SUNADA'S METHOD AND THE COVERING SPECTRUM

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ABSTRACT. In 2004, Sormani and Wei introduced the covering spectrum: a geometric invariant that isolates part of the length spectrum of a Riemannian manifold. In their paper they observed that certain Sunada isospectral manifolds share the same covering spectrum, thus raising the question of whether the covering spectrum is a spectral invariant. In the present paper we describe a group theoretic condition under which Sunada's method gives manifolds with identical covering spectra. When the group theoretic condition of our method is not met, we are able to construct Sunada isospectral manifolds with distinct covering spectra in dimension 3 and higher. Hence, the covering spectrum is not a spectral invariant. The main geometric ingredient of the proof has an interpretation as the minimum-marked-length-spectrum analogue of Colin de Verdière's classical result on constructing metrics where the first k eigenvalues of the Laplace spectrum have been prescribed.

1. INTRODUCTION

Two widely studied geometric invariants of a closed connected Riemannian manifold (M, g) are the Laplace spectrum and the length spectrum. The *Laplace spectrum* (or *spectrum*) is the non-decreasing sequence of eigenvalues, considered with multiplicities, of the Laplace operator acting on the space of smooth functions of M. While the spectrum is known to encode some geometric information such as the dimension, volume and total scalar curvature, it is by now well established that the spectrum does not uniquely determine the geometry of a Riemannian manifold (e.g., [Mi], [Vi], [G2], [Sz], [Sch], [Sut]). In particular, in 1985 Sunada devised a method that allows one to construct an abundance of isospectral manifolds by exploiting certain finite group actions [Sun], which were originally studied by Gassmann [Gas]. The *length spectrum* is the collection of lengths of the smoothly closed geodesics in (M, g), where the multiplicity of a length is counted according to the number of free homotopy classes containing a geodesic of that length. If we ignore multiplicities, the resulting set of non-negative numbers is known as the *weak* (or *absolute*) *length spectrum*. As with the Laplace spectrum, it is known that the length spectrum does not uniquely characterize the geometry of a manifold.

A classical pursuit in geometry, dynamics and mathematical physics is to understand the mutual influences of the Laplace and length spectra of a Riemannian manifold. By using the Poisson summation formula one can show that any two flat tori are isospectral if and only if they share the same length spectrum. The work of Colin de Verdière shows that for a generic Riemannian manifold—i.e., a manifold equipped with a "bumpy" metric [A]—the weak length spectrum is determined by its Laplace spectrum [CdV1]. Furthermore, Chazarain [Ch] and Duistermaat and Guillemin [DuGu] demonstrated that for an arbitrary Riemannian manifold, the *weak* length spectrum contains the singular support of the trace of its wave group (a spectrally determined tempered distribution). Hence, the Laplace spectrum always determines a non-trivial subset of the weak length spectrum of a Riemannian manifold. However, the interesting and important question of whether the weak length spectrum is actually a spectral invariant remains open.

In [SW] Sormani and Wei introduced the covering spectrum: a geometric invariant that is related to the length spectrum of a Riemannian manifold (M, g), and that "roughly measures the size of the one dimensional holes in the space". In [SW] the covering spectrum is computed by considering a certain family $\{\widetilde{M}^\delta\}_{\delta>0}$ of regular coverings of M, where $\widetilde{M}^\delta$ covers $\widetilde{M}^\epsilon$ for $\epsilon > \delta$, and selecting the values of δ where the isomorphism type of the cover changes. That is, we look for "jumps" in the "step function" $\delta \mapsto \widetilde{M}^\delta$. When viewed within this framework the definition actually applies to all complete length spaces, and Sormani and Wei demonstrated that the covering spectrum is well-behaved under Gromov-Hausdorff convergence.

In this paper we present a slightly different, yet compatible, definition of the covering spectrum that is applicable to any metric space (see Section 3). In short, we form the covering spectrum of M by assigning a non-negative real number $r(N/M)$ to *every* non-trivial covering N of M. In the case that (M, g) is a compact Riemannian manifold this number is half the length of the shortest closed geodesic that has a lift to N that is not a closed loop; see Corollary 4.5 (cf. [SW, Lemma 4.9]). For example, if $\widetilde{M}$ is the universal cover of M, then $r(\widetilde{M}/M)$ is half the systole of M. The covering spectrum of (M, g), denoted $\mathrm{CovSpec}(M, g)$, is then defined to be the collection of all $r(N/M)$ as N ranges over all non-trivial covers of M. Thus, $2\,\mathrm{CovSpec}(M, g)$ is the portion of the weak length spectrum consisting of those lengths that are "seen" by some covering space as the length of the shortest closed geodesic having a non-closed lift.

As an example, consider the flat 3×2 torus $T^2 = S^1(3) \times S^1(2)$, where $S^1(c) = \mathbb{R}/c\mathbb{Z}$ denotes the circle of circumference c. Then one can easily verify that $\mathrm{CovSpec}(T^2) = \{1, \frac{3}{2}\}$ [SW, Exa. 2.5]. In this case we see that the covering spectrum consists entirely of half the successive minima of the corresponding lattice $3\mathbb{Z} \times 2\mathbb{Z}$. However, as we show in Example 4.7 this is not the case for all flat tori.

Having identified the covering spectrum as a geometrically determined finite part of the weak length spectrum, one may wonder about the mutual influences between the covering spectrum and Laplace spectrum. Along these lines, Sormani and Wei found that so-called Komatsu pairs
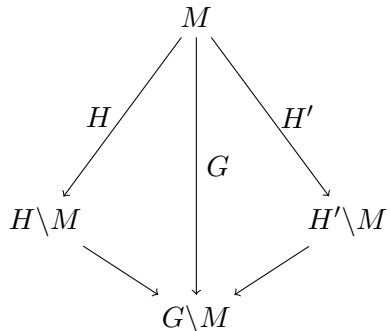
of Sunada isospectral manifolds share the same covering spectrum [SW, Ex. 10.5]. However, as we will show in Section 8, their claim [SW, Example 10.3] that certain pairs of isospectral Heisenberg manifolds due to Gordon have distinct covering spectra is false [SW2], thus keeping alive the question of whether the covering spectrum is a spectral invariant.

By engaging in both a geometric and a group theoretic analysis of the covering spectrum and its relation to the Sunada condition, we provide a negative answer to this question. Specifically, we show that in dimension 3 and higher there are Sunada isospectral manifolds with distinct covering spectra. In dimension 4 these include certain isospectral flat tori due to Conway and Sloane.

2. Main results and overview

In this section, we state our main results and give an overview of the material covered in later sections. We first recall Sunada's construction.

Let G be a finite group, and let H and H' be subgroups. Suppose that G acts on a closed connected manifold M, and suppose that the action is free in the sense that every non-trivial element of G acts without fixed points. We then consider the quotient manifolds $H\backslash M$ and $H'\backslash M$, which both cover $G\backslash M$, as indicated in the following diagram of covering spaces, where the labels at regular coverings are their deck transformation groups.

$$
\begin{array}{ccc}
 & M & \\
H \swarrow & \downarrow G & \searrow H' \\
H\backslash M & & H'\backslash M \\
 & \searrow \quad \downarrow \quad \swarrow & \\
 & G\backslash M &
\end{array}
$$

2.1. **Theorem** (Sunada [Sun], Pesce [Pes3]). *The following are equivalent:*

(1) *for every conjugacy class C of G we have $\#(H \cap C) = \#(H' \cap C)$;*

(2) *for every G-invariant Riemannian metric on M, the quotient Riemannian manifolds $H\backslash M$ and $H'\backslash M$ have the same Laplace spectrum.*

The fact that (1) implies (2) is Sunada's theorem, and it is the only implication that will be used in this paper (see [Br], [GMW, Sec. 3] for alternative proofs and comments). Many of the examples of isospectral manifolds in the literature can be understood within the framework of Sunada's theorem and its generalizations (e.g., [DG], [B1], [B2], [Pes1], [Sut] and [Ba]). The group theoretic condition (1) is known by several names in the literature: Perlis [Per] says that H and H' are Gassmann equivalent after Gassmann [Gas], who first used this condition in 1926, and spectral geometers (e.g., [Br]) frequently say that H and H' satisfy the Sunada

condition. Others say that H and H' are almost conjugate [Bu, GK], or linearly equivalent [DL] or that H and H' induce the same permutation representation [GW]. In this paper we will express that condition (1) holds by saying that H and H' are *Gassmann-Sunada* equivalent, or that (G, H, H') is a *Gassmann-Sunada triple*.

The statement that (2) implies (1) in Theorem 2.1 follows from the work of Pesce [Pes3], and it shows that the Gassmann-Sunada condition is not only sufficient, but also necessary if we want it to ensure that $H\backslash M$ and $H'\backslash M$ are isospectral for all possible choices of compatible Riemannian metrics in the diagram above.

In Section 6 we establish the following analogue of Sunada's method for the covering spectrum.

2.2. Theorem. *Let G, H, H' and M be as above. If M is simply connected and of dimension at least 3 then, the following are equivalent:*

(3) *for all subsets S, T of G that are stable under conjugation we have*

$$\langle H \cap S \rangle = \langle H \cap T \rangle \iff \langle H' \cap S \rangle = \langle H' \cap T \rangle;$$

(4) *for every G-invariant Riemannian metric on M, the quotient Riemannian manifolds $H\backslash M$ and $H'\backslash M$ have the same covering spectrum.*

More generally, if M is simply connected, then (3) implies (4), and if M has dimension at least 3, then (4) implies (3).

In the last statement of Theorem 2.2, the condition that M is simply connected cannot be omitted. We do not know if (4) implies (3) for all 2-dimensional manifolds. In order to understand the relevance of condition (3) in this result, and to explain the ingredients of the proof, we introduce the purely algebraic concept of a *length map* on a group.

2.3. Definition. A *length map* on a group G is a map $m\colon G \to \mathbb{R}_{\geq 0}$ from G to the set of non-negative real numbers that satisfies

(i) $m(g) > 0$ for all $g \in G - \{1\}$;

(ii) $m(g) = m(hgh^{-1})$ for all $g, h \in G$;

(iii) $m(g^k) \leq |k| m(g)$ for all $k \in \mathbb{Z}$ and $g \in G$.

Taking $k = 0$ and $k = -1$ we see that (iii) implies that $m(1) = 0$ and $m(g^{-1}) = m(g)$ for all $g \in G$. The motivation for the preceding definition is found in the following.

2.4. Definition ([SW, Def. 4.5]). Let (M, g) be a *compact* Riemannian manifold, and let $\pi_1(M)$ be its fundamental group with respect to some base point. The *minimum marked length map* $m_g\colon \pi_1(M) \to \mathbb{R}_{\geq 0}$ assigns to each $\gamma \in \pi_1(M)$ the length of the shortest closed geodesic in the free homotopy class determined by γ. Equivalently, we can set $m_g(\gamma) = \min_{x \in \widetilde{M}} d(x, \gamma \cdot x)$, where $(\widetilde{M}, \tilde{g})$ is the universal Riemannian cover of (M, g), d is the metric structure induced by $\tilde{g}$ and (after a choice of base point) the group $\pi_1(M)$ acts on $\widetilde{M}$ via deck transformations (see [Sp, Sec. 6]).

We refer the reader to [SW, Lem. 4.4] for the reason why each free homotopy class in the definition above has a shortest closed geodesic. Since the conjugacy classes of $\pi_1(M)$ naturally correspond to the free homotopy classes of loops in M, we see that m_g is a length map. Since M is compact, the image of m_g is closed and discrete in $\mathbb{R}_{\geq 0}$ [SW, Lem. 4.6].

Given a length map m on a group G we define the subgroup $\mathrm{Fil}_m^\delta G = \langle g \in G : m(g) < \delta \rangle$ of G for each $\delta > 0$. When $\delta < \epsilon$, we have $\mathrm{Fil}_m^\delta G \subset \mathrm{Fil}_m^\epsilon G$, so this defines an increasing filtration, denoted $\mathrm{Fil}_m^\bullet G$, of G by normal subgroups indexed by the positive real numbers. We define its *jump set* $\mathrm{Jump}(\mathrm{Fil}_m^\bullet G)$ to be the set of all $\delta > 0$ such that for all $\epsilon > \delta$ we have $\mathrm{Fil}_m^\delta G \neq \mathrm{Fil}_m^\epsilon G$. We will see that this notion of the jump set of the filtration defined by a length map provides the connection between conditions (3) and (4) of Theorem 2.2. More specifically, in Section 4 we will show the following.

2.5. **Proposition.** *The covering spectrum of a compact Riemannian manifold (M, g) is given by*
$$\mathrm{CovSpec}(M, g) = \frac{1}{2} \mathrm{Jump}(\mathrm{Fil}_{m_g}^\bullet \pi_1(M)).$$

In Lemma 6.3 we will see that condition (3) of Theorem 2.2 holds if and only if for every length map m on G the restrictions to H and to H' have identical jump sets. When (3) holds, we will say that H and H' are *jump equivalent* or that (G, H, H') is a *jump triple*. Combining Lemma 6.3 with Proposition 2.5 it is not hard to show that (3) implies (4) for simply connected M. In order to show the other, more difficult, implication of Theorem 2.2, we will assume that (3) does not hold and then proceed to construct a Riemannian metric on M so that the resulting minimum marked length map has a special property. In order to construct this metric it will be necessary to understand the extent to which length maps actually arise from minimum marked length maps of Riemannian manifolds. That is, we would like to know which length maps on the fundamental group of a given manifold M arise as the minimum marked length map associated to some Riemannian metric on M.

In Section 5 we take up this line of inquiry and we establish Theorem 5.1, which can be summarized as follows.

