SPECIAL L-VALUES OF t-MOTIVES: A CONJECTURE

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Abstract. We propose a conjecture on special values of L-functions in a function field context with positive characteristic coefficients.

For M a uniformizable t-motive with everywhere good reduction we conjecture a relation between the value of the Goss L-function $L(M^\vee, s)$ at $s = 0$ and the uniformization of the abelian t-module associated with M.

When M is a power of the Carlitz t-motive the conjecture specializes to a theorem of Anderson and Thakur on Carlitz zeta values. Beyond this case we present numerical evidence.

1. Introduction: Three flavors of special values

Of the three flavors of special values of L-functions that are to be discussed now, only the third is logically relevant to the rest of the paper. The first two are here to provide some context.

1.1. **Number field base, characteristic zero coefficients.**

Let K be a number field and $\bar{K}$ an algebraic closure of K.

Definition 1 (see [16]). A *strictly compatible system* of ℓ-adic representations of $\mathrm{Gal}(\bar{K}/K)$ is a collection $\rho = (\rho_\ell)_\ell$ of continuous homomorphisms $\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(V_\ell)$, one for every rational prime ℓ, where the V_ℓ are finite dimensional $\mathbf{Q}_\ell$-vector spaces, and subject to the condition that there exists a finite set S of places of K such that:

(i) for all places $v \notin S$ and for all ℓ coprime with v the representation ρ_ℓ is unramified at v;

(ii) for such ℓ and v the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ has rational coefficients and does not depend on ℓ.

A natural source of strictly compatible systems is ℓ-adic cohomology: consider the system $(V_\ell)_\ell$ where $V_\ell := H^i(X_{\bar{K},\text{ét}}, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}} \mathbf{Q}$ are the ℓ-adic cohomology groups of a smooth and projective variety X over K, or their Tate twists, and even subquotients of these constructed using correspondences defined over K. By [5] these form strictly compatible systems. Any system of representations that is isomorphic to such a system is said to *come from geometry.*

To any strictly compatible system ρ coming from geometry and any finite set of places S as above one associates an L-function as follows. First define for every finite place v that is not in S the polynomial

$$P_v(X) := \det\left(1 - X\rho_\ell(\mathrm{Frob}_v)\right) \in \mathbf{Q}[X]$$

using any ℓ which is coprime with v. Then define the L-function of ρ away from S by the Euler product

$$L_S(\rho, s) := \prod_{v \notin S} P_v(Nv^{-s})^{-1} \tag{1}$$

where the product ranges over all finite places of K not in S and where $Nv \in \mathbf{Z}_{>0}$ denotes the norm of the place v. By [5] this converges to a complex analytic function for $\Re(s)$ sufficiently large.

For any ρ coming from geometry and $n \in \mathbf{Z}$ such that $L(\rho, s)$ can be holomorphically continued to a neighborhood of $s = n$ we say that the complex number $L(\rho, n)$ is a *special value*. More generally, if $L(\rho, s)$ can be meromorphically continued to a neighborhood of $s = n$ we also call the leading coefficient of the Laurent series expansion of $L(\rho, s)$ around $s = n$ a special value.

There is a large zoo of theorems and conjectures concerning these special values: Euler's $\zeta(2) = \pi^2/6$, the class number formula, the Birch and Swinnerton-Dyer conjecture, to name just a few. A very general conjecture due to Beilinson [4] and reformulated by Scholl [15] expresses all special values (up to a rational factor) in terms of periods of mixed motives. (*see also* the excellent survey [10].)

1.2. **Function field base, characteristic zero coefficients.** Of course the above definition of an L-function associated to a strictly compatible system of ℓ-adic representations makes perfect sense if K is not a number field but the function field of a curve over a finite field with q elements.

Only the relation between special values and periods disappears from the picture, because if ρ comes from geometry then there exists a rational function $f \in \mathbf{Q}(T)$ such that $L(\rho, s) = f(q^{-s})$. In particular: if $s = n$ is not a pole of $L(\rho, s)$ then $L(\rho, n) \in \mathbf{Q}$.

(The interpretation of this rational number in terms of arithmetic geometry and algebraic K-theory is a very interesting problem [14] [12], but it is not the topic of this note.)

1.3. **Function field base, characteristic p coefficients.** After having discussed the two flavors that we will *not* be concerned with, we now come to the central topic of this paper.

Let us start with an example of a special value of this third flavor. Let $A := \mathbf{F}_q[t]$ be the polynomial ring in one variable t over a finite field $\mathbf{F}_q$ of q elements. Write A_+ for the set of monic elements of A. The infinite sum

$$\zeta(n) := \sum_{f \in A_+} f^{-n}$$

converges in $\mathbf{F}_q((t^{-1}))$ for every $n \in \mathbf{Z}_{>0}$. For example, if $q = 2$ then one easily computes by hand

$$\zeta(1) \in 1 + t^{-2} + t^{-3} + t^{-4}\mathbf{F}_2[[t^{-1}]].$$

Using unique factorization in A we obtain an expression as an infinite convergent Euler product:

$$\zeta(n) = \prod_f (1 - f^{-n})^{-1},$$

where the product runs over the monic irreducible elements. These $\zeta(n)$ with $n > 0$ are examples of special values about which our conjecture will say something. In fact, for these examples the conjecture specializes to a theorem due to Anderson and Thakur [3].

