ADMISSIBLE CONSTANTS FOR GENUS 2 CURVES

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Abstract. S.-W. Zhang recently introduced a new adelic invariant φ for curves of genus at least 2 over number fields and function fields. We calculate this invariant when the genus is equal to 2.

1. Introduction

Let X be a smooth projective geometrically connected curve of genus $g \geq 2$ over a field k which is either a number field or the function field of a curve over a field. Assume that X has semistable reduction over k. For each place v of k, let Nv be the usual local factor connected with the product formula for k.

In a recent paper [11] S.-W. Zhang proves the following theorem:

Theorem 1.1. *Let $(\omega, \omega)_a$ be the admissible self-intersection of the relative dualizing sheaf of X. Let $\langle \Delta_\xi, \Delta_\xi \rangle$ be the height of the canonical Gross-Schoen cycle on X^3. Then the formula:*

$$(\omega, \omega)_a = \frac{2g-2}{2g+1} \left(\langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \varphi(X_v) \log Nv \right)$$

holds, where the $\varphi(X_v)$ are local invariants associated to $X \otimes k_v$, defined as follows:

- *if v is a non-archimedean place, then:*

$$\varphi(X_v) = -\frac{1}{4}\delta(X_v) + \frac{1}{4}\int_{R(X_v)} g_v(x,x)((10g+2)\mu_v - \delta_{K_{X_v}}) \, ,$$

  where:
  - *$\delta(X_v)$ is the number of singular points on the special fiber of $X \otimes k_v$,*
  - *$R(X_v)$ is the reduction graph of $X \otimes k_v$,*
  - *g_v is the Green's function for the admissible metric μ_v on $R(X_v)$,*
  - *K_{X_v} is the canonical divisor on $R(X_v)$.*

  In particular, $\varphi(X_v) = 0$ if X has good reduction at v;
- *if v is an archimedean place, then:*

$$\varphi(X_v) = \sum_\ell \frac{2}{\lambda_\ell} \sum_{m,n=1}^{g} \left| \int_{X(\bar{k}_v)} \phi_\ell \omega_m \bar{\omega}_n \right|^2 ,$$

  where ϕ_ℓ are the normalized real eigenforms of the Arakelov Laplacian on $X(\bar{k}_v)$ with eigenvalues $\lambda_\ell > 0$, and $(\omega_1, \ldots, \omega_g)$ is an orthonormal basis for the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2}\int_{X(\bar{k}_v)} \omega \, \bar{\eta}$ on the space of holomorphic differentials.

The author is supported by a VENI grant from the Netherlands Organisation for Scientific Research (NWO). He thanks the Max Planck Institut für Mathematik in Bonn for its hospitality during a visit.

Apart from giving an explicit connection between the two canonical invariants $(\omega,\omega)_a$ and $\langle \Delta_\xi, \Delta_\xi \rangle$, Zhang's theorem has a possible application to the effective Bogomolov conjecture, *i.e.*, the question of giving effective positive lower bounds for $(\omega,\omega)_a$. Indeed, the height of the canonical Gross-Schoen cycle $\langle \Delta_\xi, \Delta_\xi \rangle$ is known to be non-negative in the case of a function field in characteristic zero, and should be non-negative in general by a standard conjecture of Gillet-Soulé (*op. cit.*, Section 2.4). Further, the invariant φ should be non-negative, and Zhang proposes, in the non-archimedean case, an explicit lower bound for it which is positive in the case of non-smooth reduction (*op. cit.*, Conjecture 1.4.2). Note that it is clear from the definition that φ is non-negative in the archimedean case; in fact it is positive (*op. cit.*, Remark after Proposition 2.5.3).

Besides $\varphi(X_v)$, Zhang also considers the invariant $\lambda(X_v)$ defined by:

$$\lambda(X_v) = \frac{g-1}{6(2g+1)}\varphi(X_v) + \frac{1}{12}(\varepsilon(X_v) + \delta(X_v)) \, ,$$

where:

- if v is a non-archimedean place, the invariant $\delta(X_v)$ is as above, and:
$$\varepsilon(X_v) = \int_{R(X_v)} g_v(x,x)((2g-2)\mu_v + \delta_{K_{X_v}}) \, ,$$
- if v is an archimedean place, then:
$$\delta(X_v) = \delta_F(X_v) - 4g\log(2\pi)$$
with $\delta_F(X_v)$ the Faltings delta-invariant of the compact Riemann surface $X(\bar{k}_v)$, and $\varepsilon(X_v) = 0$.