2.6. **Theorem.** *Let M be a closed manifold of dimension at least three and let $S \subset \pi_1(M)$ be a finite set. If $m : S \to \mathbb{R}_{\geq 0}$ is the restriction to S of a length map on $\pi_1(M)$, then there exists a Riemannian metric g on M such that $m_g(s) = m(s)$ for any $s \in S$. That is, m can be extended to the minimum marked length map associated to some metric g on M.*

Hence, in the case where M is a closed manifold with finite fundamental group and dimension at least 3, the above demonstrates that there is no difference between length maps on $\pi_1(M)$ and the minimum marked length maps associated to M.

In Theorem 5.1—the detailed statement of the above—we actually prove that we have some control over the extension of m. More specifically, we demonstrate that we are able to prescribe which free homotopy classes contain the shortest closed geodesics (see Remark 5.2). In Section 6, we will use this to prove Theorem 2.2.

Theorem 5.1 also provides a way to characterize the initial segments of the so-called minimum marked length spectrum of a Riemannian metric on a closed manifold of dimension at least three. Since this is perhaps of independent interest we will formulate this after the following definition.

2.7. **Definition** (cf. [GtM]). Let (M, g) be a Riemannian manifold with associated minimum marked length map $m_g : \pi_1(M) \to \mathbb{R}_{\geq 0}$. The value $m_g([\sigma])$ depends only on the class of the loop σ in the set of unoriented free homotopy classes $\mathcal{F}(M)$ of loops in M; whereby the *unoriented free homotopy class* of σ we mean the collection of all loops freely homotopic to σ or its inverse $\bar{\sigma}$. Hence, we obtain an induced map $\mathcal{F}(M) \to \mathbb{R}_{\geq 0}$, which is again denoted by m_g, and in this incarnation it is known as the *minimum marked length spectrum*. Alternatively, one can define the minimum marked length spectrum as the set of ordered pairs $(m_g(c), c)$ where each length $m_g(c)$ is "marked" by the unoriented free homotopy class $c \in \mathcal{F}(M)$. This set of pairs is a subset of the marked length spectrum as defined in [Gt], for example.

2.8. **Theorem.** *Suppose that M is a closed connected manifold of dimension at least three. Let $C = (c_1, c_2, \ldots, c_k)$ be a sequence of distinct elements of $\mathcal{F}(M)$ where the first element c_1 is trivial. Then for every sequence $0 = l_1 < l_2 \leq \cdots \leq l_k$ of real numbers the following are equivalent:*

(5) *the sequence $l_1, \ldots, l_k$ is C-admissible (see Definition 5.4 and Example 5.5);*

(6) *there is a Riemannian metric g on M such that the minimum marked length map m_g satisfies $m_g(c_i) = l_i$ for all i and $m_g(c) \geq l_k$ for all $c \in \mathcal{F}(M) - \{c_1, \ldots, c_k\}$.*

In particular, there is a metric g on M such that the systole is achieved in the unoriented free homotopy class c_2.

The statement that (5) implies (6) in the above is a minimum-marked-length-spectrum analog of a classical theorem due to Colin de Verdière [CdV2], which states that given a connected closed manifold M of dimension at least 3 and a finite sequence $a_1 = 0 < a_2 \leq a_3 \leq \cdots \leq a_k$ there is a Riemannian metric g on M such that the sequence gives exactly the first k eigenvalues, counting multiplicities, of the associated Laplacian. However, unlike Colin de Verdière's result, the sequence $0 = l_1 < l_2 \leq l_3 \cdots \leq l_k$ in the above cannot be chosen arbitrarily: it will depend on our choice of the sequence $C = (c_1, c_2, \ldots, c_k)$ in $\mathcal{F}(M)$. The above theorem then tells us that given such a choice for C, the C-admissibility of $0 = l_1 < l_2 \leq l_3 \cdots \leq l_k$ is a necessary and sufficient condition for the existence of a metric g such that the ith smallest value of the minimum marked length spectrum is $m_g(c_i) = l_i$ for $i = 1, \ldots, k$.

With Theorem 5.1 in place, we will complete the proof of Theorem 2.2 in Section 6. In Section 7 we present some group theoretic context and results concerning Gassmann-Sunada triples (condition (1) of Theorem 2.1) and jump triples (condition (3) of Theorem 2.2). There are some key differences in the behavior of jump triples and Gassmann-Sunada triples. For instance, we have $[G : H] = [G : H']$ for every Gassmann-Sunada triple (G, H, H'), but not for every jump triple. The behavior with respect to dividing out normal subgroups is also different:

if N is a normal subgroup of G that is contained in $H \cap H'$, then H and H' are Gassmann-Sunada equivalent in G if and only if H/N and H'/N are Gassmann-Sunada equivalent in G/N. However, H/N and H'/N can be jump equivalent in G/N even when H and H' are not jump equivalent in G. This implies that the condition that M is simply connected cannot be dropped from Theorem 2.2.

While most small examples of Gassmann-Sunada triples turn out to be jump triples (see Example 7.4) we will show how to adapt some well-known constructions of Gassmann-Sunada triples to obtain Gassmann-Sunada triples that are not jump triples.

Finally, in Section 8 we draw the following conclusion from our analysis in the previous sections.

2.9. Theorem. *For each $n \geq 3$ there are closed Riemannian manifolds of dimension n with identical Laplace spectra and distinct covering spectra.*

As we will see in Example 8.1, these will include certain isospectral flat tori of dimension 4 due to Conway and Sloane [CS].

We conclude the paper with a study of the covering spectrum of Heisenberg manifolds that refutes [SW, Example 10.3].

3. The covering spectrum of a metric space

The notion of the covering spectrum was defined by Sormani-Wei [SW] for complete length spaces. In this section we provide an alternate definition that works for any metric space. For compact length spaces—the main focus of this paper—our definition coincides with the definition given by Sormani and Wei, and for non-compact complete length spaces the only difference is that 0 is sometimes an element of our covering spectrum. In order to present this definition, we first review some terminology from covering space theory.

Let X be a topological space. By a *space over X* we mean a topological space Y along with a continuous map $p\colon Y \to X$. For such a space Y over X and $Z \subset X$ we say that Y is *trivial over Z*, or it *evenly covers Z*, if $p^{-1}(Z)$ is a disjoint union of open subspaces of $p^{-1}(Z)$ that are each mapped homeomorphically onto Z via the map p. A *covering space* of X is a space Y over X that is locally trivial; that is, each point $x \in X$ has an open neighborhood U so that Y is trivial over U. A covering space Y over X is *trivial* if the covering map $Y \to X$ is a homeomorphism.

3.1. Definition. Let (X, d) be a metric space and let $p\colon Y \to X$ be a covering space of X. For $\delta \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ we say that Y is *δ-trivial* over X if for each $x \in X$ the cover Y is trivial over the open ball of radius δ centered at x. The *covering radius* $r(Y/X) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ of Y over X is given by

$$r(Y/X) = \sup \{\delta \colon Y \text{ is } \delta\text{-trivial over } X\}.$$

3.2. Remark. Clearly, the diameter, $\mathrm{diam}(X)$, places an upper bound on the covering radius of any covering space of X. That is, $r(Y/X) \leq \mathrm{diam}(X)$ for any cover $Y \to X$. Also, if X is

a compact metric space, then the Lebesgue number lemma tells us that the covering radius of any non-trivial cover will be non-zero. However, in the next example we see that non-compact metric spaces can admit covers that have zero covering radius.

3.3. **Example** (Zero covering radius)**.** Let X be the metric *subspace* of the Euclidean plane equipped with the standard metric that one gets by connecting a sequence of circles with radius going to zero as in the picture:
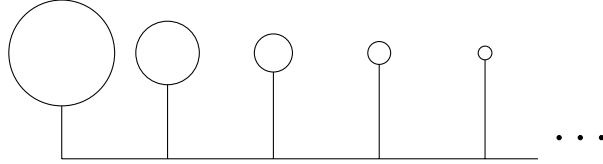
Then for any non-trivial covering space Y of X the covering radius $r(Y/X)$ is the infimum of the radii of those circles that are not evenly covered. This space X has a simply connected universal cover, which therefore has covering radius zero over X.

3.4. **Remark.** In Lemma 4.1 below we show that if (X, d) is a locally path connected space, then $p : Y \to X$ is actually an $r(Y/X)$-trivial cover. In general this may fail; see the example below. In particular, we can have $r(Y/X) = \mathrm{diam}(X)$ without $Y \to X$ being $\mathrm{diam}(X)$-trivial.

3.5. **Example** (Infinite covering radius)**.** Let us define a metric subspace X of the Euclidean plane that is locally path connected at all but a single point, together with a non-trivial double cover that is trivial over every bounded subset of X. We take a union of infinitely many circles, all tangent to each other at the same point x, with the radius of the circles going to infinity, where in each circle we omit an open interval lying around the point x whose size is shrinking to zero as we go through the infinite sequence of circles. Thus, what remains of each circle is a closed interval, and the end points of these intervals give a sequence converging to x. Then we also add the single point x to our space. Informally one might say that X has a loop that only closes on an infinite scale.
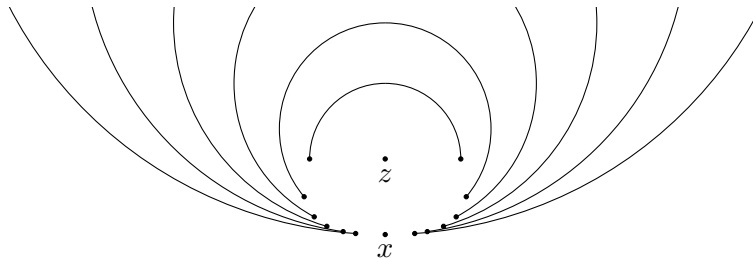
Now take any point z inside the circles that is not in X. Then the plane with z removed has a connected double cover, and we restrict this cover to X. We leave it to the reader to check that this covering has the stated properties.

One can make this space into a path connected example by connecting a single end point of each component to the point x. One can also make a similar bounded example: a space within

the open disk of radius 1 in the plane, with a double cover that is δ-trivial for all $\delta < 1$ but not 1-trivial.

3.6. **Remark** (Non-constant degree). Given a covering $p\colon Y \to X$ the degree of a point $x \in X$ is the cardinality of the fiber $p^{-1}(x)$ over x, which may be infinite. If this cardinality is the same for all $x \in X$, then we say that the covering has constant degree. Since Y is locally trivial over X this degree is locally constant on X, so it is constant on the connected components of X. When considering the covering radius we will often restrict to the case that the covering has constant degree. To see how to express the covering radius in the general case, note that for any covering $p\colon Y \to X$ that is not of constant degree the fibers of the degree map partitions X into at least two open subsets X_i, where i ranges over a suitable index set I, and for each i the space $Y_i = p^{-1}(X_i)$ is a cover of X_i of constant degree. Then the covering radius of Y over X is the infimum of the following set:

$$r(Y/X) = \inf\{d(X_i, X_j)\colon\ i \neq j\} \cup \{r(p^{-1}(X_i)/X_i)\colon\ i \in I\}.$$

For example, if X consists of two points and Y is a non-trival covering space of X then $r(Y/X)$ is the distance between the two points of X.

3.7. **Definition.** The *covering spectrum*, denoted $\mathrm{CovSpec}(X)$, of a metric space X is the set of all $r(Y/X)$ as Y varies over all non-trivial coverings of X of constant degree.

We will show in the next section that for connected locally path connected spaces the covering spectrum is the jump set of a length map on the fundamental group. We will also see that for compact Riemannian manifolds, the prime object of interest in this paper, the definition above is equivalent to the definition given by Sormani-Wei [SW]; see Remark 4.3. We conclude this section with some general remarks.

The covering spectrum of X is a subset of $\mathbb{R}_{\geq 0} \cup \{\infty\}$. It is empty if and only if all constant degree covers of X are trivial. In Example 3.3 the covering spectrum is the set of diameters of the circles together with the element 0. If we do not use the Euclidean distance, but instead view X as a length space by taking the path-length metric, then the covering spectrum consist of half the circumferences of the circles together with the element 0. In Example 3.5 the covering spectrum is $\{\infty\}$ and that of its bounded analog is $\{1\}$. The same is true for their path connected versions. These topological spaces cannot be made into length spaces, so this is a situation in which the definition of the covering spectrum as given in [SW] does not apply.

If X is connected then one may drop the condition "of constant degree" from the definition. Discrete metric spaces have an empty covering spectrum, and more generally, the covering spectrum of a union of disjoint open subspaces of a metric space is the union of their covering spectra.

4. The covering spectrum and the fundamental group

In this section we see how to express the covering spectrum in terms of additional structure on the fundamental group. The main result of this section is a proof of Proposition 2.5. First, we recall some terminology.

A *path* in a topological space X is a continuous map $\sigma\colon [0,1] \to X$. We say that σ is a *loop* if $\sigma(0) = \sigma(1)$, and we then say that the loop is *based at* $\sigma(0)$. If $p\colon Y \to X$ is a covering map and σ is a path in X, then a *lift* of σ to Y is a path $\widetilde{\sigma}$ in Y such that $p \circ \widetilde{\sigma} = \sigma$. The lemma of *unique path lifting* states that the map $\widetilde{\sigma} \mapsto \widetilde{\sigma}(0)$ gives a bijection between the set of lifts of σ to Y and the fiber $p^{-1}(\sigma(0))$ over the starting point of σ. Given a path σ its *inverse* $\overline{\sigma}$ is the path given by $\overline{\sigma}(t) = \sigma(1 - t)$, and if τ is also a path in X and $\sigma(1) = \tau(0)$, then $\sigma * \tau$ is the *composed path* sending t to $\sigma(2t)$ if $t \in [0, 1/2]$ and to $\tau(2t - 1)$ when $t \in [1/2, 1]$; that is, we travel along σ followed by τ. When ρ is a path in X with $\rho(0) = \tau(1)$, we recall that $(\sigma * \tau) * \rho$ is path homotopic to $\sigma * (\tau * \rho)$; hence, there is no ambiguity in writing $[\sigma * \tau * \rho]$ for the path homotopy class of both paths. If σ is a loop in X based at x, then we denote its homotopy class in the fundamental group $\pi_1(X, x)$ by $[\sigma]$. The group operation in $\pi_1(X, x)$ is given by $[\sigma][\tau] = [\sigma * \tau]$ and the inverse by $[\sigma]^{-1} = [\overline{\sigma}]$.