Now we generalize this example and turn to strictly compatible systems of Galois representations. For every non-zero prime ideal $\lambda \subset \mathbf{F}_q[t]$ consider the λ-adic completion $\mathbf{F}_q(t)_\lambda$ of $\mathbf{F}_q(t)$. Let K be a finite separable extension of $\mathbf{F}_q(t)$. Let $\rho = (\rho_\lambda)$ be a family of representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on finite dimensional $\mathbf{F}_q(t)_\lambda$-vector spaces, one for each prime ideal λ of $\mathbf{F}_q[t]$. We call ρ a *strictly compatible system* if there exists a finite set S of places of K such that

(i) for every finite place $v \notin S$ and for all λ not under v the representation ρ_λ is unramified at v;
(ii) for these λ and v the characteristic polynomial of Frobenius at v has coefficients in $\mathbf{F}_q(t)$ and is independent of λ.

For every finite place v of K define $\mathcal{N}v \in \mathbf{F}_q[t]$ to be a monic generator of the norm from K to $\mathbf{F}_q(t)$ of the ideal corresponding to v.

Now by analogy with (1) we define for every finite $v \notin S$

$$P_v(X) := \det(1 - X\rho_\lambda(\mathrm{Frob}_v)) \in \mathbf{F}_q(t)[X]$$

using any λ not below v and

$$L_S(\rho, n) := \prod_{v \notin S} P_v(\mathcal{N}v^{-n})^{-1},$$

the product being over the finite places v that are not in S. This converges to an element of $\mathbf{F}_q((t^{-1}))$ for all sufficiently large *integers* n.

For example, if K is $\mathbf{F}_q(t)$, and ρ the family of trivial representations then with $S = \emptyset$ we have

$$L(\rho, n) = \zeta(n).$$

In this context, a natural source of strictly compatible systems are t-motives, and our conjecture will have something to say about the special value $L(\rho, n)$ provided that ρ comes from a uniformizable t-motive with everywhere good reduction. (These notions will be explained in §2.)

To demand that ρ comes from a uniformizable t-motive is very natural, but the condition that it has everywhere good reduction (which is equivalent with saying that S can be taken to consist of only "infinite" places of K) is an ugly condition that should eventually be removed. Unfortunately at present there are almost no examples with bad reduction where the numerical data allows us to make reasonable conjectures.

Remark 1. We speak about a "special value" $L(\rho, n)$, but we have not defined $L(\rho, s)$ for any non-integral argument s. Goss [9] has shown that there is in fact an analytic function $L(\rho, s)$ of which the $L(\rho, n)$ are particular values (the tricky part is defining the domain of such a function). We will not use this.

Finally, we should point out that in a recent preprint of Vincent Lafforgue [11] formulas for certain classes of special values in terms of extensions of shtukas have been proven. We hope to discuss the precise relation between his Theorems and our Conjectures in a future paper.

2. Preliminaries

2.1. **Base and coefficients, notation.** To produce strictly compatible systems of Galois representations over function fields it is very useful to separate the base field from the coefficient rings. So we will look at representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ with K a function field containing $\mathbf{F}_q$ on vector spaces over completions $\mathbf{F}_q(t)_\lambda$ of the a priori unrelated rational function field $\mathbf{F}_q(t)$.

Eventually, to have a meaningful notion of L-functions we will fix an injective morphism $\mathbf{F}_q(t) \to K$, but we will *not* identify $\mathbf{F}_q(t)$ with the image.

(Such separation is impossible in the number field case, in the same way that trying to adapt Weil's intersection-theoretical proof of the Riemann Hypothesis for curves X over finite fields to $\mathrm{Spec}(\mathbf{Z})$ breaks down in the first step: the construction of the surface $X \times X$.)

2.2. **Table of notation.**

- $\mathbf{F}_q$: a fixed field with q elements.
- *Base rings:*
  $K_\infty :=$ $\mathbf{F}_q((\theta^{-1}))$, the field of Laurent series in θ^{-1} over $\mathbf{F}_q$;
  $K :=$ a subfield of K_∞ that has finite degree over $\mathbf{F}_q(\theta)$;
  $O_K :=$ the integral closure of the polynomial ring $\mathbf{F}_q[\theta]$ inside K;
  $\mathbf{C}_\infty :=$ the completion of an algebraic closure of K_∞;
  $K^{\mathrm{sep}} :=$ the separable closure of K in $\mathbf{C}_\infty$;
  $K^{\mathrm{perf}} :=$ the perfection of K.
- *Coefficient rings:*
  $A :=$ $\mathbf{F}_q[t]$, polynomial ring in a variable t;
  $A_\lambda :=$ $\varprojlim_n A/\lambda^n$, the λ-adic completion of A, where λ is a non-zero prime ideal of A;
  $F :=$ $\mathbf{F}_q(t)$, the fraction field of A;
  $F_\lambda :=$ $A_\lambda \otimes_A F$;
  $F_\infty :=$ $\mathbf{F}_q((1/t))$.
- *Relation between base and coefficients:* $i : \mathbf{F}_q[t] \to K$: the $\mathbf{F}_q$-algebra homomorphism that maps t to θ.

