(9,2), (10,3), (9,4), (10,4), (27,13), (28,13)\}$$

or

$$(n, k) = (2k, k) \quad \text{with} \quad k \in \{2, 3, 4, 5, 8, 11, 13, 14, 18, 63\}.$$

Let $(n, k) = (9, 2)$. Then $s \leq 1$ and we take $p = 3$ which satisfies (9.1). Let $(n, k) = (10, 3)$. Then $s \leq 2$ and we take $p = 5$ which satisfies (9.1). The pairs $(9, 4), (10, 4), (27, 13), (28, 13)$ are similarly excluded by taking $p = 7, 7, 23, 23$, respectively.

Let $(n, k) = (2k, k)$ with $k \in \{2, 3, 4, 8, 11, 13, 14, 18, 63\}$. Now we take p to be the largest prime $\leq 2k$ and we check that $p \nmid (2k+1) \cdots (2k + [.8k])$. For $k = 5$ we choose $p = 5$. $\square$

Section 10. The rational case

Let u, α, β and a be as in Theorem 1.5. Let

$$G(x) = a_n \frac{x^n}{(\alpha)_{\beta,n+u}} + a_{n-1}\frac{x^{n-1}}{(\alpha)_{\beta,n-1+u}} + \cdots + a_1 \frac{x}{(\alpha)_{\beta,1+u}} + a_0 \frac{1}{(\alpha)_{\beta,u}}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and we put

$$G_1(x) = (\alpha)_{\beta,n+u} G(x).$$

Thus

$$G_1(x) = a_n x^n + a_{n-1}(\alpha + (n+u-1)\beta)x^{n-1} + \cdots +$$

$$a_1(\alpha + (n+u-1)\beta)\cdots(\alpha+(u+1)\beta)x + a_0(\alpha+(n+u-1)\beta)\cdots(\alpha+u\beta).$$

We begin with the following result.

Lemma 10.1 *Let* $1 \le k \le n/2$ *and*

$$\beta \le 2\alpha + 2 \quad \text{if} \quad k = 1. \tag{10.1}$$

Assume that the prime p *satisfies*

$$p \ge \begin{cases} (k+u-1)\beta + \alpha + 1 & \text{if} \quad u \ne 0 \\ (k+u-1)\beta + \alpha + 2 & \text{if} \quad u = 0 \end{cases} \tag{10.2}$$

and

$$p \mid (\alpha + (n+u-k)\beta)\cdots(\alpha+(n+u-1)\beta), p \not| a_0 a_n.$$

Then $G(x)$ *has no factor of degree* k.

Proof. If suffices to show that $G_1(x)$ has no factor of degree k. Denote ν_p by ν. We consider the Newton function $H(x)$ with respect to p of the polynomial $h(x)$ obtained from $G_1(x)$ by putting $a_0 = a_1 \cdots = a_n = 1$. By Corollary 2.1 and $p|(\alpha + (n+u-k)\beta)_{\beta,k}$ it suffices to show that the slope ϕ of its right most edge is less than $1/k$. We have

$$\phi = \max_{1\le j\le n} \frac{\nu((\alpha+u\beta)_{\beta,n}) - \nu((\alpha+(j+u)\beta)_{\beta,n-j})}{j}$$

$$= \max_{1\le j\le n} \frac{\nu((\alpha+u\beta)\cdots(\alpha+(j+u-1)\beta))}{j} \le \max_{1\le j\le n} \frac{\nu((\alpha+(j+u-1)\beta)!)}{j}.$$

Now

$$\nu((\alpha+(j+u-1)\beta)!) < \frac{\alpha+(j+u-1)\beta}{p-1}.$$

Thus

$$\phi < \frac{\alpha+(j+u-1)\beta}{(p-1)j} = \frac{\alpha+(u-1)\beta}{(p-1)j} + \frac{\beta}{p-1}.$$

