Continuity properties of Markov semigroups and their restrictions to invariant L^1-spaces

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Abstract

We consider Markov semigroups on the cone of positive finite measures on a complete separable metric space. Such a semigroup extends to a semigroup of linear operators on the vector space of measures that typically fails to be strongly continuous for the total variation norm. First we characterise when the restriction of a Markov semigroup to an invariant L^1-space is strongly continuous. Aided by this result we provide several characterisations of the subspace of strong continuity for the total variation norm. We prove that this subspace is a projection band in the Banach lattice of finite measures, and consequently obtain a direct sum decomposition.

1 Introduction

Markov operators on the cone of positive finite measures are additive and positively homogeneous operators on this cone that preserve mass, i.e. the total variation norm of measures. A Markov semigroup is a semigroup of Markov operators. They naturally occur in probability theory and the theory of Markov processes [12, 18]. Moreover, one encounters such semigroups also in the setting of measure-valued structured population models (cf. e.g. [6, 7] and an application to cell growth and division in [5]). Here the measure represents the constitution of the population at each time.

The Markov semigroups that are obtained in both settings are hardly ever continuous for the total variation norm $\|\cdot\|_{\mathrm{TV}}$ on the space of finite measures $\mathcal{M}(S)$ on the underlying measurable space (S,Σ), typically a complete separable metric space with its Borel σ-algebra. Notable exceptions are Markov jump processes [12, 13], which yield strongly continuous semigroups in $\mathcal{M}(S)$ for $\|\cdot\|_{\mathrm{TV}}$ when (S,Σ) is merely a measurable space as above [29]. This may have motivated other researchers to consider the more restrictive setting of strongly continuous Markov semigroups on L^1-spaces with respect to particular positive measures (see e.g. [19, 27, 28]). In view of the above mentioned applications this setting seems to be too restrictive however.

In this paper we consider Markov semigroups $(P(t))_{t\geq 0}$ on the positive finite Borel measures $\mathcal{M}^+(S)$ on a complete separable metric space (S,d). The positive operators $P(t)$ naturally extend to bounded linear operators $\overline{P(t)}$ on the Banach lattice $(\mathcal{M}(S), \|\cdot\|_{\mathrm{TV}}$. We address two closely related questions: when $(P(t))_{t\geq 0}$ leaves invariant a cone $\Gamma \subset \mathcal{M}^+(S)$ such that the measures in Γ are all absolutely continuous with respect to a single measure μ, i.e. $\Gamma = L^1_+(S,\mu)$, it induces a semigroup of nonexpansive linear operators on $L^1(S,\mu)$ that are isometries on $L^1_+(S,\mu)$. The first question is then to characterise when this induced semigroup is strongly continuous. This is achieved in Theorem 4.6, partially using an argument inspired by [16], under the assumption that for each $\mu \in \mathcal{M}^+(S)$, the map $t \mapsto P(t)\mu : \mathbb{R}_+ \to \mathcal{M}^+(S)$ is continuous for the relative topology on $\mathcal{M}^+(S)$ of the weak*-topology on $C_b^*(S)$. It was shown in [9, 10] that this topology is metrisable by means of the norm on $\mathrm{BL}(S)^*$, the dual of the bounded Lipschitz functions on S. See also [17] for further exploration of this property.

Also under this assumption and the additional assumption that each operator $P(t) : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ is continuous for this weak*-topology, we then characterise in Theorem 5.6 and Theorem 5.7 the subspace $\mathcal{M}(S)^0_{\mathrm{TV}}$ of $\mathcal{M}(S)$ that consists of all measures μ for which $t \mapsto P(t)\mu$ is continuous for $\|\cdot\|_{\mathrm{TV}}$. To that end we exploit results of [14] on modules of Banach algebras with approximate identity and properties of Bochner integration in the Banach space $\mathcal{S}_{\mathrm{BL}}$, which is the closure of $\mathcal{M}(S)$ in $\mathrm{BL}(S)^*$. These properties are of separate interest. We state and prove them in Section 2.3. A consequence of this characterisation is that $\mathcal{M}(S)^0_{\mathrm{TV}}$ is dense in $\mathcal{M}(S)$ for the $\mathcal{S}_{\mathrm{BL}}$-topology. In particular it is non-trivial and not 'too small'. Moreover, it turns out to be a *projection band* in the Banach lattice $\mathcal{M}(S)$ (Proposition 6.1), hence it is complemented. This complement is characterised and will not be $(P(t))_{t\geq 0}$-invariant in general (unfortunately). An additional result of our approach is a generalisation of classical result by Wiener and Young ([30]) for general Markov semigroups (Theorem 6.7).

Some notational conventions. We write (Ω, Σ) to denote a measurable space, $\mathcal{M}^+(\Omega)$ to denote the cone of positive finite measures on Σ and $\mathcal{M}(\Omega)$ the real vector space of all signed finite measures. Throughout this paper (S,d) will denote a *complete separable* metric space, viewed as a measurable space with respect to its Borel σ-algebra, with at least two elements. We write $\mathbb{1}_E$ for the indicator function of $E \subset S$. $\mathbb{1}_S$ will be simplified to $\mathbb{1}$. For $f : \Omega \to \mathbb{R}$ measurable and $\mu \in \mathcal{M}(\Omega)$ we write $\langle \mu, f\rangle$ for $\int_\Omega f\, d\mu$.

2 Preliminaries on spaces of measures

$\mathcal{M}(\Omega)$ endowed with the total variation norm $\|\cdot\|_{\mathrm{TV}}$ is a Banach space. Let $\mu, \nu \in \mathcal{M}(\Omega)$. μ is absolutely continuous with respect to ν, $\mu \ll \nu$, if $|\mu|(E) = 0$

for every $E \in \Sigma$ for which $|\nu|(E) = 0$. So $\mu \ll \nu$ if and only if $|\mu| \ll |\nu|$.

Let $\mu \in \mathcal{M}(\Omega), \nu \in \mathcal{M}^+(\Omega)$, then $\mu \ll \nu$ if and only if $\mu(E) = 0$ for every $E \in \Sigma$ such that $\nu(E) = 0$, which is easy to prove.

Since S is a complete separable metric space, it is a standard result that every $\mu \in \mathcal{M}(S)$ is inner regular, i.e. for every Borel set E in S, there are compact $K_n \subset E$, such that $|\mu|(K_n) \to |\mu|(E)$.

Lemma 2.1. *Let $\mu \in \mathcal{M}(S), \nu \in \mathcal{M}^+(S)$. Then the following are equivalent:*

(i) $\mu \ll \nu$

(ii) $\mu(K) = 0$ *for all compact* K *in* S *such that* $\nu(K) = 0$.

Proof. *(i)* $\Rightarrow$ *(ii)*: Trivial. *(ii)* $\Rightarrow$ *(i)*: Let E be a Borel set in S such that $\nu(E) = 0$. Then $\nu(K) = 0$ for all compact K such that $K \subset E$, hence $\mu^+(K) = \mu^-(K)$ for all compact $K \subset E$. Since μ^+ and μ^- are inner regular on all Borel sets, there are compact sets $K_n \subset E$, such that $\lim_{n\to\infty} \mu^+(K_n) = \mu^+(E)$ and $\lim_{n\to\infty} \mu^-(K_n) = \mu^-(E)$. So $\mu^+(E) = \mu^-(E)$ and $\mu(E) = \mu^+(E) - \mu^-(E) = 0$. □

2.1 Space of measures viewed as Banach lattice

We refer to [2], [21] and [32] for the basic theory on Riesz spaces and Banach lattices.

$\mathcal{M}(\Omega)$ is an ordered vector space for the partial ordering defined by

$$\mu \leq \nu \text{ whenever } \mu(E) \leq \nu(E) \text{ for all } E \in \Sigma.$$

$\mathcal{M}(\Omega)$ is a Riesz space, where the least upper bound of μ and ν is given by

$$\mu \vee \nu(E) := \sup\{\mu(A) + \nu(E \setminus A) \mid A \in \Sigma,\ A \subset E\},$$

and the greatest lower bound is given by

$$\mu \wedge \nu(E) := \inf\{\mu(A) + \nu(E \setminus A) \mid A \in \Sigma,\ A \subset E\}.$$

Note that $|\mu| \leq |\nu|$ implies $\mu \ll \nu$. The positive and negative part of $\mu \in \mathcal{M}(\Omega)$ as introduced in measure theory, μ^+ and μ^-, correspond to the concepts of positive and negative part in a Riesz space: $\mu^+ = \mu \vee 0$, $\mu^- = (-\mu)^+$ and $|\mu| = \mu^+ + \mu^-$. $\mu, \nu \in \mathcal{M}(\Omega)$ are mutually singular, $\mu \perp \nu$, if there is a $U \in \Sigma$, such that $\mu(E) = \mu(E \cap U)$ and $\nu(E) = \mu(E \setminus U)$ for every $E \in \Sigma$. Mutual singularity of $\mu, \nu \in \mathcal{M}(\Omega)$ corresponds to the concept of disjointness in a Riesz space: μ and ν are disjoint, $\mu \perp \nu$, whenever $|\mu| \wedge |\nu| = 0$. $\mathcal{M}(\Omega)$ is a Dedekind complete Riesz space ([21, 1.1 Example vi)]).

$\mathcal{M}(\Omega)$ is a Banach lattice for the total variation norm: $\|\mu\|_{\mathrm{TV}} = |\mu|(\Omega)$, and $\|\cdot\|_{\mathrm{TV}}$ is an L-norm: $\|\mu + \nu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{TV}} + \|\nu\|_{\mathrm{TV}}$ for all $\mu, \nu \in \mathcal{M}^+(\Omega)$. As in all Banach lattices, the lattice operations are continuous for the norm topology. (see e.g. [21, Proposition 1.1.6]).

We will now define some concepts in Riesz spaces that we will need later on: Let X be a Riesz space. A subspace I of X is an *ideal* of X if $|x| \leq |y|$ for some $y \in I$ implies $x \in I$. An ideal B of X is a *band* of X if $\sup(A) \in B$ for every subset $A \subset B$ which has a supremum in X. A band B of X is a *projection band* if there exists a bounded linear projection $P : X \to B$, such that $0 \leq Px \leq x$ for all $x \in X_+$. In this case $X = B \oplus B^\perp$, where $B^\perp := \{x \in X : x \perp y \text{ for all } y \in B\}$.

In a remark in [2] (under definition 4.20) it is shown that every L-space has order continuous norm as a consequence of [21, Theorem 2.4.2]. Furthermore, in a Banach lattice with order continuous norm, every closed ideal is a projection band ([21, Corollary 2.4.2]). These statements imply

Theorem 2.2. *Every closed ideal in $\mathcal{M}(\Omega)$ is a projection band.*

2.2 The space $\mathcal{S}_{\mathrm{BL}}$

In this section we recall some definitions and results from [17]. $\mathrm{BL}(S)$ denotes the Banach space of bounded real-valued Lipschitz functions for the metric d, endowed with the norm $\|f\|_{\mathrm{BL}} := |f|_{\mathrm{Lip}} + \|f\|_\infty$, where

$$|f|_{\mathrm{Lip}} := \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in S, x \neq y. \right\}.$$

The Dirac functionals $\delta_x(f) := f(x)$ for $x \in S$ are in $\mathrm{BL}(S)^*$. We denote the usual dual norm on $\mathrm{BL}(S)^*$ by $\|\cdot\|_{\mathrm{BL}}^*$.

$\mathrm{BL}(S)$ is in fact isometrically isomorphic to the dual of a separable Banach space $\mathcal{S}_{\mathrm{BL}}$, which can be defined as the closure of the finite linear span of the δ_x, $x \in S$, in $\mathrm{BL}(S)^*$. Then, as shown in [9, Lemma 6], each $\mu \in \mathcal{M}(S)$ defines a unique element in $\mathrm{BL}(S)^*$, which we will also denote by μ, by sending $f \in \mathrm{BL}(S)$ to $\langle \mu, f \rangle = \int_S f \, d\mu$. A function $f \in \mathrm{BL}(S)$ defines a bounded linear functional on $\mathcal{S}_{\mathrm{BL}}$ by sending ϕ to $\phi(f)$.