(The classical counterpart to this last map is the canonical morphism from $\mathbf{Z}$ to any commutative ring.)

2.3. ***t*-motives and Galois representations.** Let R be a commutative ring containing $\mathbf{F}_q$.

Definition 2. A *σ-module* of *rank r* over R is a pair (M, σ) of a projective $R \otimes_{\mathbf{F}_q} A$-module M of rank r and a map $\sigma : M \to M$ such that

(i) σ is A-linear;
(ii) $\sigma(xm) = x^q \sigma(m)$ for all $x \in R$ and $m \in M$.

A morphism from (M_1, σ_1) to (M_2, σ_2) is a homomorphism $f : M_1 \to M_2$ of $R \otimes_{\mathbf{F}_q} A$-modules such that $\sigma_2 \circ f = f \circ \sigma_1$.

We will often suppress the σ from the notation and write M for a σ-module (M, σ).

If (M_1, σ_1) and (M_2, σ_2) are σ-modules then we define their tensor product to be the σ-module $(M_1 \otimes_{R \otimes A} M_2, \sigma_1 \otimes \sigma_2)$. Similarly one can define symmetric and exterior powers. In particular, given a σ-module M one can consider its determinant $\det(M)$ which is a σ-module of rank one.

If $R \to S$ is an $\mathbf{F}_q$-algebra homomorphism and M a σ-module over R then we denote by M_S the σ-module over S obtained by extension of scalars:

$$M_S = (M \otimes_R S, m \otimes s \mapsto \sigma(m) \otimes s^q).$$

Definition 3. A σ-module (M, σ) over a field L is said to be *non-degenerate* if $\det(\sigma) : \det(M) \to \det(M)$ is non-zero. A σ-module M over R is said to be non-degenerate if M_L is non-degenerate for all R-fields $R \to L$.

Let M be a σ-module over K. For every non-zero prime ideal λ of A we have that

$$T_\lambda(M) := \varprojlim_n \left(M_{K^{\mathrm{sep}}}/\lambda^n M_{K^{\mathrm{sep}}}\right)^{\sigma=1}$$

is naturally an A_λ-module with a continuous action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ and

$$V_\lambda(M) := T_\lambda(M) \otimes_{A_\lambda} F_\lambda$$

is naturally an F_λ-vector space with a continuous action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. In general these need not be finitely generated, yet one easily verifies:

Proposition 1. *If σ is non-degenerate then for all but finitely many λ the dimension of $V_\lambda(M)$ equals the rank of M.* □

So far we have not used i which relates the base and the coefficients. Recall that $\theta = i(t)$.

Definition 4. An *effective t-motive* over K is a non-degenerate σ-module M over K such that $\det(M)$ is isomorphic with the σ-module $(K[t]e, e \mapsto \alpha(t-\theta)^n e)$ for some $\alpha \in K^\times$ and $n \geq 0$.

The family of Galois representations associated with an effective t-motive forms a strictly compatible system:

Proposition 2 (Thm 3.3 of [7])**.** *Let M be an effective t-motive over K of rank r. Then* $\dim V_\lambda(M) = r$ *for all λ. Moreover, there exists a finite set S of places of K such that*

(i) *for every place $v \notin S$ and for all non-zero prime ideals λ coprime with i^*v the representation $V_\lambda(M)$ is unramified at v;*
(ii) *for these λ and v the characteristic polynomial of Frobenius at v has coefficients in A and is independent of λ.* □

Example 1. Let C be the *Carlitz* t-motive over K. This is the rank one effective t-motive given by

$$C = (K[t]e, e \mapsto (t-\theta)e).$$

Let v be a finite place of K (*i.e.* v does not lie above the place $\theta = \infty$ of $\mathbf{F}_q(\theta)$.) Let $f \in \mathbf{F}_q[\theta]$ be a monic generator of the ideal in $\mathbf{F}_q[\theta]$ corresponding to the norm of v in $\mathbf{F}_q(\theta) \subset K$. One verifies that

(i) the representation $V_\lambda(C)$ is unramified at v for all λ coprime with i^*v;
(ii) for such λ we have that Frob_v acts as $f(t)^{-1} \in \mathbf{F}_q(t)$.

So C plays the role of the Lefschetz motive $\mathbf{Q}(-1)$.

2.4. **Abelian t-modules.** Denote by $K[\tau]$ the ring whose elements are polynomial expressions $\sum a_i\tau^i$ with $a_i \in K$ and where multiplication is defined through the rule $\tau a = a^q\tau$ for $a \in K$. The ring $K[\tau]$ is canonically isomorphic with the endomorphism ring of the $\mathbf{F}_q$-vector space scheme $\mathbf{G}_a$ over K.

