An Introduction to Gradient Flows in Metric Spaces

Philippe Clément

June 15, 2009

Preface

These Lecture Notes grew out of a seminar held at the Mathematical Institute of the University of Leiden. The aim of this seminar, organized by Onno van Gaans and the author, was to study the very interesting book of L. Ambrosio, N. Gigli and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, [AGS]. In these Notes we restrict ourselves to the case of "Gradient Flows" induced by an "Evolution Variational Inequality" (EVI) involving only the distance function of the metric space, a functional defined on the metric space and a real number as in [AGS], (4.0.13).

In Section 1 it is shown how a differential equation in a real Hilbert space governed by the subdifferential of a (quasi-)convex functional can be rewritten as an EVI. Uniqueness of solutions to the corresponding initial-value problem is obtained in Section 2 as a consequence of a priori estimates requiring only the lower semicontinuity of the functional.

In Section 3 the problem of existence of solutions to the initial-value problem is addressed in the case where the metric space is a real Hilbert space and the functional ϕ is (quasi-)convex. This section serves as a motivation for the study of the beautiful theorem of Ambrosio, Gigli and Savaré [AGS, Theorem 4.0.4], which establishes the well-posedness of EVI, under general assumptions which strictly generalize the assumptions of the Hilbert space case. This is done in Section 4 (Theorems 4.1 and 4.2). In contrast to [AGS] we have adopted a "Crandall–Liggett" approach for the proof of the existence of solutions. This part is based on a joint work with Wolfgang Desch [CD2]. Although this approach does not provide the optimal rate of convergence, for appropriate initial values, it seems somewhat simpler than the approach in [AGS]. For the sake of completeness we include the statement and the proof of results of [AGS]. Moreover, we refer the reader to the last sentence of Part I of the Introduction of [AGS]. Section 5 is devoted to some applications of the theory to spaces of Probability measures, as in [AGS].

Acknowledgements

The author would like to thank Luigi Ambrosio and Giuseppe Savaré for kindly answering some questions related to their book. Moreover, he also thanks Sjoerd Verduyn Lunel who gave him the opportunity to hold a seminar, jointly with Onno van Gaans, on this subject at the Mathematical Institute of Leiden University. Students who attended the seminar are also thanked for their comments, in particular Igor Stojkovic and Jan Maas. These Notes have also been used as a support for a series of lectures given at the EPFL (Lausanne, Switzerland) during the Fall 2008. The author would like to express his gratitude to the organizers of these lectures, Jacques Rappaz and Marco Picasso. Finally, he would also like to thank Wolfgang Desch for reading and improving parts of the manuscript, as well as Jan K. Kowalski for his careful typing of the manuscript.

1 A notion of "gradient flow" on a metric space

The aim of this section is to introduce a notion of "gradient flow" on a metric space which generalizes a class of gradient (semi-)flows on a Hilbert space. We recall that a flow on a (nonempty) set X is a map $\Phi : I \times X \to X$, $I := \mathbb{R}$, satisfying

$$\Phi(0, x) = x \quad \text{and} \quad \Phi(s, \Phi(t, x)) = \Phi(s + t, x) \tag{1.1}$$

for all $x \in X$ and $t, s \in I$. If $I := \mathbb{R}_+ = [0, \infty)$ then the map Φ is called a *semi-flow* on X. Given a (semi-)flow Φ on X we can define a family $\{S(t)\}_{t \in I}$ of maps (operators) of X into itself by setting

$$S(t)x := \Phi(t, x), \quad t \in I, \quad x \in X. \tag{1.2}$$

This family of operators in X satisfies the (semi-)group property

$$\begin{cases} S(t + s) = S(t) \circ S(s), \quad t, s \in I, \\ S(0) = I_X, \end{cases} \tag{1.3}$$

where I_X stands for the identity operator in X.

Conversely, a family of operators $\{S(t)\}_{t \in I}$ in X satisfying (1.3) induces a (semi-)flow Φ on X by setting

$$\Phi(t, x) := S(t)x, \quad t \in I, \quad x \in X. \tag{1.4}$$

Observe that if Φ is a flow then the operators $\{S(t)\}_{t \in \mathbb{R}}$ are bijective and $(S(t))^{-1} = S(-t)$, $t \in \mathbb{R}$.

If (X, d) is a metric space, a (semi-)flow Φ on X is called *continuous* if the map $(t, x) \mapsto \Phi(t, x)$ is continuous. Obviously if a flow Φ is continuous the *orbits* $I \ni t \mapsto S(t)x \in X$ are continuous. A (semi-)group $(S(t))_{t \in I}$ of operators on X is called a C_0-*(semi-)group* if its orbits are continuous.

We recall that a map $F : (X_1, d_1) \to (X_2, d_2)$, where (X_i, d_i), $i = 1, 2$, are metric spaces, is called *Lipschitz continuous* ($F \in \mathrm{Lip}(X_1; X_2)$ or $\mathrm{Lip}(X_1)$ whenever $X_1 = X_2$) if there exists $k \geq 0$ such that

$$d_2(F(x), F(y)) \leq k d_1(x, y) \tag{1.5}$$

for all $x, y \in X_1$. The smallest number k for which (1.5) holds is called the *Lipschitz constant* of F and will be denoted by $[F]_{\mathrm{Lip}}$. Observe that

$$[F]_{\mathrm{Lip}} = \sup\Big\{ \frac{d_2(F(x), F(y))}{d_1(x, y)} : x, y \in X,\ x \neq y \Big\}.$$

In what follows we shall only be interested in (semi-)flows on X such that the corresponding $\{S(t)\}_{t \in I}$ are Lipschitz continuous. If in addition the Lipschitz constants $[S(t)]_{\mathrm{Lip}}$ are uniformly bounded on bounded subsets of I, the corresponding (semi-)flow is continuous. If for some $\omega \in \mathbb{R}$ $[S(t)]_{\mathrm{Lip}} \leq e^{\omega t}$ for all $t \in I$, the (semi-)group $\{S(t)\}_{t \in I}$

will be called a (semi-)group of *quasi-contractions*, of *contractions* if $\omega = 0$ (or a *quasi-contractive*, resp. *contractive*, (semi-)group of operators).

The collection of all C_0-semi-groups $\{S(t)\}_{t\geq 0}$ on a metric space (X,d) which satisfy

$$[S(t)]_{\mathrm{Lip}} \leq e^{\omega t} \quad \text{for some } \omega \in \mathbb{R} \text{ and all } t \geq 0$$

will be denoted by $Q_\omega(X)$.

