SHARP BOUNDS FOR SYMMETRIC AND ASYMMETRIC DIOPHANTINE APPROXIMATION

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Abstract. In 2004, J.C. Tong found bounds for the approximation quality of a regular continued fraction convergent of a rational number, expressed in bounds for both the previous and next approximation. We sharpen his results with a geometric method and give both sharp upper and lower bounds. We also calculate the asymptotic frequency that these bounds occur.

1. Introduction

In 1891, Hurwitz showed in [5] that for every irrational number x there exist infinitely many co-prime integers p and q, with $q > 0$, such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}} \frac{1}{q^2},$$

where the constant $1/\sqrt{5}$ is "best possible," in the sense that it cannot be replaced by a smaller constant.

Let x be a real irrational number, with regular continued fraction (RCF) expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}} = [a_0; a_1, a_2, \ldots, a_n, \ldots]. \tag{1}$$

Here $a_0 \in \mathbb{Z}$ is such, that $x - a_0 \in [0, 1)$, and $a_n \in \mathbb{N}$ for $n \geq 1$. Finite truncation in (1) yields the convergents p_n/q_n, $n \geq 0$, i.e.,

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n], \quad \text{for } n \geq 1.$$

The partial coefficients a_n can be found from the regular continued fraction map $T : [0, 1) \to [0, 1)$, defined by

$$T(x) := \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \, x \neq 0; \quad T(0) := 0,$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

Borel showed in [2] that for all $n \geq 1$,

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{5}}, \tag{2}$$

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where the *approximation coefficients* Θ_n of x are defined by

$$\Theta_n = \Theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right|, \qquad \text{for } n \geq 0. \tag{3}$$

Hurwitz' result is a direct consequence of Borel's result, and a classical theorem by Legendre, which states that if p and q are two co-prime integers with $q > 0$, satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{2} \frac{1}{q^2},$$

then there exists an $n \in \mathbb{N}$, such that $p = p_n$ and $q = q_n$.

Over the last century Borel's result (2) has been sharpened in various ways. For example, in [4], [10], and [1], it was shown that

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{a_{n+1}^2 + 4}}, \qquad \text{for } n \geq 0,$$

while J.C. Tong showed in [13] that the "conjugate property" holds

$$\max\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} > \frac{1}{\sqrt{a_{n+1}^2 + 4}}, \qquad \text{for } n \geq 0.$$

Also various other results on Diophantine approximations have been obtained, starting with Dirichlet's observation from [9], that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}, \qquad \text{for } n \geq 0,$$

which lead to various results in symmetric and asymmetric Diophantine approximation; see e.g. [14], [15], [7], and [8].

Defining for x irrational the sequence C_n, $n \geq 0$, by

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{C_n q_n q_{n+1}}, \qquad \text{for } n \geq 0, \tag{4}$$

Tong derived in [15] and [16] various properties of the sequence $(C_n)_{n \geq 0}$, and of the related sequence $(D_n)_{n \geq 0}$, where

$$D_n = [a_{n+1}; a_n, \ldots, a_1] \cdot [a_{n+2}; a_{n+3}, \ldots], \qquad \text{for } n \geq 0.$$

It is easy to show that $C_n = 1 + \dfrac{1}{D_n}$.

Recently, Tong obtained in [17] the following theorem, which "contains" to some extend all previous results.

Theorem 1.1. *Let* $x = [\, a_0; a_1,\, a_2, \ldots,\, a_n, \ldots]$ *be an irrational number and*

$$D_n = [\, a_{n+1};\, a_n, \ldots,\, a_1\,][\, a_{n+2};\, a_{n+3}, \ldots,], \quad n \geq 0.$$

If $r > 1$ *and* $R > 1$ *are two real numbers and*

$$\begin{aligned} M_{\text{Tong}} = \ & \tfrac{1}{2} \ \Bigg(\frac{1}{r} + \frac{1}{R} + a_n a_{n+1} \left(1 + \frac{1}{r}\right) \left(1 + \frac{1}{R}\right) \\ + & \sqrt{\left[\frac{1}{r} + \frac{1}{R} + a_n a_{n+1} \left(1 + \frac{1}{r}\right) \left(1 + \frac{1}{R}\right)\right]^2 - \frac{4}{rR}} \Bigg), \end{aligned} \tag{5}$$

then

(i) *$D_{n-2} < r$ and $D_n < R$ imply $D_{n-1} > M_{\text{Tong}}$;*
(ii) *$D_{n-2} > r$ and $D_n > R$ imply $D_{n-1} < M_{\text{Tong}}$.*

These bounds are not sharp. Tong also derives a similar result for the sequence C_n which seems incorrect.

In Section 3 we prove the following result.

Theorem 1.2. *Let $r, R > 1$ be reals and put*

$$F = \frac{r(a_{n+1}+1)}{a_n(a_{n+1}+1)(r+1)+1} \quad \text{and} \quad G = \frac{R(a_n+1)}{(a_n+1)a_{n+1}(R+1)+1}.$$

If $D_{n-2} < r$ and $D_n < R$, then there are three possibilities for the minimum of D_{n-1}.

(i) *If $r - a_n \geq G$ and $\frac{1}{a_n+1} \leq R - a_{n+1} < F$, then*

$$D_{n-1} > \min\left\{\frac{a_{n+1}+1}{R-a_{n+1}}, \frac{a_n+1}{G}\right\}.$$

(ii) *If $\frac{1}{a_{n+1}+1} \leq r - a_n < G$ and $R - a_{n+1} \geq F$, then*

$$D_{n-1} > \min\left\{\frac{a_n+1}{r-a_n}, \frac{a_{n+1}+1}{F}\right\}$$

(iii) *In all other cases*

$$D_{n-1} > M_{\text{Tong}}.$$

These bounds are sharp.

We prove a similar theorem for the case that $D_{n-2} > r$ and $D_n > R$ in Section 4. In Section 5 we calculate the asymptotic frequency that simultaneously $D_{n-2} > r$ and $D_n > R$. Finally we correct Tong's result for C_n in Section 6 and give sharp bound for this case.