By [17, Theorem 3.9 and Corollary 3.10], $\mathcal{M}^+(S)$ is a closed convex cone of $\mathcal{S}_{\mathrm{BL}}$, and $\mathcal{M}(S)$ is a $\|\cdot\|_{\mathrm{BL}}^*$-dense subspace of $\mathcal{S}_{\mathrm{BL}}$.

The following lemma follows from [9, Theorem 6 and Theorem 8]:

Lemma 2.3. *Let $\mu_n, \mu \in \mathcal{M}^+(S)$. Then $\|\mu_n - \mu\|_{\mathrm{BL}}^* \to 0$ if and only if $\int_S f \, d\mu_n \to \int_S f \, d\mu$ for all $f \in C_b(S)$.*

Moreover, the restriction of the weak-star topology on $C_b(S)^*$ to $\mathcal{M}^+(S)$ *equals* the restriction of the norm topology on $\mathcal{S}_{\mathrm{BL}}$ to $\mathcal{M}^+(S)$ by [9, Theorem 18].

Let
$$\mathcal{S}_{\mathrm{BL}}^+ := \{\phi \in \mathcal{S}_{\mathrm{BL}} : \phi(f) \geq 0 \text{ for all } f \in \mathrm{BL}(S), f \geq 0\}.$$
Then $\mathcal{S}_{\mathrm{BL}}^+ = \mathcal{M}^+(S)$ by [17, Corollary 4.2].

When $\mathcal{M}(S)$ and $\mathcal{M}^+(S)$ are equipped with the $\|\cdot\|_{\mathrm{BL}}^*$-topology, we write $\mathcal{M}(S)_{\mathrm{BL}}$ and $\mathcal{M}^+(S)_{\mathrm{BL}}$ respectively. When we use the $\|\cdot\|_{\mathrm{TV}}$-topology, we write $\mathcal{M}(S)_{\mathrm{TV}}$ and $\mathcal{M}^+(S)_{\mathrm{TV}}$.

2.3 Bochner integration of $\mathcal{S}_{\mathrm{BL}}$-valued functions

In this section we give some results on functions $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$ which are strongly measurable in the sense of Bochner. We will make use of the Monotone Class Theorem for functions, which we state here for convenience (see e.g.[31, Theorem II.4]).

Theorem 2.4. *Let $\mathcal{E}$ be a π-system for S and let $\mathcal{H}$ be a vector space of functions from S to $\mathbb{R}$ such that*

1. *$\mathcal{H}$ contains the indicator function $\mathbb{1}_E$ of every $E \in \mathcal{E}$, and $\mathcal{H}$ contains $\mathbb{1}_S$,*
2. *if $(f_n)_n$ is a sequence of elements of $\mathcal{H}$ with $f_n \geq 0$ and $f_n \uparrow f$, where f is bounded, then $f \in \mathcal{H}$.*

Then $\mathcal{H}$ contains every bounded real-valued function which is measurable with respect to the σ-algebra generated by $\mathcal{E}$.

Proposition 2.5. *Let $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$. Then the following conditions are equivalent:*

(i) *p is strongly measurable.*

(ii) *For each bounded measurable $f : S \to \mathbb{R}$, the map $\Omega \to \mathbb{R} : \omega \mapsto \langle p(\omega), f\rangle$ is measurable.*

(iii) *For each Borel measurable $E \subset S$, the map $\Omega \to \mathbb{R} : \omega \mapsto p(\omega)(E)$ is measurable from Ω to $\mathbb{R}$.*

Proof. *(i)$\Rightarrow$ (ii)*: Let $\mathcal{H}$ the vector space of measurable functions h from S to $\mathbb{R}$, such that $\omega \mapsto \langle p(\omega), h\rangle$ is measurable from Ω to $\mathbb{R}$. Let $\mathcal{C}$ be the π-system of closed sets in S. Our aim is to show that $\mathcal{H}$ and $\mathcal{C}$ satisfy the conditions of Theorem 2.4. Then it follows that $\mathcal{H}$ contains every bounded Borel measurable function on S.

Since p is strongly measurable, it is weakly measurable. Let C be a closed set in S and let $g_n(x) := \max(1 - nd(x, C), 0)$. Then $g_n \in \mathrm{BL}(S) \cong \mathcal{S}_{\mathrm{BL}}^*$, hence $G_n : \omega \mapsto \langle p(\omega), g_n\rangle$ is measurable from Ω to $\mathbb{R}$. Since C is closed, $g_n(x) \to \mathbb{1}_C(x)$ for every $x \in S$. Fix $\omega \in \Omega$. Then all g_n are in $L^1(p(\omega))$, thus $\mathbb{1}_C$ is in $L^1(p(\omega))$ and

$$\lim_{n\to\infty} G_n(\omega) = \langle p(\omega), \mathbb{1}_C\rangle$$

by the Lebesgue Dominated Convergence Theorem. So the function

$$\omega \to p(\omega)(C) = \langle p(\omega), \mathbb{1}_C\rangle$$

is the pointwise limit of measurable functions, hence measurable, which implies that $\mathbb{1}_C \in \mathcal{H}$ for all closed $C \subset S$. Suppose $h_n \in \mathcal{H}$ such that $0 \leq h_n \uparrow h \leq M$, for some function $h : \Omega \to \mathbb{R}$, bounded by $M > 0$. Then by assumption $H_n : \omega \mapsto \langle p(\omega), h_n\rangle$ is measurable for all $n \in \mathbb{N}$. Fix $\omega \in \Omega$. By the Lebesgue Monotone Convergence Theorem $h \in L^1(p(\omega))$ and $\lim_{n\to\infty} H_n(\omega) = \langle p(\omega), h\rangle$. This implies that the function

$$\omega \mapsto \langle p(\omega), h\rangle$$

is the pointwise limit of measurable functions, hence measurable. So $h \in \mathcal{H}$ and the conditions of Theorem 2.4 are satisfied.

(ii)⇒ (iii): Let $E \subset S$ be measurable, then $\mathbb{1}_E$ is a bounded measurable function from S to $\mathbb{R}$.

(iii)⇒(i): By assumption $\omega \mapsto \langle p(\omega), g\rangle$ is measurable, for all simple functions g on S. Let $h \in \mathrm{BL}^+(S)$. Then there are simple functions h_n such that $0 \le h_n \uparrow h$. By the Lebesgue Monotone Convergence Theorem, $\langle p(\omega), h_n\rangle \to \langle p(\omega), h\rangle$ for every $\omega \in \Omega$. So $\omega \mapsto \langle p(\omega), h\rangle$ is the pointwise limit of measurable functions, hence measurable. For general $h \in \mathrm{BL}(S)$, we can write $h = h^+ - h^-$, and thus $\omega \mapsto \langle p(\omega), h\rangle$ is the difference of two measurable functions, hence measurable. So p is weakly measurable. Since $\mathcal{S}_{\mathrm{BL}}$ is separable, p is strongly measurable by Pettis' Theorem. □

If $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$ is Bochner integrable with respect to $\mu \in \mathcal{M}^+(\Omega)$, then $\nu := \int_\Omega p(\omega)d\mu(\omega)$ defines an element in $\mathcal{S}_{\mathrm{BL}}$. Since $\mathcal{S}_{\mathrm{BL}}^+ = \mathcal{M}^+(S)$ is a closed convex cone in $\mathcal{S}_{\mathrm{BL}}, \nu$ is in $\mathcal{S}_{\mathrm{BL}}^+$.

Proposition 2.6. *Let $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$ be Bochner integrable with respect to μ in $\mathcal{M}^+(\Omega)$, and define $\nu := \int_\Omega p(\omega)d\mu(\omega)$. Then*

$$\int_S f\, d\nu = \int_\Omega \langle p(\omega), f\rangle d\mu(\omega), \tag{1}$$

for any bounded measurable $f : S \to \mathbb{R}$.

Proof. Step 1. (1) *holds for all $f \in \mathrm{BL}(S)$.*
We can view f as element of $\mathcal{S}_{\mathrm{BL}}^*$. Since p is Bochner integrable with respect to μ, the map $\Omega \to \mathbb{R} : \omega \mapsto \langle p(\omega), f\rangle$ is in $L^1(\mu)$. So we get by [8, Theorem 6] that

$$\begin{aligned} \int_S f\, d\nu &= \langle \nu, f\rangle = \Big\langle \int_\Omega p(\omega)d\mu(\omega), f\Big\rangle \\ &= \int_\Omega \langle p(\omega), f\rangle d\mu(\omega). \end{aligned}$$

Step 2. (1) *holds for all $f = \mathbb{1}_C$, $C \subset S$ closed.*
Let $f_n \in \mathrm{BL}(S)$ be defined as $f_n(x) := \max(1 - nd(x, C), 0)$. Then f_n is bounded by $\mathbb{1}$, and $f_n(x) \to \mathbb{1}_C(x)$ for all $x \in S$, so by Lebesgue Dominated Convergence Theorem we have that for all $\omega \in \Omega$, $\langle p(\omega), f_n\rangle \to \langle p(\omega), \mathbb{1}_C\rangle = [p(\omega)](C)$. Since $f_n \in \mathcal{S}_{\mathrm{BL}}^*$, $\omega \mapsto \langle p(\omega), f_n\rangle$ is in $L^1(\mu)$. Also

$$\|\langle p(\omega), f_n\rangle\| \le \|p(\omega)\|_{\mathrm{TV}} = \|p(\omega)\|_{\mathrm{BL}}^*,$$

for all $\omega \in \Omega$ and $n \in \mathbb{N}$, and by assumption $\omega \mapsto \|p(\omega)\|_{\mathrm{BL}}^*$ is in $L^1(\mu)$. Hence by the Lebesgue Dominated Convergence Theorem

$$\int_\Omega \langle p(\omega), f_n\rangle d\mu(\omega) \to \int_\Omega [p(\omega)](C)d\mu(\omega).$$

By Step 1

$$\int_\Omega \langle p(\omega), f_n\rangle d\mu(\omega) = \int_S f_n d\nu,$$

for all $n \in \mathbb{N}$. And again by the Lebesgue Dominated Convergence Theorem we can conclude that $\int_S f_n d\nu \to \nu(C)$. So $\int_\Omega [p(\omega)](C) d\mu(\omega) = \nu(C)$ for all C closed.

Step 3. (1) holds for all bounded measurable $f : S \to \mathbb{R}$.

Now we want to apply Theorem 2.4. Let $\mathcal{H}$ be the vector space of bounded measurable functions $f : S \to \mathbb{R}$, such that $\int_\Omega \langle p(\omega), f\rangle d\mu(\omega) = \int_S f d\nu$. Note that these expressions are well defined: f is bounded and measurable, so it follows from Proposition 2.5 that $\Omega \to \mathbb{R} : \omega \mapsto \langle p(\omega), f\rangle$ is in $L^1(\mu)$.

By Step 2 $\mathbb{1}_C \in \mathcal{H}$ for all $C \subset S$ closed. Now let $f_n \in \mathcal{H}$ with $0 \leq f_n \uparrow f \leq M < \infty$, for some function f and some $M > 0$. Then by the Lebesgue Monotone Convergence Theorem, $\langle p(\omega), f_n\rangle \to \langle p(\omega), f\rangle$ for all $\omega \in \Omega$, and $\int_S f_n d\nu \to \int_S f d\nu$. Since $\langle p(\omega), f_n\rangle$ is bounded from above by a constant not depending on n and ω, we can apply the Lebesgue Dominated Convergence Theorem to get that

$$\int_\Omega \langle p(\omega), f_n\rangle d\mu(\omega) \to \int_\Omega \langle p(\omega), f\rangle d\mu(\omega).$$

Since $f_n \in \mathcal{H}$ we can conclude that $\int_\Omega \langle p(\omega), f\rangle d\mu(\omega) = \int_S f d\nu$, hence $f \in \mathcal{H}$. By Theorem 2.4 we obtain that $\mathcal{H}$ contains every bounded real-valued Borel measurable function. □

Corollary 2.7. *Let $p : \Omega \to \mathcal{S}^+_{\mathrm{BL}}$ be Bochner integrable with respect to μ in $\mathcal{M}^+(\Omega)$. Then*

$$\left[\int_\Omega p(\omega) d\mu(\omega)\right](E) = \int_\Omega p(\omega)(E) d\mu(\omega)$$

for any Borel measurable $E \subset S$.