1.1 "Lipschitz flows" on a Banach space

Standard examples of quasi-contractive flows on a real Banach space $(X, \|\cdot\|)$ are "Lipschitz flows". Let $d(x,y) := \|x-y\|$ be the metric induced by the norm $\|\cdot\|$ and let $F \in \mathrm{Lip}(X)$. As is well known, for every $x \in X$ there exists a unique function $u_x : \mathbb{R} \to X$ continuously (strongly) differentiable satisfying

$$\begin{cases} \dot{u}(t) = F(u(t)), & t \in \mathbb{R}, \\ u(0) = x, \end{cases} \tag{1.6}$$

where $\dot{u}(t)$ denotes the (strong) derivative of u at t, i.e.

$$\lim_{h\to 0} \left\| \tfrac{1}{h}(u(t+h) - u(t)) - \dot{u}(t) \right\| = 0.$$

Defining $\Phi(t,x) = u_x(t)$, $t \in \mathbb{R}$, one verifies that Φ is a flow on X. Clearly the orbits $t \mapsto \Phi(t,x)$ are continuous. Moreover the corresponding operators $\{S(t)\}_{t\in\mathbb{R}}$ satisfy

$$[S(t)]_{\mathrm{Lip}} \leq e^{|t|[F]_{\mathrm{Lip}}}, \quad t \in \mathbb{R}. \tag{1.7}$$

In particular, the flow Φ is quasi-contractive hence continuous.

The example $X = \mathbb{R}^2$ equipped with the euclidean norm $|\cdot|_2$ and

$$F(x) = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad a \in \mathbb{R},$$

shows that equality in (1.7) may hold for every $t \in \mathbb{R}$. On the other hand, if

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

we have

$$[S(t)]_{\mathrm{Lip}} = 1 < e^{|t|[F]_{\mathrm{Lip}}}$$

for all $t \in \mathbb{R} \setminus \{0\}$, since $[F]_{\mathrm{Lip}} = 1$.

In order to improve estimate (1.7) one needs to distinguish the cases $t \geq 0$ and $t \leq 0$ (even if $X = \mathbb{R}$!). We first consider the case when $X = H$ is a Hilbert space with innerproduct $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$.

Lemma 1.1. *Let $G : H \to H$ be such that there exists $\omega_+ \in \mathbb{R}$ for which*

$$\langle G(x) - G(y), x - y \rangle \leq \omega_+ |x-y|^2, \quad x, y \in H, \tag{1.8}$$

holds. Let $T > 0$.

Suppose that $u_i : [0,T] \to H$, $i = 1,2$, are continuous on $[0,T]$ and differentiable in $(0,T)$. If

$$\frac{du_i}{dt}(t) = G(u_i(t)), \quad t \in (0,T), \quad i = 1,2,$$

then we have

$$|u_1(t) - u_2(t)| \le e^{\omega_+(t-s)}|u_1(s) - u_2(s)|, \quad 0 \le s \le t \le T. \tag{1.9}$$

Proof. Set $v(t) := e^{-2\omega_+ t}|u_1(t) - u_2(t)|^2$ for $t \in (0,T)$. Then v is differentiable on $(0,T)$ and

$$\begin{aligned}\frac{dv}{dt}(t) = -2\omega_+ e^{-2\omega_+ t}|u_1(t) - u_2(t)|^2 + 2e^{-2\omega_+ t} \cdot \langle u_1(t) - u_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \\ \le e^{-2\omega_+ t} \cdot |u_1(t) - u_2(t)|^2 \cdot (-2\omega_+ + 2\omega_+) = 0.\end{aligned}$$

Hence v is nonincreasing on $(0,T)$ and by continuity on $[0,T]$. The same holds for $t \mapsto \sqrt{v(t)}$ which implies (1.9). □

Remarks. 1. If G in Lemma 1.1 is Lipschitz continuous, then (1.8) holds with $\omega_+ \le [G]_{\text{Lip}}$. This implies (1.7) for $t \ge 0$. If $F(x) = -\omega x$ with $\omega \ge 0$ then $\omega_+ = -\omega$ and $[F]_{\text{Lip}} = +\omega$.

2. If $\omega_+ = 0$ the function G satisfies

$$\lambda|x - y| \le \big|\lambda(x - y) - (G(x) - G(y))\big| \quad \text{for all} \quad \lambda > 0 \quad \text{and} \quad x, y \in H. \tag{1.10}$$

Indeed,

$$\begin{aligned}&\big|\lambda(x - y) - (G(x) - G(y))\big|^2 \\ &\quad = \lambda^2|x - y|^2 - 2\lambda \langle x - y, G(x) - G(y)\rangle + |G(x) - G(y)|^2 \ge \lambda^2|x - y|^2.\end{aligned}$$

A map $G : E \to E$ satisfying (1.10) where $(E, |\cdot|)$ is a Banach space is called *dissipative*. Conversely, if $G : H \to H$ is dissipative then G satisfies (1.8) with $\omega_+ = 0$. Indeed, as above from (1.10) we get

$$-2\lambda \langle x - y, G(x) - G(y)\rangle + |G(x) - G(y)|^2 \ge 0$$

for every $\lambda > 0$, hence dividing by λ and letting $\lambda \to \infty$ we obtain (1.8) with $\omega_+ = 0$.

3. An operator $A : H \to H$ such that $-A$ satisfies (1.8) with $\omega_+ = 0$ is called *monotone*. Hence an operator A in H is monotone iff $-A$ is dissipative. An operator $B : E \to E$, where $(E, |\cdot|)$ is a Banach space, is called *accretive* iff $-B$ is dissipative. (In a Hilbert space accretive is equivalent to monotone.)

4. Condition (1.8) can be rephrased as $G - \omega_+ I_H$ being dissipative.

Problem 1.1. Prove Lemma 1.1 when H is a real Banach space $(E, |\cdot|)$ and condition (1.8) is replaced by: $G - \omega_+ I_E$ is dissipative for some $\omega_+ \in \mathbb{R}$. Use this lemma to prove (1.7).

Problem 1.2. Let $(X, |\cdot|)$ be a finite-dimensional real Banach space and let $F : X \to X$ be continuous and such that $F - \alpha I_X$ is dissipative for some $\alpha \in \mathbb{R}$. Prove that for each $x \in X$ the problem

$$\begin{cases} \dot{u}(t) = F(u(t)), & t \ge 0 \\ u(0) = x \end{cases}$$

possesses a unique solution $u : [0,\infty) \to X$ which is continuously differentiable. Show that the corresponding semi-group $\{S(t)\}_{t\ge 0}$ satisfies $[S(t)]_{\text{Lip}} \le e^{\alpha t}$, $t \ge 0$.