We may assume that

$$p \le \alpha + (j+u-1)\beta$$

otherwise $\nu((\alpha+(j+u-1)\beta)!) = 0$. Thus

$$j \ge \frac{p-\alpha-(u-1)\beta}{\beta}$$

and

$$\phi < \frac{\alpha\beta+(u-1)\beta^2}{(p-1)(p-\alpha-(u-1)\beta)} + \frac{\beta}{p-1} = \frac{\beta p}{(p-1)(p-\alpha-(u-1)\beta)}.$$

It suffices to show that

$$\frac{\beta p}{(p-1)(p-\alpha-(u-1)\beta)} \le \frac{1}{k},$$

i.e.,

$$p^2 - ((k+u-1)\beta+\alpha+1)p + \alpha + (u-1)\beta \ge 0$$

which is satisfied by (10.2) and (10.1). □

It has been proved in [30], [31] that for $\chi = \alpha + (n+u-1)\beta$ and $\epsilon > 0$, there exists an effectively computable number $C > 0$ depending only on ϵ such that

$$(10.3) \qquad P := P((\alpha + (n+u-k)\beta)_{\beta,k}) > Ck \log\log \chi \text{ if } \chi > k(\log k)^{\epsilon}.$$

The height $H(a)$ of a non-zero rational number a, in its reduced form, is defined as the maximum of the absolute values of its numerator and denominator. We write $N(a)$ for the absolute value of the numerator of a and $D(a)$ for the denominator of a. Now we combine Lemma 10.1 and (10.3) with the Prime number theorem for arithmetic progressions to prove the following result which implies Theorem 1.5 immediately.

Theorem 10.1 *Let* $|a_n| = 1,\ 2 \leq k \leq \frac{n}{2}$ *and assume that* $G(x)$ *has a factor of degree* k*. Then there exist effectively computable absolute constants* $k_0, C_1 > 0$ and C_2 *such that for* $k \geq k_0$ and $P(a_0) < C_2 k \log\log k$, *we have*

$$H(a) \geq C_1 \log\log k.$$

Proof. In fact we shall prove the more precise assertion that there exist effectively computable absolute constants $C_3 > 0$ and $C_4 > 0$ such that for $k \geq k_0$,

$$N(a) \geq C_3 k \log\log k$$

whenever

$$D(a) \leq C_4 \log\log k.$$

Assume that $G(x)$ has a factor of degree k. Put $\chi = \alpha + (n + u - 1)\beta$ and suppose that

$$\chi > k \log k.$$

Then we see from (10.3) with $\epsilon = 1$ that

$$P > C_5 k \log \log k$$

where $C_5 > 0$ is an effectively computable absolute constant. Now we choose $C_2 < C_5$ and apply Lemma 10.1 and (10.2) to conclude that

$$P \leq (k + u - 1)\beta + \alpha + 2.$$

The assertion follows by combining the above lower and upper bound for P and choosing $C_4 < C_5/2$.

Thus we may suppose that

$$\chi \leq k \log k. \tag{10.4}$$

We observe that

$$\alpha + (n + u - k)\beta \geq (k + u - 1)\beta + \alpha + \beta,$$

and that $\beta \geq 2$ if $u = 0$. Therefore, in view of Lemma 10.1, it suffices to show that there is a prime among

$$\alpha + (n + u - k)\beta, \cdots, \alpha + (n + u - 1)\beta.$$

This follows by the Prime number theorem for arithmetic progressions with error term and (10.4). $\square$

As an application of an explicit version of Theorem 1.5 we prove in Theorem 10.2 that the so-called Hermite polynomial $G_{1/2}$ is irreducible for every

n and in Theorem 10.3 that the Hermite polynomial $G_{3/2}$ is irreducible or has one linear factor. We have been informed that these results have been proved independently by C.E. Finch and N. Saradha.

If $u = 0, \alpha = 1$ and $\beta = 2$, we have the following result for the Hermite polynomial G(1/2).