3 Markov semigroups

We start by introducing the concept of Markov operators.

Definition 3.1. A *Markov operator* is a map $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$, such that

(MO1) P is additive and $\mathbb{R}_+$-homogeneous,

(MO2) $\|P\mu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{TV}}$ for all $\mu \in \mathcal{M}^+(S)$.

Since $\mathcal{M}(S)_{\mathrm{TV}}$ is a Banach lattice, condition *(MO1)* ensures that a Markov operator P extends to a positive *bounded* linear operator on $\mathcal{M}(S)_{\mathrm{TV}}$ given by $P\mu := P(\mu^+) - P(\mu^-)$. The operator norm of this extension is

$$\|P\| = \sup\{\|P\mu\|_{\mathrm{TV}} : \mu \in \mathcal{M}^+(S),\ \|\mu\|_{\mathrm{TV}} \leq 1\} = 1$$

according to *(MO2)*. Since $\mathrm{Id} : \mathcal{M}(S)_{\mathrm{TV}} \to \mathcal{M}(S)_{\mathrm{BL}}$ is continuous with operator norm equal to 1, *(MO2)* implies that $P : \mathcal{M}(S)_{\mathrm{TV}} \to \mathcal{M}(S)_{\mathrm{BL}}$ is nonexpansive and an isometry on the positive cone.

A *Markov semigroup* is a semigroup $(P(t))_{t\geq 0}$ of Markov operators.

While strong continuity of $(P(t))_{t\geq 0}$ with respect to $\|\cdot\|_{\mathrm{TV}}$ is rare, we will see that strong continuity with respect to $\|\cdot\|^*_{\mathrm{BL}}$ is not.

Thus, we call the Markov semigroup $(P(t))_{t\geq 0}$ *strongly continuous*, when $t \mapsto P(t)\mu : \mathbb{R}_+ \to \mathcal{M}(S)_{\mathrm{BL}}$ is continuous for each $\mu \in \mathcal{M}^+(S)$. Then by linearity, this continuity holds for all $\mu \in \mathcal{M}(S)$.

Lemma 3.2. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup. Then the following are equivalent:*

(i) $(P(t))_{t\geq 0}$ *is strongly continuous,*

(ii) $\langle P(t)\mu, f\rangle \to \langle \mu, f\rangle$ *as* $t \downarrow 0$ *for all* $\mu \in \mathcal{M}^+(S), f \in C_b(S)$.

(iii) $\langle P(t)\mu, f\rangle \to \langle \mu, f\rangle$ *as* $t \downarrow 0$ *for all* $\mu \in \mathcal{M}(S), f \in C_b(S)$.

Proof. *(i)⇒ (ii)*: Follows from Lemma 2.3, since $\mu, P(t)\mu \in \mathcal{M}^+(S)$.
(ii)⇒ (iii): Follows from the decomposition $\mu = \mu^+ - \mu^-$.
(iii)⇒ (i): This is a direct consequence of [9, Theorem 6]. □

We will show that certain actions of $\mathbb{R}_+$ on S provide us with an important class of examples of Markov semigroups.

A semigroup of continuous maps on S is a family of maps $(\Phi_t)_{t\geq 0}$, such that $\Phi_t : S \to S$ is continuous, $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and $\Phi_0 = \mathrm{Id}_S$ for all $s, t \in \mathbb{R}_+$. $(\Phi_t)_{t\geq 0}$ is strongly continuous if the map $\mathbb{R}_+ \to S : t \mapsto \Phi_t(x)$ is continuous for all $x \in S$.

Proposition 3.3. *Let $(\Phi_t)_{t\geq 0}$ be a semigroup of continuous maps on S. Then $P(t)\mu := \mu \circ \Phi_t^{-1}$ defines a Markov semigroup $(P(t))_{t\geq 0}$, such that $P(t) : \mathcal{M}^+(S)_{\mathrm{BL}} \to \mathcal{M}^+(S)_{\mathrm{BL}}$ is continuous for all $t \in \mathbb{R}_+$. This Markov semigroup is strongly continuous if and only if $(\Phi_t)_{t\geq 0}$ is strongly continuous.*

Proof. Let $\mu \in \mathcal{M}^+(S)$. It is easily verified that $P(t)\mu \in \mathcal{M}^+(S)$, $P(t)P(s)\mu = P(t+s)\mu$ for all $s, t \in \mathbb{R}_+$ and $P(0) = \mathrm{Id}$, and that $(P(t))_{t\geq 0}$ satisfies *(MO1)*-*(MO2)*.

Fix $t \in \mathbb{R}_+$. Let $\mu_n, \mu \in \mathcal{M}^+(S)$ such that $\|\mu_n - \mu\|^*_{\mathrm{BL}} \to 0$ as $n \to \infty$. Then for all $f \in C_b(S)$, $\int_S f\, d\mu_n \to \int_S f\, d\mu$ by Lemma 2.3. Hence

$$\langle P(t)\mu_n, f\rangle = \langle \mu_n, f \circ \Phi_t\rangle \to \langle \mu, f \circ \Phi_t\rangle = \langle P(t)\mu, f\rangle.$$

Therefore $\|P(t)\mu_n - P(t)\mu\|^*_{\mathrm{BL}} \to 0$, hence $P(t) : \mathcal{M}^+(S)_{\mathrm{BL}} \to \mathcal{M}^+(S)_{\mathrm{BL}}$ is continuous.

Suppose $(\Phi_t)_{t\geq 0}$ is strongly continuous and let $f \in C_b(S)$. Then $f \circ \Phi_t(x) \to f(x)$ as $t \downarrow 0$. Also, $|f \circ \Phi_t(x)| \leq \|f\|_\infty \mathbb{1} \in L^1(\mu)$, hence by the Lebesgue Dominated Convergence Theorem

$$\langle P(t)\mu, f\rangle = \langle \mu, f \circ \Phi_t\rangle \to \langle \mu, f\rangle,$$

so $(P(t))_{t\geq 0}$ is strongly continuous by Lemma 3.2.

Now suppose $(P(t))_{t\geq 0}$ is strongly continuous and let $x \in S$. Then $P(t)\delta_x = \delta_{\Phi_t(x)} \to \delta_x$ as $t \downarrow 0$. It follows from [17, Lemma 3.5] that for all $y, z \in S$, $d(y, z) = \frac{2\|\delta_y - \delta_z\|^*_{\mathrm{BL}}}{2 - \|\delta_y - \delta_z\|^*_{\mathrm{BL}}}$, so

$$d(\Phi_t(x), x) = \frac{2\|P(t)\delta_x - \delta_x\|^*_{\mathrm{BL}}}{2 - \|P(t)\delta_x - \delta_x\|^*_{\mathrm{BL}}} \to 0,$$

as $t \downarrow 0$. Hence $(\Phi_t)_{t\geq 0}$ is strongly continuous. □

In [17, Section 5] it is shown that if, in addition to the conditions above, the maps $\Phi_t : S \to S$ are Lipschitz, then the Markov semigroup $(P(t))_{t\geq 0}$ can be extended to a semigroup of bounded linear operators $(\overline{P(t)})_{t\geq 0}$ on $\mathcal{S}_{\mathrm{BL}}$. Moreover, $(\overline{P(t)})_{t\geq 0}$ is strongly continuous if $(\Phi_t)_{t\geq 0}$ is strongly continuous and $\limsup_{t\downarrow 0} |\Phi_t|_{\mathrm{Lip}} < \infty$.

4 Restriction to invariant L^1-spaces

Let $\mu \in \mathcal{M}^+(S)$. For $f \in L^1(\mu)$ we define $j_\mu(f) = f\,d\mu$. Then j_μ is a linear map from $L^1(\mu)$ into $\mathcal{M}(S)$.

Lemma 4.1. *The following properties hold:*

(i) j_μ *is an isometric embedding of* $L^1(\mu)$ *into* $\mathcal{M}(S)_{\mathrm{TV}}$*, i.e.* $\|j_\mu(f)\|_{\mathrm{TV}} = \|f\|_1$ *for all* $f \in L^1(\mu)$.

(ii) j_μ *is a continuous embedding of* $L^1(\mu)$ *into* $\mathcal{S}_{\mathrm{BL}}$*, with* $\|j_\mu(f)\|^*_{\mathrm{BL}} = \|f\|_1$ *for all* $f \in L^1_+(\mu)$ *and* $\|j_\mu(f)\|^*_{\mathrm{BL}} \leq \|f\|_1$ *for all* $f \in L^1(\mu)$.

The proof is straightforward.

Let $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ be a Markov operator. Suppose that P leaves $j_\mu(L^1_+(\mu))$ invariant. Then P induces an additive and positively homogeneous map $T : L^1_+(\mu) \to L^1_+(\mu)$:

$$Tf := j_\mu^{-1} \circ P \circ j_\mu(f).$$

Because $L^1(\mu)$ is a Banach lattice, T extends to a positive bounded linear operator on $L^1(\mu)$, which we will also denote by T, and

$$\|T\| = \sup\{\|Tf\| \mid f \in L^1_+(\mu),\ \|f\| \leq 1\} = 1 \tag{2}$$

by Lemma 4.1 and *(MO2)*. T will be called the operator (in $L^1(\mu)$) induced by P.

At this point we would like to note, that

Lemma 4.2. *If* $\mu, \nu \in \mathcal{M}^+(S)$ *satisfy* $\mu \ll \nu$*, then* $P\mu \ll P\nu$.

Proof. There exists $f \in L^1_+(\nu)$, such that $j_\nu(f) = \mu$. There are $f_n \in L^\infty_+(\mu)$ with $\|f_n - f\|_1 \to 0$. According to Lemma 4.1,

$$\|Pj_\nu(f_n) - Pj_\nu(f)\|_{\mathrm{TV}} \leq \|j_\nu(f_n) - j_\nu(f)\|_{\mathrm{TV}} = \|f_n - f\|_1 \to 0.$$

Furthermore, $0 \leq j_\nu(f_n) \leq \|f_n\|_\infty \nu$. Hence by positivity of P, $0 \leq Pj_\nu(f_n) \leq \|f_n\|_\infty P\nu$. Therefore $Pj_\nu(f_n) \ll P\nu$, hence $Pj_\nu(f_n) \in L^1_+(P\nu)$ for all $n \in \mathbb{N}$. Because $L^1_+(P\nu)$ is closed in $\mathcal{M}^+(S)_{\mathrm{TV}}$, $Pj_\nu(f) \in L^1(P\nu)$ as well, thus $P\mu \ll P\nu$. □

Corollary 4.3. *P leaves* $j_\mu(L^1_+(\mu))$ *invariant if and only if* $P\mu \ll \mu$.

Proof. Clearly, if P leaves $j_\mu(L^1_+(\mu))$ invariant, then in particular $P\mu \ll \mu$. The proof in the opposite direction follows from Lemma 4.2: if $f \in L^1_+(\mu)$, then $0 \le j_\mu(f) \ll \mu$, hence $Pj_\mu(f) \ll P\mu \ll \mu$. $\square$

Crucial in our approach is the following general topological closed graph theorem (cf. [22], (14.1.2), p.313):

Proposition 4.4. *Let f map the topological space S into the topological space T. If f is closed and T is compact, then f is continuous.*

Moreover, we use the following characterisation of relatively weakly compact subsets of L^1 (e.g. [1], Theorem 5.2.9, p. 109):

Theorem 4.5 (Dunford-Pettis)**.** *Let (Ω, Σ, μ) be a σ-finite measure space. In addition let $\mathcal{F}$ be a bounded set in $L^1(\mu)$. Then the following conditions on $\mathcal{F}$ are equivalent:*

(i) $\mathcal{F}$ is relatively weakly compact.