If (M,σ) is an effective t-motive over K then M is naturally a left $K[\tau]$ module through $\tau m := \sigma(m)$. Now consider the functor

$$E_M : \{K\text{-algebras}\} \to \{A\text{-modules}\} : R \mapsto \mathrm{Hom}_{K[\tau]}(M,R),$$

where R is a left $K[\tau]$-module through $\tau r := r^q$. This functor is representable by an affine A-module scheme.

Conversely, given an A-module scheme E over K define

$$M_E := \mathrm{Hom}_{K-\mathrm{gr.sch.}}(E, \mathbf{G}_a),$$

which is naturally a left $A \otimes_{\mathbf{F}_q} K[\tau]$-module.

Theorem 1 (§1 of [1], §10 of [17]). *The functors $M \mapsto E_M$ and $E \mapsto M_E$ form a pair of quasi-inverse anti-equivalences between the categories of effective t-motives M over K that are finitely generated as left $K[\tau]$-modules and the category of A-module schemes E over K that satisfy*

(i) *for some $d \geq 0$ the group schemes $E_{K^{\mathrm{perf}}}$ and $\mathbf{G}^d_{a,K^{\mathrm{perf}}}$ are isomorphic;*
(ii) *$t-\theta$ acts nilpotently on $\mathrm{Lie}(E)$;*
(iii) *M_E is finitely generated as a $K[t]$-module.* □

Definition 5. An $\mathbf{F}_q[t]$-module scheme E satisfying the above three conditions is called an *abelian t-module* of dimension d. An abelian t-module of dimension one is called a *Drinfeld module*.

Question 1. Is the underlying group scheme of an abelian t-module isomorphic to $\mathbf{G}_a^d$ over K?

For Drinfeld modules this is indeed the case, since the only form of $\mathbf{G}_a$ that has infinite endomorphism ring is $\mathbf{G}_a$ itself (*see* [13], *see also* §10 of [17]).

The tangent space at the identity of E can be expressed in terms of M_E as follows:

Proposition 3 (*see* [1]). $\mathrm{Lie}_E(K) = \mathrm{Hom}_K(M_E/K\sigma(M_E), K)$.

Also the Galois representations associated with M_E can be expressed in terms of E. If $\lambda = (f) \subset A$ a non-zero prime ideal then define the λ-adic Tate module of E to be

$$V_\lambda(E) := (\varprojlim_n E[f^n](K^{\mathrm{sep}})) \otimes_{A_\lambda} F_\lambda.$$

If M is the effective t-motive associated with E then we have

Proposition 4. $V_\lambda(M_E) \cong \mathrm{Hom}(V_\lambda(E), F_\lambda)$. □

2.5. **Uniformization.**

Proposition 5 (*see* §2 of [1]). *Let E be an abelian t-module over K.*

(i) *There exists a unique entire A-module homomorphism $\exp_E : \mathrm{Lie}_E(\mathbf{C}_\infty) \to E(\mathbf{C}_\infty)$ that is tangent to the identity;*
(ii) *The kernel of $\exp_E$ is a finitely generated free discrete sub-A-module in $\mathrm{Lie}_E(\mathbf{C}_\infty)$.*

When $\exp_E$ is surjective this yields an analytic description of the A-module $E(\mathbf{C}_\infty)$ as the quotient of $\mathrm{Lie}_E(\mathbf{C}_\infty)$ by a discrete submodule.

Denote by M the t-motive associated with E. The following theorem characterizes the E such that $\exp_E$ is surjective:

Theorem 2. *The following are equivalent:*

(i) $\exp_E$ *is surjective;*
(ii) *the rank of* $\ker \exp_E$ *equals the rank of* M*;*
(iii) *for all* λ *the restriction of the Galois representation* $\rho_\lambda : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}(V_\lambda(M))$ *to* $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ *has finite image.*

When these equivalent statements hold we say that E (or M) is *uniformizable*.

Proof. The equivalence of (i) and (ii) is part of Theorem 4 of [1], the equivalence of (iii) and (i) is part of Theorem 5.12 of [7]. □

Examples of uniformizable effective t-motives are provided by the following:

Proposition 6. (i) *Drinfeld modules are uniformizable;*
(ii) *The tensor product of two uniformizable effective t-motives is uniformizable;*
(iii) *Subquotients of uniformizable effective t-motives are uniformizable.*

Proof. The first claim is shown in [6]. The other two follow at once from the third characterization in Theorem 2. □

2.6. **Good reduction.** Let M be an effective t-motive over K.

Theorem 3. *The following are equivalent:*

(i) *there exists a non-degenerate σ-module $\mathcal{M}$ over O_K and an isomorphism $\alpha : \mathcal{M}_K \to M$;*
(ii) $(H_\lambda(M,\sigma))_\lambda$ *forms a strictly compatible system with exceptional set S consisting uniquely of infinite places of K.*

Moreover, if it exists the pair $(\mathcal{M}, \alpha)$ is unique up to a unique isomorphism.

If these equivalent statements hold we say that M has *everywhere good reduction* and we call $\mathcal{M}$ a *good model* for M.