Theorem 10.2. *Let $u = 0, \alpha = 1, \beta = 2$ and $\mid a_0 \mid = \mid a_n \mid = 1$. Then $G(x)$ is irreducible.*

Proof. Let $1 \leq k \leq \frac{n}{2}$ and assume that $G(x)$ has a factor of degree k. We observe that (10.1) is satisfied. Next we see from Lemma 10.1 and (10.2) that

$$(10.5) \qquad P((2n-2k+1)(2n-2k+3)\cdots(2n-1)) \leq 2k-1.$$

Since all numbers are odd, it implies

$$\omega((2n-2k+1)_{2,k}) \leq \pi(2k-1) - 1.$$

Now we derive from Lemma 2.13 that $k \leq 4$, since $2n - 2k + 1 \geq 2k + 1$.

Let $2 \leq k \leq 4$. By deleting the multiples of primes p with $5 \leq p \leq 2k-1$ in the product $(2n-2k+1)(2n-2k+3)\cdots(2n-1)$, we see that it has a term which is a power of 3 that is at most $2k-2$. Therefore $2k-2 \geq 2n-2k+1$. This is a contradiction with $n \geq 2k$.

If $k = 1$, then $2n - 1 = 1$ by (10.5) contradicting $n \geq 2k$. □

Now we turn to the case $u = 1, \alpha = 1$ and $\beta = 2$ which concerns another Hermite polynomial. We prove

Theorem 10.3. *Let* $u = 1, \alpha = 1, \beta = 2$ and $\mid a_0 \mid = \mid a_n \mid = 1$. *Then* $G(x)$ *has no factor of degree* ≥ 2.

Proof. Let $2 \leq k \leq \frac{n}{2}$ and assume that $G(x)$ has a factor of degree k. Now, as in the proof of Theorem 10.2, by Lemma 10.1 and (10.2), it suffices to show that

$$P((2n-2k+3)\cdots(2n+1)) \geq 2k+3.$$

We observe that

$$N := 2n - 2k + 3 \geq 2k + 3. \tag{10.6}$$

and N is odd. Let $k \geq 5$. By Lemma 2.13 and (10.6), we may suppose that

$$P(N(N+2)\cdots(N+2k-2)) = 2k+1. \tag{10.7}$$

Hence, by (10.6) and the fact that $2k+1$ is an odd prime, we have $N+2k-2 \geq 3(2k+1)$, hence $N \geq 4k+5$. Let α' be given by $N = \alpha' d$ with $d = 2$. Thus $\alpha' = N/2 \geq 2k+2$. We take $m = \pi(2k) - \pi_2(k)$ in [17] Theorem 1 = [15] Lemma 7.1.2 and we may suppose that $W \leq m$. Note that

(10.8). $$\frac{\text{gcd }(N, k-1)}{N} \leq \frac{k-1}{4k+5}$$

Thus there exists a t such that

$$t \geq k - m - \pi_2(k) - 1 = k - \pi(2k) - 1,$$

and, by (10.8),

$$2^t < \frac{k-1}{4k+5} \cdot \frac{(k-2)!2^{-\text{ord}_2((k-2)!)}}{(2k+3)(2k+4)\dots(3k-\pi(2k)+1)}.$$

Thus

$$2^{k-\pi(2k)} < \frac{(k-1)!2^{1-\mathrm{ord}_2((k-2)!)}}{(4k+5)(2k+3)(2k+4)\cdots(3k-\pi(2k)+1)}. \tag{10.9}$$

Hence

(10.10)

$$2^{k-\pi(2k)} < \frac{k!\cdot 2^{-\mathrm{ord}_2((k-2)!)}}{k(2k+2)(2k+3)\cdots(3k-\pi(2k)+1)} < \frac{(k!)^2 2^{-\mathrm{ord}_2((k-2)!)}}{(2k-\pi(2k))!k}.$$

Lemma 2.11 implies

$$(k!)^2 \le 2\pi k k^{2k} e^{-2k} e^{\frac{1}{6k}} \tag{10.11}$$

and we write

$$2k-\pi(2k) = 2k\left(1-\frac{\pi(2k)}{2k}\right) \tag{10.12}$$

$$\ge 2k\left(1-\frac{1}{\log(2k)}-\frac{1.2762}{\log^2(2k)}\right) = 2k(1-\phi-\theta) = 2k(1-\psi)$$

where

$$\theta = \frac{1.2762}{\log^2(2k)}, \phi = \frac{1}{\log\ 2k}, \psi = \phi+\theta.$$