(ii) For every sequence A_n of disjoint measurable sets

$$\lim_{n\to\infty} \sup_{f\in\mathcal{F}} \int_{A_n} |f|\, d\mu = 0.$$

The fundamental result of this section is:

Theorem 4.6. *Let $(P(t))_{t\ge 0}$ be a strongly continuous Markov semigroup. Let $\mu \in \mathcal{M}^+(S)$ be such that $j_\mu(L^1_+(\mu))$ is $(P(t))_{t\ge0}$-invariant. Then the following statements are equivalent:*

(i) The semigroup $(T(t))_{t\ge0}$ on $L^1(\mu)$ induced by $(P(t))_{t\ge0}$ is strongly continuous and positive, and consists of isometries on $L^1_+(\mu)$.

(ii) The map $t \mapsto P(t)\mu$ is continuous from $\mathbb{R}_+$ to $\mathcal{M}(S)_{\mathrm{TV}}$.

(iii) There exists $\tau > 0$ such that for any sequence A_n of disjoint Borel measurable subsets of S,

$$\lim_{n\to\infty} \sup_{0\le t\le\tau} P(t)\mu(A_n) = 0. \tag{3}$$

Proof. (i) ⇒(ii). From Lemma 4.1 it follows that for $s, t \in \mathbb{R}_+$

$$\|P(t)\mu - P(s)\mu\|_{\mathrm{TV}} = \|P(t)j_\mu(\mathbb{1}) - P(s)j_\mu(\mathbb{1})\|_{\mathrm{TV}} = \|T(t)\mathbb{1} - T(s)\mathbb{1}\|_1.$$

By assumption $t \mapsto T(t)\mathbb{1}$ is continuous from $\mathbb{R}_+$ to $L^1(\mu)$, hence $t \mapsto P(t)\mu$ is continuous from $\mathbb{R}_+$ to $\mathcal{M}(S)_{\mathrm{TV}}$.

(ii)⇒(iii). For all $s, t \in \mathbb{R}_+$ we know by Lemma 4.1 that

$$\|T(t)\mathbb{1} - T(s)\mathbb{1}\|_1 = \|P(t)\mu - P(s)\mu\|_{\mathrm{TV}}.$$

By assumption $t \mapsto P(t)\mu$ is continuous from $\mathbb{R}_+$ to $\mathcal{M}(S)_{\mathrm{TV}}$, so $t \mapsto T(t)\mathbb{1}$ is continuous from $\mathbb{R}_+$ to $L^1(\mu)$.

Let $\tau > 0$. By continuity the partial orbit $\{T(t)\mathbb{1} | 0 \leq t \leq \tau\}$ is norm compact, hence weakly compact. According to Theorem 4.5, for any sequence of disjoint measurable sets A_n,

$$0 = \lim_{n\to\infty} \sup_{0\leq t\leq\tau} \int_{A_n} |T(t)\mathbb{1}| d\mu = \lim_{n\to\infty} \sup_{0\leq t\leq\tau} P(t)\mu(A_n).$$

(iii)⇒(i). Lemma 4.1, Markov operator property *(MO2)* and (2) yield that each $T(t)$ is an isometry on $L^1_+(\mu)$ and $\|T(t)\| = 1$ for all $t \geq 0$. We write L^1_w to denote the space $L^1(\mu)$ with the weak topology and $(\mathcal{S}_{\mathrm{BL}})_w$ to denote $\mathcal{S}_{\mathrm{BL}}$ with the weak topology.

Step 1. For any step function f, $t \mapsto T(t)f : [0,\tau] \to L^1(\mu)$ has a closed graph in $[0,\tau] \times L^1_w$.

It suffices to prove the statement for $f = \mathbb{1}_E$, the indicator function of a measurable $E \subset S$. To that end, observe that the map $j_\mu : L^1(\mu) \to \mathcal{S}_{\mathrm{BL}}$ is continuous, hence continuous for the weak topologies in $L^1(\mu)$ and $\mathcal{S}_{\mathrm{BL}}$ ([4, Theorem VI.1.1]). The map

$$\psi_E : [0,\tau] \to \mathcal{S}_{\mathrm{BL}} : t \mapsto P(t)j_\mu(\mathbb{1}_E)$$

is norm continuous, hence continuous for the weak topology in $\mathcal{S}_{\mathrm{BL}}$. Thus its graph is closed in $[0,\tau] \times (\mathcal{S}_{\mathrm{BL}})_w$. We conclude that $t \mapsto T(t)\mathbb{1}_E$ must have a closed graph in $[0,\tau] \times L^1_w$.

Step 2. $j_\mu^{-1}(\psi_E([0,t]))$ is compact in L^1_w.

First of all, it is weakly closed by continuity of $j_\mu : L^1_w \to (\mathcal{S}_{\mathrm{BL}})_w$ and compactness of $\psi_E([0,t])$ in $(\mathcal{S}_{\mathrm{BL}})_w$. According to Theorem 4.5 it suffices to show that for any sequence of disjoint measurable subsets A_n of S,

$$\lim_{n\to\infty} \sup_{0\leq t\leq\tau} \int_{A_n} T(t)\mathbb{1}_E d\mu = 0.$$

We have $\mathbb{1}_E \leq \mathbb{1}$, thus $0 \leq T(t)\mathbb{1}_E \leq T(t)\mathbb{1}$ by positivity of $T(t)$. Therefore,

$$\int_{A_n} T(t)\mathbb{1}_E d\mu \leq \int_{A_n} T(t)\mathbb{1} d\mu = P(t)\mu(A_n).$$

Condition (3) now completes Step 2.

Step 3. There is a norm-dense subspace D of $L^1(\mu)$, such that $t \mapsto T(t)f$ is norm-continuous for $f \in D$.

Then $(T(t))_{t\geq 0}$ is strongly continuous on $L^1(\mu)$, because $\|T(t)\| = 1$ for all t (e.g. [11, Proposition I.5.3]). The proof of this step mimicks that of [11, Theorem I.5.8] ('a weakly continuous semigroup in a Banach space is strongly continuous'). According to Step 1, Step 2 and Proposition 4.4, $t \mapsto T(t)f : \mathbb{R}_+ \to L^1(\mu)$ is weakly continuous for any step function $f \geq 0$. By separability of S and Pettis' Theorem we conclude that for any step function f, $t \mapsto T(t)f$ is measurable in the sense of Bochner. It is integrable over $[0,\tau]$, because $\|T(t)f\|_1 \leq \|f\|_1$. Thus we can define as Bochner integral in $L^1(\mu)$:

$$f_r := \frac{1}{r}\int_0^r T(t)f dt, \qquad 0 < r \leq \tau.$$

Fix $r > 0$ and let $0 \leq s \leq r$. Then for any step function f,

$$\begin{aligned} \|T(s)f_r - f_r\|_1 &= \frac{1}{r} \left\| \int_s^{s+r} T(t)f dt - \int_0^r T(t)f dt \right\|_1 \\ &= \frac{1}{r} \left\| \int_r^{s+r} T(t)f dt - \int_0^s T(t)f dt \right\|_1 \leq \|f\|_1 \frac{2s}{r}. \end{aligned}$$

Thus $\|T(t)f_r - f_r\|_1 \to 0$ as $s \downarrow 0$. Because $t \mapsto T(t)f$ is weakly continuous, $f_{r_n} \to f$ weakly as $r_n \downarrow 0$. The step functions are norm dense, hence weakly dense in $L^1(\mu)$. Norm closure and weak closure agree on convex sets. Therefore

$$D := \text{span}\{f_r : f \text{ step function}, \ 0 < r \leq \tau\}$$

is a norm dense subspace of $L^1(\mu)$. □

Not every strongly continuous Markov semigroup which leaves $j_\mu(L^1_+(\mu))$ invariant for some $\mu \in \mathcal{M}^+(S)$ satisfies one of the equivalent conditions of Theorem 4.6, as the following example will show. Let m denote the Lebesgue measure on $\mathbb{R}^n$. The diffusion semigroup $(T_d(t))_{t\geq 0}$ on $L^1(m) = L^1(\mathbb{R}^n, m)$ is given by

$$T_d(t)f(x) := \int_{\mathbb{R}} h_d(x-y,t)f(y)dm(y), \text{ for } t > 0,$$

where the diffusion kernel h_d is given by

$$h_d(x,t) = (4\pi dt)^{-n/2}e^{-|x|^2/4dt}.$$

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, then one can show that $x \mapsto g_\mu(x) = \int_{\mathbb{R}} h_d(x-y,t)f(y)d\mu(y)$ is in $L^1(m)$, and hence defines a measure $g_\mu dm$. We can extend $T_d(t)$ to $\mathcal{M}(\mathbb{R}^n)$, by defining $T_d(t)\mu$ to be $g_\mu dm$. Then $T_d(t)$ is linear, leaves $\mathcal{M}^+(\mathbb{R}^n)$ invariant, and $\|T_d(t)\mu\|^*_{\text{BL}} \leq \|\mu\|^*_{\text{BL}}$ for all $\mu \in \mathcal{M}(\mathbb{R}^n)$, so $T_d(t)$ can be extended to a bounded linear operator on $\mathbb{R}^n_{\text{BL}}$. Moreover, $(T_d(t))_{t\geq 0}$ is strongly continuous there. Hence the restriction of $(T_d(t))_{t\geq 0}$ to $\mathcal{M}^+(\mathbb{R}^n)$ is a strongly continuous Markov semigroup. Note that $T_d(t)\mu \ll m$ for every $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $t > 0$. Now let $f \in L^1(\mathbb{R}^n)$ such that $f > 0$ almost everywhere, and set $\mu = f\,dm + \delta_0$. Then $T_d(t)(\mu) \ll \mu$ for all $t \geq 0$, so $(T_d(t))_{t\geq 0}$ leaves $L^1_+(\mu)$ invariant by Corollary 4.3. But $T_d(t)\mu \in L^1(m)$ for all $t > 0$, hence $t \mapsto T_d(t)\mu$ cannot be continuous from $\mathbb{R}_+$ to $\mathcal{M}(\mathbb{R}^n)_{\text{TV}}$, since $L^1(m)$ is closed in $\mathcal{M}(\mathbb{R}^n)_{\text{TV}}$ and $\mu \notin L^1(m)$, so condition *(ii)* (hence *(i)* and *(iii)*) of Theorem 4.6 is not satisfied.

5 Strong continuity for total variation norm

Let $(P(t))_{t\geq 0}$ be a strongly continuous Markov semigroup on S. It extends to a positive semigroup of bounded linear operators on $\mathcal{M}(S)_{\text{TV}}$ as we have seen. Typically the latter is not strongly continuous. In this section we will give several characterisations of the closed invariant subspace of $\mathcal{M}(S)_{\text{TV}}$ on which $(P(t))_{t\geq 0}$ is strongly continuous with respect to $\|\cdot\|_{\text{TV}}$, i.e. the space

$$\mathcal{M}(S)^0_{\text{TV}} := \{\mu \in \mathcal{M}(S) : t \mapsto P(t)\mu \text{ is continuous from } \mathbb{R}_+ \text{ to } \mathcal{M}(S)_{\text{TV}}\}.$$

Our approach is based on that of Gulick et al. [14]. There the following situation is considered: A locally compact group G acts as a group of homeomorphisms $(\Phi_g)_{g\in G}$ on a locally compact Hausdorff space X, sending $x \in X$ to $\Phi_g(x)$. This induces an action $(P(g))_{g\in G}$ on the Banach space of bounded Radon measures on X, $\mathcal{M}(X)$, endowed with total variation norm, given by $P(g)\mu(E) := \mu(\Phi_{g^{-1}}E)$. The subspace of $\mathcal{M}(X)$, consisting of measures μ such that $g \mapsto P(g)\mu$ is continuous from G to $\mathcal{M}(X)$ is then identified using convolution of certain functions on G with Radon measures on X, and this identification is used to provide several characterisations of this subspace (see also [20]).