Proof. This follows easily from Theorem 1.1 of [8]. □

2.7. **The L-function of an effective t-motive.** Let M be an effective t-motive over K. Let S be an exceptional set of places of K for the strictly compatible system of Galois representations $\rho = (\rho_\lambda)_\lambda$ associated with M.

Let v be a finite place of K corresponding to a prime ideal $I \subset O_K$. Denote by $\mathcal{N}v \in A$ the unique monic generator of the inverse image image under $i : A \to \mathbf{F}_q[\theta]$ of the norm of I in $\mathbf{F}_q[\theta]$.

For any finite v that is not in S define

$$P_v(X) := \det(1 - X\rho_\lambda(\mathrm{Frob}_v)) \in A[X]$$

using any λ such that $i(\lambda)$ is coprime with v and

$$L_S(M,n) := \prod_{v \notin S} P_v(\mathcal{N}v^{-n})^{-1},$$

the product being over the finite places v that are not in S. This converges to an element of F_∞ for all sufficiently large integers n.

Now assume (for simplicity) that M has everywhere good reduction and let $\mathcal{M}$ be a model for M over O_K. Let v be a finite place of K and $k(v)$ the residue class field of v. Let $d(v)$ be the degree of $k(v)$ over $\mathbf{F}_q$. Note that $\sigma^{d(v)}$ is a *linear* endomorphism of $\mathcal{M}_{k(v)}$. The Euler factors in $L(M,n)$ can be computed as follows:

Proposition 7. $P_v(X) = \det(1 - X\sigma^{d(v)} | \mathcal{M}_{k(v)})$. □

2.8. **The L-function of a t-motive.** The category $t\mathcal{M}_{\text{eff}}$ of effective t-motives over K with its tensor product is an A-linear tensor category, but it is not closed under duals.

After formally inverting the object C it embeds into an A-linear *rigid* tensor category $t\mathcal{M}$. The objects of this latter category are called *t-motives*. They are formal expressions $M \otimes C^{\otimes n}$ with M a t-motive and $n \in \mathbf{Z}$, and morphisms are defined as

$$\mathrm{Hom}_{t\mathcal{M}}(M_1 \otimes C^{\otimes n_1}, M_2 \otimes C^{\otimes n_2}) := \mathrm{Hom}_{t\mathcal{M}_{\text{eff}}}(M_1 \otimes C^{\otimes n_1+n}, M_2 \otimes C^{\otimes n_2+n})$$

for n sufficiently large so that both n_1+n and n_2+n become non-negative. This is independent of n because for every pair M_1, M_2 of effective t-motives there is a canonical isomorphism

$$\mathrm{Hom}_{t\mathcal{M}_{\text{eff}}}(M_1, M_2) = \mathrm{Hom}_{t\mathcal{M}_{\text{eff}}}(M_1 \otimes C, M_2 \otimes C).$$

Given a t-motive M there exists a dual t-motive $M^\vee$, and the operations $(-)^\vee$ and $\otimes$ satisfy all the usual properties from representation theory. (Proofs and more details can be found in §2 of [18].)

Since the functors

$$V_\lambda : t\mathcal{M}_{\text{eff}} \to \{\mathrm{Gal}(K^{\text{sep}}/K)\text{-representations}/F_\lambda\}$$

respect the tensor product, they extend to $t\mathcal{M}$. In particular, this allows us to define L-functions for t-motives.

We have that $L(M \otimes C, n+1) = L(M,n)$, which allows us to shift special values around in a way that is more or less obvious when working with t-motives but rather non-trivial when working with abelian t-modules. This is one of the reasons that we consider t-motives in this paper, rather than working uniquely with abelian t-modules. Another reason is given by the notion of good reduction, which is quite straight-forward on the t-motives side, but rather subtle on the abelian t-modules side.

2.9. **Convergence.** So far we have ignored questions of convergence. The following proposition guarantees that the special values that occur in our conjecture will be well-defined.

Proposition 8. *If M is an effective t-motive over K that is finitely generated as a $K[\tau]$-module, then the Euler product for $L(M^\vee, 0)$ converges.*

Proof. As one might expect, the proof is based upon bounds for the $1/t$-adic valuations of eigenvalues of Frobenius.

Consider the $K^{\text{sep}}((t^{-1}))$-vector space $M((t^{-1})) := M \otimes_{K[t]} K^{\text{sep}}((t^{-1}))$. The action of σ on M extends to action on $M((t^{-1}))$ that is F_∞-linear and satisfies $\sigma(xm) = x^q\sigma(m)$ for all $x \in K^{\text{sep}}$ and $m \in M((t^{-1}))$. Since σ is not linear it does not make sense to speak about eigenvalues of σ, yet by [18, §5.1] the valuations

$\lambda_1, \lambda_2, \cdots, \lambda_r$ of the eigenvalues of a matrix representing σ relative to a chosen basis of M do not depend on the chosen basis. (In other words: the newton polygon of the characteristic polynomial is a well-defined invariant of $(M((t^{-1})), \sigma)$.)

Claim. $\lambda_i < 0$ for all i.