The significance of this invariant is that if $\deg\det R\pi_*\omega$ denotes the (non-normalized) geometric or Faltings height of X one has a simple expression:

$$\deg\det R\pi_*\omega = \frac{g-1}{6(2g+1)}\langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \lambda(X_v)\log Nv$$

for $\deg\det R\pi_*\omega$, as follows from the Noether formula:

$$12\deg\det R\pi_*\omega = (\omega,\omega)_a + \sum_v (\varepsilon(X_v) + \delta(X_v))\log Nv \, .$$

Now assume that X has genus $g = 2$. Our purpose is to calculate the invariants $\varphi(X_v)$ and $\lambda(X_v)$ explicitly. For the λ-invariant we obtain:

- if v is non-archimedean, then:
$$10\lambda(X_v) = \delta_0(X_v) + 2\delta_1(X_v) \, ,$$
where $\delta_0(X_v)$ is the number of non-separating nodes and $\delta_1(X_v)$ is the number of separating nodes in the special fiber of $X \otimes k_v$;
- if v is archimedean, then:
$$10\lambda(X_v) = -20\log(2\pi) - \log\|\Delta_2\|(X_v) \, ,$$
where $\|\Delta_2\|(X_v)$ is the normalized modular discriminant of the compact Riemann surface $X(\bar{k}_v)$ (see below).

Thus, the $\lambda(X_v)$ are precisely the well-known local invariants corresponding to the discriminant modular form of weight 10 [6] [9] [10]. In particular we have:

$$\deg\det R\pi_*\omega = \sum_v \lambda(X_v)\log Nv$$

and we recover the fact that the height of the canonical Gross-Schoen cycle vanishes for X.

2. The non-archimedean case

Let k be a complete discretely valued field. Let X be a smooth projective geometrically connected curve of genus 2 over k. Assume that X has semistable reduction over k. In this section we give the invariants $\varphi(X)$ and $\lambda(X)$ of X.

The proof of our result is based on the classification of the semistable fiber types in genus 2 and consists of a case-by-case analysis. The notation we employ for the various fiber types is as in [8]. We remark that there are no restrictions on the residue characteristic of k.

Theorem 2.1. *The invariant $\varphi(X)$ is given by the following table, depending on the type of the special fiber of the regular minimal model of X:*

Type	δ_0	δ_1	ε	φ
I	0	0	0	0
$II(a)$	0	a	a	a
$III(a)$	a	0	$\frac{1}{6}a$	$\frac{1}{12}a$
$IV(a,b)$	b	a	$a+\frac{1}{6}b$	$a+\frac{1}{12}b$
$V(a,b)$	$a+b$	0	$\frac{1}{6}(a+b)$	$\frac{1}{12}(a+b)$
$VI(a,b,c)$	$b+c$	a	$a+\frac{1}{6}(b+c)$	$a+\frac{1}{12}(b+c)$
$VII(a,b,c)$	$a+b+c$	0	$\frac{1}{6}(a+b+c)+\frac{1}{6}\frac{abc}{ab+bc+ca}$	$\frac{1}{12}(a+b+c)-\frac{5}{12}\frac{abc}{ab+bc+ca}$

For $\lambda(X)$ the formula:

$$10\lambda(X) = \delta_0(X) + 2\delta_1(X)$$

holds.

Let us indicate how the theorem is proved. Let r be the effective resistance function on the reduction graph $R(X)$ of X, extended bilinearly to a pairing on $\mathrm{Div}(R(X))$. By Corollary 2.4 of [2] the formula:

$$\varphi(X) = -\frac{1}{4}(\delta_0(X) + \delta_1(X)) - \frac{3}{8}r(K,K) + 2\varepsilon(X)$$

holds, where K is the canonical divisor on $R(X)$. The invariant $r(K,K)$ is calculated by viewing $R(X)$ as an electrical circuit. The invariant ε is calculated on the basis of explicit expressions for the admissible measure and admissible Green's function; see [7] and [8] for such computations. The results we find are as follows:

Type	δ_0	δ_1	$r(K,K)$	ε
I	0	0	0	0
$II(a)$	0	a	$2a$	a
$III(a)$	a	0	0	$\frac{1}{6}a$
$IV(a,b)$	b	a	$2a$	$a+\frac{1}{6}b$
$V(a,b)$	$a+b$	0	0	$\frac{1}{6}(a+b)$
$VI(a,b,c)$	$b+c$	a	$2a$	$a+\frac{1}{6}(b+c)$
$VII(a,b,c)$	$a+b+c$	0	$2\frac{abc}{ab+bc+ca}$	$\frac{1}{6}(a+b+c)+\frac{1}{6}\frac{abc}{ab+bc+ca}$

The values of φ follow.

The formula for $\lambda(X)$ is verified for each case separately.

3. The archimedean case

Let X be a compact and connected Riemann surface of genus 2. In this section we calculate the invariants $\varphi(X)$ and $\lambda(X)$ of X. Let $\mathrm{Pic}(X)$ be the Picard variety of X, and for each integer d denote by $\mathrm{Pic}^d(X)$ the component of $\mathrm{Pic}(X)$ of degree d. We have a canonical theta divisor Θ on $\mathrm{Pic}^1(X)$, and a standard hermitian metric $\|\cdot\|$ on the line bundle $\mathcal{O}(\Theta)$ on $\mathrm{Pic}^1(X)$. Let ν be its curvature form. We have:

$$\int_{\mathrm{Pic}^1(X)} \nu^2 = \Theta.\Theta = 2\,.$$

Let K be a canonical divisor on X, and let $\mathbf{P}$ be the set of 10 points P of $\mathrm{Pic}^1(X) - \Theta$ such that $2P \equiv K$. Denote by $\|\theta\|$ the norm of the canonical section θ of $\mathcal{O}(\Theta)$. We let:

$$\|\Delta_2\|(X) = 2^{-12} \prod_{P \in \mathbf{P}} \|\theta\|^2(P)\,,$$

the normalized modular discriminant of X, and we let $\|H\|(X)$ be the invariant of X defined by:

$$\log \|H\|(X) = \frac{1}{2} \int_{\mathrm{Pic}^1(X)} \log \|\theta\|\, \nu^2\,.$$

These two invariants were introduced in [1].

Theorem 3.1. *For the φ-invariant and the λ-invariant of X, the formulas:*

$$\varphi(X) = -\frac{1}{2}\log \|\Delta_2\|(X) + 10 \log \|H\|(X)$$

and

$$10\lambda(X) = -20\log(2\pi) - \log\|\Delta_2\|(X)$$

hold.

The key to the proof is the following lemma. Let Φ be the map:

$$X^2 \to \mathrm{Pic}^1(X)\,, \qquad (x,y) \mapsto [2x-y]\,.$$

Lemma 3.2. *The map Φ is finite flat of degree* 8.

Proof. Let $y \mapsto y'$ be the hyperelliptic involution of X. We have a commutative diagram:

$$\begin{array}{ccc} X^2 & \xrightarrow{\Phi} & \mathrm{Pic}^1(X) \\ {\scriptstyle\alpha}\downarrow & & \uparrow{\scriptstyle\beta} \\ X^2 & \xrightarrow{\Phi^\vee} & \mathrm{Pic}^3(X) \end{array}$$

where α and β are isomorphisms, with:

$$\begin{array}{ccc} \alpha\colon X^2 \to X^2\,, & \Phi^\vee\colon X^2 \to \mathrm{Pic}^3(X)\,, & \beta\colon \mathrm{Pic}^3(X) \to \mathrm{Pic}^1(X)\,, \\ (x,y) \mapsto (x,y')\,, & (x,y) \mapsto [2x+y]\,, & [D] \mapsto [D-K]\,. \end{array}$$