2. The natural extension

From now on we only consider $x \in [0,1) \setminus \mathbb{Q}$, so with $a_0 = 0$. Define the space $\Omega = [0,1) \times [0,1]$ and define $\mathcal{T} : \Omega \to \Omega$ as

$$\mathcal{T}(x,y) = \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{a_1(x)+y}\right).$$

The following theorem was obtained in 1977 by Nakada et al [12]; see also [11] and [6].

Theorem 2.1. *Let ν be the probability measure on Ω with density $d(x,y)$, given by*

$$d(x,y) = \frac{1}{\log 2} \frac{1}{(1+xy)^2}, \quad (x,y) \in \Omega; \tag{6}$$

then ν is the invariant measure for $\mathcal{T}$. Furthermore, the dynamical system $(\Omega, \nu, \mathcal{T})$ is an ergodic system.

The system $(\Omega, \nu, \mathcal{T})$ is the natural extension of the ergodic dynamical system $([0,1), \mu, T)$, where μ is the so-called Gauss-measure, the probability measure on $[0,1)$ with density

$$d(x) = \frac{1}{\log 2} \frac{1}{1+x}, \quad x \in [0,1).$$

This natural extension plays a key role in the proofs of various results in this paper.

Write T_n and V_n for the "future" respectively "past" of $\frac{p_n}{q_n}$,

$$T_n = [0; a_{n+1}, a_{n+2}, \dots] \quad \text{and} \quad V_n = [0; a_n, \dots, a_1],$$

and we find

$$(T_n, V_n) = \mathcal{T}^n(x, 0).$$

It is easy to show that

$$D_{n-2} = \frac{(a_n + T_n)V_n}{1 - a_n V_n}, \quad D_{n-1} = \frac{1}{T_n V_n}, \quad \text{and} \quad D_n = \frac{(a_{n+1} + V_n)T_n}{1 - a_{n+1} T_n}. \tag{7}$$

The following result on the distribution of the sequence $(T_n, V_n)_{n \geq 0}$ is a consequence of the Ergodic Theorem, and was originally obtained by W. Bosma et al in [3], see also Chapter 4 of [6].

Theorem 2.2. *For almost all $x \in [0,1)$ the two-dimensional sequence*

$$(T_n, v_n) = \mathcal{T}^n(x, 0), \quad n \geq 0,$$

is distributed over Ω according to the density-function $d(t, v)$, as given in (6).

Consequently, for any Borel measurable set $B \subseteq \Omega$ with $\lambda^2(\delta B) = 0$ one has that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} I_B(T_n, V_n) = \nu(B), \tag{8}$$

where λ^2 denotes Lebesgue measure on Ω, and I_B is the indicator function of B. We use this result frequently throughout this paper. For example, since $D_n = \frac{1}{T_{n+1} V_{n+1}}$, we immediately have the following result.

Proposition 2.1. *For almost all $x \in [0,1)$, and for all $R \geq 1$, we have that the limit*

$$\lim_{n \to \infty} \frac{1}{n} \#\{1 \leq j \leq n; D_j(x) \leq R\}$$

exists, and equals

$$H(R) = 1 - \frac{1}{\log 2} \left(\log \left(\frac{R+1}{R} \right) + \frac{\log R}{R+1} \right). \tag{9}$$

Consequently, for almost all $x \in [0,1)$ one has that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} D_n(x) = \infty.$$

Proof. Due to (7) and (8), for almost every x the asymptotic frequency that $D_{n-1} \leq R$ is given by

$$\frac{1}{\log 2} \int_{t=\frac{1}{R}}^{1} \int_{v=\frac{1}{Rt}}^{1} \frac{\mathrm{d}v \, \mathrm{d}t}{(1+tv)^2};$$

see Figure 1. This integral equals

$$\frac{1}{\log 2}\left[\log 2 - \log\frac{R+1}{R} - \frac{1}{R+1}\log R\right].$$

From this (9) follows. To calculate the expectation of D_n we use that $h(x) = H'(x)$ is given by

$$h(x) = \frac{1}{\log 2}\frac{\log x}{(x+1)^2}.$$

So

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} D_k(x) = \lim_{t\to\infty}\int_1^t \frac{1}{\log 2}\frac{x\log x}{(x+1)^2} = \infty.$$

□

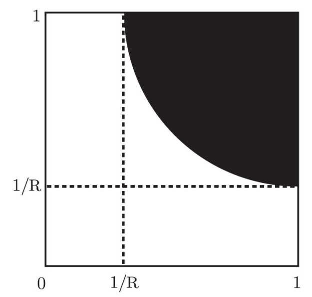

FIGURE 1. The curve $D_{n-1} = R$ on Ω. On the black part $D_{n-1} \le R$.

Apart from metric results on the D_n's , the natural extension $(\Omega, \nu, \mathcal{T})$ is also very handy to obtain various Borel-type results on the D_n's.

For $a, b \in \mathbb{N}$ consider the rectangle $\Delta_{a,b} = \left[\frac{1}{b+1}, \frac{1}{b}\right) \times \left[\frac{1}{a+1}, \frac{1}{a}\right) \subset \Omega$. On this rectangle holds $a_n = a$ and $a_{n+1} = b$, i.e. $(T_n, V_n) \in \Delta_{a,b}$ if and only if $a_n = a$ and $a_{n+1} = b$. We use a and b as abbreviation for a_n and a_{n+1} respectively if we are working within such a rectangle.

Setting $t = T_n$ and $v = V_n$ and using that $D_{n-1} = \frac{1}{tv}$, it follows that on $\Delta_{a,b}$

$$\frac{\partial D_{n-1}}{\partial t} < 0 \quad \text{and} \quad \frac{\partial D_{n-1}}{\partial v} < 0.$$

So

$$a_n a_{n+1} \le D_{n-1} \le (a_n+1)(a_{n+1}+1). \tag{10}$$

We define two functions from $[0, 1)$ to $\mathbb{R}$

$$f_{a,r}(t) = \frac{r}{a(r+1)+t} \quad \text{and} \quad g_{b,R}(t) = \frac{R}{t} - b(R+1). \tag{11}$$

From (7) it follows for $(T_n, V_n) \in \Delta_{a,b}$ that

$$\begin{aligned} D_{n-2} < r &\quad \text{if and only if} \quad v < f_{a,r}(t), \\ D_n < R &\quad \text{if and only if} \quad v < g_{b,R}(t). \end{aligned}$$

In Figure 2 we show the possible intersection points of the graphs of $f_{a,r}$ and $g_{b,R}$ with the boundary of $\Delta_{a,b}$. In view of this we introduce the following notation

$$F = \frac{r(a_{n+1}+1)}{a_n(a_{n+1}+1)(r+1)+1} \quad \text{and} \quad G = \frac{R(a_n+1)}{(a_n+1)a_{n+1}(R+1)+1}, \tag{12}$$

where $F = f_{a,r}\left(\frac{1}{b+1}\right)$ and $g_{b,R}(G) = \frac{1}{a+1}$.