1.2 "Gradient flows" on a Hilbert space

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with corresponding norm $|\cdot|$ and let $\varphi : H \to \mathbb{R}$. Recall that φ is called *Fréchet differentiable at* $x \in H$ if there exists $x^* \in \mathcal{L}(H; \mathbb{R}) = H^*$ (bounded linear functionals on H) such that

$$\varphi(x+h) - \varphi(x) = x^*(h) + o(|h|), \quad \text{where} \quad \lim_{|h| \to 0} \frac{o(|h|)}{|h|} = 0.$$

If such an x^* exists it is unique and is called the *gradient* of φ at x ($\operatorname{grad} \varphi(x)$). In view of the Riesz representation theorem there exists a unique element $y \in H$ such that $\langle y, h \rangle = x^*(h)$ for every $h \in H$. Moreover $\|x^*\|_{H^*} = |y|$. The element y will also be called the gradient of φ at x (Hilbert gradient) and will be denoted by $\nabla \varphi(x)$. If φ is (Fréchet-)differentiable at every $x \in H$ and the map $\nabla \varphi : H \to H$ is continuous, we shall say that φ is *continuously differentiable*, in notation $\varphi \in C^1(H; \mathbb{R})$. If moreover $\nabla \varphi \in \operatorname{Lip}(H)$ we shall use the notation $\varphi \in C^{1,1}(H; \mathbb{R})$.

Let $\varphi \in C^{1,1}(H; \mathbb{R})$ and let $\{S(t)\}_{t \in \mathbb{R}}$ be the group of operators associated with $F = \nabla \varphi$. Observe that for every $x \in H$ the map $t \mapsto S(t)x$ is continuously differentiable as well as the map $t \mapsto \varphi(S(t)x)$. Moreover

$$\frac{d}{dt} \varphi(S(t)x) = \left\langle \nabla \varphi(S(t)x), \frac{d}{dt}(S(t)x) \right\rangle = |\nabla \varphi(S(t)x)|^2 \geq 0, \quad t \in \mathbb{R}.$$

Hence the map $t \mapsto \varphi(S(t)x)$ is nondecreasing. If we consider the group of operators $\{\widetilde{S}(t)\}_{t \in \mathbb{R}}$ associated with $-\nabla \varphi$ then we have $t \mapsto \varphi\{\widetilde{S}(t)x\}_{t \in \mathbb{R}}$ nonincreasing, since $\widetilde{S}(t) = S(-t)$, $t \in \mathbb{R}$ (φ is a "Lyapunov function" for the flow). Both flows can be called "gradient" flows in H.

In the sequel we shall systematically consider (semi-)flows associated with $-\nabla \varphi$. Hence we shall consider problem (1.6) with $F = -\nabla \phi$.

Our goal is to reformulate problem (1.6) in terms of the function ϕ and the metric d only. The following lemma is useful in this respect.

Lemma 1.2. *Let* $\psi : H \to \mathbb{R}$ *be convex and Fréchet differentiable at* $x \in H$. *Let* $y \in H$. *Then the following assertions are equivalent*:

i) $y = \nabla \psi(x)$,

ii) $\langle y, h \rangle + \psi(x) \leq \psi(x+h)$ *for every* $h \in H$.

Remark. For a function $\psi : D(\psi) \subset H \to \mathbb{R}$ and $x \in D(\psi)$ we say that $y \in H$ is a *subgradient* of ψ at x if

$$\langle y, z - x \rangle + \psi(x) \leq \psi(z) \quad \text{for every} \quad z \in D(\psi). \tag{1.11}$$

The collection of all subgradients of ψ at x is called the *subdifferential* of ψ at x and is denoted by $\partial \psi(x)$. In other words, $(x, y) \in \partial \psi$ iff (1.11) holds. $\partial \psi$ can be viewed as a multivalued operator in H. It may be "empty".

Proof. i) $\Longrightarrow$ ii) Let $x_1, x_2 \in H$. The convexity of ψ implies the convexity of the map $t \mapsto \psi(x_1 + t x_2)$. It follows that the difference quotient

$$0 < t \mapsto \frac{\psi(x_1 + t x_2) - \psi(x_1)}{t}$$

is nondecreasing. Choosing $x_1 = x$ and $x_2 = h$ we obtain

$$\langle y, h\rangle = \langle \nabla\psi(x), h\rangle = \lim_{t\downarrow 0}\frac{\psi(x+th)-\psi(x)}{t} = \inf_{t>0}\frac{\psi(x+th)-\psi(x)}{t} \le \psi(x+h)-\psi(x).$$

ii) $\Longrightarrow$ i) Replacing h by th in ii) with $t>0$ we obtain

$$\langle y, h\rangle \le \frac{\psi(x+th)-\psi(x)}{t}.$$

Taking the limit as $t\to 0$ we get $\langle y, h\rangle \le \langle \nabla\psi(x), h\rangle$. Replacing h by $-h$ we obtain equality. Choosing $h = y - \nabla\psi(x)$ we arrive at i). □

Remark. The implication ii) $\Longrightarrow$ i) does not require the convexity of the function ϕ.

As a corollary we see that if $u \in C^1((a,b);H)$ for some $a, b \in \mathbb{R}$ with $a < b$ and $\psi : H \to \mathbb{R}$ is everywhere Fréchet differentiable and convex, then

$$\dot u(t) = -\nabla\psi(u(t)), \quad t \in (a,b), \tag{1.12}$$

iff

$$\frac{1}{2}\frac{d}{dt}(d(u(t),z))^2 + \psi(u(t)) \le \psi(z) \quad \text{for every} \quad z\in H,\ t\in(a,b). \tag{1.13}$$

Indeed, by Lemma 1.2 (1.12) is equivalent to

$$\langle -\dot u(t), z-u(t)\rangle + \psi(u(t)) \le \psi(z) \quad \text{for every} \quad z\in H,\ t\in(a,b),$$

which is equivalent to (1.13), since $t\mapsto |u(t)-z|^2$ is differentiable and

$$\frac{d}{dt}|u(t)-z|^2 = 2\,\langle \dot u(t), u(t)-z\rangle.$$

Notice that (1.13) is formulated in terms of ψ and the metric d only.

