Multiplicity Free Subgroups of Semi-Direct Products

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Abstract

Recently, Aizenbud and Gourevitch and, independently, Sun and Zhu, see [5], have shown that the pairs of groups (O(p,q), O$(p-1,q)$), (U(p,q), U$(p-1,q)$), (GL$(n,\mathbb{R})$, GL$(n-1,\mathbb{R})$) and their complex counterparts (O$(n,\mathbb{C})$, O$(n-1,\mathbb{C})$) and (GL$(n,\mathbb{C})$, GL$(n-1,\mathbb{C})$) are so-called multiplicity free pairs. In this note we extend this result to two types of semi-direct products. In particular we show that the pairs (U$(p,q) \ltimes \mathbb{C}^{p+q}$, U$(p,q)$) and (U$(p,q) \ltimes H_{p+q}$, U$(p,q)$) are multiplicity free pairs. Here H_{p+q} denotes the Heisenberg group of real dimension $2(p+q)+1$.

1 Introduction

Boerner proved (see [1]) that the pairs of compact groups (O(n), O$(n-1)$) and (U(n), U$(n-1)$) are multiplicity free pairs: any irreducible unitary representation of O(n) (U(n)) decomposes, when restricted to O$(n-1)$ (U$(n-1)$) multiplicity free into irreducible unitary representations of O$(n-1)$ (U$(n-1)$). This result was extended to the Riemannian symmetric pairs (O$(n,1)$, O(n)) and (U$(n,1)$, U(n)) by Koornwinder, using the same definition of multiplicity free. We refer to [3]. If in the pair of groups (G,H) the subgroup H is not compact, then we need a more general definition of the notion "multiplicity free". Let therefore G be a unimodular Lie group with finitely many connected components and let H be a closed unimodular subgroup of G. Denote by π an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$ and let ρ be one of the subgroup H on V. We denote by

π_∞ and ρ_∞ the representations of G and H on the spaces $\mathcal{H}_\infty$ and V_∞ of C^∞ vectors of π and ρ respectively and set

$$\mathrm{Hom}_H\,(\pi_\infty,\,\rho_\infty)$$

for the space of continuous linear mappings $A : \mathcal{H}_\infty \to V_\infty$ satisfying

$$A\,\pi_\infty(h) = \rho_\infty(h)\,A$$

for all $h \in H$.

Definition 1.1 *We shall say that the pair of groups (G, H) is a multiplicity free pair if for any irreducible unitary representation π of G and any irreducible unitary representation ρ of H one has*

$$\dim \mathrm{Hom}_H\,(\pi_\infty,\,\rho_\infty) \leq 1.$$

Recall that a pair (G, H) is called a *generalized Gelfand pair* if for any irreducible unitary representation π of G one has $\dim \mathrm{Hom}_H\,(\pi_\infty,\, id) \leq 1$, where id is the trivial one-dimensional representation of H.

The following proposition is well-known.

Proposition 1.2

(i) *If the pair* $(G \times H,\ \mathrm{diag}\,(H \times H))$ *is a generalized Gelfand pair, then* (G, H) *is a multiplicity free pair.*

(ii) *If both G and H are type* I *groups, then the converse of* (i) *is true.*

For the definition of type I groups, see [4], for properties of generalized Gelfand pairs, see [7].

The reproducing distributions associated to the pair $(G \times H,\ \mathrm{diag}\,(H \times H))$ are in one-to-one correspondence with *positive-definite* distributions T on G satisfying $T(hgh^{-1}) = T(g)$ for all $h \in H$. We shall need the criterion of Thomas, see [7] Corollary 8.8, that reads here as follows.

Proposition 1.3 *Let τ be an involutive automorphism of G leaving H invariant. If for all (extremal) positive-definite* $\mathrm{Ad}\,(H)$*-invariant distributions T one has $T^\tau = \check{T}$, then (G, H) is a multiplicity free pair.*

Recall that $\check{T}$ is defined by $\langle \check{T},\, \varphi \rangle = \langle T,\, \check{\varphi} \rangle$ $(\varphi \in C_c^\infty(G))$, where $\check{\varphi}(x) = \varphi(x^{-1})$ $(x \in G)$.

It is sufficient to check this criterion for $\mathrm{Ad}\,(H)$-invariant eigendistributions of $\mathcal{Z}(\mathfrak{g})$, the algebra of bi-$G$-invariant differential operators on G.