Note that for $r - a < \frac{1}{b+1}$ the graph of $f_{a,r}$ has no intersection with the rectangle $\Delta_{a,b}$. Similarly, for $R - b < \frac{1}{a+1}$ the graph of $g_{b,R}$ does not intersect this rectangle $\Delta_{a,b}$.

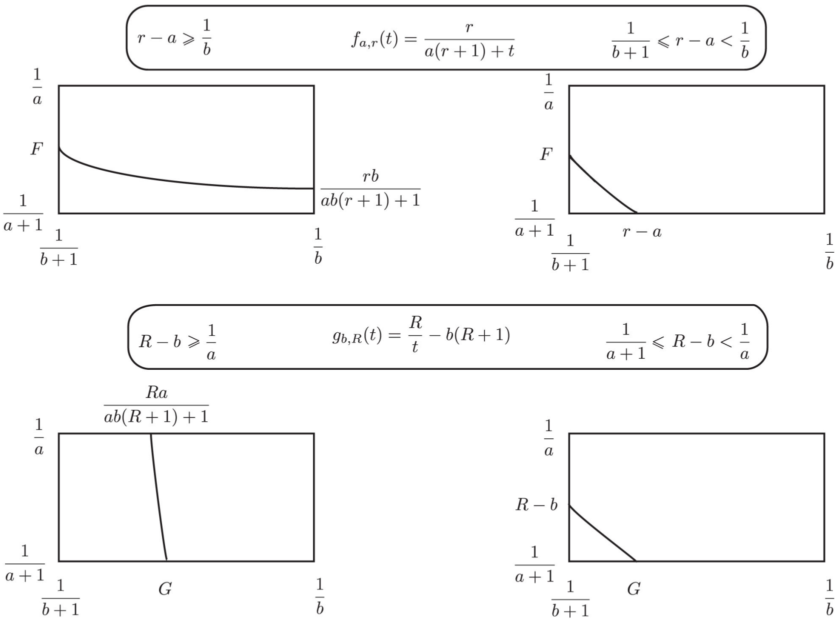

FIGURE 2. The possible intersection point of the graphs of f and g and the boundary of the rectangle $\Delta_{a,b}$.

3. THE CASE WHERE $D_{n-2} < r$ AND $D_n < R$

We assume that both D_{n-2} and D_n are smaller than some given reals r and R. We now prove Theorem 1.2 from the Introduction.

Proof. We determine the shape of the region where $D_{n-2} < r$ and $D_n < R$ and find the minimum for D_{n-1} on this region. In Figure 4 we show all possibilities. If $r - a < \frac{1}{b+1}$, then $D_{n-2} > r$ and similarly if $R - b < \frac{1}{a+1}$, then $D_n > R$ on $\Delta_{a,b}$. So in these cases there is no region in $\Delta_{a,b}$ where $D_{n-2} < r$ and $D_n < R$; also see Figure 4 (v).

If $r - a \geq G$ and $\dfrac{1}{b+1} \leq R - a_{n+1} < F$, then then the graph of $f_{a,r}$ lies above the graph of $g_{b,R}$ on the rectangle we consider; see Figure 4 (i) and (ii). From (10) we find that the minimum for D_{n-1} is found in one of the intersection points of the graph of $g_{b,R}$ and the boundary of the rectangle $\Delta_{a,b}$

$$\left(\frac{1}{b+1}, R - b\right) \text{ or } \left(G, \frac{1}{a+1}\right).$$

Using (7), $a = a_n$ and $b = a_{n+1}$ we find

$$D_{n-1} > \min\left\{\frac{a_{n+1}+1}{R-a_{n+1}}, \frac{a_n+1}{G}\right\}.$$

Likewise, if $\frac{1}{b+1} \le r - a < G$ and $R - b \ge F$, then the graph of $g_{b,R}$ lies above that of $f_{a,r}$; see Figure 4 (iii) and (iv). The minimum for D_{n-1} is found in one of the points

$$\left(r-a, \frac{1}{a+1}\right) \text{ or } \left(\frac{1}{b+1}, F\right).$$

So

$$D_{n-1} > \min\left\{\frac{a_n+1}{r-a_n}, \frac{a_{n+1}+1}{F}\right\}.$$

In all other relevant cases the graphs of $f_{a,r}$ and $g_{b,R}$ intersect inside the rectangle and the minimum for D_{n-1} is obtained in this point. The intersection point is given by

$$\left(\frac{-L+R-r+w}{2b(R+1)}, \frac{2br(R+1)}{L+R-r+w}\right),$$

with

$$L = ab(r+1)(R+1) \text{ and } w = \sqrt{4LR + (r-R+L)^2}.$$

The corresponding minimum for D_{n-1} in this point is

$$M = \frac{1}{\frac{-L+R-r+w}{2b(R+1)} \cdot \frac{2br(R+1)}{L+R-r+w}} = \frac{-L-R+r-w}{r(L-R+r-w)}.$$

We show that this constant M is exactly M_{Tong} as defined in (5). First rewrite M_{Tong} in terms of L and w

$$M_{\text{Tong}} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{R} + \frac{L}{Rr} + \frac{w}{Rr}\right).$$

Rewriting M and using $w^2 = 4LR + (r-R+L)^2$ yields

$$\begin{aligned} M &= \frac{-L-R+r-w}{r(L-R+r-w)} \cdot \frac{L-R+r+w}{L-R+r+w} \\ &= \frac{-L^2+r^2-2Rr+R^2-2Lw-w^2}{L^2r+2Lr^2+r^3-2LRr-2Rr^2+R^2r-rw^2} \\ &= \frac{-2L^2-2Lw-2Lr-2LR}{-4RrL} \\ &= \frac{1}{2}\left(\frac{1}{r} + \frac{1}{R} + \frac{L}{Rr} + \frac{w}{Rr}\right) = M_{\text{Tong}}. \end{aligned}$$

These bounds can not be improved since the minimum is attained at the extreme point. □

Example. We take $r = 2.9$ and $R = 3.6$; see Figure 3.