Next we consider a slightly more general situation. Set $e(x) = \frac{1}{2}|x|^2$, $x\in H$. We have

$$\nabla e(x) = x, \quad e(x-y) = \tfrac{1}{2}(d(x,y))^2, \quad x, y \in H. \tag{1.14}$$

Proposition 1.1. *Let $\phi : H \to \mathbb{R}$ be everywhere Fréchet differentiable and such that $\phi - \alpha e$ is convex for some $\alpha\in\mathbb{R}$. Let J be a nonempty interval of $\mathbb{R}$ and let $u\in C^1(J;H)$. Then the following assertions are equivalent:*

$$\dot u(t) = -\nabla\phi(u(t)), \quad t\in J, \tag{1.15}$$

$$\frac{1}{2}\frac{d}{dt}(d(u(t),z))^2 + \frac{\alpha}{2}(d(u(t),z))^2 + \phi(u(t)) \le \phi(z) \quad \textit{for every} \quad z\in H \textit{ and } t\in J. \tag{1.16}$$

Remark. (1.16) is called an *evolution variational inequality.*

Proof. Set $\psi := \phi - \alpha e$. (1.15) is equivalent to $\nabla\psi(u(t)) = -\dot u(t) - \alpha u(t)$ which is equivalent to

$$\langle -\dot u(t), z-u(t)\rangle - \langle \alpha u(t), z-u(t)\rangle + \psi(u(t)) \le \psi(z) \quad \text{for every} \quad z\in H,$$

by Lemma 1.2. Using the definition of ψ we get

$$\frac{1}{2}\frac{d}{dt}(d(u(t),z))^2 + \frac{\alpha}{2}|u(t)|^2 - \alpha\,\langle u(t), z\rangle + \frac{\alpha}{2}|z|^2 + \phi(u(t)) \le \phi(z), \quad z\in H.$$

Since $|u(t)|^2 - 2\,\langle u(t), z\rangle + |z|^2 = (d(u(t),z))^2$ we are done. □

Remark. As in Lemma 1.2, the implication (1.16) $\Longrightarrow$ (1.15) does not require the convexity of $\phi - \alpha e$.

Next we show that if $\phi \in C^{1,1}(H;\mathbb{R})$ then there exists $\alpha \in \mathbb{R}$ such that $\phi - \alpha e$ is convex. We recall

Lemma 1.3. *Let $\psi : H \to \mathbb{R}$ everywhere Fréchet differentiable. Then ψ is convex iff $\nabla\psi$ is monotone, i.e.*

$$\langle \nabla\psi(x_1) - \nabla\psi(x_2), x_1 - x_2 \rangle \geq 0 \quad \textit{for all} \quad x_1, x_2 \in H.$$

Proof. (Only if) Let ψ be convex and let $x_1, x_2 \in H$. Set $y_i = \nabla\psi(x_i)$, $i = 1, 2$. From Lemma 1.2 we obtain $\langle y_i, h\rangle + \psi(x_i) \leq \psi(x_i + h)$, $i = 1, 2$, $h \in H$. For $i = 1$ choose $h = x_2 - x_1$, and $h = x_1 - x_2$ for $i = 2$. Adding both inequalities we get

$$\langle y_1 - y_2, x_2 - x_1\rangle \leq 0 \quad \text{or} \quad \langle y_2 - y_1, x_2 - x_1\rangle \geq 0.$$

(If) Let $\nabla\psi$ be monotone and let $x, y \in H$, $t \in \mathbb{R}$. Set

$$\alpha(t) := \psi((1-t)x + ty) - (1-t)\psi(x) - t\psi(y).$$

Then $\alpha(0) = \alpha(1) = 0$ and α is differentiable,

$$\alpha'(t) = \langle \nabla\psi((1-t)x + ty), y - x\rangle + \psi(x) - \psi(y).$$

Let $t_1 < t_2$. Observe that $[(1-t_2)x + t_2 y] - [(1-t_1)x + t_1 y] = (t_2 - t_1)(y - x)$. We have

$$\begin{aligned}\alpha'(t_2) - \alpha'(t_1) = \langle &\nabla\psi((1-t_2)x + t_2 y) - \nabla\psi((1-t_1)x + t_1 y), \\ &[(1-t_2)x + t_2 y] - [(1-t_1)x + t_1 y]\rangle \cdot \frac{1}{t_2 - t_1} \geq 0.\end{aligned}$$

Hence α' is nondecreasing. If α had a positive maximum $\xi \in (0,1)$, then $\alpha'(\xi) = 0$ and in view of the mean value theorem α would be nonincreasing for $t < \xi$ and nondecreasing for $t > \xi$. A contradiction. Hence $\alpha(t) \leq 0$ for $t \in [0,1]$ and ψ is convex. □

If $\phi \in C^{1,1}(H;\mathbb{R})$, then $\langle \nabla\phi(x_1) - \nabla\phi(x_2), x_1 - x_2\rangle \geq -[\nabla\phi]_{\text{Lip}}|x_1 - x_2|^2$, $x_1, x_2 \in H$. Hence $\nabla(\phi - \alpha e)$ is monotone for $\alpha \leq -[\nabla\phi]_{\text{Lip}}$ and $\phi - \alpha e$ is convex in view of Lemma 1.3 at least for $\alpha \leq -[\nabla\phi]_{\text{Lip}}$.

We summarize the situation in the next proposition.

Proposition 1.2. *Let $\phi : H \to \mathbb{R}$ be such that $\phi - \alpha e$ is convex for some $\alpha \in \mathbb{R}$. If $\phi \in C^{1,1}(H;\mathbb{R})$, then for every $x \in H$ there exists a unique function $u \in C^1(\mathbb{R}; H)$ satisfying* (1.16)*, equivalently* (1.15)*, with $J = \mathbb{R}$ together with $u(0) = x$. Moreover, if $u_1, u_2 \in C^1(\mathbb{R}; H)$ satisfy* (1.16) *with $J = \mathbb{R}$ then*

$$d(u_1(t), u_2(t)) \leq e^{-\alpha(t-s)} d(u_1(s), u_2(s))$$

for every $s < t$, $s, t \in \mathbb{R}$.

Remark. Simple examples show that (even in case $H = \mathbb{R}$) Proposition 1.2 does not hold when the condition $\phi \in C^{1,1}(H;\mathbb{R})$ is replaced by $\phi \in C^1(H;\mathbb{R})$. However, if we restrict the domain of definition of the function u to $[0,\infty)$, i.e. we consider semi-flows instead of flows, Proposition 1.2 holds. See Problem 1.2 when $\dim H < \infty$ and Section 3 for the infinite-dimensional case.

1.3 Absolutely continuous curves in a metric space

The condition $u \in C^1(\mathbb{R}; H)$ prevents us to formulate Proposition 1.2 in a general (complete) metric space. The aim of this section is to introduce a weaker notion which will appear to be appropriate for this goal. We recall first some definition.

Definition 1.1. Let (X, d) be a metric space and $a, b \in \mathbb{R}$ with $a < b$. A function $u : [a, b] \to X$ is called *absolutely continuous* on $[a, b]$ if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that, for all positive integers n and all families $(a_1, b_1), \dots, (a_n, b_n)$ of disjoint open subintervals of $[a, b]$ of total length at most δ, we have

$$\sum_{k=1}^{n} d(u(a_k), u(b_k)) \le \varepsilon. \tag{1.17}$$

The collection of all such functions is denoted by $\mathrm{AC}([a, b]; X)$.

Observe that $\mathrm{AC}([a, b]; X) \subset C([a, b]; X)$.