Adopting this approach to our setting is not straightforward: Instead of a group G as in [14], we consider a semigroup $\mathbb{R}_+$, which implies that actions need not be invertible. Also, in [14] an action of the group on the underlying space X is considered, which induces an action on $\mathcal{M}(X)$. While we look, more generally, at actions of $\mathbb{R}_+$ on $\mathcal{M}(S)$ directly, that contain those coming from an underlying action on S by Proposition 3.3. Furthermore, in [14] X must be locally compact, since measures on X are defined there by constructing certain functionals on $C_0(X)$; in our setting S needs to be a separable complete metric space, but not necessarily locally compact. We can however overcome these difficulties by using the Banach space $\mathcal{S}_{\mathrm{BL}}$ and the theory of integrating functions with values in $\mathcal{S}_{\mathrm{BL}}^+ = \mathcal{M}^+(S)$ as developed in Section 2.3 and prove analogous characterisations of $\mathcal{M}(S)^0_{\mathrm{TV}}$ as those in [14] and [20].

These characterisations will help in identifying when the restriction of $(P(t))_{t\geq 0}$ to invariant L^1-spaces is strongly continuous.

Let A be a Banach algebra with multiplication $*$. A net (e_α) in A is an *approximate identity* of A, if $\lim_\alpha e_\alpha * f = f$ and $\lim_\alpha f * e_\alpha = f$ for all $f \in A$. It is a *bounded* approximate identity if the net is bounded. A Banach space M is a *Banach module* over A if there exists a bilinear map $* : A \times M \to M$ having the following properties:

(BM1) $(f * g) * m = f * (g * m)$ for all $f, g \in A, m \in M$.

(BM2) $\|f * m\|_M \leq \|f\|_A \|m\|_M$ for all $f \in A, m \in M$.

Proposition 5.1. *([14, Corollary 2.3]) Let A be a Banach algebra with bounded approximate identity (e_α). If M is a Banach module over A, then $A * M := \{a * m : a \in A, m \in M\}$ is a closed subspace of K. In particular, for $m \in M$, $m \in A * M$ if and only if $\lim_\alpha e_\alpha * m = m$.*

The latter characterisation of elements in $A*M$ makes clear that $A*M$ is indeed a vector subspace of M.

Proposition 5.2. *The Banach space $L^1(\mathbb{R}_+)$ is a commutative Banach algebra with multiplication defined by convolution:*

$$f * g(t) := \int_0^t f(t-s)g(s)\,ds,$$

with bounded approximate identity (e_n) given by $e_n = n\mathbb{1}_{[0,\frac{1}{n}]}$.

The proof is straightforward, observing that $L^1(\mathbb{R}_+)$ is canonically contained as closed subspace in the commutative Banach algebra $L^1(\mathbb{R})$ with convolution.

For a strongly continuous Markov semigroup $(P(t))_{t\geq 0}$, $t\mapsto P(t)\mu, \mathbb{R}_+\to\mathcal{S}_{\mathrm{BL}}$ is continuous for each $\mu\in\mathcal{M}(S)$ (though $P(t):\mathcal{M}(S)_{\mathrm{BL}}\to\mathcal{M}(S)_{\mathrm{BL}}$ need not be continuous) and

$$\|P(t)\mu\|_{\mathrm{BL}}^{*}\leq\|P(t)\mu\|_{\mathrm{TV}}\leq\|\mu\|_{\mathrm{TV}},$$

Thus $P(\cdot)\mu\in C_b(\mathbb{R}_+,\mathcal{S}_{\mathrm{BL}})$ and we can define for $f\in L^1(\mathbb{R}_+)$ and $\mu\in\mathcal{M}(S)$

$$f*_P\mu:=\int_{\mathbb{R}_+}f(s)P(s)\mu\,ds$$

as Bochner integral in $\mathcal{S}_{\mathrm{BL}}$. Clearly $(f,\mu)\mapsto f*_P\mu$ is a bilinear map from $L^1(\mathbb{R}_+)\times\mathcal{M}(S)$ to $\mathcal{S}_{\mathrm{BL}}$. Because $\mathcal{S}_{\mathrm{BL}}^{+}$ is closed and convex in $\mathcal{S}_{\mathrm{BL}}$, $f*_P\mu\in\mathcal{S}_{\mathrm{BL}}^{+}=\mathcal{M}^+(S)$, when $f\in L^1_+(\mathbb{R}_+)$ and $\mu\in\mathcal{M}^+(S)$. By writing $f\in L^1(\mathbb{R}_+)$ and $\mu\in\mathcal{M}(S)$ as difference of positive and negative parts $f^{\pm}$ and $\mu^{\pm}$ respectively, it follows that

$$f*_P\mu=f^+*_P\mu^+-f^-*_P\mu^+-f^+*_P\mu^-+f^-*_P\mu^-. \tag{4}$$

So $(f,\mu)\mapsto f*_P\mu$ is a bilinear map from $L^1(\mathbb{R}_+)\times\mathcal{M}(S)$ into $\mathcal{M}(S)$.

The right translation semigroup $(R_+(t))_{t\geq 0}$ on $L^1(\mathbb{R}_+)$ is given by:

$$R_+(t)f(s):=\begin{cases}f(s-t), & \text{if } s\geq t,\\ 0, & \text{if } 0\leq s\leq t.\end{cases}.$$

It is a strongly continuous positive semigroup on $L^1(\mathbb{R}_+)$.

Proposition 5.3. *The following holds for all $f\in L^1(\mathbb{R}_+),\mu\in\mathcal{M}(S)$:*

(i) Let $P(t):\mathcal{M}^+(S)_{\mathrm{BL}}\to\mathcal{M}^+(S)_{\mathrm{BL}}$ be continuous. Then

$$P(t)(f*_P\mu)=f*_P(P(t)\mu)=(R_+(t)f)*_P\mu,$$

(ii) $f*_P\mu(E)=\int_{\mathbb{R}_+}f(t)P(t)\mu(E)\,dt$ for all Borel sets E in S.

Proof. It suffices to prove *(i)* for $f\in L^1_+(\mathbb{R}_+)$ and $\mu\in\mathcal{M}^+(S)$. The general statement follows then from (4). For such f and μ, because $P(t):\mathcal{M}^+(S)_{\mathrm{BL}}\to\mathcal{M}^+(S)_{\mathrm{BL}}$ is continuous, Theorem A.1 implies that

$$P(t)(f*_P\mu)=P(t)\int_{\mathbb{R}_+}f(s)P(s)\mu\,ds=\int_{\mathbb{R}_+}f(s)P(t+s)\mu\,ds=f*_P(P(t)\mu).$$

The map $s\mapsto f(s)P(t+s)\mu$ is Bochner integrable from $\mathbb{R}_+$ to $\mathcal{S}_{\mathrm{BL}}$. Using the fact that Lebesgue measure on $\mathbb{R}$ is invariant under translation,

$$\int_{\mathbb{R}_+}f(s)P(t+s)\mu\,ds=\int_{\mathbb{R}_+}(R_+(t)f)(s)P(s)\mu\,ds=(R_+(t)f)*_P\mu.$$

The statement in *(ii)* follows from (4) and Corollary 2.7. □

From this point on we will implicitly assume that $P(t):\mathcal{M}^+(S)_{\mathrm{BL}}\to\mathcal{M}^+(S)_{\mathrm{BL}}$ is continuous for every $t>0$.

Proposition 5.4. *Let $f, g \in L^1(\mathbb{R}_+)$ and $\mu \in \mathcal{M}(S)$, then*

(i) $(f * g) *_P \mu = f *_P (g *_P \mu)$

(ii) $\|f *_P \mu\|_{\mathrm{TV}} \leq \|f\|_1 \|\mu\|_{\mathrm{TV}}$.

Proof. We first prove *(i)*. We use Fubini's Theorem for Bochner integration ([15, Theorem 3.7.13]) and Proposition 5.3:

$$\begin{aligned}
(f * g) *_P \mu &= \int_{\mathbb{R}_+} \left\{ \int_{\mathbb{R}_+} f(s) g(t-s)\, ds \right\} P(t)\mu \, dt \\
&= \int_{\mathbb{R}_+} f(s) \left\{ \int_{\mathbb{R}_+} g(t-s) P(t)\mu \, dt \right\} ds \\
&= \int_{\mathbb{R}_+} f(s) (R_+(s) g) *_P \mu \, ds \\
&= \int_{\mathbb{R}_+} f(s) P(s) (g *_P \mu)\, ds = f *_P (g *_P \mu).
\end{aligned}$$

For $f \in L^1_+(\mathbb{R}_+)$ and $\nu \in \mathcal{M}^+(S)$, $f *_P \nu \in \mathcal{M}^+(S)$ and

$$\begin{aligned}
\|f *_P \nu\|_{\mathrm{TV}} = \|f *_P \nu\|^*_{\mathrm{BL}} &\leq \int_{\mathbb{R}_+} \|f(t) P(t)\nu\|^*_{\mathrm{BL}} \, dt \\
&= \int_{\mathbb{R}_+} f(t) \|P(t)\nu\|_{\mathrm{TV}} \, dt = \|\nu\|_{\mathrm{TV}} \|f\|_1
\end{aligned}$$

by using property *(MO2)*. For general $f \in L^1(\mathbb{R}_+)$ and $\mu \in \mathcal{M}(S)$ we then obtain

$$\begin{aligned}
\|f * \mu\|_{\mathrm{TV}} &\leq (\|f^+\|_1 + \|f^-\|_1)\|\mu^+\|_{\mathrm{TV}} + (\|f^+\|_1 + \|f^-\|_1) * \|\mu^-\|_{\mathrm{TV}} \\
&= \|f\|_1 (\|\mu^+\|_{\mathrm{TV}} + \|\mu^-\|_{\mathrm{TV}}) = \|f\|_1 \|\mu\|_{\mathrm{TV}},
\end{aligned}$$

by using (4) and the fact that $\mathcal{M}(S)$ and $L^1(\mathbb{R}_+)$ are L-spaces. □

Put $L^1(\mathbb{R}_+) *_P \mathcal{M}(S) := \{f * \mu : f \in L^1(\mathbb{R}_+), \mu \in \mathcal{M}(S)\}$.

Then we have, by Proposition 5.2, Proposition 5.4 and Proposition 5.1, the following result:

Corollary 5.5. *$\mathcal{M}(S)_{\mathrm{TV}}$ is a Banach module over $L^1(\mathbb{R}_+)$. In particular, $L^1(\mathbb{R}_+) *_P \mathcal{M}(S)$ is a non-trivial closed subspace of $\mathcal{M}(S)_{\mathrm{TV}}$.*

This closed subspace equals the subspace of strong continuity of $P(t)$ with respect to $\|\cdot\|_{\mathrm{TV}}$:

Theorem 5.6. *For $\mu \in \mathcal{M}(S)$ the following are equivalent:*

(i) $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$*, i.e.* $t \mapsto P(t)\mu : \mathbb{R}_+ \to \mathcal{M}(S)_{\mathrm{TV}}$ *is continuous.*

(ii) $\mu \in L^1(\mathbb{R}_+) *_P \mathcal{M}(S)$.

(iii) If E is a Borel set in S such that $P(t)\mu(E) = 0$ for almost every $t \in [0, \infty)$, then $\mu(E) = 0$.

(iv) There exists $\nu \in \mathcal{M}^+(S)^0_{\mathrm{TV}}$ *such that* $j_\nu(L^1(\nu))$ *is* $(P(t))_{t\geq 0}$*-invariant and* $\mu \in j_\nu(L^1(\nu))$*.*

Proof. *(i)⇒ (ii)*: Let $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$. By Proposition 5.1 it is sufficient to show that $e_n * \mu \to \mu$. Let $\epsilon > 0$. Since $t \mapsto P(t)\mu : \mathbb{R}_+ \to \mathcal{M}(S)_{\mathrm{TV}}$ is continuous, there exists an $N \in \mathbb{N}$, such that $\|P(t)\mu - \mu\|_{\mathrm{TV}} \leq \epsilon$ for all $t \in [0, \frac{1}{N}]$. For $n \in \mathbb{N}$

$$e_n * \mu - \mu = n \int_0^{\frac{1}{n}} P(t)\mu - \mu \, dt$$

is defined as Bochner integral in $\mathcal{S}_{\mathrm{BL}}$.