The finite generation of M as $K[\tau]$-module guarantees that there exists a finite dimensional K^{sep}-vector subspace $V \subset M((t^{-1}))$ such that

$$\cup_{i\geq 0} \cup_{j\geq 0} t^{-i} K^{\mathrm{sep}} \sigma^j(V) \text{ is dense in } M((t^{-1})). \tag{2}$$

But from the classification [18, §5.1] it follows that there exists a positive integer n and a basis of $M((t^{-1}))$ such that the action of σ^n with respect to that basis is given by

$$\begin{pmatrix} t^{-n\lambda_1} & 0 & \cdots & 0 \\ 0 & t^{-n\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t^{-n\lambda_r} \end{pmatrix}.$$

Hence for (2) to hold with a finite dimensional V one needs that $\lambda_i < 0$ for all i, which proves the claim.

To finish the proof it now suffices to observe that for almost all places v the Newton polygon of σ does not change under reduction mod v. □

Remark 2. The converse holds as well: if M is an effective t-motive over K then the Euler product defining $L(M^\vee, 0)$ converges if and only if M is finitely generated over $K[\tau]$. This follows essentially from [18, Theorem 5.3.1].

3. The conjecture

For a t-module E over K define

$$W_E := \mathrm{Lie}_E / (t-\theta)\, \mathrm{Lie}_E$$

and write w for the canonical projection $\mathrm{Lie}_E \to W_E$. Note that $W_E(K_\infty)$ carries naturally the structure F_∞-vector space (coming from the action of A) as well as a that of a K_∞-vector space and that the two structures coincide under the identification "$t = \theta$."

Now assume that the t-motive M associated with E has everywhere good reduction and let $(\mathcal{M}, \alpha : \mathcal{M}_K \xrightarrow{\sim} M)$ be a good model. We define $E(O_K) \subset E(K)$ to be the image of the map

$$\mathrm{Hom}_{O_K[\tau]}(\mathcal{M}, O_K) \to \mathrm{Hom}_{K[\tau]}(M, K) = E(K)$$

induced by α. Also we define $\mathrm{Lie}_E(O_K)$ as the image of

$$\mathrm{Hom}_{O_K}(\mathcal{M}/\sigma\mathcal{M}, O_K) \to \mathrm{Hom}_K(M/\sigma M, K) = \mathrm{Lie}_E(K).$$

and $W_E(O_K) \subset W_E(K)$ as the image of $\mathrm{Lie}_E(O_K)$ under w.

Conjecture 1. *Let E be a uniformizable abelian t-module over K such that the associated t-motive M has has everywhere good reduction.*

There exists a sub-A-module $Z \subset \mathrm{Lie}_E(K_\infty)$ of rank $\dim W_E$ such that $\exp_E(Z) \subset E(O_K)$ and such that

$$\bigwedge_A^{\dim W_E} w(Z) = L(E, 0) \cdot \left(\bigwedge_A^{\dim W_E} W_E(O_K) \right)$$

as A-lattices inside the 1*-dimensional* F_∞*-vector space* $\bigwedge_{K_\infty}^{\dim W_E} W_E(K_\infty)$.

Remark 3. $L(E,0) = L(M^\vee, 0)$.

Theorem 4 ([3])**.** *For* $M = C^{\otimes n}$ *the conjecture holds.* □

Proposition 9. *If the conjecture holds for* M_1 *and* M_2 *then it also holds for* $M_1 \oplus M_2$. □

4. Numerical experiments

Given a t-motive M and an n such that the Euler product defining $L(M,n)$ converges one can numerically approximate

$$L(M,n) \in \mathbf{F}_q((t^{-1}))$$

simply by multiplying all Euler factors at places of degree $\leq d$. The proof of Proposition 8 yields hard error estimates for this approximation. This bound is linear in d and hence this algorithm will compute $L(M,n)$ modulo $t^{-X}\mathbf{F}_q[[t^{-1}]]$ in a running time that is exponential in X.

Since the conjecture does not predict the module Z, or does not even give bounds on the "height" of generators of Z, it does not lend itself to numerical falsification. Yet we have systematically found that when working with M of low (naive) height there is always a Z of low (naive) height for which the conjecture holds numerically to relatively high precision.

Before we state some of these numerical examples we introduce the *logarithm* of an abelian t-module, which we will need to produce candidate modules Z in some of these examples.

4.1. The logarithm of an abelian t-module.

Let $E = (\mathbf{G}_a^d, \phi)$ be an Abelian t-module over K_∞. If we identify $\mathrm{Lie}_E(\mathbf{C}_\infty)$ and $E(\mathbf{C}_\infty)$ with $\mathbf{C}_\infty^d$ in the obvious way then $\exp_E : \mathrm{Lie}_E(\mathbf{C}_\infty) \to E(\mathbf{C}_\infty)$ can be expressed as a power series

$$\exp_E = \sum_{i=0}^{\infty} e_i \tau^i$$

with $e_i \in M_d(K_\infty)$ and $e_0 = 1$. We claim that there is a unique power series

$$\log_E = \sum_{i=0}^{\infty} l_i \tau^i$$

with $l_i \in M_d(K_\infty)$ and $l_0 = 1$ such that

$$\exp_E \log_E = 1. \qquad (3)$$

Indeed, if $n > 0$ then comparing coefficients of τ^n in (3) yields

$$l_n + e_1\tau(l_{n-1}) + \cdots + e_n\tau^n(l_0),$$

where $\tau(b)$ is the matrix obtained from b by raising every entry to the q-th power. This last expression gives a recursion for the l_i that shows that there is a unique power series $\log_E$ satisfying (3).