We recall a fundamental result of real analysis.

Theorem 1.1.

i) *Let $u \in \mathrm{AC}([a, b]); \mathbb{R})$. Then u is differentiable a.e. in (a, b), $u' \in L^1(a, b)$ and*

$$\int_s^t u'(r)\, dr = u(t) - u(s) \quad \textit{for all} \quad a \le s < t \le b. \tag{1.18}$$

ii) *Let $f \in L^1(a, b)$. Then the function $t \mapsto u(t) = \int_a^t f(r)\, dr$ is absolutely continuous on $[a, b]$ and $u'(t) = f(t)$ a.e. in (a, b).*

Remark ([ABHN, Corollary 1.2.7]). The following generalization of Theorem 1.1 holds. Let X be a reflexive Banach space (in particular a Hilbert space).

i) If $u \in \mathrm{AC}([a, b]; X)$ then u is strongly differentiable a.e. in (a, b), $u' \in L^1(a, b; X)$ and (1.18) holds where the integral is a Bochner integral.

ii) If $f \in L^1(a, b); X)$, $u(t) := \int_a^t f(s)\, ds$, $t \in [a, b]$, then $u \in \mathrm{AC}([a, b]; X)$ and $u'(t) = f(t)$ a.e. in (a, b).

The following characterization of absolute continuity will be very useful.

Theorem 1.2 (see Appendix A1)**.** *Let $u : [a, b] \to X$, (X, d) a metric space. Then $u \in \mathrm{AC}([a, b]; X)$ iff there exists $m \in L^1(a, b)$, $m \ge 0$, such that*

$$d(u(s), u(t)) \le \int_s^t m(r)\, dr \quad \textit{for all} \quad a \le s < t \le b. \tag{1.19}$$

Moreover, if $u \in \mathrm{AC}([a, b]; X)$,

$$|\dot{u}|(t) := \lim_{h \to 0} \frac{d(u(t + h), u(t))}{|h|}$$

exists for almost all $t \in (a, b)$, $|\dot{u}| \in L^1(a, b)$,

$$d(u(s), u(t)) \le \int_s^t |\dot{u}|(r)\, dr, \quad a \le s \le t \le b,$$

and if m satisfies (1.19), then $|\dot{u}|(r) \le m(r)$ a.e.

The function $t \mapsto |\dot{u}|(t) \in \mathbb{R}_+$ is called the *metric derivative* of u.

Remark. If $u \in \mathrm{AC}([a,b];X)$ then $u \in \mathrm{BV}([a,b];X)$ (bounded variation) and $\int_a^b |\dot{u}|(t)\,dt = \mathrm{Var}(u;[a,b])$ the (total) variation of u on $[a,b]$.

If $u \in \mathrm{AC}([a,b];X)$ then it is easy to verify that the function $t \mapsto v(t) := d(u(t),z)$, $z \in X$, belongs to $\mathrm{AC}([a,b];\mathbb{R})$, as well as $t \mapsto (v(t))^2$. It follows that Proposition 1.1 holds if the condition $u \in C^1(J;X)$ is replaced by $u \in \mathrm{AC}([a,b];X)$ and $t \in J$ in (1.15), (1.16) is replaced by: for almost all $t \in (a,b)$.

In §2 the problem of "uniqueness" of solutions to an evolution variational inequality for a general metric space will be studied. In §3 the problem of "existence" will be treated in Hilbert spaces for $\phi : H \to (-\infty,+\infty]$ convex lower semicontinuous and in §4 the case of a complete metric space will be considered where the notion of convexity will be generalized.

Problem 1.3 ([R]). Let $(H,\langle\cdot,\cdot\rangle)$ be a real Hilbert space. A subset A of $H \times H$ is called *cyclically monotone* if for every finite cyclic sequence $x_0, x_1, \ldots, x_n = x_0$ in $D(A)$ (i.e. $x \in H$ such that there exists $y \in H$ with $(x,y) \in A$) and every sequence $y_1, \ldots, y_n$ with $(x_i, y_i) \in A$, $1 \le i \le n$, we have

$$\sum_{i=1}^{n} \langle y_i, x_i - x_{i-1}\rangle \ge 0.$$

i) Show that if $\phi : H \to (-\infty,+\infty]$ is not identically $+\infty$, the subdifferential $\partial\phi$ of ϕ is cyclically monotone.

ii) Show that if $A \subset H \times H$ is not empty and cyclically monotone, there exists $\phi : H \to (-\infty,+\infty]$ not identically $+\infty$, lower semicontinuous and convex such that $A \subset \partial\phi$.

Hint: Take $(x_0,y_0) \in A$ and for any $x \in H$ set

$$\phi(x) := \sup\big\{\langle y_n, x - x_n\rangle + \langle y_{n-1}, x_n - x_{n-1}\rangle + \ldots + \langle y_0, x_1 - x_0\rangle : \\ (x_i,y_i) \in A,\ i = 1,\ldots,n,\ n \in \mathbb{N}_+\big\}.$$

2 Uniqueness and a-priori estimates

The aim of this section is to define an *Evolution variational inequality* on a metric space (X,d) and to establish an a priori estimate for its solutions. Uniqueness will follow from this estimate.

The Evolution variational inequality (EVI) will be defined in terms of a given function $\phi : X \to (-\infty,+\infty]$, a real number α and the metric d.

A function $\phi : X \to (-\infty,+\infty]$ is called *proper* if its effective domain $D(\phi) := \{x \in X : \phi(x) < \infty\}$ is not empty. A proper function ϕ is called *lower semicontinuous* (l.s.c.) at $x \in X$ if for every sequence $\{x_n\} \subset X$ converging to x we have $\phi(x) \le \varliminf_{n\to\infty} \phi(x_n)$. For $x \in D(\phi)$, ϕ is l.s.c. at x iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi(y) \ge \phi(x) - \varepsilon$ for $y \in X$ such that $d(x,y) \le \delta$. A function ϕ is everywhere l.s.c. iff for every $c \in \mathbb{R}$ $\{x \in X : \phi(x) \le c\}$ is closed in X. We recall that a l.s.c. function on a compact metric space is bounded from below and achieves its minimum.

For the definition of solution to an EVI on a metric space we need the notion of (locally) absolutely continuous function. Let I be a nonempty interval of $\mathbb{R}$. A function $u : I \to X$ is called *locally absolutely continuous*, in notation $u \in \mathrm{AC}_{\mathrm{loc}}(I; X)$, if $u \in \mathrm{AC}([a, b]; X)$ for every $a, b \in I$ with $a < b$ and $[a, b] \subset I$. We recall that if $u \in \mathrm{AC}([a, b]; X)$, then for every $z \in X$ the function $t \mapsto \big(d(u(t), z)\big)^2$ belongs to $\mathrm{AC}([a, b]; \mathbb{R})$.