Recently, Aizenbud and Gourevitch and, independently, Sun and Zhu, see [5], have shown that the pairs $(\mathrm{O}(p,q),\ \mathrm{O}(p-1,q))$, $(\mathrm{U}(p,q),\ \mathrm{U}(p-1,q))$, $(\mathrm{GL}(n,\mathbb{R}),\ \mathrm{GL}(n-1,\mathbb{R}))$ and their complex counterparts $(\mathrm{O}(n,\mathbb{C}),\ \mathrm{O}(n-1,\mathbb{C}))$ and $(\mathrm{GL}(n,\mathbb{C}),\ \mathrm{GL}(n-1,\mathbb{C}))$ are multiplicity free pairs. We shall extend this result to two types of semi-direct products related to these pairs, and thereby heavily lean on the paper [5] by Sun and Zhu.

2 Semi-direct products

Let H stand for one of the groups $\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{O}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{C})$ and let $n=p+q$. Denote by $E=E_n$ one of the spaces $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^n\times\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{C}^n\times\mathbb{C}^n$ respectively. The group H acts on E mostly in the standard way; in the cases $H=\mathrm{GL}(n,\mathbb{F})$ with $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ the action is however given by

$$h\cdot(x,y)=(h\cdot x,\ {}^{t}h^{-1}\cdot y)\quad (x,y\in\mathbb{F}^n,\ h\in\mathrm{GL}(n,\mathbb{F})).$$

Let us consider the semi-direct product $G=H\ltimes E$. Elements of G are written as pairs $g=(h,v)$ $(h\in H,\ v\in E)$ and the product in G is given by

$$(h,v)\,(h',v')=(hh',\ v+h\cdot v')\quad (h,h'\in H; v,v'\in E).$$

In particular $(h,v)^{-1}=(h^{-1},\ -h^{-1}\cdot v)$.

The main theorem of this section says:

Theorem 2.1 *The pair $(H\ltimes E,\ H)$ is a multiplicity free pair.*

Define involutions τ on G as follows.

- For $H=\mathrm{O}(p,q)$ or $H=\mathrm{O}(n,\mathbb{C})$ take $\tau=id$.
- For $H=\mathrm{U}(p,q)$ take $\tau(h,v)=(\overline{h},\ \overline{v})$ $(h\in H,\ v\in\mathbb{C}^n)$, where "bar" means complex conjugation.
- For $H=\mathrm{GL}(n,\mathbb{F})$ $(\mathbb{F}=\mathbb{R}$ or $\mathbb{C})$ take $\tau(h,(x,y))=(\,{}^{t}h^{-1},\ (y,x))$ $(h\in H;\ x,y\in\mathbb{F}^n)$.

In accordance with Proposition 1.3 we shall consider distributions T on G that are positive-definite and satisfy the relation

$$T(h_1hh_1^{-1},h_1\cdot v)=T(h,v) \tag{2.1}$$

for all $h_1\in H$. And we have to show that such distributions T have the property

$$T(h^{-1},\ -h^{-1}\cdot v)=T^{\tau}(h,v). \tag{2.2}$$

This would imply Theorem 2.1. In proving (2.2) we use a fundamental result from [5] (Theorem 6.4), that we formulate here as a lemma.

Lemma 2.2 *Let T be a distribution satisfying* (2.1)*. Then one has*

$$T^{\tau}(h^{-1},\, -v) = T(h, v).$$

To complete the proof of Theorem 2.1 we thus have to show that any positive-definite distribution T satisfying (2.1) also satisfies $T(h,\, h \cdot v) = T(h, v)$. This property occurs to be true without the requirement of being positive-definite. We have:

Proposition 2.3 *Let T be a distribution on $H \times E$ with the property*

$$T(h_1 h h_1^{-1},\, h_1 \cdot v) = T(h, v)$$

for all $h_1 \in H$. Then T satisfies $T(h,\, v) = T(h,\, h \cdot v)$.

As said, this proposition makes the proof of Theorem 2.1 complete. If T satisfies (2.1) and is a regular distribution, a continuous function say, then the property $T(h,\, v) = T(h,\, h \cdot v)$ follows at once from (2.1). For general distributions we have to be more careful. The following elementary lemma is of great help.

Lemma 2.4 *Let H be a unimodular Lie group and let σ be a distribution on $H \times H$ invariant under the action*

$$(h,\, h_0) \mapsto (yhy^{-1},\, yh_0) \quad (h, h_0, y \in H).$$

Then there exists a distribution s on H such that

$$\sigma(\alpha) = s(\beta_\alpha)$$

for $\alpha \in C_c^\infty(H \times H)$, where β_α is given by

$$\beta_\alpha(h) = \int_H \alpha(yhy^{-1}, y)\, dy \quad (h \in H)$$

and dy denotes a Haar measure on H.