It suffices to prove that $\Phi^\vee$ is finite flat of degree 8. Let $p\colon X^{(3)} \to \mathrm{Pic}^3(X)$ be the natural map; then p is a $\mathbf{P}^1$-bundle over $\mathrm{Pic}^3(X)$, and $\Phi^\vee$ has a natural injective lift to $X^{(3)}$. A point D on $X^{(3)}$ is in the image of this lift if and only if D, when seen as an effective divisor on X, contains a point which is ramified for the morphism $X \to \mathbf{P}^1$ determined by the fiber $|D|$ of p in which D lies. Since every morphism $X \to \mathbf{P}^1$ associated to a D on $X^{(3)}$ is ramified, the map $\Phi^\vee$ is surjective. As every morphism $X \to \mathbf{P}^1$ associated to a D on $X^{(3)}$ has only finitely many ramification points, the map $\Phi^\vee$ is quasi-finite, hence finite since $\Phi^\vee$ is proper. As X^2 and $\mathrm{Pic}^3(X)$ are smooth and the fibers of $\Phi^\vee$ are equidimensional, the map $\Phi^\vee$ is flat. By Riemann-Hurwitz the generic $X \to \mathbf{P}^1$ associated to a D on $X^{(3)}$ has 8 simple ramification points. It follows that the degree of $\Phi^\vee$ is 8. □

Let $G\colon X^2 \to \mathbf{R}$ be the Arakelov-Green's function of X, and let Δ be the diagonal divisor on X^2. We have a canonical hermitian metric on the line bundle $\mathcal{O}(\Delta)$ on X^2 by putting $\|1\|(x,y) = G(x,y)$, where 1 is the canonical section of $\mathcal{O}(\Delta)$. Denote by h_Δ the curvature form of $\mathcal{O}(\Delta)$. We have:

$$\int_{X^2} h_\Delta^2 = \Delta.\Delta = -2\,.$$

Restricting $\mathcal{O}(\Delta)$ to a fiber of any of the two natural projections of X^2 onto X and taking the curvature form we obtain the Arakelov $(1,1)$-form μ on X. We have $\int_X \mu = 1$ and:

$$\int_X \log G(x,y)\,\mu(x) = 0$$

for each y on X. Let (ω_1, ω_2) be an orthonormal basis of $\mathrm{H}^0(X, \omega_X)$, the space of holomorphic differentials on X. We can write explicitly:

$$h_\Delta(x,y) = \mu(x) + \mu(y) - i\sum_{k=1}^{2}(\omega_k(x)\bar{\omega}_k(y) + \omega_k(y)\bar{\omega}_k(x))$$

and:

$$\mu(x) = \frac{i}{4}\sum_{k=1}^{2}\omega_k(x)\bar{\omega}_k(x)\,.$$

By [11, Proposition 2.5.3] we have:

$$\varphi(X) = \int_{X^2} \log G\, h_\Delta^2\,.$$

We compute the integral using our results from [4] and [5]. Let W be the divisor of Weierstrass points on X, and let $p_1\colon X^2 \to X$ be the projection onto the first coordinate. The divisor W is reduced effective of degree 6. According to [3, p. 31] there exists a canonical isomorphism:

$$\sigma \colon \Phi^* \mathcal{O}(\Theta) \xrightarrow{\cong} \mathcal{O}(2\Delta + p_1^* W)$$

of line bundles on X^2, identifying the canonical sections on both sides. In [4, Proposition 2.1] we proved that this isomorphism has a constant norm over X^2. Thus, the curvature forms on both sides are equal:

$$\Phi^* \nu = 2h_\Delta + 6\mu(x) \quad \text{on } X^2 \,.$$

Squaring both sides of this identity we get:

$$h_\Delta^2 = \frac{1}{4} \Phi^*(\nu^2) - 6 h_\Delta \mu(x) \,,$$

since $\mu(x)^2 = 0$. Denote by $S(X)$ the norm of σ. Then we have:

$$2 \log G(x,y) + \sum_w \log G(x,w) = \log \|\theta\|(2x - y) + \log S(X)$$

for generic $(x,y) \in X^2$, where w runs through the Weierstrass points of X. By fixing y and integrating against $\mu(x)$ on X we find that:

$$\log S(X) = - \int_X \log \|\theta\|(2x-y)\, \mu(x) \,.$$

By integrating against h_Δ^2 on X^2 we obtain:

$$2\varphi(X) + \sum_w \int_{X^2} \log G(x,w)\, h_\Delta^2 = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x-y)\, h_\Delta^2 \,.$$