Definition 2.1. Let $\phi : X \to (-\infty, +\infty]$ be proper and l.s.c. and let $\alpha \in \mathbb{R}$. A function $u \in C([0, \infty); X) \cap \mathrm{AC}_{\mathrm{loc}}((0, \infty); X)$ satisfying

$$u(0) \in \overline{D(\phi)}, \quad u(t) \in D(\phi) \quad \text{for every} \quad t > 0,$$

for every $z \in D(\phi)$

$$\text{(EVI)} \qquad \frac{1}{2}\frac{d}{dt}\big(d(u(t), z)\big)^2 + \frac{\alpha}{2}\big(d(u(t), z)\big)^2 + \phi(u(t)) \le \phi(z) \quad \text{a.e. in } (0, \infty),$$

is called a *solution* to the Evolution Variational Inequality (EVI). The value $u(0)$ is called the *initial value* of u.

Remark. Observe that if u is such a solution then for any $h > 0$ the function $v(t) := u(t + h)$, $t \ge 0$, is also a solution to (EVI) with initial value $u(h)$.

We have the following

A priori estimate 2.1. *Suppose u and v are two solutions of EVI. Then the following estimate holds*:

$$\text{(2.1)} \qquad d(u(t), v(t)) \le e^{-\alpha(t-s)}\, d(u(s), v(s)) \quad \textit{for all} \quad 0 \le s < t < \infty.$$

In particular, two solutions of EVI with the same initial values coincide, if they exist.

Proof. Suppose u and v are two solutions to EVI, let $0 < a < b < \infty$ and let $z \in D(\phi)$. The function $[a, b] \ni t \mapsto \phi(u(t))$ is l.s.c., hence Borel measurable and bounded from below. From EVI it follows that this function is bounded from above by a Lebesgue integrable function, hence $\int_a^b |\phi(u(t))|\, dt < \infty$. Integrating EVI on $[a, b]$ we obtain

$$\text{(2.2)} \quad \frac{1}{2}\big(d(u(b), z)\big)^2 - \frac{1}{2}\big(d(u(a), z)\big)^2 + \frac{\alpha}{2}\int_a^b \big(d(u(t), z)\big)^2\, dt + \int_a^b \phi(u(t))\, dt \\ \le (b - a)\phi(z), \quad \text{for every} \quad z \in D(\phi).$$

Similarly for v.

Set $g(t) := \frac{1}{2}\, e^{2\alpha t} d^2(u(t), v(t))$, $t \ge 0$. Clearly $g \in C[0, \infty)$. We want to prove $t \mapsto g(t)$ nonincreasing on $[0, \infty)$. It suffices to show

$$\text{(2.3)} \qquad -\int_0^\infty g(t)\eta'(t)\, dt \le 0 \quad \text{for every nonnegative } \eta \in C_c^1(0, \infty).$$

Let η be as in (2.3). Extend η by 0 on $(-\infty, 0]$ and let $h_0 > 0$ be such that $\eta(t) = 0$ for $-\infty < t \le h_0$. We have for $h \in (0, h_0)$

$$-\int_0^\infty g(t)\,\frac{1}{h}\,(\eta(t) - \eta(t - h))\, dt = \int_0^\infty \frac{1}{h}\,(g(t + h) - g(t))\eta(t)\, dt.$$

Now

$$g(t+h)-g(t)=\frac{1}{2}\big[e^{2\alpha(t+h)}-e^{2\alpha t}\big]d^2\big(u(t+h),v(t+h)\big)$$
$$+\frac{1}{2}\,e^{2\alpha t}\big[d^2\big(u(t+h),v(t+h)\big)-d^2\big(u(t),v(t+h)\big)\big]$$
$$+\frac{1}{2}\,e^{2\alpha t}\big[d^2\big(u(t),v(t+h)\big)-d^2\big(u(t),v(t)\big)\big]=I_1+I_2+I_3.$$

From (2.2) with $b:=t+h$, $a:=t$ and $z:=v(t+h)$ we obtain

$$I_2\le\frac{1}{2}\,e^{2\alpha t}\Big\{2h\phi(v(t+h))-\alpha\int_t^{t+h}d^2\big(u(r),v(t+h)\big)\,dr-2\int_t^{t+h}\phi(u(r))\,dr\Big\}.$$

Similarly with u replaced by v in (2.2), $b:=t+h$, $a:=t$ and $z:=u(t)$ we obtain

$$I_3\le\frac{1}{2}\,e^{2\alpha t}\Big\{2h\phi(u(t))-\alpha\int_t^{t+h}d^2\big(v(r),u(t)\big)\,dr-2\int_t^{t+h}\phi(v(r))\,dr\Big\}.$$

Using the nonnegativity of η we arrive at

$$\int_0^\infty\eta(t)\,\frac{1}{h}\,\big(g(t+h)-g(t)\big)\,dt$$
$$\le\int_0^\infty\eta(t)\,\frac{1}{2}\,e^{2\alpha t}\Big\{\Big[\frac{1}{h}(e^{2\alpha h}-1)d^2\big(u(t+h),v(t+h)\big)\Big]$$
$$+2\Big[\phi(v(t+h))-\frac{1}{h}\int_t^{t+h}\phi(u(r))\,dr-\frac{\alpha}{2}\,\frac{1}{h}\int_t^{t+h}d^2\big(u(r),v(t+h)\big)\,dr\Big]$$
$$+2\Big[\phi(u(t))-\frac{1}{h}\int_t^{t+h}\phi(v(r))\,dr-\frac{\alpha}{2}\,\frac{1}{h}\int_t^{t+h}d^2\big(v(r),u(t)\big)\,dr\Big]\Big\}\,dt.$$

Since $\phi(v(\cdot+h))$ (resp. $\frac{1}{h}\int_t^{t+h}\phi(u(r))\,dr$, $\frac{1}{h}\int_t^{t+h}\phi(v(r))\,dr$) tends to $\phi\circ v(\cdot)$ (resp. $\phi\circ u(\cdot)$, $\phi\circ v(\cdot)$) in $L^1_{\mathrm{loc}}(0,\infty)$ as $h\to 0$, we get

$$-\int_0^\infty g(t)\eta'(t)\,dt=\lim_{h\to 0}-\frac{1}{h}\int_0^\infty g(t)\big(\eta(t)-\eta(t-h)\big)\,dt$$
$$\le\int_0^\infty\eta(t)\,\frac{1}{2}\,e^{2\alpha t}\Big\{2\alpha d^2\big(u(t),v(t)\big)+2\phi(v(t))-2\phi(u(t))-\alpha d^2\big(u(t),v(t)\big)$$
$$+2\phi(u(t))-2\phi(v(t))-\alpha d^2\big(u(t),v(t)\big)\Big\}=0.\ \square$$

2.1 Integral formulation of EVI

The proof of the A priori estimate 2.1 motivates the following definition.