By a *filtration* $\mathrm{Fil}^\bullet G$ of a group G we mean a family of subgroups $\mathrm{Fil}^i G$ of G, with i ranging over an ordered index set I, such that $\mathrm{Fil}^i G \subset \mathrm{Fil}^j G$ when $i < j$. The *jump set* of the filtration is the subset of I given by

$$\mathrm{Jump}(\mathrm{Fil}^\bullet G) = \{i \in I\colon \mathrm{Fil}^i G \neq \mathrm{Fil}^j G \text{ for all } j \in I \text{ with } i < j\}.$$

That is, $i \in \mathrm{Jump}(\mathrm{Fil}^\bullet G)$ if and only if $\mathrm{Fil}^i G$ is a *proper subset* of $\mathrm{Fil}^j G$ for all $j > i$ (cf. Example 4.7). In this paper our index set I will always be the set of positive real numbers (with the usual ordering) and the subgroups of the filtration will always be normal subgroups of G.

For a metric space (X, d) with base point x_0, we now define a filtration $\mathrm{Fil}^\bullet \pi_1(X, x_0)$ on the fundamental group $\pi_1(X, x_0)$ of X. This filtration will determine the covering radius of any covering of X and hence encodes the covering spectrum. For $\delta > 0$ we let $\mathrm{Fil}^\delta \pi_1(X, x_0)$ be the subgroup of $\pi_1(X, x_0)$ generated by the elements of the form $[\alpha * \sigma * \overline{\alpha}]$, where σ is a non-trivial loop contained inside some open δ-ball and α is a path from x_0 to $\sigma(0)$. These generators are precisely the homotopy classes of loops based at x_0 that are *freely* homotopic to a loop completely contained in some open δ-ball. We note that in [SW, Section 2] the same filtration is considered for complete length spaces, where $\mathrm{Fil}^\delta \pi_1(X, x_0)$ is denoted by $\pi_1(X, \delta, x_0)$. The next lemma expresses the covering radius (see Definition 3.1) in terms of this filtration.

4.1. **Lemma.** *Let (X, d) be a connected and locally path connected metric space, with base point x_0, and let Y be a non-trivial covering space of X. Then Y is $r(Y/X)$-trivial over X. Moreover, $r(Y/X)$ is the maximum of all $\delta > 0$ such that for every loop σ in X based at x_0 with $[\sigma] \in \mathrm{Fil}^\delta \pi_1(X, x_0)$, every lift to Y of σ is a closed loop.*

Proof. Recall first that for each open subset $U \subset X$ the cover Y is trivial over U if and only if every lift to Y of a loop in U is a loop in Y; [Sp, Lemma 2.4.9].

To show the first statement, let B be an open ball of radius $r(Y/X)$ in X centered at some point $x \in X$, and let σ be a loop whose image is completely contained inside B. Then since the image of σ is compact we see that there is a $\delta < r(Y/X)$ such that the image of σ is completely contained inside the open ball of radius δ centered at x. By the definition of the covering radius $r(Y/X)$, the covering $Y \to X$ is δ-trivial, hence we see that σ lifts to a closed loop. It now follows from our initial remark that Y is an $r(Y/X)$-trivial cover of X.

Next, we note that the connected components of Y are open, and they are themselves coverings of X. For each component of Y one may choose a point y_0 in the fiber of this component over x_0. With [Sp, Lemma 2.5.11, 2.4.9] one then shows that this component is δ-trivial over X if and only if $\mathrm{Fil}^\delta \pi_1(X, x_0)$ is contained in the image of $\pi_1(Y, y_0)$ under the map induced by the covering map. Thus, the second statement follows. □

It follows from the lemma that the covering spectrum of a locally path connected space does not contain the element ∞ (cf. Example 3.5).

4.2. **Proposition.** *If (X, d) is a connected and locally path connected metric space, then*

$$\mathrm{CovSpec}(X) - \{0\} = \mathrm{Jump}(\mathrm{Fil}^\bullet \pi_1(X, x_0)).$$

Proof. Suppose that $\delta \in \mathrm{CovSpec}(X) - \{0\}$. Then $\delta \neq \infty$ by Lemma 4.1, and by definition there is a covering $p\colon Y \to X$ with $r(Y/X) = \delta$. Again by the lemma, the collection of homotopy classes of loops based at x_0 all of whose lifts to Y are themselves loops contains $\mathrm{Fil}^\delta \pi_1(X, x_0)$, but it does not contain the set $\mathrm{Fil}^\epsilon \pi_1(X, x_0)$ for any $\epsilon > \delta$. It follows that δ is a jump for the filtration, so the inclusion "$\subset$" holds.

Now, let $\delta > 0$. Following [SW, Def. 2.3] we see with [Sp, Theorem 2.5.13] that there is a connected and locally path connected covering space $p : \widetilde{X}^\delta \to X$ such that $p_\#(\pi_1(\widetilde{X}^\delta)) = \mathrm{Fil}^\delta \pi_1(X, x_0) \subset \pi_1(X, x_0)$. Hence, $\mathrm{Fil}^\delta \pi_1(X, x_0)$ consists of exactly those classes of loops whose lifts to $\widetilde{X}^\delta$ are all loops. With the lemma, or by using [Sp, Lemma 2.5.11, 2.4.9] again, we see that for all $\delta \in \mathrm{Jump}(\mathrm{Fil}^\bullet \pi_1(X, x_0))$, we have $r(\widetilde{X}^\delta/X) = \delta$ and "$\supset$" follows. □

4.3. **Remark.** The notion of the covering spectrum was first defined by Sormani and Wei in [SW] for so-called complete length spaces, and the proposition above shows that their definition exactly gives the set $\mathrm{Jump}(\mathrm{Fil}^\bullet \pi_1(X, x_0)) = \mathrm{CovSpec}(X) - \{0\}$. The only difference with our notion of covering spectrum for complete length spaces is that for non-compact X we sometimes have 0 in the covering spectrum such as for the space in Example 3.3 (endowed with shortest path length metric). We note that under our definition, $0 \in \mathrm{CovSpec}(X)$ if and only if there is a covering space $Y \to X$ such that X has open subsets of arbitrary small diameter that are not evenly covered.

To conclude this section we will make the filtration of the fundamental group more explicit for the case of compact Riemannian manifolds. We will identify generators for the normal

subgroups in the filtration: the classes of short closed geodesics. In later sections, this will allow us to compute the covering spectra of some familiar classes of manifolds. In particular, in Example 8.3 we will see how to compute the covering spectrum of a Heisenberg manifold.

Let M be a manifold and let g be a Riemannian metric on M. Choose a base point $x_0 \in M$. In Section 2 we defined the map $m_g \colon \pi_1(M, x_0) \to \mathbb{R}_{\geq 0}$ by sending a class to the infimum of all lengths of loops that are freely homotopic to it.

4.4. **Proposition.** *For every connected manifold M with Riemannian metric g and base point $x_0 \in M$ and every $\delta > 0$ we have*

$$\mathrm{Fil}^\delta \, \pi_1(M, x_0) = \langle \gamma \in \pi_1(M, x_0) \colon \; m_g(\gamma) < 2\delta \rangle.$$

Proof. Suppose that σ is a loop based at x_0 in M so that $m_g([\sigma]) < 2\delta$. Then σ is freely homotopic to a loop τ of length below 2δ. It is clear that $\mathrm{Im}(\tau)$ is contained in the open ball $B_\delta(\tau(0))$ of radius δ centered at $\tau(0)$. If α is the path from x_0 to $\tau(0)$, given by the moving base point during such a homotopy, then $[\sigma] = [\alpha * \tau * \overline{\alpha}] \in \mathrm{Fil}^\delta \, \pi_1(M, x_0)$. This shows "$\supset$".

For the other inclusion, recall that $\mathrm{Fil}^\delta \, \pi_1(M, x_0)$ is generated by elements of the form $[\alpha * \sigma * \overline{\alpha}]$ where α is a path from x_0 to a point $x_1 \in M$ and σ is a loop based at x_1 that is contained in $B_\delta(x_2)$ for some $x_2 \in X$. Since $B_\delta(x_2)$ is locally simply connected we can assume that σ has finite length. The image of σ is compact, so it is in fact contained in $B_{\delta-\epsilon}(x_2)$ for some $\epsilon > 0$. Choose any path α_1 from x_1 to x_2. We can now break σ into finitely many sections of length less than ϵ, and by connecting the division points with x_2 by paths of length smaller than $\delta - \epsilon$ we may write $[\overline{\alpha_1} * \sigma * \alpha_1] \in \pi_1(M, x_2)$ as a finite product $[\sigma_1] \cdots [\sigma_n]$ with each σ_i a loop based at x_2 of length less than $2(\delta - \epsilon) + \epsilon < 2\delta$. Writing $\beta = \alpha * \alpha_1$ we now have

$$[\alpha * \sigma * \overline{\alpha}] = [\beta * \sigma_1 * \overline{\beta}][\beta * \sigma_2 * \overline{\beta}] \cdots [\beta * \sigma_n * \overline{\beta}]$$

in $\pi_1(M, x_0)$, and since we have $m_g([\beta * \sigma_2 * \overline{\beta}]) < 2\delta$ for each i, our other inclusion follows. $\square$

4.5. **Corollary** (cf. [SW, Lemma 4.9]). *Let (M, g) be a compact Riemannian manifold and let $p \colon N \to M$ be a non-trivial covering space. Then the covering radius $r(N/M)$ is half the length of the shortest closed geodesic σ in M that has a lift to N that is not a closed loop.*

Proof. By [Sp, Lemma 2.4.9] and the fact that the image of m_g is discrete (see Definition 2.4), there is a shortest closed geodesic in M that has a lift to Y that is not a closed loop. Let σ be such a geodesic and let $\delta = m_g([\sigma])$ be its length. For every $\epsilon > \delta$ the image of σ lies within an open ball of radius $\epsilon/2$, so the covering p is not $\epsilon/2$-trivial, and $r(N/M) \leq \delta/2$. On the other hand, for any loop in M shorter than δ all lifts to N are closed loops. Proposition 4.4 then tells us that all loops in M whose homotopy class in $\mathrm{Fil}^{\delta/2} \, \pi_1(M)$ have only closed loops as lifts to N, and with Lemma 4.1 this gives $r(N/M) \geq \delta/2$. $\square$

Proof of Proposition 2.5. This is an immediate consequence of Propositions 4.2 and 4.4 and the definitions of $\mathrm{Fil}^\bullet$ and $\mathrm{Fil}^\bullet_{m_g}$. $\square$

The moral of Proposition 2.5 is that it allows us to compute the covering spectrum of a Riemannian manifold in an easy way as the jump set of the filtration associated to a length map. Under the hypotheses of the proposition, the length map has a closed discrete image in $\mathbb{R}_{\geq 0}$, and the group it is defined on, the fundamental group, is finitely generated. We now describe how to compute the jump set in this scenario.

4.6. **Jump set algorithm** (cf. p. 54 of [SW]). Let G be a group that is *finitely generated* and let $m : G \to \mathbb{R}_{\geq 0}$ be a length map with *closed discrete* image. Then $\mathrm{Jump}(\mathrm{Fil}_m^\bullet G)$ can be computed as follows. Let

$$
\begin{aligned}
\delta_1 &= \inf\{m(g) \colon g \in G, g \neq 1\} \\
\delta_2 &= \inf\{m(g) \colon g \in G, g \notin \langle g \in G \colon m(g) \leq \delta_1 \rangle\} > \delta_1 \\
\delta_3 &= \inf\{m(g) \colon g \in G, g \notin \langle g \in G \colon m(g) \leq \delta_2 \rangle\} > \delta_2 \\
&\cdots
\end{aligned}
$$

where we continue until $\langle g \in G \colon m(g) \leq \delta_r \rangle = G$. The process terminates because G is finitely generated: if M is the largest value attained by m on some finite set of generators of G, then m assumes only finitely many values up to M, and the δ_i are among them. Then we have

$$
\mathrm{Jump}(\mathrm{Fil}_m^\bullet G) = \{\delta_1, \ldots, \delta_r\}.
$$

In particular we recover the result that the covering spectrum of a compact manifold is always finite [SW, Theorem 3.4]. As an application we show in the following example how to compute the covering spectrum of a flat torus.

4.7. **Example** (Flat tori). Let E be a Euclidean space, i.e., a vector space over $\mathbb{R}$ of finite dimension n with an inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{L}$ be a lattice in E of full rank, and let T be the flat torus $T = E/\mathcal{L}$, where the Riemannian metric is induced by $\langle \cdot, \cdot \rangle$. Then we can identify the fundamental group of T with $\mathcal{L}$, and the minimum marked length map m on $\mathcal{L}$ is given by $m(l) = \|l\| = \sqrt{\langle l, l \rangle}$. We then have $\mathrm{CovSpec}(T) = \frac{1}{2}\mathrm{Jump}(\mathrm{Fil}_m^\bullet \mathcal{L})$, where for $\delta > 0$ we have $\mathrm{Fil}_m^\delta \mathcal{L} = \langle l : \|l\| < \delta \rangle$.