By continuity, $t \mapsto P(t)\mu - \mu : [0, \frac{1}{n}] \to \mathcal{M}(S)_{\mathrm{TV}}$ is strongly measurable and bounded, hence Bochner integrable, so we can also view the integral $n \int_0^{\frac{1}{n}} P(t)\mu - \mu \, ds$ as a Bochner integral in $\mathcal{M}(S)_{\mathrm{TV}}$. Since $\mathcal{M}(S)_{\mathrm{TV}}$ embeds continuously in $\mathcal{S}_{\mathrm{BL}}$, the two integrals are the same.

Moreover,

$$\|e_n *_P \mu - \mu\|_{\mathrm{TV}} \leq n \int_0^{\frac{1}{n}} \|P(t)\mu - \mu\|_{\mathrm{TV}} \, dt \leq \epsilon,$$

for all $n \geq N$.

(ii)⇒ (i): Let $\mu = f *_P \nu \in L^1(\mathbb{R}_+) *_P \mathcal{M}(S)$. Let $t, s \geq 0$. According to Proposition 5.3 and Proposition 5.4,

$$\begin{aligned} \|P(t)\mu - \mu\|_{\mathrm{TV}} &= \|(R_+(t)f) *_P \nu - f *_P \nu\|_{\mathrm{TV}} \\ &\leq \|R_+(t)f - f\|_1 \|\nu\|_{\mathrm{TV}}. \end{aligned}$$

Since $(R_+(t))_{t\geq 0}$ is strongly continuous on $L^1(\mathbb{R}_+)$, $\|R_+(t)f - f\|_1 \to 0$ as $t \downarrow 0$ and thus $\|P(t)\mu - \mu\|_{\mathrm{TV}} \to 0$. So $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$.

Thus from now on we can identify $\mathcal{M}(S)^0_{\mathrm{TV}}$ with $L^1(\mathbb{R}_+) *_P \mathcal{M}(S)$.

(i)⇒ (iii): Let $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ and let E be a Borel set in S. Then $t \mapsto P(t)\mu(E)$ is continuous, hence if $P(t)\mu(E) = 0$ for almost every $t \in [0, \infty)$, then $\mu(E) = 0$.

(iii)⇒ (iv) Let $f \in L^1(\mathbb{R}_+)$, such that $f(t) > 0$ for almost every $t \in [0, \infty)$. Define $\nu = f *_P |\mu|$. Suppose $\nu(E) = 0$ for a Borel set E in S, then $P(t)|\mu|(E) = 0$ for almost every $t \in [0, \infty)$. By positivity of $P(t)$, $|P(t)\mu|(E) \leq P(t)|\mu|(E) = 0$ for almost every $t \in [0, \infty)$, hence $\mu(E) = 0$ and $\mu \ll \nu$. Furthermore, by Proposition 5.3,

$$P(t)\nu(E) = \int_{\mathbb{R}_+} f(s)P(t+s)|\mu|(E) \, ds = 0,$$

hence $P(t)\nu \ll \nu$. According to Corollary 4.3, $(P(t))_{t\geq 0}$ leaves $j_\nu(L^1(\nu))$ invariant, and $\mu \in j_\nu(L^1(\nu))$.

(iv)⇒(i): Since $\nu \in L^1(\mathbb{R}_+) *_P \mathcal{M}(S) = \mathcal{M}(S)^0_{\mathrm{TV}}$, $t \mapsto P(t)\nu : \mathbb{R}_+ \to \mathcal{M}(S)_{\mathrm{TV}}$ is continuous. Then Theorem 4.6 implies that the semigroup $(T(t))_{t\geq 0}$ in $L^1(\nu)$ induced by $(P(t))_{t\geq 0}$ is strongly continuous. By assumption there is an $f \in L^1(\nu)$ such that $j_\nu(f) = \mu$. Then

$$\|P(t)\mu - \mu\|_{\mathrm{TV}} = \|T(t)f - f\|_1 \to 0,$$

as $t \downarrow 0$. □

The following theorem gives some useful conditions for showing that $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$.

Theorem 5.7. *Let $\mu \in \mathcal{M}(S)$. Then the following are equivalent:*

(i) $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$.

(ii) For all compact K in S, $t \mapsto P(t)\mu(K)$ is continuous.

(iii) If K in S compact and $P(t)\mu(K) = 0$ for almost every $t \in [0,\infty)$, then $\mu(K) = 0$.

(iv) There is a $\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ such that $\mu \ll \nu$.

Proof. *(i)⇒ (ii)*: Since $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$, $t \mapsto P(t)\mu(E)$ is continuous for all Borel sets E in S.

(ii)⇒ (iii): Let K in S be compact, such that $P(t)\mu(K) = 0$ for almost every $t \in [0,\infty)$. Then, by continuity of $t \mapsto P(t)\mu(K)$, $\mu(K) = 0$.

(iii)⇒ (iv): Let $f \in L^1(\mathbb{R}_+)$, such that $f(t) > 0$ for almost every $t \in [0,\infty)$. Define $\nu := f *_P |\mu|$. Let K in S be compact, such that $\nu(K) = 0$, then $P(t)|\mu|(K) = 0$ for almost every $t \in [0,\infty)$. By positivity of $P(t)$, $|P(t)\mu|(K) \leq P(t)|\mu|(K) = 0$ for almost every $t \in [0,\infty)$, hence $\mu(K) = 0$. Thus $\mu \ll \nu$ by Lemma 2.1.

(iv)⇒ (i): Let $f \in L^1(\mathbb{R}_+)$, such that $f(t) > 0$ for almost every $t \in [0,\infty)$. Define $\rho := f *_P |\nu| \in \mathcal{M}^+(S) \cap L^1(\mathbb{R}_+) *_P \mathcal{M}(S)$. Now, let E be a Borel set in S such that $\rho(E) = 0$. Then $P(t)|\nu|(E) = 0$ for almost every $t \in [0,\infty)$. Note that $|\nu|(E) = 0$ if and only if $\nu(F) = 0$ for all Borel sets $F \subset E$. Let $F \subset E$ be Borel, then by positivity of $P(t)$,

$$|P(t)\nu(F)| \leq P(t)|\nu|(F) = 0,$$

for almost every $t \in [0,\infty)$. Since $\nu \in L^1(\mathbb{R}_+) *_P \mathcal{M}(S)$, $t \mapsto P(t)\nu(F)$ is continuous, so $\nu(F) = 0$. So $\mu \ll \nu \ll \rho$.

Also, by Proposition 5.3, we have for every $s \geq 0$

$$P(s)\rho(E) = \int_{\mathbb{R}_+} f(t)P(t+s)|\nu|(E)\,dt = 0,$$

since $P(t)|\nu|(E) = 0$ for every $t \geq 0$. So $P(t)\rho \ll \rho$ for all $t \geq 0$, and $\mu \ll \rho$. By Corollary 4.3 $(P(t)0_{t\geq 0}$ leaves $j_\rho(L^1(\rho))$ invariant, and $\mu \in j_\rho(L^1(\rho))$. Now we can apply Theorem 5.6. □

Corollary 5.8. *Let $\mu \in \mathcal{M}^+(S)$. If there is a $\tau > 0$ such that $\mu \ll P(t)\mu$ for all $t \in [0,\tau]$, then $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$.*

Proof. Let $E \subset S$ be measurable such that $P(t)\mu(E) = 0$ for almost every $t \in [0,\infty)$. Then there is a $t \in [0,\tau]$ such that $P(t)\mu(E) = 0$, and then $\mu(E) = 0$, since $\mu \ll P(t)\mu$. Hence $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ by Theorem 5.6. □

An important consequence of the characterisations in Theorem 5.6 is:

Proposition 5.9. $\mathcal{M}(S)^0_{\mathrm{TV}}$ *is dense in* $\mathcal{M}(S)_{\mathrm{BL}}$*, hence in* $\mathcal{S}_{\mathrm{BL}}$.

Proof. Let $\mu \in \mathcal{M}(S)$ and $\epsilon > 0$. Then there is a $\tau > 0$ such that $\|P(t)\mu - \mu\|_{\mathrm{BL}}^* < \epsilon$ for all $t \in [0, \tau]$. By Theorem 5.6 $e_n *_P \mu \in \mathcal{M}(S)_{\mathrm{TV}}^0$.

$$\begin{aligned}\|e_n *_P \mu - \mu\|_{\mathrm{BL}}^* &= n\|\int_0^{\frac{1}{n}} P(t)\mu - \mu\, dm\|_{\mathrm{BL}}^* \\ &\leq n\int_0^{\frac{1}{n}} \|P(t)\mu - \mu\|_{\mathrm{BL}}^*\, dm < \epsilon,\end{aligned}$$

for all $t \in [0, \tau]$. So $\|e_n *_P \mu - \mu\|_{\mathrm{BL}}^* \to 0$, and $\mathcal{M}(S)_{\mathrm{TV}}^0$ is dense in $\mathcal{M}(S)_{\mathrm{BL}}$. □

However, whenever the Markov semigroup arises from an underlying semigroup of continuous maps on S, $\mathcal{M}(S)_{\mathrm{TV}}^0$ cannot be too large:

Proposition 5.10. *Let $(\Phi_t)_{t\geq 0}$ be a strongly continuous semigroup of continuous maps on S, and let $(P(t))_{t\geq 0}$ be the associated strongly continuous Markov semigroup. Then $\mathcal{M}(S)_{\mathrm{TV}}^0 = \mathcal{M}(S)$ if and only if $\Phi_t = \mathrm{Id}$ for every $t \in \mathbb{R}_+$.*

Proof. Suppose $\Phi_t = \mathrm{Id}$ for every $t \in \mathbb{R}_+$. Then $P(t)\mu = \mu$ for every $t \in \mathbb{R}_+$ and $\mu \in \mathcal{M}(S)$, hence $\mathcal{M}(S)_{\mathrm{TV}}^0 = \mathcal{M}(S)$. Suppose $\mathcal{M}(S) = \mathcal{M}(S)_{\mathrm{TV}}^0$, and let $x \in S$. Then

$$\|\delta_{\Phi(t)x} - \delta_x\|_{\mathrm{TV}} = \|P(t)\delta_x - \delta_x\|_{\mathrm{TV}} \downarrow 0,$$

as $t \downarrow 0$. Hence there is a $\tau > 0$ such that $\delta_{\Phi_t(x)} = \delta_x$ for all $t \in [0, \tau)$, and then by continuation $\delta_{\Phi_t(x)} = \delta_x$ for all $t \in \mathbb{R}_+$, so $\Phi_t(x) = x$ for all $t \in \mathbb{R}_+$. □

However, there do exist non-trivial strongly continuous Markov semigroups $(P(t))_{t\geq 0}$ such that $\mathcal{M}(S)_{\mathrm{TV}}^0 = \mathcal{M}(S)$; in [29, Section 5] a C_0-semigroup on $\mathcal{M}(\Omega)_{\mathrm{TV}}$, with (Ω, Σ) a general measurable space, is constructed, which under certain conditions is a Markov semigroup.

6 Decomposition of the space of measures

6.1 Absolute continuous and singular measures

For $\mu \in \mathcal{M}(\mathbb{R})$, define $\mu_t(E) := \mu(E - t)$, $t \in \mathbb{R}$. It is a classical result by Plessner [26] that $\|\mu_t - \mu\|_{\mathrm{TV}} \to 0$ as $t \to 0$ if and only if μ is absolutely continuous with respect to the Lebesgue measure m. Then the Lebesgue-Radon-Nikodym Decomposition Theorem implies that every μ in $\mathcal{M}(\mathbb{R})$ can be uniquely decomposed into $\mu_a + \mu_s$, where $\mu_a \in L^1(\mathbb{R}, m)$, and μ_s is singular with respect to m.