Now we estimate from below $(2k(1-\psi))^{2k(1-\psi)}$. We have

$$(2k)^{2k(1-\psi)} = (2k)^{2k-2k\phi-2k\theta} = (2k)^{2k}e^{-2k}(2k)^{-2k\theta}.$$

Further

$$(1-\psi)^{1-\psi} > e^{-\psi}.$$

Thus

$$(2k(1-\psi))^{2k(1-\psi)} > (2k)^{2k}e^{-2k}(2k)^{-2k\theta}e^{-2k\psi}. \tag{10.13}$$

By (10.12), (10.13) and Lemma 2.11,

$$(2k-\pi(2k))! \geq 2\sqrt{\pi k(1-\psi)}(2k(1-\psi))^{2k(1-\psi)}e^{-2k(1-\psi)} \tag{10.14}$$

$$\geq \sqrt{\pi k(1-\psi)}k^{2k}2^{2k+1}e^{-4k}(2k)^{-2k\theta}.$$

Finally, we observe from [15] Lemma 3.1.6 that

$$2^{-\mathrm{ord}_2((k-2)!)} \leq 2^{-k+3}(k-2).$$

Therefore, by (10.10), (10.11) and (10.14),

$$2^{k-\pi(2k)} < \sqrt{\pi k}2^{-3k+3}e^{2k}(2k)^{2k\theta}\frac{e^{\frac{1}{6k}}}{\sqrt{1-\psi}}.$$

Let $k \geq 165$. Then it follows that

$$2^k < 16 \cdot 2^{-3k}e^{2k}(2k)^{2k\theta}2^{2k\psi}\sqrt{k}.$$

Therefore

$$\left(\frac{16}{e^2}\right)^k < 16(2k)^{2k\theta}2^{2k\psi}\sqrt{k}.$$

Taking k-th roots on both the sides and using $k \geq 165$, we get $2.16 < \frac{16}{e^2} < 2.15$ which gives a contradiction. Consequently $k \leq 164$.

Let $5 \leq k \leq 164$. By using the exact values of the order function, the $\pi(x)$ function and $k!$, we calculate that (10.9) is possible only if

$$k \in \{7, 9, 10\}.$$

The values 7 and 10 are excluded, since $2k+1$ has to be prime.

Let $k = 9$. By deleting multiples of primes p with $7 \leq p \leq 2k+1$ in the product

$$N(N+2)\cdots(N+2k-2), \tag{10.15}$$

as well as the terms with the highest power of 5 and of 3, we see from (10.7) that there is at least one term in the product (10.15) which divides $3 \cdot 5$. Therefore, by (10.15), $4k + 5 \leq N \leq 15$. This gives a contradiction.

Thus $k \in \{2, 3, 4\}$. We see from Lemma 2.1(g) that (10.7) is only impossible if $k = 2, N = 25$. This case is excluded by Lemma 2.12 with $p = 3, r = 1$.

$\square$

Remark The possibility of $G(x)$ having a linear factor cannot be excluded in the case $u = 1, \alpha = 1$ and $\beta = 2$. For this, as in [1], we consider $G_1(x)$ with $2n + 1$ a power of 3, $a_{n-2} = \ldots = a_2 = 0, a_n = a_0 = 1$ and $a_{n-1} = -X, a_1 = -(Y + 1)$ where X and Y are integers satisfying

$$(2n + 1)3^{n-1}X + cY = 3^n$$

and

$$c = (2n + 1)(2n - 1) \cdots 3.$$

It is possible to find X and Y satisfying the equation since

$$\gcd((2n + 1)3^{n-1}, c) \mid 3^n.$$

It follows immediately that $G(x)$ has a factor $x - 3$.

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