Those jumps of the filtration where the rank of the sublattice increases, are the so-called *successive minima* of the lattice. Counting these jumps with a *multiplicity*, which is the increase in rank of the sublattice at the jump, we see that there are n successive minima. However, the covering spectrum of T can have more than n elements: there can also be jumps where the rank does not increase.

For instance, when E is 5-dimensional consider a lattice $\mathcal{L}$ in E spanned by orthogonal vectors $e_1, \ldots, e_5$ where $1 \leq \|e_1\| < \cdots < \|e_5\| < \frac{\sqrt{5}}{2}$. With the procedure above we see that the covering spectrum of the flat torus $E/\mathcal{L}$ is given by $\frac{1}{2}\{\|e_1\|, \|e_2, \|, \|e_3\|, \|e_4\|, \|e_5\|\}$. Now let $v = \frac{1}{2}(e_1 + \cdots + e_5)$ and consider the lattice $\mathcal{L}' = \langle \mathcal{L}, v \rangle$. The lattice $\mathcal{L}$ is a sublattice of $\mathcal{L}'$ of index 2 and any vector in $\mathcal{L}'$ that is not in $\mathcal{L}$ is of the form $v + w$, where $w \in \mathcal{L}$, and its length is at least $\frac{\sqrt{5}}{2}$. It follows that $\mathrm{CovSpec}(E/\mathcal{L}') = \mathrm{CovSpec}(E/\mathcal{L}) \cup \{\frac{1}{2}\|v\|\}$.

We recall from the introduction that the covering spectrum of $M = E/\mathcal{L}'$ is also measuring when a particular family $\{\widetilde{M}^\delta\}_{\delta>0}$ of regular covers of M changes isomorphism type. These coverings, as defined in [SW, Def. 3.1], can be given by $\widetilde{M}^\delta = E/\operatorname{Fil}^\delta \mathcal{L}'$. The above example highlights the difference between isomorphism type as coverings and as topological spaces. Indeed, if $\frac{1}{2}\|e_5\| < \delta \leq \frac{1}{2}\|v\|$, then $\widetilde{M}^\delta = E/\mathcal{L}$, and if $\delta > \frac{1}{2}\|v\|$, then $\widetilde{M}^\delta = M$. These are both 5-tori, but they are distinct regular covers of M.

We revisit the covering spectra of flat tori in Section 8.

5. Constructing metrics with a prescribed minimum marked length map

The purpose of this section is to give a proof of Theorems 2.6 and 2.8. As we noted in Definition 2.4, every Riemannian metric g on a closed manifold M gives rise to a length map $m_g\colon \pi_1(M) \to \mathbb{R}_{\geq 0}$, known as the minimum marked length map, sending the homotopy class of a loop to the infimum of the lengths of the loops that are freely homotopic to it. The general problem underlying this section is to understand the nature of those length maps that arise as the minimum marked length map associated to some Riemannian metric. Do these length maps satisfy additional properties not yet implied by the length map axioms of Definition 2.3?

The next result, whose proof will take up most of this section, implies that this is not the case if we restrict our attention to properties involving only finitely many elements. In addition to helping us establish Theorems 2.6 and 2.8, the following result is also instrumental in the proof of Theorem 2.2 as we will see in Section 6.

5.1. **Theorem.** *Let M be a closed and connected manifold of dimension at least 3. Let S be a finite subset of $\pi_1(M)$ and suppose that $m\colon S \to \mathbb{R}_{\geq 0}$ is a map satisfying:*

 (i) *$m(x) > 0$ for all non-trivial $x \in S$;*
 (ii) *$m(x) \leq |k|m(y)$ for all $x, y \in S$ and $k \in \mathbb{Z}$ such that x is conjugate to y^k.*

Then there is a Riemannian metric g on M such that $m_g(x) = m(x)$ for all $x \in S$. In addition, for every $B > 0$ the metric g may be chosen so that m_g satisfies:

 (1) *for all $x \in \pi_1(M)$ for which the set $\{|k|m_g(y)\colon y \in S,\ k \in \mathbb{Z}$ and x is conjugate to $y^k\}$ is non-empty with infimum $l_x < B$ we have $m_g(x) = l_x$;*
 (2) *for all other $x \in \pi_1(M)$ we have $m_g(x) \geq B$.*

5.2. **Remark.** By Definition 2.3 conditions (i) and (ii) are clearly necessary for the existence of an extension of m to a length map on $\pi_1(M)$. The theorem above implies that they are also sufficient, and that this extension can be taken to be m_g for a suitable metric g. Conditions (1) and (2) describe the degree of control we have over the behavior of m_g outside S. In particular, this control allows us to ensure that in (M, g) the unoriented free homotopy classes (see Definition 2.7) containing geodesics of length less than B can only be found *among* those classes corresponding to elements of the form y^k for some $y \in S$ and $k \in \mathbb{Z}$: condition (1) determines the collection of such "short" classes precisely. We will use this in the proof of Theorem 2.8.

Let M be a manifold and let $\mathcal{M}$ denote the space of all Riemmannian metrics on M. Given two metrics $g, h \in \mathcal{M}$ we will say that g is *larger than* h, denoted $g \succeq h$, if for any tangent vector $v \in TM$ we have $g(v, v) \geq h(v, v)$. Clearly, $\preceq$ defines a partial order on $\mathcal{M}$. The set $\mathcal{M}$ is closed under addition, and under multiplication by everywhere positive smooth functions on M.

Proof of Theorem 5.1. First, we note that by increasing B we can reduce to the case that $B > m(s)$ for all $s \in S$.

Let the manifold $S^1 = \mathbb{R}/\mathbb{Z}$ be the standard circle. Since the dimension of M is at least 3, we can represent the free homotopy classes of the elements $s \in S$ by smooth embeddings $\sigma_s\colon S^1 \to M$ with pairwise disjoint images. The tubular neighborhood theorem [Hi, Thm. 5.2, Ch. 4] says that for each $s \in S$ there is a smooth vector bundle N_s over S^1 together with a diffeomorphism $i_s\colon N_s \to T_s \subset M$ onto an open subset T_s of M, whose composition with the zero section $S^1 \to N_s$ is the map σ_s. The loops σ_s have positive distance between each other, so by shrinking the neighborhoods, if necessary, we can also assume that the tubular neighborhoods $T_s = \operatorname{Im}(i_s)$ are disjoint.

Let us consider the structure of such a tubular neighborhood T_s. If n is the dimension of M, then the bundle N_s on S^1 has rank $n - 1$. There are up to isomorphism exactly two vector bundles of rank r over S^1: one orientable and one non-orientable [Hi, Ch. 4, Sec. 4, Ex. 2], [Ra, Chp. 5]. In view of choosing metrics later, let B^{n-1} be the standard open unit ball around 0 in $\mathbb{R}^{n-1}$, which is diffeomorphic to each fiber of the vector bundle N_s over S^1. Now consider the quotient $(B^{n-1} \times \mathbb{R})/\mathbb{Z}$, where $n \in \mathbb{Z}$ acts by sending $((x_1, x_2, \ldots, x_{n-1}), t)$ to $((x_1, x_2, \ldots, x_{n-1}), t + n)$ if N_s is orientable, and to $((-x_1, x_2, \ldots, x_{n-1}), t + n)$ if N_s is non-orientable. Then it follows that there is a commutative diagram:

$$
\begin{array}{ccc}
t \mapsto (0,t) & S^1 & \sigma_s \\
& \downarrow \text{0-section} & \\
(B^{n-1} \times \mathbb{R})/\mathbb{Z} \xrightarrow{\ \sim\ } & N_s \xrightarrow[i_s]{\ \sim\ } & T_s
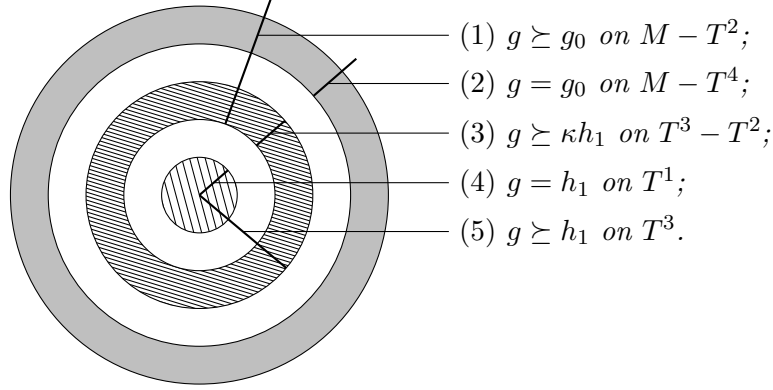\end{array}
$$

where the horizontal maps are diffeomorphisms.

In both the orientable and non-orientable case, the diffeomorphisms give rise to the following additional structure on T_s. First, the standard Riemannian metric on $(B^{n-1} \times \mathbb{R})/\mathbb{Z}$ gives a Riemannian metric h_0 on T_s that has the property that for every $k \in \mathbb{Z}$ each loop within T_s homotopic to σ_s^k has length at least $|k|$, with equality for the loop σ_s^k itself. Second, we obtain a smooth map $r_s\colon T_s \to [0, 1)$ by sending the point associated to $(x_1, \ldots, x_{n-1}, t)$ to the length of the vector $(x_1, \ldots, x_{n-1}) \in B^{n-1}$. Note that r_s is 0 on $\operatorname{Im}(\sigma_s)$.

For each $s \in S$ we will consider five open neighborhoods $T_s^i = \{x \in T_s : r_s(x) < i/5\}$ on $\text{Im}(\sigma_s)$ for $i = 1, \ldots, 5$, where $T_s^5 = T_s$. We write $T = \bigcup_s T_s$ and $T^i = \bigcup_s T_s^i$, and we let $r : T \to [0, 1)$ be the smooth map given by r_s on each T_s.

Let h_1 be the metric on T whose restriction to the tubular neighborhood T_s is given by $m(s) \cdot h_0$. Then with respect to the metric h_1, the loop σ_s has length $m(s)$, for each $s \in S$. Now, let $\kappa \geq 1$ be a constant such that with respect to $\kappa \cdot h_1$ the distance between $T - T^3$ and T_2 is at least B. Since M is compact, we may fix a Riemannian metric g_0 on M such that every non-contractible loop in M has length at least B; that is, we may choose g_0 so that its systole is at least B.

5.3. **Lemma.** *With the notation and assumptions as above, there is a Riemannian metric g on M that satisfies the following properties:*

(1) $g \succeq g_0$ *on* $M - T^2$;

(2) $g = g_0$ *on* $M - T^4$;

(3) $g \succeq \kappa h_1$ *on* $T^3 - T^2$;

(4) $g = h_1$ *on* T^1;

(5) $g \succeq h_1$ *on* T^3.

Proof of Lemma. First consider the metric $h_2 = g_0 + \kappa h_1$ on T: it clearly satisfies $h_2 \succeq g_0$ on T. Now, let $f_1 : [0, 1] \to [0, 1]$ be a smooth function with $f_1(t) = 1$ for $t \leq 3/5$ and $f_1(t) = 0$ for $t \geq 4/5$. We define the metric $\hat{g}$ by gluing two metrics on open subsets of M that coincide on their intersection $T - T^4$:

$$\hat{g} = \begin{cases} (f_1 \circ r)h_2 + (1 - (f_1 \circ r))g_0 & \text{on } T \\ g_0 & \text{on } M - T^4 \end{cases}$$

On M we have $\hat{g} \succeq g_0$, and on T^3 we have $\hat{g} = h_2 \succeq \kappa \cdot h_1 \succeq h_1$.

Now let $f_2 : [0, 1] \to [0, 1]$ be a smooth function such that $f_2(t) = 1$ for $t \leq 1/5$ and $f_2(t) = 0$ for $t \geq 2/5$ and set

$$g = \begin{cases} (f_2 \circ r)h_1 + (1 - (f_2 \circ r))\hat{g} & \text{on } T \\ \hat{g} & \text{on } M - T^2 \end{cases}$$

Then g satisfies properties (1)–(5).   □

We claim that the metric g produced above has the desired properties.

Indeed, let $x \in \pi_1(M)$, and suppose c is a loop of shortest length in the free homotopy class associated to x: c will necessarily have length $m_g(x)$. Then we are in at least one of the following three cases.

Case A. If $\mathrm{Im}(c) \subset T^3$, then there is a unique $s \in S$ such that $\mathrm{Im}(c) \subseteq T_s^3$. Then c is freely homotopic within T_s to σ_s^k for some $k \neq 0 \in \mathbb{Z}$. We have $g \succeq h_1$ on T_s^3, so c has length at least $|k|m(s)$ with respect to g.

Case B. If $\mathrm{Im}(c) \subset M - T^2$, then we use that $g \succeq g_0$ on $M - T^2$ to see that c has length at least B.

Case C. If $\mathrm{Im}(c)$ contains points in T^2 and in $M - T^3$, then the length of c is at least the distance measured with metric g between T^2 and $M - T^3$. But since $g \succeq \kappa h_1$ on $T^3 - T^2$ the definition of κ shows that this distance is at least B.