If $a_n = a_{n+1} = 1$, then $r - a_n = 1.9, R - a_{n+1} = 2.6, F \approx 0.66$ and $G \approx 0.71$. Since $R - a_{n+1} > F$ we do not have case (i) of Theorem 1.2. Since $r - a_n > G$ we are

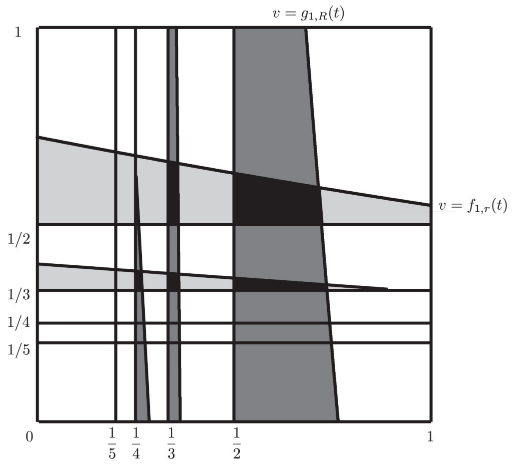

FIGURE 3. Example with $r = 2.9$ and $R = 3.6$. The regions where $D_{n-2} < 2.9$ are light grey, the regions where $D_n < 3.6$ are dark grey. The intersection where both $D_{n-2} < 2.9$ and $D_n < 3.6$ is black.

not in case (ii) either. So in this case $D_{n-1} > M_{\text{Tong}} \approx 2.30$. For the following combinations the minimum is also given by M_{Tong}

$$\begin{aligned}
a_n = 1 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\text{Tong}} &\approx 4.04.\\
a_n = 2 \text{ and } a_{n+1} = 1 : D_{n-1} > M_{\text{Tong}} &\approx 4.04.\\
a_n = 2 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\text{Tong}} &\approx 7.48.\\
a_n = 2 \text{ and } a_{n+1} = 3 : D_{n-1} > M_{\text{Tong}} &\approx 10.92.
\end{aligned}$$

If $a_n = 1$ and $a_{n+1} = 3$, then $F \approx 0.70$ and $G \approx 0.50$ So $r - a_n > G$ and $\frac{1}{a_{n+1}} < R - a_{n+1} < F$. Thus

$$\begin{aligned}
D_{n-1} &> \min\left\{\frac{a_{n+1}+1}{R-a_{n+1}}, \frac{a_n+1}{G}\right\}\\
&\approx \min\{6.67, 7.94\} = 6.67 \gg M_{\text{Tong}} \approx 5.76.
\end{aligned}$$

For all other values of a_n and a_{n+1} either $D_{n-2} > r$ or $D_n > R$, or both.

4. THE CASE WHERE $D_{n-2} > r$ AND $D_n > R$

In this section we study the case that D_{n-2} and D_n are respectively larger than given reals r and R. Now there are six different configurations of the graphs of $f_{a,r}$ and $g_{b,R}$; see Figure 4.

Theorem 4.1. *If $D_{n-2} > r$ and $D_n > R$ for given reals $r, R > 1$, then there are six possibilities for the maximum of D_{n-1}.*

(i) *If $r - a_n \geq \frac{1}{a_{n+1}}$ and $R - a_{n+1} < F$, then*

$$D_{n-1} < \frac{a_{n+1}+1}{F}.$$

(ii) *If $G \leq r - a_n < \frac{1}{a_{n+1}}$ and $R - a_{n+1} < F$, then*

$$D_{n-1} < \max\left\{\frac{a_n+1}{r-a_n}, \frac{a_{n+1}+1}{F}\right\}.$$

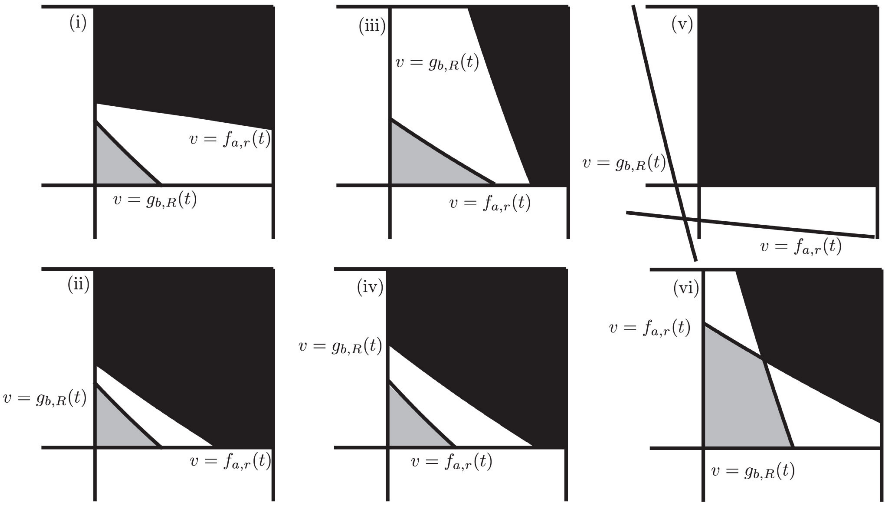

FIGURE 4. The possible configurations of the graphs of $f_{a,r}$ and $g_{b,R}$. On the black parts holds $D_{n-2} > r$ and $D_n > R$, on the grey parts holds $D_{n-2} < r$ and $D_n < R$. For the latter case (i) and (ii) give the same bound on D_{n-1}, as do (iii) and (iv).