Given an $x \in E(K_\infty)$ it is not necessarily true that the infinite sum $\log_E(x)$ converges, but when it does converge then clearly $\exp_E(\log_E(x)) = x$.

4.2. $L(E,0)$ **with** E **a Drinfeld module.** If $E=(E,\varphi)$ is a Drinfeld module then $W_E=\mathrm{Lie}_E$ and hence one-dimensional. So if E has everywhere good reduction the conjecture predicts that

$$\exp_E(L(E,0)e)\in E(O_K) \tag{4}$$

where $e\in\mathrm{Lie}_E(K_\infty)$ is a generator defined over O_K.

If E has rank 1 over $K=\mathbf{F}_q(\theta)$ and has everywhere good reduction then it is necessarily of the form

$$E=(\mathbf{G}_a, t\mapsto\theta+\alpha\tau)$$

with $\alpha\in\mathbf{F}_q^\times$. We have

$$L(E,0)=\sum_{f\in A_+}\frac{\alpha^{\deg(f)}}{f}$$

and for these (4) is known. (If $\alpha=1$ this is Theorem 4 with $n=1$. For other values of α one reduces to this case by a change of variable $t':=\alpha^{-1}t$.)

If the rank of E is higher than one and if E does not have CM then the methods of the proof break down completely since there is no explicit description of $L(E,0)$ as an infinite sum, only as an Euler product.

However, $L(E,0)$ can be approximated numerically.

Example 2. Let $q=2$ and $E=(\mathbf{G}_a, t\mapsto\theta+\tau+\tau^2)$ over $K=\mathbf{F}_2(\theta)$. This Drinfeld module does not have complex multiplication over K^{sep}. We have

$$\begin{aligned} L(E,0) \;\in\; & 1+t^{-2}+t^{-3}+t^{-5}+t^{-7}+t^{-9}+ \\ & t^{-10}+t^{-17}+t^{-18}+t^{-19}\mathbf{F}_2[[t^{-1}]]\subset F_\infty. \end{aligned}$$

If we identify $E(K)=\mathbf{G}_a(K)=K$ then one verifies that $E(O_K)=O_K$. Using the natural generator $e\in\mathrm{Lie}_E(K_\infty)$ we compute

$$\exp_E(L(E,0)\,e)\in 1+\theta^{-19}\mathbf{F}_2[[\theta^{-1}]]\subset K_\infty,$$

so $\exp_E(L(E,0)\,e)$ is at least very close to an element of $E(O_K)$.

Similarly but now $q=3$ and $E=(\mathbf{G}_a, t\mapsto\theta+\theta\tau-\tau^2)$. We find that

$$\exp_E(L(E,0)\,e)\in 1+\theta^{-12}\mathbf{F}_3[[\theta^{-1}]].$$

We have computed hundreds of such examples over $\mathbf{F}_2(\theta)$, $\mathbf{F}_3(\theta)$ and $\mathbf{F}_5(\theta)$ (but to a slightly lower precision than the examples above), and in all of them $\exp_E(L(E,0)e)$ coincided with a polynomial in θ (not always the constant polynomial 1), within the computed precision.

Finally a rank 3 example:

Example 3. Take $q=2$ and $E=(\mathbf{G}_a, t\mapsto\theta+\tau+\tau^3)$. Then

$$\exp_E(L(E,0)\,e)\in 1+\theta^{-12}\mathbf{F}_2[[\theta^{-1}]].$$

4.3. $L(M,2)$ **with** M **the** t**-motive of a Drinfeld module of rank** 2. Let E be a Drinfeld module of rank 2 and $M=M(E)$. We have that $M^\vee\cong M\otimes\det(M)^\vee$, so if we put

$$\tilde{M}:=M\otimes C^{\otimes 2}\otimes\det(M)^\vee$$

then

$$L(M,2)=L(\tilde{M}^\vee,0).$$

Let $\tilde{E}$ be the t-module corresponding to $\tilde{M}$. Then $\tilde{E}$ has dimension 3 and the maximal quotient $w:\mathrm{Lie}_{\tilde{E}}\to W_{\tilde{E}}$ on which $t-\theta$ acts trivially is two-dimensional.