We can translate this to our setting: let $\Phi_t(x) = x + t$, then $(\Phi_t)_{t\in\mathbb{R}}$ defines a strongly continuous group of continuous mappings $\Phi_t : \mathbb{R} \to \mathbb{R}$. This defines a strongly continuous Markov group $(P_\Phi(t))_{t\in\mathbb{R}}$, by $P_\Phi(t)\mu = \mu \circ \Phi_t^{-1}$, by Proposition 3.3. Note that we only formulated Proposition 3.3 for semigroups, but it can easily adapted for groups. Plessner's result implies that the subspace of strong continuity $\mathcal{M}(\mathbb{R})_{\mathrm{TV}}^0$ equals $L^1(\mathbb{R})$, and every $\mu \in \mathcal{M}(\mathbb{R})$ can be uniquely decomposed into $\mu_a + \mu_s$, where $\mu_a \in \mathcal{M}(\mathbb{R})_{\mathrm{TV}}^0$ and μ_s is singular with respect to every $\nu \in \mathcal{M}(\mathbb{R})_{\mathrm{TV}}^0$. We will generalise this decomposition in our setting.

As in the previous section we assume $(P(t))_{t\geq 0}$ is a strongly continuous Markov semigroup on S, such that $P(t) : \mathcal{M}^+(S)_{\mathrm{BL}} \to \mathcal{M}^+(S)_{\mathrm{BL}}$ is continuous for all $t > 0$.

Proposition 6.1. *$\mathcal{M}(S)^0_{\mathrm{TV}}$ is a projection band in $\mathcal{M}(S)_{\mathrm{TV}}$.*

Proof. We first show that $\mathcal{M}(S)^0_{\mathrm{TV}}$ is an ideal. Let $\mu, \nu \in \mathcal{M}(S)$ such that $0 \leq |\mu| \leq |\nu|$ and $\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}$. Then $|\nu| \in \mathcal{M}(S)^0_{\mathrm{TV}}$ by Theorem 5.7. Since $\mu \ll |\nu|$, $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$, again by Theorem 5.7. Hence $\mathcal{M}(S)^0_{\mathrm{TV}}$ is a closed ideal in $\mathcal{M}(S)_{\mathrm{TV}}$, hence a projection band by Theorem 2.2. □

So we can write

$$\mathcal{M}(S) = \mathcal{M}(S)^0_{\mathrm{TV}} \oplus (\mathcal{M}(S)^0_{\mathrm{TV}})^{\perp}, \tag{5}$$

by Theorem [21, Theorem 1.2.9].

We will show that $(\mathcal{M}(S)^0_{TV})^{\perp} = \mathcal{M}(S)^s_{\mathrm{TV}}$, where

$$\mathcal{M}(S)^s_{\mathrm{TV}} := \{\mu \in \mathcal{M}(S) : \mu^+ \perp P(t)\mu^+, \mu^- \perp P(t)\mu^- \text{ for almost every } t \geq 0\}.$$

Our approach is based on that by Liu and Van Rooij [20].

Proposition 6.2. *Let $\mu \in \mathcal{M}(S)$. Then the following are equivalent:*

(i) $\mu \in \mathcal{M}(S)^s_{\mathrm{TV}}$.

(ii) $\mu \perp \nu$ *for every* $\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}$

(iii) For all $\nu \in \mathcal{M}(S)$, $\mu \perp P(t)\nu$ *for almost every* $t \in [0, \infty)$.

Proof. (i)⇒ (ii): Let $\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}$, then $|\nu| \in \mathcal{M}(S)^0_{\mathrm{TV}}$ by Theorem 5.7. By the Lebesgue-Radon-Nikodym Theorem, there are unique $\mu^+_a, \mu^+_s \in \mathcal{M}(S)^+$, such that $\mu^+ = \mu^+_a + \mu^+_s$, $\mu^+_a \ll |\nu|$ and $\mu^+_s \perp |\nu|$. Then $\mu^+_a \in \mathcal{M}(S)^0_{\mathrm{TV}}$ by Theorem 5.7. By assumption, $\mu^+ \perp P(t)\mu^+$ for almost every $t \in [0, \infty)$. Suppose $\mu^+ \perp P(t)\mu^+$, then there is a Borel set U, such that $\mu^+(E) = \mu^+(E \cap U)$ and $P(t)\mu^+(U) = 0$ for all Borel sets E. So

$$0 \leq \mu^+_a(E \backslash U) \leq \mu^+(E \backslash U) = 0,$$

hence $\mu^+_a(E) = \mu^+_a(E \cap U)$ for all Borel sets E, and

$$0 \leq P(t)\mu^+_a(U) \leq P(t)\mu^+(U) = 0,$$

so $P(t)\mu^+_a \perp \mu^+_a$.

Hence $\mu^+_a \perp P(t)\mu^+_a$ for almost every $t \in [0, \infty)$. $\{\mu^+_a\}^{\perp}$ is a band in $\mathcal{M}(S)_{\mathrm{TV}}$, hence closed. Since $t \mapsto P(t)\mu^+_a : \mathbb{R}_+ \to \mathcal{M}(S)_{\mathrm{TV}}$ is continuous, $\mu^+_a \in \{\mu^+_a\}^{\perp}$, hence $\mu^+_a = 0$. This implies that $\mu^+ = \mu^+_s$, so $\mu^+ \perp |\nu|$, and therefore $\mu^+ \perp \nu$.

In a similar way we can prove that $\mu^- \perp \nu$, hence $\mu \perp \nu$.

(ii)⇒ (iii): Let $\nu \in \mathcal{M}(S)$ and define $\rho := f *_P |\nu| \in L^1(\mathbb{R}_+) *_P \mathcal{M}(S)$, where $f \in L^1(\mathbb{R}_+)$, such that $f(t) > 0$ for almost every $t \in [0, \infty)$. Then $\rho \in \mathcal{M}(S)^0_{\mathrm{TV}}$ by Theorem 5.6. Then $\mu \perp \rho$, hence there is a Borel set $U \subset S$, such that $\mu(E) = \mu(E \cap U)$ and $\rho(U) = 0$ for all Borel sets E in S. Thus $P(t)|\nu|(U) = 0$ for almost every $t \in [0, \infty)$. Then positivity of $(P(t))_{t\geq 0}$ implies that for almost

every $t \in [0, \infty)$, $|P(t)\nu|(U) = 0$, hence $|P(t)\nu| \perp \mu$. So $P(t)\nu \perp \mu$ for almost every $t \in [0, \infty)$.

(iii)$\Rightarrow$ (i): By assumption, $\mu \perp P(t)\mu^+$ and $\mu \perp P(t)\mu^-$ for almost every $t \in [0, \infty)$. Hence $|\mu| \perp P(t)\mu^+$ and $|\mu| \perp P(t)\mu^-$, so $\mu^+ \perp P(t)\mu^+$ and $\mu^- \perp P(t)\mu^-$ for almost every $t \in [0, \infty)$. □

Corollary 6.3. $\mathcal{M}(S)^s_{\mathrm{TV}} = (\mathcal{M}(S)^0_{\mathrm{TV}})^{\perp}$.

This implies that $\mathcal{M}(S)^s_{\mathrm{TV}}$ is a projection band by [21, Proposition 1.2.7].

As in [20] we call $\mu \in \mathcal{M}(S)$ *absolutely continuous* with respect to $(P(t))_{t\geq 0}$ if $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ and *singular* with respect to $(P(t))_{t\geq 0}$ if $\mu \in \mathcal{M}(S)^s_{\mathrm{TV}}$. This terminology is based on the fact that $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ if and only if there is a $\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ such that $\mu \ll |\nu|$ by Theorem 5.7, and $\mu \in \mathcal{M}(S)^s_{\mathrm{TV}}$ if and only if μ and ν are singular for every $\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ by Theorem 5.6.

An immediate consequence of (5) and Corollary 6.3 is the following:

Proposition 6.4. *Every $\mu \in \mathcal{M}(S)$ has a unique decomposition $\mu = \mu_a + \mu_s$, with $\mu_a \in \mathcal{M}(S)^0_{\mathrm{TV}}$, and $\mu_s \in \mathcal{M}(S)^s_{\mathrm{TV}}$.*

We denote the band projections on $\mathcal{M}(S)^0_{\mathrm{TV}}$ and $\mathcal{M}(S)^s_{\mathrm{TV}}$ by P_0 and P_s respectively. Then P_0, P_s are positive bounded linear operators on $\mathcal{M}(S)_{\mathrm{TV}}$, with $\|P_0\| \leq 1$ and $\|P_s\| \leq 1$, and $P_0\mu = \mu_a$, $P_s\mu = \mu_s$.

While $\mathcal{M}(S)^0_{\mathrm{TV}}$ is invariant under $(P(t))_{t\geq 0}$, $\mathcal{M}(S)^s_{\mathrm{TV}}$ need not be, as the following example shows: Let $S = \mathbb{R}_+$ with euclidean metric. Define $\Phi_t(x) = \max(x - t, 0)$, for $t, x \in \mathbb{R}_+$. Then $(\Phi_t)_{t\geq 0}$ is a strongly continuous semigroup of continuous maps on S, hence it defines, by Proposition 3.3, a strongly continuous Markov semigroup $(P(t))_{t\geq 0}$ given by $P(t)\mu := \mu \circ \Phi_t^{-1}$. Let $x > 0$, then clearly $\delta_x \perp P(t)\delta_x$ for all $t > 0$, hence $\delta_x \in \mathcal{M}(S)^s_{\mathrm{TV}}$. However, for $t \geq x$, $P(t)\delta_x = \delta_0$, and δ_0 is in $\mathcal{M}(S)^0_{\mathrm{TV}}$, and not in $\mathcal{M}(S)^s_{\mathrm{TV}}$, since $P(t)\delta_0 = \delta_0$ for all $t \in \mathbb{R}_+$.

For each $\mu \in \mathcal{M}(S)$, we can define $d(\mu, \mathcal{M}(S)^0_{\mathrm{TV}})$ to be the distance of μ to $\mathcal{M}(S)^0_{\mathrm{TV}}$ with respect to $\|\cdot\|_{\mathrm{TV}}$. Clearly, $\mu \in \mathcal{M}(S)^0_{\mathrm{TV}}$ if and only if $d(\mu, \mathcal{M}(S)^0_{\mathrm{TV}}) = 0$.

Lemma 6.5. *Let $\mu \in \mathcal{M}(S)$. Then $d(\mu, \mathcal{M}(S)^0_{\mathrm{TV}}) = \|\mu_s\|_{\mathrm{TV}}$.*

Proof. '$\leq$': $\mu = \mu_a + \mu_s$, so $\|\mu - \mu_a\|_{\mathrm{TV}} = \|\mu_s\|_{\mathrm{TV}}$. Hence

$$d(\mu, \mathcal{M}(S)^0_{\mathrm{TV}}) = \inf_{\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}} \|\mu - \nu\|_{\mathrm{TV}} \leq \|\mu - \mu_a\|_{\mathrm{TV}} = \|\mu_s\|_{\mathrm{TV}}.$$

'$\geq$': Let $\nu \in \mathcal{M}(S)^0_{\mathrm{TV}}$. Then

$$\|\mu_s\|_{\mathrm{TV}} = \|P_s\mu\|_{\mathrm{TV}} = \|P_s\mu - P_s\nu\|_{\mathrm{TV}} \leq \|\mu - \nu\|_{\mathrm{TV}},$$

which implies that $\|\mu_s\|_{\mathrm{TV}} \leq d(\mu, \mathcal{M}(S)^0_{\mathrm{TV}})$. □

Lemma 6.6. *Let $\mu \in \mathcal{M}(S)$. The function $t \mapsto \|P_sP(t)\mu\|_{\mathrm{TV}}$ is non-increasing.*

Proof. It suffices to show that $\|P_sP(t)\mu\|_{\mathrm{TV}} \le \|P_s\mu\|_{\mathrm{TV}}$ for all $t \in \mathbb{R}_+$.