To finish the proof, we first notice that σ_s^k has length $|k|m(s)$ by property (4) of the lemma. Now, suppose $x \in \pi_1(M)$, with associated loop c (as above), is such that the set $\mathcal{S}_x = \{|k|m(s) : s \in S, k \in \mathbb{Z}$ with σ_s^k freely homotopic to $c\}$ is nonempty with infimum l_x. If this infimum is below B, then we must be in case A, and $l_x = m_g(x)$. In particular, for $x \in S$ we know that l_x is less than B by our assumption at the start of the proof, and $l_x = m(x)$ by the assumption on m in the Theorem, and it follows that $m(x) = m_g(x)$. If the set $\mathcal{S}_x$ is non-empty and its infimum is at least B, then $m_g(x) \geq B$ in all three cases. If $\mathcal{S}_x$ is empty, then we cannot be in case A and we also deduce that $m_g(x) \geq B$. This completes the proof of Theorem 5.1.   □

Proof of Theorem 2.6. In the first part of Remark 5.2 we established that conditions (i) and (ii) in the hypotheses of Theorem 5.1 are equivalent to m being the restriction of a length map on $\pi_1(M)$. Hence, we see that Theorem 2.6 is just the first statement of Theorem 5.1.   □

Before proving Theorem 2.8 we recall that its moral is that we can find a metric g with a prescribed value and location for the first k elements of the minimum marked length spectrum (see Definition 2.7). Now, given distinct unoriented free homotopy classes $c_1, c_2, \ldots, c_k$ and a minimum marked length map m_g, the length map axioms imply certain relations between the values $l_i = m_g(c_i)$. Examining conditions (ii) and (1) in the statement of Theorem 5.1 and keeping the moral of Theorem 2.8 in mind we are led to make the following definition.

5.4. **Definition.** Let M be a closed manifold and let $\mathcal{F}(M)$ denote its collection of unoriented free homotopy classes. Given a finite sequence $C = (c_1, c_2, \ldots, c_k)$ of distinct elements of $\mathcal{F}(M)$ with c_1 the trivial class, we will say that a sequence $l_1 = 0 < l_2 \leq l_3 \leq \cdots \leq l_k$ is *C-admissible* if it satisfies the following conditions:

(1) for $i, j = 2, \ldots, k$ we have $l_i \leq nl_j$ whenever $c_i = c_j^n$ for some $n \in \mathbb{Z}$;
(2) for $i = 2, \ldots, k$ we have $l_i \geq \frac{1}{|n|}l_k$ whenever $c_i^n \notin \{c_1, \ldots, c_n\}$ for some non-zero $n \in \mathbb{Z}$.

5.5. **Example** (*C-admissibility*). One can quickly verify that a sequence $l_1 = 0 < l_2 \leq l_3 \leq \cdots \leq l_k$ such that $2l_2 \geq l_k$ is *C*-admissible for any choice of $C = (c_1, c_2, \ldots, c_k)$ in the definition above. Also, we note that if $\{c_1, \ldots, c_n\}$ is a subset that is closed under taking powers, the definition of *C*-admissibility reduces to condition (1) (cf. Remark 5.2).

Proof of Theorem 2.8. To prove that (5) implies (6) first recall from homotopy theory that two homotopy classes $x, y \in \pi_1(M)$ give rise to the same unoriented free homotopy class in $\mathcal{F}(M)$ if and only if x is conjugate to y or to y^{-1} in $\pi_1(M)$. As a consequence, any minimum marked length map $m_g : \pi_1(M) \to \mathbb{R}_{\geq 0}$ induces a map $m_g : \mathcal{F}(M) \to \mathbb{R}_{\geq 0}$ called the minimum marked length spectrum (cf. Definition 2.7).

Now, as in the statement of Theorem 2.8, let $C = (c_1, \ldots, c_k)$ be a sequence of k distinct unoriented free homotopy classes, where c_1 is trivial and let $0 = l_1 < l_2 \leq \cdots \leq l_k$ be a C-admissible sequence. For each $i = 1, \ldots, k$ choose $x_i \in \pi_1(M)$ within the class c_i. Let $S = \{x_1, \ldots, x_k\} \subset \pi_1(M)$ and define $m \colon S \to \mathbb{R}_{\geq 0}$ by $m(x_j) = l_j$. Then it follows from condition (1) of C-admissibility that m satisfies the hypotheses of Theorem 5.1. Hence, choosing $B \geq l_k = \max_j m(x_j)$ in Theorem 5.1 it follows from condition (2) of C-admissibility that we obtain a metric g with the desired properties.

To prove that (6) implies (5), let g be a metric on M such that $l_1 = m_g(c_1) = 0 < l_2 = m_g(c_2) \leq l_3 = m_g(c_3) \leq \cdots \leq l_k = m_g(c_k)$ and $m_g(c) \geq l_k$ for all $c \in \mathcal{F}(M) - \{c_1, \ldots, c_k\}$. Then an examination of the length map axioms allows us to conclude that the sequence $l_1 = 0 < l_2 \leq \cdots \leq l_k$ is C-admissible. $\square$

5.6. **Remark.** The statement and proofs of Theorems 5.1 and 2.8 also hold when M is a closed manifold of dimension 2 and the sequence of free homotopy classes associated to the elements of S can be represented by disjoint simple closed loops in M: a condition that is always met in dimension 3 and higher. It is worth noting that Colin de Verdière's result [CdV2] on prescribing the Laplace spectrum fails in dimension 2 precisely because not every finite collection of free homotopy classes can be represented by a collection of pairwise disjoint simple closed curves [Ber, p. 415].

We close this section by noting that in the case where $\pi_1(M)$ is finite, we may take S to be all of $\pi_1(M)$ in Theorem 5.1 and $m \colon S \to \mathbb{R}_{\geq 0}$ to be any length map in the sense of Definition 2.3. Then since every finitely generated group is the fundamental group of some closed 4-manifold [GoSt, Exercise 4.6.4(b)] we obtain the following result, which says that every length map on a finite group is the minimum marked length map of some Riemannian manifold.

5.7. **Corollary.** *Let* $m \colon G \to \mathbb{R}_{\geq 0}$ *be a length map on a finite group* G. *Then there exists a Riemannian manifold* (M, g) *and an isomorphism* $\varphi \colon G \to \pi_1(M)$ *so that* $m = m_g \circ \varphi$.

6. A GROUP THEORETIC CRITERION FOR EQUALITY OF COVERING SPECTRA

The aim of this section is to establish Theorem 2.2. First, we identify how the covering spectrum of a covering of a Riemannian manifold can be computed in terms of the base space.

Let $p \colon (M, g) \to (N, h)$ be a Riemannian covering; that is, p is a covering map that is also a local isometry. Fixing a base point m_0 of M and $p(m_0)$ of N, we have an injective group homomorphism $p_\# \colon \pi_1(M) \to \pi_1(N)$ given by $[\sigma] \mapsto [p \circ \sigma]$. Since p is a local isometry, the lengths of σ and $p \circ \sigma$ are the same, so we have the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(M) & \xrightarrow{\ p_\# \ } & \pi_1(N) \\
 & m_g \searrow \quad \swarrow m_h & \\
 & \mathbb{R}_{\geq 0} &
\end{array}$$

From this the following observation is immediate.

6.1. **Lemma.** $\mathrm{CovSpec}(M) = \frac{1}{2}\,\mathrm{Jump}(\mathrm{Fil}^\bullet_{m_h}\,\mathrm{Im}(p_\#))$.

Hence, if $p : (M, g) \to (N, h)$ and $p : (M', g') \to (N, h)$ are Riemannian coverings of a common base space (N, h), then $\mathrm{CovSpec}(M, g) = \mathrm{CovSpec}(M', g')$ if and only if $\mathrm{Jump}(\mathrm{Fil}^\bullet_{m_h} H) = \mathrm{Jump}(\mathrm{Fil}^\bullet_{m_h} H')$, where $H = \mathrm{Im}(p_\#)$ and $H' = \mathrm{Im}(p'_\#)$ are subgroups of $G = \pi_1(N)$.

6.2. **Remark.** A word of warning is in order here. Suppose that m is a length map on a group G and H is a subgroup of G, then the restriction of m to H is a length map on H that gives rise to the filtration $\mathrm{Fil}^\bullet_m H$. Note however that $\mathrm{Fil}^\delta_m H$ may not be the same as $(\mathrm{Fil}^\delta_m G) \cap H$. The first group consists of finite products of elements of length below δ where all elements are in H, and the second group consists of such products where the elements are in G and only the product is required to be in H. Thus, to obtain the correct covering spectrum of M out of the length map m_h on N one has to first restrict m_h to $\mathrm{Im}(p_\#)$, rather than restricting the filtration of $\pi_1(N)$ to $\mathrm{Im}(p_\#)$.

In order to prepare for the proof of Theorem 2.2, we first give an interpretation of condition (3) of the theorem in terms of length maps.

6.3. **Lemma.** *Let G be a finite group, and let H and H' be subgroups. Then the following are equivalent:*

    (1) *for all subsets $S \subset T \subset G$ that are stable under conjugation by elements of G we have*

$$\langle H \cap S \rangle = \langle H \cap T \rangle \iff \langle H' \cap S \rangle = \langle H' \cap T \rangle$$

    (2) *for every length map m on G we have $\mathrm{Jump}(\mathrm{Fil}^\bullet_m H) = \mathrm{Jump}(\mathrm{Fil}^\bullet_m H')$.*

6.4. **Definition.** We will say that H and H' are *jump equivalent* subgroups of G or that (G, H, H') is a *jump triple*, if the conditions in the lemma above are met.

In the next section we will see how the notion of jump equivalence is related to other equivalence relations of subgroups.

6.5. **Remark.** There is some freedom in choosing the range of conjugacy stable subsets of G that S and T vary over in condition (1). For instance, one obtains an equivalent condition when one lets S and T vary over all conjugacy stable subsets of G, so that condition (1) in the lemma is equivalent to condition (3) of Theorem 2.2.

Proof of Lemma. Suppose that (1) holds and that m is a length map on G. For every $\delta > 0$ the subset $S_\delta = \{g \in G : m(g) < \delta\}$ is stable under conjugation. We have $\{h \in H : m(h) < \delta\} = H \cap S_\delta$, so

$$\delta \in \text{Jump}(\text{Fil}_m^\bullet H) \iff \langle H \cap S_\delta \rangle \neq \langle H \cap S_\epsilon \rangle \text{ for all } \epsilon > \delta.$$

By condition (1) we see that this is equivalent to the same statement for H', and (2) follows.

To show that (2) implies (1) suppose that $S, T \subset G$ are conjugacy stable subsets of G, and that $S \subset T$. We will construct a length map $m \colon G \to \mathbb{R}_{\geq 0}$ that depends on S and T so that condition (2) for this length map implies that the equivalence in (1) holds.

We may assume that S and T are closed under taking inverses. Define m by

$$m(g) = \begin{cases} 0 & \text{if } g = 1; \\ 2 & \text{if } g \in S - \{1\}; \\ 3 & \text{if } g \in T - (S \cup \{1\}); \\ 4 & \text{otherwise.} \end{cases}$$

Then m satisfies the length map axioms of Definition 2.3, and we have $(H \cap S) \cup \{1\} = \{h \in H : m(h) < 3\}$, and $(H \cap T) \cup \{1\} = \{h \in H : m(h) < 4\}$. It follows that

$$\langle H \cap S \rangle = \langle H \cap T \rangle \iff \text{Fil}_m^3 H = \text{Fil}_m^4 H \iff 3 \notin \text{Jump}(\text{Fil}_m^\bullet H).$$

The same holds when we replace H by H', so indeed condition (2) for m implies the equivalence in (1). $\square$

With these preliminaries out of the way we now turn to the proof of Theorem 2.2.

Proof of Theorem 2.2. First, we notice that since G acts freely on M we see that the natural projection $q \colon M \to G\backslash M$ is a regular covering with G serving as the group of deck transformations. Let us choose a base point in M for the fundamental group $\pi_1(M)$, which also determines a base point in the quotient of M by any subgroup of G. By [Sp, Cor. 2.6.3] there is a surjective homomorphism $\varphi \colon \pi_1(G\backslash M) \to G$ with $\ker(\varphi) = q_\#(\pi_1(M))$. Hence, φ is an isomorphism if and only if M is simply-connected. We also note that, letting $p \colon H\backslash M \to G\backslash M$ and $p' \colon H'\backslash M \to G\backslash M$ denote the covering maps, φ maps $p_\#(\pi_1(H\backslash M))$ and $p'_\#(\pi_1(H'\backslash M))$ onto H and H' respectively.

We now prove that if M is simply-connected, then (3) implies (4). Assume that the triple (G, H, H') satisfies condition (3). Then, since $\varphi \colon \pi_1(G\backslash M) \to G$ is an isomorphism, we see that condition (3) holds for the triple $(\pi_1(G\backslash M), p_\#(\pi_1(H\backslash M)), p'_\#(\pi_1(H'\backslash M)))$. The result now follows by applying Lemma 6.3 and Lemma 6.1 to the coverings $p \colon M \to H\backslash M$ and $p' \colon M \to H'\backslash M$.

We now assume that M is a manifold of dimension at least 3 and prove that in this case condition (4) implies condition (3).

Assuming condition (3) does not hold, our goal will be to construct a Riemannian metric on $G\backslash M$ such that the length map attains certain prescribed values on a suitably chosen

finite set of conjugacy classes of $\pi_1(G\backslash M)$. Pulling this metric back to M will then give a G-invariant Riemannian metric on M so that the quotient manifolds $H\backslash M$ and $H'\backslash M$ have distinct covering spectra.

Note first that $\pi_1(G\backslash M)$ is finitely generated, and that G is finite, so $\ker\varphi$ is generated as a group by a finite subset K of $G - \{1\}$; see e.g. [Ha, Cor. 7.2.1]. By the assumption that condition (3) does not hold, one sees that there are subsets S and T of $G - \{1\}$ both stable under conjugation by elements of G, and taking inverses, so that we have $S \subset T$, and $\langle H \cap S\rangle = \langle H \cap T\rangle$, but $\langle H' \cap S\rangle \neq \langle H' \cap T\rangle$ (switch H and H' if necessary). Now choose any lift T' of T to $\pi_1(G\backslash M)$, i.e., any subset $T' \subset \pi_1(G\backslash M)$ so that φ maps T' bijectively to T. Let $S' = \{t \in T'\colon \varphi(t) \in S\}$, which is a lift of S.