From the conjecture we should therefore expect to express $L(M,2) = L(\tilde{E},0)$ as a two by two determinant. Here is an explicit example:

Example 4. Let $q = 2$ and $E = (\mathbf{G}_a, t \mapsto \theta + \tau + \tau^2)$. Then there is an $O_K[t]$-basis for $\mathcal{M}$ on which σ is expressed as

$$\begin{pmatrix} 1 & \theta + t \\ 1 & 0 \end{pmatrix}.$$

Note that $\det(M) = C$, so the action of σ on the obvious basis for $\tilde{M} = M \otimes C$ is given by

$$\begin{pmatrix} \theta + t & \theta^2 + t^2 \\ \theta + t & 0 \end{pmatrix}.$$

From this the corresponding t-module $\tilde{E}$ can be computed. It is given by $\tilde{E} = (\mathbf{G}_a^3, \varphi)$, where φ is determined by

$$\varphi(t) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ \theta x_1 + \theta x_2 + x_3 + \tau(x_1) \\ \theta^2 x_1 + \tau(x_2) \end{pmatrix}.$$

The quotient $w : \mathrm{Lie}_{\tilde{F}} \to W_{\tilde{F}}$ takes the explicit form

$$w \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 + \xi_2 \\ \theta\xi_1 + \xi_3 \end{pmatrix}$$

Now let $z_1 = (1,0,0)$ and $z_2 = (0,0,1)$ in $\tilde{E}(O_K)$. Then $\log_{\tilde{E}}(z_1)$ and $\log_{\tilde{E}}(z_1)$ are well-defined elements of $\mathrm{Lie}_{\tilde{E}}(K_\infty)$ (the defining infinite sums converge) and the ratio of the determinant

$$w(\log_{\tilde{E}}(z_1)) \wedge w(\log_{\tilde{E}}(z_2)) \in \wedge^2 \mathrm{Lie}_E(K_\infty)$$

with

$$L(\tilde{E},0)((1,0) \wedge (0,1))$$

is computed to lie in $1 + \theta^{-31}\mathbf{F}_2[[\theta^{-1}]]$. So the conjecture seems to hold with Z the module generated by $\log_{\tilde{E}}(z_1)$ and $\log_{\tilde{E}}(z_2)$.

4.4. $L((Sym^2 M)^\vee, 0)$ **with** M **the** t**-motive of a rank** 2 **Drinfeld module.** Let M be the t-motive of a rank 2 Drinfeld module. Then $\mathrm{Sym}^2 M$ is the t-motive of a rank 3 and dimension 3 t-module E. The quotient $\mathrm{Lie}_E / (t-\theta)\,\mathrm{Lie}_E$ is two-dimensional.

Example 5. Let $q = 3$ and M the t-motive of the Drinfeld module $(\mathbf{G}_a, t \mapsto \theta - \tau + \tau^2)$ over $\mathbf{F}_3(\theta)$. The action of σ on a suitable basis of $\mathrm{Sym}^2 M$ is given by

$$\begin{pmatrix} 1 & t - \theta & t^2 + \theta t + \theta^2 \\ 1 & \theta - t & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

and the corresponding t-module is $E = E_{\mathrm{Sym}^2 M} = (\mathbf{G}_a^3, \varphi)$, where φ is given by

$$\varphi(t) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \theta x_1 - x_1^q - x_3^q \\ -\theta x_1 - \theta^2 x_3 - x_3^q + x_3^{q^2} \\ x_1 + x_2 - \theta x_3 \end{pmatrix}.$$

The quotient $w_E : \mathrm{Lie}_E \to W_E = \mathrm{Lie}_E / (t-\theta)\,\mathrm{Lie}_E$ is

$$w \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 + \theta\xi_3 \end{pmatrix}.$$

and we find that with Z the module generated by $\log_E(1,0,0)$ and $\log_E(0,1,0)$ the conjecture is compatible with the computed approximation

$$\begin{aligned} L(E,0) \quad \in \quad & 1 + t^{-3} + t^{-5} + t^{-6} + t^{-7} - t^{-8} + t^{-11} - t^{-12} + t^{-13} \\ & -t^{-15} + t^{-16} - t^{-17} - t^{-18} + t^{-19} + t^{-20}\mathbf{F}_2[[t^{-1}]]. \end{aligned}$$

5. A challenge

Let $f \in A$ be irreducible and $\chi : (A/f)^\times \to \bar{\mathbf{F}_q}^\times$ be a group homomorphism. Extend χ to a multiplicative map $A \to \bar{\mathbf{F}_q}$ in the obvious way. Anderson [2] has given an expression for

$$L(\chi, 1) := \sum_{f \in A_+} \frac{\chi(f)}{f} \in \bar{\mathbf{F}_q}((1/t))$$

in terms of Carlitz logarithms. So one can certainly say something about some special values related to t-motives with bad reduction.

Yet here is a challenge: let E be the Drinfeld module $(\mathbf{G}_a, t \mapsto \theta + \theta^{-1}\tau + \tau^2)$ over $\mathbf{F}_2(\theta)$. Let v be the place $\theta = 0$ of bad reduction. Find an expression for

$$\begin{aligned} L_{\{v,\infty\}}(E,0) \quad \in \quad & 1 + t^{-7} + t^{-9} + t^{-10} + t^{-11} + t^{-13} + \\ & t^{-14} + t^{-15} + t^{-17} + t^{-18} + t^{-19}\mathbf{F}_2[[t^{-1}]]. \end{aligned}$$

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