(iii) *If* $r - a_n < G$ *and* $R - a_{n+1} \geq \frac{1}{a_n}$,
$$D_{n-1} < \frac{a_n + 1}{G}.$$
(iv) *If* $r - a_n < G$ *and* $F \leq R - a_{n+1} < \frac{1}{a_n}$, *then*
$$D_{n-1} < \max\left\{\frac{a_{n+1}+1}{R - a_{n+1}}, \frac{a_n + 1}{G}\right\}.$$
(v) *If* $r - a_n < \frac{1}{a_{n+1}+1}$ *and* $R - a_{n+1} < \frac{1}{a_n+1}$, *then*
$$D_{n-1} > (a_n + 1)(a_{n+1} + 1).$$
(vi) *In all other cases*
$$D_{n-1} > M_{\text{Tong}}.$$

The bounds are sharp.

Proof. The proof is very similar to that of Theorem 1.2, therefore we only do the first and fifth case. If $r - a_n \geq \frac{1}{a_{n+1}}$ and $R - a_{n+1} < F$, then the graph of $f_{a,r}$ lies above the graph of $g_{b,R}$ on $\Delta_{a,b}$ and $f_{a,r}$ intersects the rectangle at the righthand side and not at the bottom. The maximum for D_{n-1} is found in one of these intersection points $\left(\frac{1}{a_{n+1}+1}, F\right)$ or $\left(\frac{1}{a_{n+1}}, \frac{r a_{n+1}}{a_n a_{n+1}(r+1)+1}\right)$. So
$$\begin{aligned} D_{n-1} &< \max\left\{\frac{a_n(a_{n+1}+1)(r+1)+1}{r}, \frac{a_n a_{n+1}(r+1)+1}{r}\right\} \\ &= \frac{a_n(a_{n+1}+1)(r+1)+1}{r} = \frac{a_n+1}{F}. \end{aligned}$$
In case the graphs of $f_{a,r}$ and $g_{b,R}$ noth do not intersect with $\Delta_{a,b}$ (this is the fifth case), then clearly we have for $(T_n, V_n) \in \Delta_{a,b}$ that $D_{n-2} > r$ and $D_n > R$. In this case D_n attains its maximum in $\left(\frac{1}{a_{n+1}+1}, \frac{1}{a_n+1}\right)$. □

Example. We again use $r = 2.9$ and $R = 3.6$; see Figure 5 and Table 1.

a_n	a_{n+1}	Case	Upper bound for D_{n-1}	Tong's upper bound
1	1	(vi)	2.30	2.30
1	2	(vi)	4.04	4.04
1	3	(i)	5.72	5.76
1	4	(i)	7.07	7.48
1	$5, 6, \ldots$	(i)	$\ldots$	$\ldots$
1	37	(i)	51.44	64.20
2	1	(vi)	4.04	4.04
2	2	(vi)	7.48	7.48
2	3	(vi)	10.92	10.92
2	4	(i)	13.79	14.36
2	$5, 6, \ldots$	(i)	$\ldots$	$\ldots$
2	42	(i)	116.00	144.97
3	1	(iii)	4.04	4.04
3	2	(iii)	7.48	7.48
3	3	(iii)	10.92	10.92
3	4	(v)	13.79	14.36
$3, 4, 5, \ldots$	$1, 2, 3$	(iii)	$\ldots$	$\ldots$
$3, 4, 5 \ldots$	$5, 6, \ldots$	(v)	$\ldots$	$\ldots$
17	29	(v)	540.00	847.79

TABLE 1. The sharp upper bounds and the Tong bounds for D_{n-1} for $r = 2.9$ and $R = 3.6$.

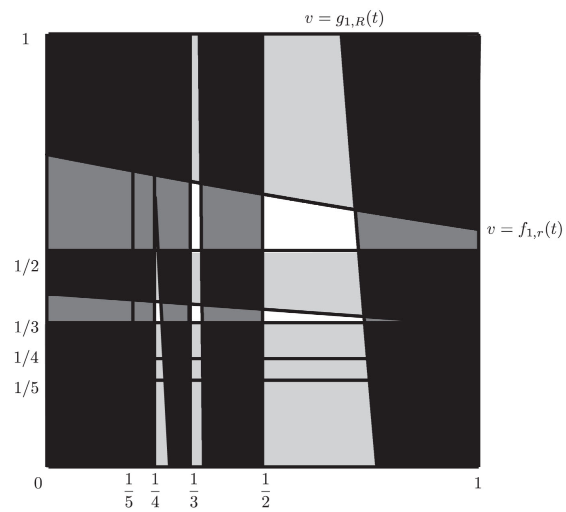

FIGURE 5. Example with $r = 2.9$ and $R = 3.6$. The regions where $D_{n-2} > 2.9$ are light grey, the regions where $D_n > 3.6$ are dark grey. The intersection where both $D_{n-2} > 2.9$ and $D_n > 3.6$ is black.

5. ASYMPTOTIC FREQUENCIES

Due to Theorem 2.1 and the ergodic theorem, the asymptotic frequency that an event occurs is equal to the measure of the area of this event in the natural extension. We calculate the measure of the region where $D_{n-2} > r$ and $D_n > R$. The same calculations can be done in the other case where $D_{n-2} < r$ and $D_n < R$, that case is actually easier. We calculate the measure in $\Delta_{a,b}$ above the graphs of $f_{a,r}$ and $g_{b,R}$ in the six cases from Theorem 4.1. Setting these measures as $m(i)_{a,b}$, we find

$$\begin{aligned} m(i)_{a,b} &= \frac{1}{\log 2}\int_{\frac{1}{b+1}}^{\frac{1}{b}}\int_{\frac{r}{a(r+1)+t}}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} = \frac{1}{\log 2}\int_{\frac{1}{b+1}}^{\frac{1}{b}} \left[\frac{-1}{t}\frac{1}{1+tv}\right]_{\frac{r}{a(r+1)+t}}^{\frac{1}{a}} \mathrm{d}t \\ &= \frac{1}{(r+1)\log 2}\left(\log\left(a+\frac{1}{b}\right) - \log\left(a+\frac{1}{b+1}\right)\right). \end{aligned}$$

For the other cases we give the results without computations.