Definition 2.2. Let $\phi:X\to(-\infty,+\infty]$ be proper and l.s.c. and let $\alpha\in\mathbb{R}$. A function $u\in C([0,\infty);X)$ is called an "*integral solution*" to EVI if for every $0<a<b$ the function $\phi\circ u\in L^1(a,b)$ and satisfies (2.2).

Proposition 2.1.

i) *A solution to EVI is an "integral solution" to EVI.*

ii) *If u and v are "integral solutions" to EVI, then they satisfy estimate* (2.1). *In particular, they coincide if* $u(0) = v(0)$.

iii) *If u is an "integral solution" to EVI and satisfies $u \in \mathrm{Lip}([a,b];X)$ for every $0 < a < b$, then u is a solution to EVI.*

Proof. Parts i) and ii) follow from the proof of A priori estimate 2.1.

iii) Let $z \in D(\phi)$ and $0 < a' < b'$. Let $u \in \mathrm{Lip}([a',b'];X)$ with $\phi \circ u \in L^1(a',b')$ satisfying (2.2) for every $a' \le a < b \le b'$. We will show that there exists a set $N \subset (a',b')$ of measure 0 such that u satisfies EVI on $(a',b') \setminus N$ and $\phi \circ u$ is bounded from above on $(a',b') \setminus N$ by a finite number C. Since $\phi \circ u \in L^1(a',b')$ and $u \in \mathrm{Lip}([a',b'];X)$ there exists such N for which every $t_0 \in (a',b') \setminus N$ is a Lebesgue point of $\phi \circ u$ in (a',b') and is a point of differentiability of the function $t \mapsto d(u(t),z)$ in (a',b'). Choosing $a = t_0 \in (a',b') \setminus N$, $b = t_0 + h$ with $0 < h < b' - t_0$, dividing (2.2) by h and letting h tend to 0 we obtain

$$\frac{d}{dt}\Big(\frac{1}{2}\, d^2(u(t_0),z)\Big) + \frac{\alpha}{2}\, d^2(u(t_0),z) + \phi(u(t_0)) \le \phi(z).$$

Setting $C_1(a',b') := \max\limits_{t\in[a',b']} (u(t),z)$, we get

$$\phi(u(t_0)) \le \phi(z) + \frac{|\alpha|}{2}\, C_1^2 + C_1 [u]_{\mathrm{Lip},[a',b']} =: C(a',b').$$

In view of the density of $(a',b') \setminus N$ in (a',b'), the continuity of u and the lowersemicontinuity of ϕ we also have $\phi(u(t)) \le C$ for every $t \in (a',b')$, hence $u(t) \in D(\phi)$, $t \in (a',b')$. Now claim iii) easily follows. □

Remark. It can be shown that an "integral solution" to EVI is actually a solution to EVI (see [AGS, Remark 4.0.5b) on page 78] and [CD1]).

3 "Existence" in case X is a Hilbert space

Let $(X, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with corresponding norm $|\cdot|$ and metric $d(\cdot,\cdot)$. Let $\phi : X \to (-\infty,\infty]$ be a proper, lower semicontinuous, function such that $\phi - \alpha e$ is convex for some $\alpha \in \mathbb{R}$.

In what follows a function ϕ such that $\phi - \alpha e$ is convex will be called *α-convex.*

Consider the Evolution Variational Inequality (EVI) defined in Section 2 associated with ϕ and α. We already know that for any $x \in \overline{D(\phi)}$ there exists at most one solution to (EVI) with initial value $u(0) = x$. The aim of this section is to prove the existence of such solution.

The proof of the existence result will be done by approximating the function ϕ by a family of functions $\phi_h \in C^{1,1}(X;\mathbb{R})$, $h \in I_\alpha$, where

$$I_\alpha := \begin{cases} (0,\infty) & \text{if } \alpha \ge 0, \\ (0,|\alpha|^{-1}) & \text{if } \alpha < 0. \end{cases} \tag{3.1}$$

The functions ϕ_h are usually called *Moreau–Yosida approximations* of ϕ. These functions converge to ϕ as h tends to zero and are $\frac{\alpha}{1+\alpha h}$-convex.

Therefore in view of the results of Section 1 one can uniquely solve (EVI) where ϕ is replaced by ϕ_h and α by $\frac{\alpha}{1+\alpha h}$. The next step consists of establishing appropriate a priori estimates independent of $h \in I_\alpha$ which allow to pass to the limit and find a solution of (EVI) with ϕ and α.

In order to define the functions ϕ_h we need some preliminary results.

3.1 Preliminaries

Lemma 3.1. *Let $\psi : X \to (-\infty, +\infty]$ be proper, l.s.c. and convex. Then there exist $b \in X$ and $c \in \mathbb{R}$ such that*

$$\psi(x) \geq \langle b, x\rangle + c, \quad x \in X. \tag{3.2}$$

Proof. Consider the epigraph of ψ defined by $\text{epi}(\psi) := \{(x,t) \in X \times \mathbb{R} : \psi(x) \leq t\}$. Since ψ is proper, $\text{epi}(\psi) \neq \varnothing$. Moreover, $\text{epi}(\psi)$ is convex in view of the convexity of ψ. We introduce the innerproduct $\langle\langle \cdot, \cdot \rangle\rangle$ in $X \times \mathbb{R}$ defined by $\langle\langle (x_1, t_1), (x_2, t_2) \rangle\rangle := \langle x_1, x_2 \rangle + t_1 t_2$. Clearly $(X \times \mathbb{R}, \langle\langle \cdot, \cdot \rangle\rangle)$ is a Hilbert space. The subset $\text{epi}(\psi)$ is closed in $X \times \mathbb{R}$ as a consequence of the lower semicontinuity of ψ. Let $x_0 \in D(\psi)$ and $t_0 < \psi(x_0)$. Then $(x_0, t_0) \notin \text{epi}(\psi)$. By the projection theorem on closed convex sets in Hilbert spaces, there exists a unique element $(\overline{x}, \overline{t}) \in \text{epi}(\psi)$ satisfying

$$\langle x - \overline{x}, x_0 - \overline{x}\rangle + (t - \overline{t})(t_0 - \overline{t}) \leq 0 \tag{3.3}$$

for every $(x,t) \in \text{epi}(\psi)$.