As we have:

$$h_\Delta^2 = 2\mu(x)\mu(y) - \sum_{k,l=1}^{2} (\omega_k(x)\bar{\omega}_l(x)\bar{\omega}_k(y)\omega_l(y) + \bar{\omega}_k(x)\omega_l(x)\omega_k(y)\bar{\omega}_l(y))$$

it follows that:

$$\int_{X^2} \log G(x,w)\, h_\Delta^2 = 0$$

for each w in W and hence we simply have:

$$2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x-y)\, h_\Delta^2 \,.$$

Using our earlier expression for h_Δ^2 this becomes:

$$2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x-y) \left(\frac{1}{4} \Phi^*(\nu^2) - 6 h_\Delta \mu(x) \right) \,.$$

It is easily verified that $h_\Delta \mu(x) = h_\Delta \mu(y) = \mu(x)\mu(y)$ and hence:

$$\int_{X^2} \log \|\theta\|(2x-y)\, h_\Delta \mu(x) = \int_{X^2} \log \|\theta\|(2x-y)\, \mu(x)\mu(y) = -\log S(X) \,.$$

From Lemma 3.2 it follows that:

$$\int_{X^2} \log \|\theta\|(2x-y)\, \Phi^*(\nu^2) = 8 \int_{\mathrm{Pic}^1(X)} \log \|\theta\|\, \nu^2 = 16 \log \|H\|(X) \,.$$

All in all we find:

$$\varphi(X) = 2 \log S(X) + 2 \log \|H\|(X) \,.$$

Let $\delta_F(X)$ be the Faltings delta-invariant of X. According to [5, Corollary 1.7] the formula:

$$\log S(X) = -16\log(2\pi) - \frac{5}{4}\log\|\Delta_2\|(X) - \delta_F(X)$$

holds, and in turn, according to [1, Proposition 4] we have:

$$\delta_F(X) = -16\log(2\pi) - \log\|\Delta_2\|(X) - 4\log\|H\|(X)\,.$$

The formula:

$$\varphi(X) = -\frac{1}{2}\log\|\Delta_2\|(X) + 10\log\|H\|(X)$$

follows.

By definition we have:

$$\lambda(X) = \frac{1}{30}\varphi(X) + \frac{1}{12}\delta_F(X) - \frac{2}{3}\log(2\pi)$$

so we obtain:

$$10\lambda(X) = -20\log(2\pi) - \log\|\Delta_2\|(X)$$

by using [1, Proposition 4] once more.

References

[1] J.-B. Bost, *Fonctions de Green-Arakelov, fonctions thêta et courbes de genre 2.* C.R. Acad. Sci. Paris Ser. I 305 (1987), 643–646.

[2] X. Faber, *The geometric Bogomolov conjecture for small genus curves.* Preprint, arXiv:0803.0855.

[3] J. D. Fay, *Theta functions on Riemann surfaces.* Lect. Notes in Math. vol. 352, Springer-Verlag 1973.

[4] R. de Jong, *Arakelov invariants of Riemann surfaces.* Doc. Math. 10 (2005), 311–329.

[5] R. de Jong, *Faltings' delta-invariant of a hyperelliptic Riemann surface.* In: G. van der Geer, B. Moonen, R. Schoof (eds.), Proceedings of the Texel Conference "The analogy between number fields and function fields", Birkhäuser Verlag 2005.

[6] Q. Liu, *Conducteur et discriminant minimal de courbes de genre* 2. Compositio Math. 94 (1994), 51–79.

[7] A. Moriwaki, *Bogomolov conjecture over function fields for stable curves with only irreducible fibers.* Compositio Math. 105 (1997), 125–140.

[8] A. Moriwaki, *Bogomolov conjecture for curves of genus* 2 *over function fields.* J. Math. Kyoto Univ. 36 (1996), 687–695.

[9] T. Saito, *The discriminants of curves of genus* 2. Compositio Math. 69 (1989), 229–240.

[10] K. Ueno, *Discriminants of curves of genus* 2 *and arithmetic surfaces.* In: Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, Kinokuniya, Tokyo 1987.

[11] S.-W. Zhang, *Gross-Schoen cycles and dualising sheaves.* Preprint, arXiv:0812.0371.

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