$$m(ii)_{a,b} = \frac{1}{\log 2}\left(\frac{1}{r+1}\log\frac{r(b+1)}{a(b+1)+1} + \log\frac{(ab+1)(r+1)}{(ab+b+1)r}\right).$$

$$m(iii)_{a,b} = \frac{1}{\log 2}\left(\frac{R}{R+1}\log\frac{G_1}{G} + \frac{1}{R+1}\log\frac{bG_1-1}{bG-1} + \log\frac{(ab+1)(a+G+1)}{(ab+b+1)(a+G_1)}\right).$$

$$\begin{aligned} m(iv)_{a,b} &= \frac{1}{\log 2}\left(\frac{R}{R+1}\log\frac{1}{G(b+1)} + \frac{1}{R+1}\log\frac{-1}{(bG-1)(b+1)}\right. \\ &\quad \left. +\log\frac{(ab+1)(a+G+1)(b+1)}{(ab+b+1)(ab+a+1)}\right). \end{aligned}$$

$$m(v)_{a,b} = \frac{1}{\log 2}\left(\log\frac{ab+1}{ab+b+1} + \log\frac{(a+1)(b+1)+1}{a(b+1)+1}\right).$$

In case (vi) there are three possibilities for the measure of the part above the graphs of f and g (depending on where the graphs intersect with $\Delta_{a,b}$). If $r-a \geq \frac{1}{b+1}$ and $R-b \geq \frac{1}{a+1}$ then we have

$$m(vi)_{a,b} = \frac{1}{\log 2}\left(\frac{1}{R+1}\log\frac{S(1-bG_1)}{G_1(1-bS)} + \frac{1}{r+1}\log\frac{ab+1}{(a+S)b} + \log\frac{(a+S)G_1}{(a+G_1)S}\right).$$

If $r-a \geq \frac{1}{b+1}$, but $R-b \leq \frac{1}{a+1}$ then we have

$$\begin{aligned} m(vii)_{a,b} = \frac{1}{\log 2} & \left(\frac{R}{R+1}\log\frac{(R+1)(a+S)}{(b+1)(R+1)S} + \frac{1}{R+1}\log\frac{a+S}{(b+1)(1-bS)}\right. \\ & \left. +\frac{1}{r+1}\log\frac{ab+1}{b(a+S)} - \log\frac{ab+a}{b+1}\right). \end{aligned}$$

Finally, if $r-a \leq \frac{1}{b+1}$ and $R-b \geq \frac{1}{a+1}$ then we have

$$\begin{aligned} m(viii)_{a,b} = \frac{1}{\log 2} & \left(\frac{R}{R+1}\log\frac{G_1}{S} + \frac{1}{R+1}\log\frac{1-bG_1}{1-bS}\right. \\ & \left. + \frac{1}{r+1}\log\frac{r}{a+S} + \log\frac{(a+S)(1+ab)(r+1)}{(a+G_1)(1+ab+b)}\right). \end{aligned}$$

Here $G_1 = \frac{Ra}{ab(R+1)+1}$ is found from $g_{b,R}(G_1) = \frac{1}{a}$ and $S = \frac{-L+R-r+w}{2b(R+1)}$ the first coordinate of the intersection point of the graphs of $f_{a,r}$ and $g_{b,R}$.

For each $a_n < \lfloor r \rfloor$ we can deal with all $a_{n+1} > \lfloor R \rfloor$ in one integral

$$\frac{1}{\log 2}\int_{t=0}^{\frac{1}{\lfloor R\rfloor+1}}\int_{v=f_{a_n,r}(t)}^{\frac{1}{a_n}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}. \tag{13}$$

Similarly for each $a_{n+1} < \lfloor R \rfloor$ and all $a_n > \lfloor r \rfloor$

$$\frac{1}{\log 2}\int_{t=G_1}^{G}\int_{v=\frac{R}{t}-b(R+1)}^{\frac{1}{\lfloor r\rfloor+1}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}+\frac{1}{\log 2}\int_{t=G}^{\frac{1}{a_{n+1}}}\int_{v=0}^{\frac{1}{\lfloor r\rfloor+1}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}. \tag{14}$$

For all $a_n > \lfloor r \rfloor$ and $a_{n+1} > \lfloor R \rfloor$ we can also use just one integral

$$\frac{1}{\log 2}\int_{t=0}^{\frac{1}{\lfloor R\rfloor+1}}\int_{v=0}^{\frac{1}{\lfloor r\rfloor+1}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}. \tag{15}$$

For every $r > 1$ and $R > 1$ the asymptotic frequency that $D_{n-2} > r$ and $D_n > R$ can be found by adding a finite number of these integrals. We have obtained the following proposition.

Proposition 5.1. *For almost all $x \in [0,1)$, and for all $r, R \geq 1$, we have that the limit*

$$\lim_{n\to\infty}\frac{1}{n}\#\{2\leq j\leq n+1; D_{j-2}>r \text{ and } D_i>R\}$$

exists and equals

$$\begin{aligned}&\sum_{a=1}^{\lfloor r\rfloor-1}\sum_{b=\lfloor R\rfloor+1}^{\infty}m(i)_{a,b}+\sum_{a=1}^{\lfloor r\rfloor-1}\sum_{b=1}^{\lfloor R\rfloor-1}m(vi)_{a,b}+M_{r,R}\\ &+\sum_{a=\lfloor r\rfloor+1}^{\infty}\sum_{b=\lfloor R\rfloor+1}^{\infty}m(v)_{a,b}+\sum_{a=\lfloor r\rfloor+1}^{\infty}\sum_{b=1}^{\lfloor R\rfloor-1}m(iii)_{a,b},\end{aligned} \tag{16}$$