We will prescribe the length map on the finite subset $K \cup T'$ of $\pi_1(G\backslash M)$. Note that this union is a disjoint union. For any subset X of $\pi_1(G\backslash M)$ let us write $c(X) = \{\gamma y \gamma^{-1}\colon \gamma \in \pi_1(G\backslash M),\ y \in X\}$ for its closure under conjugation. Since we assumed that the dimension of M is at least 3, Theorem 5.1 now implies that there is a Riemannian metric g on $G\backslash M$ such that the associated minimum marked length map $m_g\colon \pi_1(G\backslash M) \to \mathbb{R}_{\geq 0}$ satisfies

$$m_g(\gamma) \begin{cases} = 0 & \text{if } \gamma = 1; \\ = 3 & \text{if } \gamma \in c(K \cup S'); \\ = 4 & \text{if } \gamma \in c(T' - S'). \\ > 5 & \text{otherwise} \end{cases}$$

For any conjugacy stable $X \subset \pi_1(G\backslash M)$ that contains K the group $\langle X \cap \varphi^{-1}(H)\rangle$ contains $\ker\varphi = \langle K\rangle$, and its image under φ is $\langle \varphi(X) \cap H\rangle$, so we have

$$\langle X \cap \varphi^{-1}(H)\rangle = \varphi^{-1}(\langle \varphi(X) \cap H\rangle).$$

Since $\langle H \cap S\rangle = \langle H \cap T\rangle$, we see that for every δ with $4 < \delta < 5$ we have

$$\mathrm{Fil}^4_{m_g}\varphi^{-1}(H) = \langle c(K \cup S') \cap \varphi^{-1}(H)\rangle = \varphi^{-1}(\langle S \cap H\rangle)$$
$$= \varphi^{-1}(\langle T \cap H\rangle) = \langle c(K \cup T') \cap \varphi^{-1}(H)\rangle = \mathrm{Fil}^\delta_{m_g}\varphi^{-1}(H).$$

On the other hand, since $\langle H' \cap S\rangle \neq \langle H' \cap T\rangle$, we have for each $4 < \delta < 5$

$$\mathrm{Fil}^4_{m_g}\varphi^{-1}(H') = \varphi^{-1}(\langle S \cap H'\rangle) \neq \varphi^{-1}(\langle T \cap H'\rangle) = \mathrm{Fil}^\delta_{m_g}\varphi^{-1}(H').$$
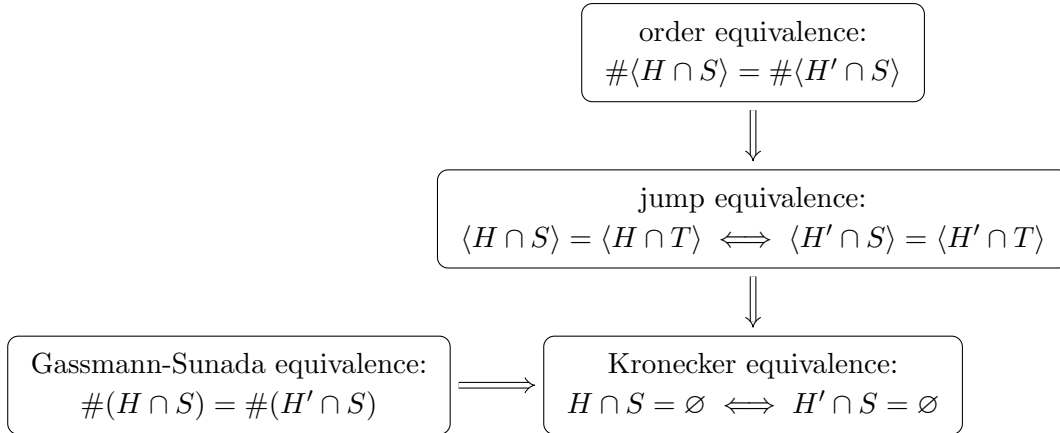
With Lemma 6.1 it now follows that 4 lies in the covering spectrum of $H'\backslash M$ but not in the covering spectrum of $H\backslash M$. This completes the proof of Theorem 2.2. $\square$

7. Jump equivalence and Gassmann-Sunada equivalence

Suppose that G is a finite group, and let H and H' be subgroups. In this entirely group theoretic section we study the notion of jump equivalence of H and H', as defined in Definition 6.4.

Like Gassmann-Sunada equivalence, jump equivalence is a strong form of the group theoretic notion corresponding to Kronecker equivalence of number fields [J]. Many instances

of Gassmann-Sunada equivalence, such as the Komatsu triples considered in [SW], satisfy a stronger condition, which we call *order equivalence*. The definitions and relations of these four equivalences are given in the diagram below, where the stated conditions are to hold for all subsets S, T of G that are closed under conjugation by elements of G.

$$\boxed{\begin{array}{c} \text{order equivalence:} \\ \#\langle H \cap S \rangle = \#\langle H' \cap S \rangle \end{array}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} \text{jump equivalence:} \\ \langle H \cap S \rangle = \langle H \cap T \rangle \iff \langle H' \cap S \rangle = \langle H' \cap T \rangle \end{array}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} \text{Gassmann-Sunada equivalence:} \\ \#(H \cap S) = \#(H' \cap S) \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{Kronecker equivalence:} \\ H \cap S = \varnothing \iff H' \cap S = \varnothing \end{array}}$$

It is not hard to show the implications in the diagram—we leave this to the reader. The main purpose of this section is to provide examples of Gassmann-Sunada equivalent subgroups that are not jump equivalent, which in the next section give rise to isospectral manifolds with distinct covering spectra. In fact, we will see that there are no relations between the four equivalences that hold for all (G, H, H') other than the ones implied by the diagram. We start with a few basic remarks.

7.1. **Remark.** One obtains an equivalent condition for jump equivalence by restricting the condition to the case where $S \subset T$, or even to the case where T is the union of S and a single conjugacy class of G. For Gassmann-Sunada equivalence and Kronecker equivalence, but not for order equivalence, one may restrict to subsets S consisting of a single conjugacy class.

7.2. **Remark** (Preserving the index). If H and H' are order equivalent or Gassmann-Sunada equivalent, then $[G : H] = [G : H']$. Thus, condition (1) in Theorem 2.1 implies that $H \backslash M$ and $H' \backslash M$ are coverings of the same degree of $G \backslash M$. For Kronecker and jump equivalence this is not necessarily so. The easiest example is the jump triple (A_4, V_4, C_2): the alternating group A_4 on four letters, its unique subgroup V_4 of order 4 and any subgroup C_2 of order 2.

7.3. **Remark** (Reduced triples). Another property that holds for Gassmann-Sunada equivalence, but not for jump equivalence is the following. Suppose that N is a normal subgroup of G that is contained in $H \cap H'$. If the only such N is the trivial group, then we say that the triple (G, H, H') is *reduced*. It is easy to see that (G, H, H') is a Gassmann-Sunada triple if and only if $(G/N, H/N, H'/N)$ is a reduced Gassmann-Sunada triple. The same is true for Kronecker equivalence. For jump equivalence the situation is different. If (G, H, H') is

a jump triple, then so is $(G/N, H/N, H'/N)$, but the converse may not hold. For instance, multiplying all groups in the example in the previous remark by C_2, the reader may check that $(C_2 \times A_4, C_2 \times V_4, C_2 \times C_2)$ is not a jump triple, while dividing out $C_2 \times \{1\}$ gives the jump triple (A_4, V_4, C_2). We will see more examples of this below.

7.4. **Example** (Small index). For many Gassmann-Sunada triples (G, H, H') there is an automorphism α of G such that $H' = \alpha(H)$ and $\alpha(g)$ is conjugate to g for each $g \in G$. For, instance this is the case for all 19 reduced Gassmann-Sunada triples (G, H, H') with $[G : H] < 16$; see [BD, Thm. 3]. Such subgroups, H and H', are order equivalent and therefore also jump equivalent in G.

7.5. **Example** (Linear groups). One way to make examples of Gassmann-Sunada triples is the following. Let k be a finite field of q elements and let V be a vector space of dimension d. Let $G = \mathrm{Gl}(V)$, let H be the stabilizer in G of a non-zero element of V, and let H' be the stabilizer of a non-zero vector in the dual space $\mathrm{Hom}_k(V, k)$. Then H and H' are Gassmann-Sunada equivalent subgroups of $\mathrm{Gl}(V)$ and H is not conjugate to H' if $d \geq 2$ and $(q, d) \neq (2, 2)$. Again, there is an automorphism α as in the previous remark, so H and H' are order equivalent and also jump equivalent in $\mathrm{Gl}(V)$.

One can obtain a Gassmann-Sunada triple that is not a jump triple out of this example by enlarging the linear group to the group $V \rtimes G$, which is the group of affine linear transformations of V. Then V is a normal subgroup of $V \rtimes G$, and $(V \rtimes G, V \rtimes H, V \rtimes H')$ is a Gassmann-Sunada triple that is not reduced if $d > 0$.

7.6. **Proposition.** *If $d \geq 2$ and $(q, d) \neq (2, 2)$, then $(V \rtimes G, V \rtimes H, V \rtimes H')$ is a Gassmann-Sunada triple that is not a jump triple.*

Proof. Let S be the subset of those elements of $V \rtimes G$ that have a conjugate in $\{0\} \rtimes H$. Since H and H' are Gassmann-Sunada equivalent in G this is also the subset of elements of $V \rtimes G$ that have a conjugate in $\{0\} \rtimes H'$. We will show that $\langle S \cap (V \rtimes H) \rangle = V \rtimes H$ and $\langle S \cap (V \rtimes H') \rangle \neq V \rtimes H'$. By taking $T = V \rtimes G$ in the defining property of jump equivalence the Proposition will then follow.

Note that the vector space V is a left module over the group ring $\mathbb{Z}[H]$. The augmentation ideal $I(H)$ in this ring is the ideal generated by all elements $1 - h$ with $h \in H$. We now claim that
$$\langle S \cap (V \rtimes H) \rangle = (I(H) \cdot V) \rtimes H.$$
To see "$\subset$" note that any element of $S \cap (V \rtimes H)$ is of the form
$$(v, \gamma)(0, h)(v, \gamma)^{-1} = ((1 - \gamma h \gamma^{-1})v, \gamma h \gamma^{-1})$$
with $h \in H$, $\gamma \in G$ and $v \in V$ satisfying $\gamma h \gamma^{-1} \in H$, and therefore $1 - \gamma h \gamma^{-1} \in I(H)$. For the other inclusion, note first that $\{0\} \times H \subset S \cap (V \rtimes H)$. By taking $\gamma = 1$ in the identity above, one sees that for every $h \in H$ and $v \in V$ we have $((1 - h)v, h) \in S \cap (V \rtimes H)$, and therefore $((1 - h)v, 1) \in S \cap (V \rtimes H)$. One then deduces "$\supset$".

This shows the claim, which by the same argument also holds with H replaced by H'. It remains to show $I(H) \cdot V = V$ and $I(H') \cdot V \neq V$. Since H' is the stabilizer of a non-zero linear map $\varphi \colon V \to k$, any element $h \in H'$ acts trivially on $V/\ker \varphi$, so $I(H') \subset \ker \varphi \neq V$.

Let us now show that $I(H) \cdot V = V$. Choose a basis $e_1, \ldots, e_d$ of V as a vector space over k so that H fixes the first basis vector. The element $h \in H$ sending e_1 to e_1 and e_i to $e_i + e_{i-1}$ for $i > 1$ satisfies $(1-h)V = \mathrm{Span}_k\{e_1, \ldots, e_{d-1}\}$. Thus, it now suffices to show that $e_d \in I(H) \cdot V$. When $d \geq 3$ this follows by what we did already, because of the freedom to choose the basis. When $d = 2$, we assumed that $q \neq 2$, so there is an element $\lambda \in k$ that is not 0 or 1. Now the element h of H given by $e_1 \mapsto e_1$ and $e_2 \mapsto \lambda e_2$ satisfies $(1-h)V = ke_2$. $\qquad \square$

Taking $q = 2$ and $d = 3$ this gives a non-reduced Gassmann-Sunada triple (G, H, H') with $[G:H] = 7$ that is not a jump triple. There are no non-trivial Gassmann-Sunada triples where $[G:H]$ is smaller.

7.7. **Example** (Todd-Komatsu method). A different method to construct Gassmann-Sunada triples is to start with two finite groups H and H' of some order n, such that for all divisors d of n the groups H and H' have the same number of elements of order d. By numbering the group elements, and considering the regular actions of the groups on themselves by left-multiplication we obtain embeddings of H and H' into S_n. Under such an embedding each element of order d is sent to a product of n/d disjoint cycles of length d. Then since two elements in S_n are conjugate exactly when they have the same cycle decomposition, it follows that for each d all elements of order d from H or H' are conjugate in S_n. This shows that the triple (S_n, H, H') is a Gassmann-Sunada triple. This triple is non-trivial (i.e., H and H' are non-conjugate subgroups of S_n) if and only if H is not isomorphic to H'.

The Komatsu [K] triples are the ones obtained in this way when H and H' are two finite groups of the same order n and with the same odd prime p as exponent; that is, $x^p = 1$ for all $x \in H$ or H'. In this case all non-trivial elements of H and H' are in the same conjugacy class of S_n, so H and H' are order equivalent and jump equivalent. Thus, as noted by Sormani and Wei [SW, Proposition 10.6] Komatsu isospectral manifolds have the same covering spectrum.

Another instance of this method was used by Todd [T] in 1949 to give a counterexample to a conjecture of Littlewood that for any Gassmann-Sunada triple (G, H, H') the groups H and H' are isomorphic. Consider the groups $H = C_8 \times C_2$ and $H' = C_8 \rtimes C_2$, where C_n denotes a cyclic group of order n and the generator of C_2 in the semidirect product H' acts on C_8 by raising all elements to the fifth power. Then H and H' are groups of order 16 that each have 1, 3, 4 and 8 elements of order 1, 2, 4 and 8 respectively, and we obtain Todd's Gassmann-Sunada triple (S_{16}, H, H'). Again, H and H' are also order equivalent. However, Todd's idea can be slightly modified to produce a Gassmann-Sunada triple that is not a jump triple.