Let $0 \le t$. First assume $\mu \in \mathcal{M}^+(S)$, then $0 \le \mu_a \le \mu$. Since $\mathcal{M}(S)^0_{\mathrm{TV}}$ is invariant under $P(t)$, $P_0P(t)\mu_a = P(t)\mu_a$, hence

$$0 \le P(t)\mu_a = P_0P(t)\mu_a \le P_0P(t)\mu.$$

Then

$$0 \le P_sP(t)\mu = P(t)\mu - P_0P(t)\mu \le P(t)\mu - P(t)\mu_a = P(t)\mu_s,$$

hence

$$\|P_sP(t)\mu\|_{\mathrm{TV}} \le \|P(t)\mu_s\|_{\mathrm{TV}} \le \|\mu_s\|_{\mathrm{TV}}. \tag{6}$$

Now let $\mu = \mu^+ - \mu^- \in \mathcal{M}(S)$. Then $P_s\mu^+ \perp P_s\mu^-$, which implies that $\|P_s\mu\|_{\mathrm{TV}} = \|P_s\mu^+\|_{\mathrm{TV}} + \|P_s\mu^-\|_{\mathrm{TV}}$. By (6)

$$\begin{aligned}\|P_sP(t)\mu\|_{\mathrm{TV}} &\le \|P_sP(t)\mu^+\|_{\mathrm{TV}} + \|P_sP(t)\mu^-\|_{\mathrm{TV}}\\ &\le \|P_s\mu^+\|_{\mathrm{TV}} + \|P_s\mu^-\|_{\mathrm{TV}} = \|P_s\mu\|_{\mathrm{TV}}.\end{aligned}$$

□

6.2 A Wiener-Young type theorem

Wiener and Young ([30]) extended the result by Plessner by showing that for all $\mu \in \mathcal{M}(\mathbb{R})$, $\limsup_{t\to 0} \|\mu_t - \mu\|_{\mathrm{TV}} = 2\|\mu_s\|_{\mathrm{TV}}$, where μ_s is the singular component of μ with respect to the Lebesgue measure.

We can generalise this result to the Markov semigroups with conditions as before. It has been generalised in several other directions: see for instance [23, 24] for a generalisation in the setting of dual semigroups of positive C_0-semigroups on Banach lattices.

Theorem 6.7. *Let $\mu \in \mathcal{M}(S)$. Then* $\limsup_{t\downarrow 0} \|P(t)\mu - \mu\|_{\mathrm{TV}} = 2\|\mu_s\|_{\mathrm{TV}}$.

Proof. Step 1. $\limsup_{t\downarrow 0} \|P(t)\mu - \mu\|_{\mathrm{TV}} = \limsup_{t\downarrow 0} \|P(t)\mu_s\|_{\mathrm{TV}} + \|\mu_s\|_{\mathrm{TV}}$ *for all $\mu \in \mathcal{M}(S)$.*
Clearly $\|P(t)\mu_a - \mu_a\|_{\mathrm{TV}} \to 0$. This implies that $\limsup_{t\downarrow 0} \|P(t)\mu - \mu\|_{\mathrm{TV}} = \limsup_{t\downarrow 0} \|P(t)\mu_s - \mu_s\|_{\mathrm{TV}}$. By Proposition 6.2, $P(t)\mu_s \perp \mu_s$ for almost every $t \in [0,\infty)$. Hence for these t $\|P(t)\mu_s - \mu_s\|_{\mathrm{TV}} = \|P(t)\mu_s\|_{\mathrm{TV}} + \|\mu_s\|_{\mathrm{TV}}$, hence

$$\limsup_{t\downarrow 0} \|P(t)\mu_s - \mu_s\|_{\mathrm{TV}} \ge \limsup_{t\downarrow 0} \|P(t)\mu\|_{\mathrm{TV}} + \|\mu_s\|_{\mathrm{TV}}.$$

Furthermore, $\|P(t)\mu_s - \mu_s\|_{\mathrm{TV}} \le \|P(t)\mu_s\|_{\mathrm{TV}} + \|\mu_s\|_{\mathrm{TV}}$ for all $t \in [0,\infty)$, hence the statement holds.
Step 2. $\limsup_{t\downarrow 0} \|P(t)\mu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{TV}}$ *for all $\mu \in \mathcal{M}(S)$.*
Let $\epsilon > 0$. Since

$$\|\mu\|_{\mathrm{TV}} = \sup\{|\int_S f\,d\mu| : f \in C_b(S), \|f\|_\infty \le 1\},$$

there is an $f \in C_b(S)$ with $\|f\|_\infty \leq 1$ and $|\|\mu\|_{\mathrm{TV}} - \int_S f\, d\mu| < \frac{\epsilon}{2}$. By strong continuity of $(P(t))_{t\geq 0}$ and Lemma 3.2 there exists a $\tau > 0$, such that

$$|\langle P(t)\mu, f\rangle - \langle \mu, f\rangle| < \frac{\epsilon}{2} \text{ for all } t \in [0,\tau).$$

Thus for $t \in [0, \tau)$ we can conclude

$$\|P(t)\mu\|_{\mathrm{TV}} \geq |\langle P(t)\mu, f\rangle| \geq \|\mu\|_{\mathrm{TV}} - \epsilon,$$

which implies that $\limsup_{t\downarrow 0} \|P(t)\mu\|_{\mathrm{TV}} \geq \|\mu\|_{\mathrm{TV}}$. By *(MO2)*

$$\limsup_{t\downarrow 0} \|P(t)\mu\|_{\mathrm{TV}} \leq \|\mu\|_{\mathrm{TV}},$$

which completes the proof. □

A Auxiliary results from measure theory

Let (Ω, Σ, μ) be a measure space and X a Banach space. Let f be an X-valued function on Ω. It is called a simple function if it is constant on each of a finite number of pairwise disjoint measurable sets E_n, such that $\bigcup_n E_n = \Omega$ and $\mu\big(\{\omega : \|f(\omega)\| > 0\}\big) < \infty$. f is weakly measurable if for each $x^* \in X^*$, $\Omega \to \mathbb{R} : \omega \mapsto x^*(f(\omega))$ is measurable. It is strongly μ-measurable if there exists a sequence of simple functions that converges pointwise in X, μ-almost everywhere. f is called essentially separably valued if there exists a μ-null set N, such that $f(\Omega \setminus N)$ is separable. Weak and strong measurability are related by Pettis' Theorem (e.g. [8, 15, 25]).

Theorem A.1 (Pettis' Measurability Theorem)**.** *Let (Ω, Σ, μ) be a σ-finite measure space, X a Banach space and f an X-valued function on Ω. The following are equivalent:*

(i) f is strongly μ-measurable,

(ii) f is weakly μ-measurable and essentially separably valued,

(iii) There exists μ-null sets N and N', $N \subset N'$, and a sequence of simple functions $f_n : \Omega \to X$, such that $f_n(\Omega) \subset f(\Omega\backslash N)\cup\{0\}$ and $f_n(\omega) \to f(\omega)$ as $n \to \infty$ for all $\omega \in \Omega \setminus N'$. Furthermore, for all $\omega \in \Omega \setminus N$ and $n \in \mathbb{N}$, either $\|f_n(\omega) - f(\omega)\| < \frac{1}{n}$ or $f_n(\omega) = 0$.

If μ is finite, then the range of each of the simple functions f_n may taken to be contained in $f(\Omega \setminus N)$.

The usual statement involves only part (i) and (ii). When examining the proof found in [15, Theorem 3.5.3, p. 73], however, one finds that one actually exhibits a sequence of simple functions as stated in part (iii), which we will use in the proof of Theorem A.3 below.

Lemma A.2. *Let X and Y be normed vector spaces and Γ a convex cone in X. If $\phi : \Gamma \to Y$ is continuous at 0 and positively homogeneous, then ϕ is bounded, i.e. there exists an $C \geq 0$ such that $\|\phi(x)\|_Y \leq C\|x\|_X$ for all $x \in \Gamma$.*

Proof. It suffices to show that there exists an $M \geq 0$ such that $\|\phi(x)\|_Y \leq M$ for all $x \in \Gamma, \|x\|_X \leq 1$. Suppose that there is no such M. Then there are $x_n \in \Gamma$ such that $\|x_n\|_X \leq 1$ and $\phi(x_n) \geq 2^n$. Define $z_n := \frac{x_n}{2^n} \in \Gamma$. Then $\|z_n\|_X \leq \frac{1}{2^n}$, so $z_n \to 0$ in Γ as $n \to \infty$ and $\phi(z_n) \to \phi(0) = 0$ by continuity of ϕ. This contradicts $\phi(z_n) \geq 1$. □

Theorem A.3. *Let (Ω, Σ, μ) be a finite measure space, X a Banach space and $f : \Omega \to X$ a μ-Bochner integrable function. Let $\Gamma \subset X$ be a closed convex cone, such that $f(\omega) \in \Gamma$ for μ-almost every $\omega \in \Omega$. Let $G : \Gamma \to X$ be continuous, positively homogeneous and additive. Then*

$$G \int_\Omega f \, d\mu = \int_\Omega G \circ f d\mu.$$

Proof. Let $N_1 \in \Sigma$ be a μ-null set, such that $f(\Omega \setminus N_1) \subset \Gamma$. Define $\tilde{f} = \mathbb{1}_{\Omega \setminus N_1} \cdot f$, then $\tilde{f}$ equals f μ-almost everywhere, and $\tilde{f}(\Omega) \subset \Gamma$. So without loss of generality we can assume that $f(\Omega) \subset \Gamma$.

f is strongly μ-measurable, so by Theorem A.3 there exist μ-null sets N, N' and simple functions $f_n : \Omega \to X$ such that the conditions in Theorem A.1.*(iii)* are satisfied. Then for every $\omega \in \Omega \setminus N'$ and every $n \in \mathbb{N}$, $\|f_n(\omega)\| < \frac{1}{n} + \|f(\omega)\| \leq 1 + \|f(\omega)\|$. Since f is μ-Bochner integrable and μ is finite, $1 + \|f(\cdot)\|$ is in $L^1(\mu)$. Hence by the vector-valued Dominated Convergence Theorem $\int_\Omega f_n \, d\mu \to \int_\Omega f \, d\mu$. Since $f_n(\Omega) \subset \Gamma$ for every $n \in \mathbb{N}$, $f(\Omega) \subset \Gamma$ and Γ is a closed convex cone, it follows from [8, Corollary II.8] that $\int_\Omega f_n d\mu, \int_\Omega f d\mu \in \Gamma$. So $G \int_\Omega f_n d\mu \to G \int_\Omega f \, d\mu$.

Let $h : \Omega \to X$ be a simple function with $h(\Omega) \subset \Gamma$. We can write $h = \sum_{k=1}^m \mathbb{1}_{E_k} x_k$, with the $E_k \in \Sigma$ pairwise disjoint, and $x_k \in \Gamma$. Then $G \circ h$ is a simple function. Since G is additive and positively homogeneous, $G \int_\Omega h \, d\mu = \int_\Omega G \circ h \, d\mu$. G is also continuous, hence it is bounded by Lemma A.2, so there exists a $C > 0$ such that $\|Gx\| \leq C\|x\|$ for all $x \in \Gamma$. Thus for every $\omega \in \Omega \setminus N'$, $\|G \circ f_n(\omega)\| \leq C\|f_n(\omega) \leq C(1 + \|f(\omega)\|)$ and $G \circ f_n(\omega) \to G \circ f(\omega)$. By the vector-valued Dominated Convergence Theorem, $\int_\Omega G \circ f_n \, d\mu \to \int_\Omega G \circ f \, d\mu$. Hence

$$G \int_\Omega f \, d\mu = \lim_{n\to\infty} G \int_\Omega f_n \, d\mu = \lim_{n\to\infty} \int_\Omega G \circ f_n \, d\mu = \int_\Omega G \circ f d\mu.$$

□

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