where

$$\begin{aligned}M_{r,R}=&\sum_{b=\lfloor R\rfloor+1}^{\infty}\left(1_{(\{r\}\geq\frac{1}{b})}m(i)_{\lfloor r\rfloor,b}+1_{(\frac{1}{b+1}<\{r\}<\frac{1}{b})}m(ii)_{\lfloor r\rfloor,b}\right)\\ +&\sum_{b=1}^{\lfloor R\rfloor-1}\left(1_{(\{r\}\leq G)}m(iii)_{\lfloor r\rfloor,b}+1_{(\{r\}\geq\frac{1}{b})}m(vi)_{\lfloor r\rfloor,b}+1_{(G<\{r\}<\frac{1}{b})}m(viii)_{\lfloor r\rfloor,b}\right)\\ +&\sum_{a=\lfloor r\rfloor+1}^{\infty}\left(1_{(\{R\}\geq\frac{1}{a})}m(iii)_{a,\lfloor R\rfloor}+1_{(\{R\}\geq\frac{1}{a})}m(vi)_{a,\lfloor R\rfloor}+1_{(F<\{R\}<\frac{1}{a})}m(vii)_{a,\lfloor R\rfloor}\right)\\ +&\sum_{a=1}^{\lfloor r\rfloor-1}\left(1_{(\{R\}\leq F)}m(i)_{a,\lfloor R\rfloor}+1_{(\{R\}\geq\frac{1}{a})}m(vi)_{a,\lfloor R\rfloor}+1_{(F<\{R\}<\frac{1}{a})}m(vii)_{a,\lfloor R\rfloor}\right)\\ +&\ 1_{(\{r\}\geq\frac{1}{\lfloor R\rfloor}\,\&\,\{R\}\leq F)}m(i)_{\lfloor r\rfloor,\lfloor R\rfloor}+1_{(G<\{r\}<\frac{1}{\lfloor R\rfloor}\,\&\,\{R\}\leq F)}m(ii)_{\lfloor r\rfloor,\lfloor R\rfloor}\\ +&\ 1_{(\{r\}<G\,\&\,\{R\}\geq\frac{1}{\lfloor r\rfloor})}m(iii)_{\lfloor r\rfloor,\lfloor R\rfloor}+1_{(\{r\}<G\,\&\,F<\{R\}<\frac{1}{\lfloor r\rfloor})}m(iv)_{\lfloor r\rfloor,\lfloor R\rfloor}\\ +&\ 1_{(\{r\}<\frac{1}{\lfloor R\rfloor+1}\,\&\,\{R\}<\frac{1}{\lfloor r\rfloor+1})}m(v)_{\lfloor r\rfloor,\lfloor R\rfloor}+1_{(\{r\}<\frac{1}{\lfloor R\rfloor+1}\,\&\,\{R\}\geq\frac{1}{\lfloor r\rfloor+1})}m(vi)_{\lfloor r\rfloor,\lfloor R\rfloor}\\ +&\ 1_{(\{r\}\geq\frac{1}{\lfloor R\rfloor}\,\&\,\{R\}<\frac{1}{\lfloor r\rfloor})}m(vii)_{\lfloor r\rfloor,\lfloor R\rfloor}+1_{(G<\{r\}<\frac{1}{\lfloor R\rfloor}\,\&\,\{R\}>\frac{1}{\lfloor r\rfloor})}m(viii)_{\lfloor r\rfloor,\lfloor R\rfloor}.\end{aligned}$$

Here 1_A is the indicator function of A, i.e.

$$1_A=\begin{cases}1 & \text{if condition } A \text{ is satisfied,}\\ 0 & \text{else.}\end{cases}$$

Note that all the infinite sums are just finite integrals. For instance the infinite sum in (16) is equal to

$$\begin{aligned}\sum_{a=\lfloor r\rfloor+1}^{\infty}\sum_{b=\lfloor R\rfloor+1}^{\infty} m(v)_{a,b} &= \frac{1}{\log 2}\int_0^{\frac{1}{\lfloor r\rfloor+1}}\int_0^{\frac{1}{\lfloor R\rfloor+1}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}\\ &= \frac{1}{\log 2}\log\left(\frac{1}{(\lfloor r\rfloor+1)(\lfloor R\rfloor+1)}\right).\end{aligned}$$

The constant $M_{r,R}$ is the measure of the regions where $D_{n-2} > r$ and $D_n > R$ in the strips $[0,1)\times\left[\frac{1}{\lfloor r\rfloor+1},\frac{1}{\lfloor r\rfloor}\right)$ and $\left[\frac{1}{\lfloor R\rfloor+1},\frac{1}{\lfloor R\rfloor}\right)\times[0,1)$. The first sum lists the regions where $a=\lfloor r\rfloor$ and $b>\lfloor R\rfloor$, the second sum lists the regions where $a=\lfloor r\rfloor$ and $b<\lfloor R\rfloor$. The third and fourth sum list the regions where $b=\lfloor R\rfloor$ and respectively $a>\lfloor r\rfloor$ and $a<\lfloor r\rfloor$. The eight single indicator functions are eight the possible measures for $\Delta_{\lfloor r\rfloor,\lfloor R\rfloor}$; depending on the exact positions of the graphs of f and g.

Example. In this example we calculate the asymptotic frequency that simultaneously $D_{n-2} > 2.9$ and $D_n > 3.6$; see Figure 5 and Table 2.

a_n	a_{n+1}	Case	asymptotic frequency
1	1	(*vi*)	0.068
1	2	(*vi*)	0.037
1	>2	(13)	0.106
2	1	(*vi*)	0.037
2	2	(*vi*)	0.019
2	3	(*vi*)	0.013
2	>3	(13)	0.044
>2	1	(14)	0.097
>2	2	(14)	0.050
>2	3	(14)	0.034
>2	>3	(15)	0.115

TABLE 2. The probabilities that $D_{n-2} > 2.9$ and $D_n > 3.6$ in the different cases.

For almost all $x\in[0,1)\setminus\mathbb{Q}$ the asymptotic frequency that simultaneously $D_{n-2} > 2.9$ and $D_n > 3.6$ is 0.619.

We can also compute the conditional probability that M_{Tong} is the sharp bound. Given that $D_{n-2} > 2.9$ and $D_n > 3.6$ the conditional probability that M_{Tong} is the sharp bound is 0.28.

6. Results for C_n.

In [17], Tong states the following theorem without a proof.

Theorem 6.1. *Let $r>1$, $R>1$ be two real numbers and*

$$L=\frac{1}{r-1}+\frac{1}{R-1}+a_na_{n+1}rR,\quad K=\frac{1}{2}\left(L+\sqrt{L^2-\frac{4}{(r-1)(R-1)}}\right).$$

Then

(i) $C_{n-2} < r, C_n < R$ *imply* $C_{n-1} > K$;

(ii) $C_{n-2} > r, C_n > R$ *imply* $C_{n-1} < K$.