Proof. Fix a function $\chi \in C_c^\infty(H)$ such that $\int_H \chi(h)\, dh = 1$. For any function $\beta \in C_c^\infty(H)$ let $\alpha \in C_c^\infty(H \times H)$ be given by

$$\alpha(h, h_0) = \chi(h_0)\, \beta(h_0^{-1} h h_0) \quad (h, h_0 \in H).$$

Then $\beta_\alpha = \beta$. Set $\alpha = r(\beta)$. Define the distribution s on H by

$$s(\beta) = \sigma(r(\beta)) \quad (\beta \in C_c^\infty(H)).$$

It is obvious that s is a distribution. Let now $\alpha \in C_c^\infty(H \times H)$ be given and consider $r(\beta_\alpha)$. We have

$$r(\beta_\alpha)(h, h_0) = \chi(h_0) \int_H \alpha(yhy^{-1}, yh_0)\, dy \quad (h, h_0 \in H)$$

and

$$\begin{aligned} \sigma(r(\beta_\alpha)) &= \int_H \int_H \chi(h_0)\, \alpha(yhy^{-1}, yh_0)\, d\sigma(h, h_0)\, dy \\ &= \int_H \int_H \chi(y^{-1}h_0)\, \alpha(h, h_0) dy\, d\sigma(h, h_0) \\ &= \sigma(\alpha), \end{aligned}$$

by the invariance property of σ. This completes the proof of the lemma.

We are now ready to give the proof of Proposition 2.3.

Proof of Proposition 2.3. Consider the mapping $H \times E \to E$ given by

$$(h, v) \mapsto h \cdot v.$$

This mapping is everywhere submersive. By a well-known theorem, proved by Harish-Chandra in [2], we have a surjective continuous linear mapping $\alpha \mapsto f_\alpha$ from $C_c^\infty(H \times H \times E)$ to $C_c^\infty(H \times E)$ defined by

$$\int_{H\times H\times E} \alpha(h; h_0, v)\, F(h; h_0 \cdot v)\, dh dh_0 dv = \int_{H\times E} f_\alpha(h, v)\, F(h, v)\, dh dv$$

for all continuous functions F on $H \times E$. Here dv stands for the Lebesgue measure on E, that is clearly seen to be invariant under the action of H. Given $y \in H$ and $\alpha \in C_c^\infty(H \times H \times E)$ set

$$(\lambda(y)\alpha)(h; h_0, v) = \alpha(yhy^{-1}; yh_0, v) \quad (h, h_0 \in H,\ v \in E).$$

Let T be a distribution on $H \times E$ satisfying (2.1). Define the distribution σ_T on $H \times H \times E$ by $\sigma_T(\alpha) = T(f_\alpha)$. Then we easily obtain

$$\sigma_T(\lambda(y)\alpha) = \sigma_T(\alpha)$$

for all $y \in H$ and all $\alpha \in C_c^\infty(H \times H \times E)$. Now apply Lemma 2.4. We get a distribution s_T on $H \times E$ such that

$$\sigma_T(\alpha) = s_T(\beta_\alpha)$$

where

$$\beta_\alpha(h, v) = \int_H \alpha(yhy^{-1}; y, v)\, dy.$$

Define for $\alpha \in C_c^\infty(H \times H \times E)$ the function α' by

$$\alpha'(h; h_0, v) = \alpha(h, h^{-1}h_0, v) \quad (h, h_0 \in H,\ v \in E).$$

Then $\alpha' \in C_c^\infty(H \times H \times E)$ and, clearly, $\beta_\alpha = \beta_{\alpha'}$. We get

$$T(f_\alpha(h, h^{-1} \cdot v)) = \sigma_T(\alpha') = s_T(\beta_{\alpha'}) = s_T(\beta_\alpha) = T(f_\alpha(h, v)).$$

This completes the proof.

3 Semi-direct products with Heisenberg groups

Consider the groups H from Section 2 and let H' stand for the groups $\mathrm{O}(p+1,q+1)$, $\mathrm{U}(p+1,q+1)$, $\mathrm{GL}(n+2,\mathbb{R})$, $\mathrm{O}(n+2,\mathbb{C})$ and $\mathrm{GL}(n+2,\mathbb{C})$ respectively. They act on the spaces $E' = E_{n+2}$. Let $\xi^0 \in E'$ stand for the following elements.