7.8. **Proposition.** *Let $N = C_4 \times C_2 = \langle a, b : a^4 = b^2 = 1, ab = ba \rangle$, and define the groups $H = N \times \langle c : c^2 = 1 \rangle$ and $H' = N \rtimes \langle c : c^2 = 1 \rangle$ of order 16, where in the semidirect product c acts on N by $a \mapsto a$ and $b \mapsto a^2 b$. Then we have*

(1) *for every $d \mid 16$ the groups H and H' have the same number of elements of order d;*
(2) *the Gassmann-Sunada triple (S_{16}, H, H') is not a jump triple;*
(3) *the Gassmann-Sunada triple $(S_{64}, H \times C_4, H' \times C_4)$ is a jump triple, but the subgroups are not order equivalent.*

Proof. Note that for the group H the square of $a^i b^j c^k$, where $i, j, k \in \mathbb{Z}$, is a^{2i}, while for H' the square of $a^i b^j c^k$ is $a^{2(i+jk)}$. Thus, in both groups it is true that given any $j, k \in \{0, 1\}$ there are exactly two $i \in \{0, 1, 2, 3\}$ such that $(a^i b^j c^k)^2 \neq 1$. One deduces that H and H' both have 8 elements of order 4 and 7 elements of order 2. Since H and H' are both of order 16 this establishes the first statement.

By the method described in 7.7, we obtain a Gassmann-Sunada triple (S_{16}, H, H'). Now it is easy to see that the elements of order 2 generate a subgroup of index 2 in H, but in H' they generate the whole group. Taking $S = \{\sigma \in S_{16} : \sigma^2 = 1\}$ and $T = S_{16}$ we see that H and H' are not jump equivalent in S_{16}.

For the third statement, note that the two subgroups are contained in three conjugacy classes C_1, C_2, C_4 of S_{64}, where C_i consists of elements of order i. We now have $\langle C_4 \cap H \rangle = H$ and $\langle C_4 \cap H' \rangle = H'$, while $[H : \langle C_2 \cap H \rangle] = 4$ and $[H' : \langle C_2 \cap H' \rangle] = 2$. Thus, we see that H and H' are jump equivalent, and Gassmann-Sunada equivalent in S_{64}, but not order equivalent. $\square$

7.9. Example (Mathieu groups)**.** Another example of a Gassmann-Sunada triple that is not a jump triple is the triple $(M_{23}, 2^4 A_7, M_{21} * 2)$ mentioned by Guralnick and Wales [GW, p. 101]. Here M_i denotes the Mathieu group in degree i. To see that the jump condition does not hold, one may check, for instance with Magma [BCP], that both subgroups $2^4 A_7$ and $M_{21} * 2$ are generated by their elements of order 3, and that $2^4 A_7$ is generated by its elements of order 2 while $M_{21} * 2$ is not.

7.10. Example (Order equivalence)**.** We conclude this section with a triple (G, H, H') where H is order equivalent to H', but not Gassmann-Sunada equivalent. Consider the action of the alternating group A_4 on a regular tetrahedron. Let E be the set of its 6 edges and choose an element $e \in E$. Then the group A_4 acts transitively on E, and the stabilizer of e is a subgroup of order 2.

Let V be the vector space over the field of 2 elements with E as a basis. Then A_4 acts on V by permuting coordinates. We now let W be 5-dimensional subspace of V where the sum of the coordinates is zero, and we put $G = W \rtimes A_4$, which is a group of order 384. We let H be the 4-dimensional subspace of W consisting of the vectors with coordinate 0 at e. To define H', choose a subset E' of E consisting of 3 edges in such a way that E' contains two non-adjacent edges, and let H' be the 4-dimensional subspace of W consisting of the vectors whose coordinate sum over E' is zero. Then one can check that H and H' are order equivalent in G, but not Gassmann-Sunada equivalent.

7.11. Example (More jump equivalent triples that are not Gassmann-Sunada equivalent)**.** In Remark 7.2 we gave an example of a jump triple that is not a Gassmann-Sunada triple. We

now demonstrate that such examples can be constructed in a rather routine fashion. Indeed, let V be a finite dimensional vector space over a finite field k, and let $G = V \rtimes \mathrm{Gl}(V)$ denote the affine transformation group of V. We will agree to call an element $v \in V \leq G$ a *translation* and any subspace $W \leq V$ a *translation subgroup*. Since $\mathrm{Gl}(V)$ acts transitively on the non-trival vectors of V it follows that the conjugacy class of an element $v \in V - \{0\}$ consists entirely of all the non-trivial translations of V. Therefore, any two non-trivial translation subgroups $V_1, V_2 \leq G$ are Kronecker equivalent in G (cf. [LMNR, Thm. 2.10(i)]).

Now, suppose $S \subset G$ is stable under conjugation. Then, we have $\langle V_i \cap S \rangle$ is trivial or all of V_i (for $i = 1, 2$) depending on whether S contains a non-trivial translation or not, and we conclude that (G, V_1, V_2) is a jump triple. If we take V_1 and V_2 to be of different dimensions (and hence of different orders), then (G, V_1, V_2) is a jump triple that is not Gassmann-Sunada.

8. Covering spectra of isospectral manifolds

In this last section, we turn to isospectral manifolds and their covering spectra. We first describe how to use the results of the previous two sections in order to prove Theorem 2.9. We then show that the isospectral lattices of Conway and Sloane can also give rise to isospectral flat tori of dimension 4 with distinct covering spectra. At the end of the section we study the covering spectra of isospectral Heisenberg manifolds and refute the claim made in [SW, Example 10.3].

Proof of Theorem 2.9. Given $n \geq 3$ we will produce isospectral Riemannian manifolds of dimension n with distinct covering spectra. In the previous section we saw that there exists a Gassmann-Sunada triple (G, H, H') such that H and H' are not jump-equivalent in G. Suppose that G can be generated by k generators: $g_1, \ldots, g_k$. For instance, $k = 2$ for the triple in part (2) of Proposition 7.8. We will now construct a closed manifold M of dimension n on which G acts freely as in [Bu, Sec. 11.4].

Let N_0 be any compact orientable surface of genus at least k. Then $\pi_1(N_0)$ has a quotient group that is free on k generators. One can see this with an explicit description of $\pi_1(N_0)$ as in [Sp, 3.8.12], or by looking through the k holes and projecting the surface to a closed disc in the plane with k points missing. Since G is a quotient of the free group on k generators, there is a surjective homomorphism $\varphi : \pi_1(N_0) \to G$. (One can also obtain an explicit homomorphism of $\pi_1(N_0)$ onto G as in [Bu, Sec. 11.4].) Now let N be the regular covering of N_0 corresponding to the normal subgroup $\ker(\varphi)$ of $\pi_1(N_0)$. Then the group of deck transformations of N over N_0 is isomorphic to G; see [Sp, Thm. 2.3.12, 2.5.13, Cor. 2.6.3]. Letting $M = N \times Y$ be the product of N with any closed manifold Y of dimension $n - 2$, then we see that M is a closed n-manifold on which G acts freely. Then, since (G, H, H') is not a jump triple, Theorem 2.2 tells us that M admits a G-invariant metric such that the quotient manifolds $H \backslash M$ and $H' \backslash M$ have distinct covering spectra, while Sunada's result says they have the same Laplace spectra. $\square$

8.1. **Example.** An alternative way to obtain examples of isospectral manifolds with distinct covering spectra of dimension 4 is by using the construction of rank 4 isospectral lattices by

Conway and Sloane [CS]. It was pointed out by Gordon and Kappeler [GK, Sec. 4.4] that these lattices arise from a Gassmann-Sunada triple. We briefly explain the construction.

Let A be the additive group of the group ring $\mathbb{Z}[V_4]$ where $V_4 = \{1, \sigma, \tau, \rho\}$ is the Klein group of order 4 with unit element 1. Then let $H = \langle 3A, \sigma + \tau + \rho, 1 + \sigma - \rho \rangle$ and $H' = \langle 3A, \sigma + \tau + \rho, 1 - \sigma + \rho \rangle$. We can view H and H' as subgroups of the semidirect product $G = A \rtimes V_4$.

A note on notation: our lattice H is denoted $\mathcal{L}$ in [GK] and $\mathcal{L}^+$ in [CS], while H' is $\mathcal{L}'$ in [GK], and reflecting across the hyperplane spanned by e_2, e_3 and e_4 gives $\mathcal{L}^-$ in [CS].

Note that the group ring $\mathbb{R}[V_4]$ over the field of real numbers has a basis as a vector space over $\mathbb{R}$ consisting of primitive idempotents on which the V_4-action is diagonalized. In other words, we can choose a basis of $\mathbb{R}[V_4]$ such that the mappings $\{1, \sigma, \rho, \tau\}$ from $\mathbb{R}[V_4]$ to $\mathbb{R}[V_4]$ are given by diagonal matrices. We now take any Euclidean structure on $\mathbb{R}[V_4]$ so that these eigenspaces are orthogonal, then G acts by isometries on $\mathbb{R}[V_4]$.

Now consider the flat torus $\mathbb{R}[V_4]/3A$ as a Riemannian manifold. The group $G/3A$ acts on $\mathbb{R}[V_4]/3A$, and its quotients by the subgroups $H/3A$ and $H'/3A$ are the flat tori $T = \mathbb{R}[V_4]/H$ and $T' = \mathbb{R}[V_4]/H'$. It is not hard to see (cf. [GK, Sec. 4.4]) that $(G/3A, H/3A, H'/3A)$ is a Gassmann-Sunada triple. Note that $G/3A$ does not act freely on $\mathbb{R}[V_4]/3A$, but $H/3A$ and $H'/3A$ do. In this context, Sunada's result is still valid [B1, B2], so T and T' are isospectral.

For a judicious choice of our Euclidean structure on $\mathbb{R}[V_4]$ we now obtain distinct covering spectra. Specifically, if we label the orthogonal idempotents $e_1 = (1 + \sigma - \tau + \rho)/4$, $e_2 = (1 + \sigma + \tau - \rho)/4$, $e_3 = (-1 + \sigma - \tau - \rho)/4$, $e_4 = (-1 + \sigma + \tau + \rho)/4$, then the following table gives three examples of Euclidean structures and the covering spectra they give rise to.

$3\langle e_1, e_1 \rangle, \ldots, 3\langle e_4, e_4 \rangle$	CovSpec(T)	CovSpec(T')
$1, 4, 10, 13$	$\{\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}\}$	$\{\sqrt{3}, \sqrt{5}, \sqrt{7}\}$
$2, 8, 14, 20$	$\{\sqrt{5}, \sqrt{9}, \sqrt{10}, \sqrt{13}\}$	$\{\sqrt{5}, \sqrt{9}, \sqrt{12}, \sqrt{13}\}$
$1, 7, 13, 19$	$\{2, \sqrt{8}, \sqrt{10}\}$	$\{2, \sqrt{8}, \sqrt{10}\}$

In the last row the covering spectra are the same, but the multiplicities in the sense of Example 4.7 or [SW, Definition 6.1] are not: they are $1, 2, 1$ (respectively) for T and $1, 1, 2$ (respectively) for T'. This table was obtained using Magma [BCP] and can be verified (tediously) by hand.

8.2. **Remark.** In fact, $H/3A$ and $H'/3A'$ are jump equivalent in $G/3A$, but since $\mathbb{R}[V_4]/3A$ is not simply connected, this does not contradict Theorem 2.2. Furthermore, $H/3A$ and $H'/3A'$ are order equivalent in $G/3A$. However, one can check that $H/9A$ and $H'/9A$ are *not* jump equivalent in $G/9A$.

8.3. **Example** (Heisenberg Manifolds). Let $(V, \langle \cdot, \cdot \rangle)$ be a $2n$-dimensional real inner product space that is endowed with a non-degenerate symplectic form ω; that is, $\omega \colon V \times V \to \mathbb{R}$ is a non-degenerate skew-symmetric form. The Heisenberg group $H(V)$ associated to V is the

manifold $H = V \times \mathbb{R}$, together with the group operation

$$(x, t)(x', t') = (x + x', t + t' + \omega(x, x')/2).$$

With this operation H becomes a non-commutative real Lie group. The identity element $e \in H$ is the pair $(0, 0)$. The center $Z = \{0\} \times \mathbb{R}$ of H is equal to the commutator subgroup $[H, H]$ of H, so H is 2-step nilpotent, and the commutator pairing $H \times H \to [H, H]$ is given by the symplectic form:

$$(x, t)(x', t')(x, t)^{-1}(x', t')^{-1} = (0, \omega(x, x')).$$

The tangent space of H at e is $V \times \mathbb{R}$, which we view as an inner product space by putting the standard inner product on $\mathbb{R}$, and taking the orthogonal direct sum with $(V, \langle \cdot, \cdot \rangle)$. We extend this inner product uniquely to a left-invariant Riemannian metric g on H . Note that the group isomorphism $V \to H/Z$ is an isometry between the Euclidean space V with its flat metric and the quotient manifold H/Z together with its induced structure of a Riemannian manifold.

Now suppose we have a lattice $\mathcal{L}$ of full rank in V, and a positive real number c such that $\omega(\mathcal{L}, \mathcal{L}) \subset c\mathbb{Z}$. Then $\Gamma = \mathcal{L} \times c\mathbb{Z}$ is a discrete subgroup of H that acts freely and by isometries on H by left multiplication. The quotient space $X = \Gamma \backslash H$ is then a Riemannian manifold, called a *Riemannian Heisenberg manifold*.