This statement is not correct; assume for instance that $C_{n-2} < 1.1$ and $C_n < 1.4$, and that $a_n = a_{n+1} = 1$. Part (1) of Tong's Theorem then implies that $C_{n-1} > 11.94$. However, by definition $C_{n-1} \in (1,2)$, so this bound is clearly wrong. We correct this result in this section. The bounds in our theorems are sharp. We start with the case that both C_{n-2} and C_n are larger than given reals, this is related to the case where D_{n-2} and D_n are smaller than given numbers.

Theorem 6.2. *Let* $t, T \in (1,2)$ *and* $C_{n-2} > t$ *and* $C_n > T$. *Put*

$$F' = \frac{a_{n+1}+1}{(a_n a_{n+1} + a_n + 1)t - 1}, \quad G' = \frac{a_n + 1}{(a_n a_{n+1} + a_{n+1} + 1)T - 1}$$

and $$K = t + T + a_n a_{n+1} tT - 2.$$

There are three possibilities for the maximum of C_{n-1}

(i) *If* $\frac{1}{t-1} - a_n \geq G'$ *and* $\frac{1}{a_n+1} \leq \frac{1}{T-1} - a_{n+1} < F'$, *then*

$$C_{n-1} < \max\left\{\frac{T}{(a_{n+1}+1)(T-1)}, 1 + \frac{G'}{a_n+1}\right\}.$$

(ii) *If* $\frac{1}{a_{n+1}+1} \leq \frac{1}{t-1} - a_n < G'$ *and* $\frac{1}{T-1} - a_{n+1} \geq F'$

$$C_{n-1} < \max\left\{\frac{t}{(a_n+1)(t-1)}, 1 + \frac{F'}{a_{n+1}+1}\right\}.$$

(iii) *In all other cases*

$$C_{n-1} < 1 + \frac{2K - 2\sqrt{K^2 - 4(t-1)(T-1)}}{4(t-1)(T-1)}.$$

Proof. The proof follows from Theorem 1.2 and the observation that $C_n = 1 + \frac{1}{D_n}$. If $C_{n-2} > t$, then $D_{n-2} = \frac{1}{C_{n-2}-1} < \frac{1}{t-1}$ and likewise if $C_n > T$, then $D_n < \frac{1}{T-1}$. We now apply Theorem 1.2 with $r = \frac{1}{t-1}$ and $R = \frac{1}{T-1}$. Rewriting F and G from (12) in terms of t and T yields F' and G'. Rewriting the conditions of Theorem 1.2 is now trivial. We show how to find the bound in case (iii), the others can be derived similarly. We use (5) for M_{Tong}.

$$\begin{aligned} C_{n-1} &< 1 + \frac{1}{M_{\text{Tong}}} \\ &= 1 + \frac{2}{t + T + a_n a_{n+1} tT - 2 + \sqrt{[t + T + a_n a_{n+1} tT - 2]^2 - 4(t-1)(T-1)}} \\ &= 1 + \frac{2}{K + \sqrt{K^2 - 4(t-1)(T-1)}} \cdot \frac{K - \sqrt{K^2 - 4(t-1)(T-1)}}{K - \sqrt{K^2 - 4(t-1)(T-1)}} \\ &= 1 + \frac{2K - 2\sqrt{K^2 - 4(t-1)(T-1)}}{4(t-1)(T-1)}. \end{aligned}$$

□

Example. Take $t = 1.1, T = 1.4$ and $a_n = a_{n+1} = 1$. We find that $F' = 0.870, G' = 0.625$ and $K = 2.04$. Since $\frac{1}{a_{n+1}} = \frac{1}{2} \not< \frac{1}{T-1} - a_{n+1} = 1\frac{1}{2}$ case (i) of Theorem 6.2 does not apply. The second case does not apply either, since $\frac{1}{t-1} - a_n = 9 \not< G'$. So we are in case (iii) and $C_n < 1.50$.

We state the next theorem without a proof, since is similar to that of Theorem 6.2; cf. Theorem 4.1.

Theorem 6.3. *Let $t, T \in (1,2)$ and $C_{n-2} < t$ and $C_n < T$. Let F', G' and K be defined as in Theorem 6.2. Then there are six possibilities for the minimum of C_{n-1}.*

(i) *If* $\frac{1}{t-1} - a_n \geq \frac{1}{a_{n+1}}$ *and* $\frac{1}{T-1} - a_{n+1} < F'$, *then*

$$C_{n-1} > 1 + \frac{F'}{a_{n+1}+1}.$$

(ii) *If* $G' \leq \frac{1}{t-1} - a_n < \frac{1}{a_{n+1}}$ *and* $\frac{1}{T-1} - a_{n+1} < F'$, *then*

$$C_{n-1} > \min\left\{\frac{t}{(a_{n+1}+1)(t-1)}, 1 + \frac{F'}{a_{n+1}+1}\right\}.$$

(iii) *If* $\frac{1}{t-1} - a_n < G'$ *and* $\frac{1}{T-1} - a_{n+1} \geq \frac{1}{a_n}$, *then*

$$C_{n-1} > 1 + \frac{G'}{a_n+1}.$$

(iv) *If* $\frac{1}{t-1} - a_n < G'$ *and* $F' \leq \frac{1}{T-1} - a_{n+1} < \frac{1}{a_n}$, *then*

$$C_{n-1} > \min\left\{\frac{T}{(a_{n+1}+1)(T-1)}, 1 + \frac{G'}{a_n+1}\right\}.$$

(v) *If* $\frac{1}{t-1} - a_n < \frac{1}{a_{n+1}+1}$ *and* $\frac{1}{T-1} - a_{n+1} < \frac{1}{a_n+1}$, *then*

$$C_{n-1} > 1 + \frac{1}{(a_n+1)(a_{n+1}+1)}.$$

(vi) *In all other cases*

$$C_{n-1} > 1 + \frac{2K - 2\sqrt{K^2 - 4(t-1)(T-1)}}{4(t-1)(T-1)}.$$

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