- If $H' = \mathrm{O}(p+1,q+1)$, $\mathrm{U}(p+1,q+1)$ or $\mathrm{O}(n+2,\mathbb{C})$, then $\xi^0 = (1,0,\ldots,0,1)$.
- If $H' = \mathrm{GL}(n+2,\mathbb{R})$ or $\mathrm{GL}(n+2,\mathbb{C})$, then $\xi^0 = (e_1, e_{n+2})$, where e_i is the notation for the i-th unit vector in $\mathbb{R}^{n+2}$ or $\mathbb{C}^{n+2}$.

The stabilizer of ξ^0 in H' can be seen as the semi-direct product $G = H \ltimes N$ where N is a Heisenberg group. The group N has the following form.

- If $H = \mathrm{O}(p,q)$ or $\mathrm{O}(n,\mathbb{C})$, then $N = \mathbb{R}^n$ or $\mathbb{C}^n$ respectively.
- If $H = \mathrm{U}(p,q)$, then N consists of elements $n(z,w)$ with $z \in \mathbb{C}^{p+q}$, $w \in i\mathbb{R}$ and multiplication

$$n(z,w)\cdot n(z,w') = n(z+z', w+w'+i\,\mathrm{Im}\,[z,\,z'])$$

where
$$[z,\,z'] = z_1\overline{z_1'} + \cdots + z_p\overline{z_p'} - z_{p+1}\overline{z_{p+1}'} - \cdots - z_{p+q}\overline{z_{p+q}'}$$
if $z = (z_1,\ldots,z_{p+q})$, $z' = (z_1',\ldots,z_{p+q}')$.

The group H acts on N by

$$h\,n(z,w)\,h^{-1} = n(h\cdot z, w) \quad (h\in H,\, n(z,w)\in N).$$

- If $H = \mathrm{GL}(n,\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), then N consists of elements $n(z,w)$ with $z = (x,y) \in \mathbb{F}^n \times \mathbb{F}^n$ and $w \in \mathbb{R}$ with multiplication

$$n(z,w)\cdot n(z',w') = n(z+z', w+w'+[z,z'])$$

where $[z,z'] = \langle x,y'\rangle - \langle y,x'\rangle$ if $z=(x,y)$, $z'=(x',y')$, and $\langle a,\,b\rangle = a_1b_1 + \cdots + a_nb_n$ if $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)$ in $\mathbb{F}^n$. The action of H on N is as before:

$$h\,n(z,w)\,h^{-1} = n(h\cdot z, w) \quad (h\in H,\, n(z,w)\in N).$$

The group $G = H \ltimes N$ is a unimodular Lie group. Elements of G are again written as pairs $g = (h,n)$ and the multiplication in G is then given by
$$(h,n)\cdot(h',n') = (hh', n\,(hn'h^{-1}))\ (h,h'\in H; n,n'\in N).$$

In particular we have $(h, n)^{-1} = (h^{-1},\, h^{-1}n^{-1}h)$. Notice that $n(z, w)^{-1} = n(-z, -w)$ in all cases.

We have the following theorem.

Theorem 3.1 *The pair $(H \ltimes N, H)$ is a multiplicity free pair.*

Proof. Observe that in the cases $\mathrm{O}p, q) \ltimes N$ and $\mathrm{O}(n, \mathbb{C}) \ltimes N$ there is nothing extra to prove; one can just apply Theorem 2.1. For the remaining cases we apply again Proposition 1.3 with the following involutions τ.

- If $H = \mathrm{U}(p, q)$, then $\tau(h,\, n(z, w)) = (\overline{h},\, n(\overline{z}, -w))$.
- If $H = \mathrm{GL}(n, \mathbb{F})$, then $\tau(h, n((x, y), w)) = ({}^{\,t}h^{-1},\, n((y, x), -w))$.

We have to show that any positive-definite distribution T satisfying

$$T(h_1 h h_1^{-1}, n(h_1 \cdot z, w)) = T(h,\, n(z, w))$$

for all $h_1 \in H$, also satisfies the relation

$$T^{\tau} = \check{T}.$$

This amounts to the relation

$$T(h^{-1},\, n(-h^{-1} \cdot z, -w)) = T^{\tau}(h,\, n(z, w)).$$

Fixing w, we see that this just amounts to the same relation as stated for $H \ltimes E$, see (2.2). This completes the proof.

In his recent paper [6], Sun has formulated and proved similar results for p-adic groups. His list of semi-direct products is even a little longer, since it involves symplectic groups H as well. This new case has an Archimedean counterpart, that is a multiplicity free pair as well (private communication by Sun).

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