Note that $Z/(\{0\} \times c\mathbb{Z})$ acts freely by isometries on X, and that the quotient X/Z, considered as a Riemannian manifold, is just the flat torus $T = V/\mathcal{L}$. The covering spectrum of T is the jump set of the length map $m_{\mathcal{L}}$ on $\mathcal{L}$ as in Example 4.7.

We will now show how to obtain the covering spectrum of X from that of T. The covering spectrum of X is the jump set of the length map $m_\Gamma(\gamma) = \inf_{h \in H} d_H(h, \gamma h)$ on Γ, where d_H is the geodesic distance on H (see Definition 2.4).

8.4. **Proposition.** *Let $\delta_T = \inf \operatorname{CovSpec}(T)$, and put $\delta_Z = m_\Gamma((0, c))$. Then we have*

$$\operatorname{CovSpec}(X) = \begin{cases} \{\delta_Z\} \cup \operatorname{CovSpec}(T) & \textit{if } \delta_Z < \delta_T; \\ \operatorname{CovSpec}(T) & \textit{otherwise.} \end{cases}$$

Proof. It is well-known for Riemannian Heisenberg manifolds that

(1) $m_\Gamma(v, kc) = m_{\mathcal{L}}(v)$ for all non-zero $v \in \mathcal{L}$, and

(2) among all free homotopy classes represented by central elements $\{0\} \times c\mathbb{Z}$ of Γ, the one that can be represented by a closed geodesic of smallest length is $(0, c)$.

For details of (1) and (2) and/or a precise calculation of the geodesics and δ_Z we refer the reader to [Ka], [G1], or [E].

To deduce the proposition, suppose first that $\delta_Z < \delta_T$. Then by property (2) the element $(0, c)$ has the smallest non-zero length in Γ. The intermediate groups of $\langle (0, c) \rangle \subset \Gamma$ now correspond exactly to the subgroups of $\mathcal{L}$, and by property (1) the filtrations induced by m_Γ and $m_{\mathcal{L}}$ (respectively) correspond as well, so this gives us the first case.

For the second case, suppose that $v \in \mathcal{L}$ with $v \neq 0$ and $\delta_T(v) \leq \delta_Z$. Then we have $(0, kc) = (v, kc)(v, 0)^{-1}$, and by property (1) both $(v, 0)$ and (v, kc) have length $\delta_T(v)$. Thus, $\langle (0, c), (v, 0) \rangle$ is contained in the smallest non-trivial group in the filtration of Γ, and the second case follows. $\qquad\square$

8.5. Remark. By [G3, Prop 2.16] any quotient of a $2n + 1$-dimensional simply connected Heisenberg Lie group endowed with a left-invariant metric by a cocompact, discrete subgroup is isometric to a Riemannian Heisenberg manifold as described above, i.e., to one of the form $\Gamma \backslash H(V)$, where $(V, \omega, \langle \cdot, \cdot \rangle)$ and $\Gamma = \mathcal{L} \times c\mathbb{Z}$ are as above.

Moreover, if two such Riemannian Heisenberg manifolds are isospectral, then they are isometric to manifolds $X = (\mathcal{L} \times c\mathbb{Z}) \backslash H(V)$ and $X' = (\mathcal{L}' \times c\mathbb{Z}) \backslash H(V)$ given with the same V, $\langle \cdot, \cdot \rangle$, ω, c, and two cocompact lattices $\mathcal{L}$ and $\mathcal{L}'$ in V such that the flat tori $T = V/\mathcal{L}$ and $T' = V/\mathcal{L}'$ are isospectral. In fact, two Heisenberg manifolds X and X' as above are isospectral if and only if $T = V/\mathcal{L}$ and $T' = V/\mathcal{L}'$ are isospectral. To see these comments, the reader may consult [G3, Prop. 2.14, 2.16] and [Pes2, Prop. III.5].

Combining the remark with the proposition we have the following.

8.6. Corollary. *Suppose that X and X' are isospectral Heisenberg manifolds as in Remark 8.5. Then X and X' have the same covering spectra if and only if T and T' have the same covering spectra.*

Proof. By statement (2) in the proof of Proposition 8.4 we see that $\delta_Z = \delta'_Z$, since both are equal to the distance in $H(V)$ from $(0, 0)$ to $(0, c)$. As the flat tori $T = V/\mathcal{L}$ and $T' = V/\mathcal{L}'$ are isospectral, they have the same length spectrum, so there is a one to one correspondence between the lengths of the vectors in $\mathcal{L}$ and the lengths of the vectors in $\mathcal{L}'$. In particular, we must have $\delta_T = \delta_{T'}$. The result now follows from Proposition 8.4. $\qquad\square$

8.7. Example (Isospectral Heisenberg manifolds with different covering spectra)**.** Let $\mathcal{L}, \mathcal{L}' \subset \mathbb{R}[V_4]$ be a pair of lattices of Conway and Sloane as in Example 8.1, that give rise to isospectral 4-dimensional flat tori with distinct covering spectra. One easily checks that $\Gamma = \mathcal{L} \times \mathbb{Z}$ and $\Gamma' = \mathcal{L}' \times \mathbb{Z}$ are discrete subgroups of $H(V)$, where $V = \mathbb{R}[V_4] = \mathbb{R}^4$ with the standard symplectic form. By Remark 8.5, $X = \Gamma \backslash H(V)$ and $X' = \Gamma' \backslash H(V)$ are isospectral, since $T = V/\mathcal{L}$ and $T' = V/\mathcal{L}'$ are isospectral. By Proposition 8.4, they have distinct covering spectra, as T and T' have distinct covering spectra.

8.8. Remark. It follows from Corollary 8.6 that the example of isospectral Heisenberg manifolds with different covering spectra given by Sormani and Wei (see [SW, Example 10.3], [SW2]) is incorrect, as their tori T and T' have the same covering spectrum.

8.9. Remark. In [SW, Definition 6.1], Sormani and Wei assign a *basis multiplicity* to each element δ of the covering spectrum of a compact Riemannian manifold M. It is the minimal number of generators of length 2δ that one needs to add to $\mathrm{Fil}^\delta \pi_1(M)$ to generate the next bigger group in the filtration. One can show that if X is a Riemannian Heisenberg manifold with

$\delta_Z \geq \delta_T$, then the basis multiplicity of δ_T in CovSpec(X) is either equal to the basis multiplicity of δ_T in CovSpec(T) or equal to one more than the basis multiplicity of δ_T in CovSpec(T). All other basis multiplicities of CovSpec(X) are unaffected; i.e., their basis multiplicity in CovSpec(X) is equal to their basis multiplicity in CovSpec(T). The isospectral Heisenberg manifolds of [G1, Example 2.4a] have the same covering spectra by Corollary 8.6, but δ_Z has a different basis multiplicity in the two examples.

References

[A]   R. Abraham, *Bumpy metrics*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), 1–3, Amer. Math. Soc., Providence, R.I., 1970.

[Ba]   W. Ballmann, *On the construction of isospectral manifolds*, Geometric methods in inverse problems and PDE control, 1–14, IMA Vol. Math. Appl., **137**, Springer, New York, 2004.

[B1]   P. Bérard, *Transplantation et isospectralité. I*, Math. Ann. **292** (1992), 547–559.

[B2]   P. Bérard, *Transplantation et isospectralité. II*, J. London Math. Soc. (2) **48** (1993), 565–576.

[Ber]   M. Berger, *A Panoramic View of Riemannian Geometry*, Springer, Berlin (2003).

[BD]   W. Bosma and B. de Smit, *Arithmetically equivalent number fields of small degree*, Algorithmic Number Theory (C. Fieker & D. Kohel, eds.), 67–79, Lecture Notes in Computer Science 2369, Springer-Verlag, 2002.

[BCP]   W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system I. The user language.*, J. Symbolic Comput. **24** (1997), 235–265.

[Br]   R. Brooks, *The Sunada method*, Contemporary Mathematics **231** (1999), 25–35.

[Bu]   P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser, Boston (1992).

[Ch]   J. Chazarain, *Formule de Poisson pour les variétés riemanniennes*, Invent. Math. **24** (1974), 65–82.

[CdV1]   Y. Colin de Verdière, *Spectre du laplcien et longueurs des géodésiques périodiques II*, Compositio. Math. **27** (1973), 159–184.

[CdV2]   Y. Colin de Verdière, *Construction de laplaciens dont une partie finie du spectre est donnée*, Ann. Sci. École Norm. Sup (4) **20** (1992), No. 4, 599–615.

[CS]   J. H. Conway and N.J.A. Sloane, *Four-dimensional lattices with the same theta series*, Int. Math. Res. Not. **1992** No. 4, 93–96.

[DL]   B. de Smit and H.W. Lenstra, *Linearly equivalent actions of solvable groups*, J. Algebra **228** (2000), 270–285.

[DG]   D. DeTurck and C. Gordon, *Isospectral Deformations II: trace formulas, metrics and potentials*, Comm. Pure and Appl. Math. **42** (1989), 1067–1095.

[DuGu]   J.J. Duistermaat and V. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79.

[E]   P. Eberlein, *Geometry of 2-step nilpotent Lie groups with a left invariant metric*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 5, 611–660.

[Gas]   F. Gassmann, *Bemerkungen zu der vorstehnden Arbeit von Hurwitz*, ('*Über Beziehungen zwischen den Primidealen eines algebraischen Körpers und den Substitutionen seiner Gruppe*'), Math. Z. **25** (1926), 124–143.

[GoSt]   R. Gompf and A. Stipsicz, 4-*Manifolds and Kirby Calculus*, American Mathematical Society, Providence, RI 1999.

[G1]   C. S. Gordon, *The Laplace spectra versus the length spectra of Riemannian manifolds*, Nonlinear Problems in Geometry (Mobile, Ala. 1985), 63–80, Contemp. Math., **51**, Amer. Math. Soc., Providence, RI, 1986.

[G2]   C. S. Gordon, *Isospectral closed Riemannian manifolds which are not locally isometric*, J. Differential Geom. **37** (1993), 639–649.

[G3]   C. S. Gordon, *Riemannian manifolds isospectral on functions but not on 1-forms*, J. Differential Geom. **24** (1986), 79–96.

[GK]   C.S. Gordon and T. Kappeler, *Isospectral potentials on flat tori, II*, Comm. Partial Differential Equations **20** (1995), 709–728.

[GMW]   C.S. Gordon, E. Makover and D. Webb, *Transplantation and jacobians of Sunada isospectral Riemann surfaces*, Adv. Math. **197** (2005), 86–119.

[Gt]   R. Gornet, *The marked length spectrum vs. the p-form spectrum of Riemannian nilmanifolds*, Comment. Math. Helv., **71** (1996), 297–329.

[GtM]   R. Gornet and M. Mast, *The minimal marked length spectrum of Riemannian two-step nilmanifolds*, Michigan Math. J. **52** (2004), 683–716.

[GW]   R. Guralnick and D. Wales, *Subgroups inducing the same permutation representation, II*, J. Algebra **96** (1985), 94–113.

[Ha]   M. Hall, *The theory of groups*, Macmillan, New York (1959).

[Hi]   M.W. Hirsch, *Differential topology*, Springer-Verlag GTM **33**, New York (1976).

[J]   W. Jehne, *Kronecker classes of algebraic number fields*, J. Number Theory **9** (1977), no. 3, 279–320.

[Ka]   A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata, **11** (1981), no. 2, 127–136.

[K]   K. Komatsu, *On the adele ring of algebraic number fields*, Kodai Math. Sem. Rep. **28** (1976), 78–84.

[LMNR] C.J. Leininger, D.B. McReynolds, W.D. Neumann and A.W. Reid, *Length and eigenvalue equivalence*, Int. Math. Res. Not. IMRN **2007** No. 24, Art. ID rnm135, 24pp.

[Mi]   J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Natl. Acad. Sci. USA **51** (1964), 542.

[Per]   R. Perlis, *On the equation $\zeta_K(s) = \zeta_{K'}(s)$*, J. Number Theory **9** (1977), 342–360.

[Pes1] H. Pesce, *Representations relativement equivalentes et variétés riemanniennes isospectrales*, Comment. Math. Helv. **71** (1996), 243–268.

[Pes2] H. Pesce, *Une formule de Poisson pour les variétés de Heisenberg*, Duke Math. J. **73** (1994), no. 1, 79–95.

[Pes3] H. Pesce, *Une réciproque généric du théorème de Sunada*, Compositio Math. **109** (1997), no. 3, 357–365.

[Ra]   A. Ranicki, *Algebraic and geometric surgery*, Oxford University Press, New York (2002).

[Sch]   D. Schueth, *Continuous families of isospectral metrics on simply connected manifolds*, Ann. of Math. **149** (1999), 287–308.

[Sp]   E. Spanier, *Algebraic Topology*, Mcgraw-Hill, New York, 1966.

[SW]   C. Sormani and G. Wei, *The covering spectrum of a compact length space*, J. Differential Geom., **67** (2004), 33-77.

[SW2]   C. Sormani and G. Wei, *Erratum to "The covering spectrum of a compact length space"*, J. Differential Geom., **74** (2006), no. 3, 523.

[Sun]   T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121** (1985), 169–186.

[Sut]   C. J. Sutton, *Isospectral simply-connected homogeneous spaces and the spectral rigidity of group actions*, Comment. Math. Helv. **77** (2002), 701–717.

[Sz]   Z.I. Szabó, *Locally non-isometric yet super isospectral spaces*, GAFA **9** (1999), 185–214.

[T]   J.A. Todd, *On a conjecture in group theory*, J. London Math. Soc. **25** (1950), 246.

[Vi]   M.F. Vignéras, *Variétés riemanniennes isospectrales et non isométriques*, Ann. of Math. 112 (1980), 21–32.

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