Choose $x = x_0$ and $t \geq \phi(x_0)$ in (3.3). Since $0 < \langle x_0 - \overline{x}, x_0 - \overline{x}\rangle$ we see that $t_0 - \overline{t}$ cannot be zero. Moreover, choosing $t > \overline{t}$ shows that $t_0 - \overline{t}$ has to be negative. Finally, choosing $x \in D(\psi)$ and $t = \psi(x)$ in (3.3) we obtain (3.2) with

$$b := \frac{1}{\overline{t} - t_0}(\overline{x} - x_0) \quad \text{and} \quad c := \overline{t} + \frac{1}{\overline{t} - t_0}\langle \overline{x}, \overline{x} - x_0 \rangle .$$

Clearly (3.2) holds for $x \in X \setminus D(\psi)$. ◻

Lemma 3.2. *Let $\phi : X \to (-\infty, +\infty]$ be proper, l.s.c. and α-convex for some $\alpha \in \mathbb{R}$. Then for every $h \in I_\alpha$ and every $x \in X$ the function*

$$\varphi(y) := \begin{cases} \frac{1}{2h}|y - x|^2 + \phi(y), & y \in D(\phi), \\ +\infty, & \textit{otherwise.} \end{cases} \tag{3.4}$$

has a unique global minimizer, which we denote by $J_h x$.

Proof. By α-convexity and Lemma 3.1 the function φ can be rewritten as

$$\varphi(y) = \Big(\alpha + \frac{1}{h}\Big)\frac{1}{2}|y|^2 + \Big\langle b - \frac{1}{h}x, y \Big\rangle + \Big(c + \frac{1}{2h}|x|^2\Big) + \psi_1(y) \tag{3.5}$$

where $\psi_1 : X \to [0, \infty]$ is proper, l.s.c. and convex. Since $\alpha + \frac{1}{h} > 0$ and $\psi_1(y) \geq 0$, it is clear that φ is bounded below. Set $\gamma := \inf_{y \in H} \varphi(y) \in \mathbb{R}$. Let $\{y_n\}_{n\geq 1} \subset D(\varphi)$ be a minimizing sequence, i.e. $\lim_{n\to\infty} \varphi(y_n) = \gamma$. We claim that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence. Let $\overline{y}$ denote the limit in X. Then by lower semicontinuity we have

$$\gamma \leq \varphi(\overline{y}) \leq \varliminf_{n\to\infty} \varphi(y_n) = \gamma.$$

It remains to prove the claim. By using the convexity of ψ_1 in (3.5), given $y, \widehat{y} \in D(\varphi)$ we have

$$\varphi(y) + \varphi(\widehat{y}) - 2\varphi\Big(\frac{y + \widehat{y}}{2}\Big) \geq \Big(\alpha + \frac{1}{h}\Big)\left[\frac{1}{2}|y|^2 + \frac{1}{2}|\widehat{y}|^2 - \Big|\frac{y+\widehat{y}}{2}\Big|^2\right] = \Big(\alpha + \frac{1}{h}\Big)\Big|\frac{y - \widehat{y}}{2}\Big|^2.$$

Since $\frac{y+\widehat{y}}{2} \in D(\varphi)$ we obtain

$$|y - \widehat{y}| \leq \left(\alpha + \frac{1}{h}\right)^{-1/2} \cdot 2 \cdot \sqrt{(\varphi(y) - \gamma) + (\varphi(\widehat{y}) - \gamma)}. \tag{3.6}$$

Replacing y by y_m and $\widehat{y}$ by y_n in (3.6) and using $\lim_{n\to\infty} \varphi(y_n) = \gamma$ we are done.

The uniqueness of the global minimizer follows from (3.6). □

We introduce some

Definitions 3.1. Let $\phi : X \to (-\infty, +\infty]$ be proper, l.s.c. and α-convex for some $\alpha \in \mathbb{R}$. Set

$$\psi := \phi - \alpha e. \tag{3.7}$$

For $h \in I_\alpha$ and $x \in X$ set

$$A_h x := \frac{1}{h}(x - J_h x). \tag{3.8}$$

We collect some properties of J_h and A_h.

Lemma 3.3. *For $h \in I_\alpha$ and $x, \widehat{x} \in X$, we have*

$$J_h x \in D(\partial\psi) \text{ and } A_h x - \alpha J_h x \in \partial\psi(J_h x), \tag{3.9}$$

$$|J_h x - J_h \widehat{x}| \leq \frac{1}{1 + \alpha h}\,|x - \widehat{x}|, \tag{3.10}$$

$$|A_h x - A_h \widehat{x}| \leq \frac{1}{h}\,\frac{2 + \alpha h}{1 + \alpha h}\,|x - \widehat{x}|, \tag{3.11}$$

$$\langle A_h x - A_h \widehat{x}, x - \widehat{x}\rangle \geq \frac{\alpha}{1 + \alpha h}\,|x - \widehat{x}|^2. \tag{3.12}$$

Proof. (3.9). From (3.4) and (3.7) we get

$$\varphi(y) = \left(\frac{1}{h} + \alpha\right)\frac{1}{2}|y|^2 - \left\langle \frac{1}{h}\,x, y \right\rangle + \frac{1}{2h}\,|x|^2 + \psi(y), \quad y \in X.$$

Set $g(y) := \frac{1}{2}\big(\frac{1}{h} + \alpha\big)|y|^2 - \big\langle \frac{1}{h}\,x, y\big\rangle + \frac{1}{2h}\,|x|^2$, $y \in X$. Then $\varphi = g + \psi$. Since $J_h x$ is a global minimizer of φ, we have for every $y \in D(\psi)$ and $t \in (0, 1)$

$$g((1 - t)J_h x + ty) + \psi((1 - t)J_h x + ty) \geq g(J_h x) + \psi(J_h x).$$

Using the convexity of ψ we obtain

$$-\frac{1}{t}\,\big(g((1 - t)J_h x + ty) - g(J_h x)\big) \leq \psi(y) - \psi(J_h x).$$

Letting t tend to zero we arrive at

$$-\langle \nabla g(J_h x), y - J_h x\rangle \leq \psi(y) - \psi(J_h x).$$

Noting that $\nabla g(z) = \big(\frac{1}{h} + \alpha\big)z - \frac{1}{h}\,x$, $z \in X$, using (3.8) and the definition of the subdifferential of ψ we obtain (3.9).

(3.10). Let $x_1, x_2 \in X$. From (3.9) we have

$$\frac{1}{h}(x_i - J_h x_i) - \alpha J_h x_i \in \partial\psi(J_h x_i), \quad i = 1, 2.$$

Using the monotonicity of $\partial\psi$ (see Problem 1.3) we get

$$\left\langle \left[-\left(\frac{1}{h} + \alpha\right) J_h x_1 + \frac{1}{h} x_1\right] - \left[-\left(\frac{1}{h} + \alpha\right) J_h x_2 + \frac{1}{h} x_2\right], J_h x_1 - J_h x_2 \right\rangle \geq 0.$$

Hence

$$(1 + \alpha h)|J_h x_1 - J_h x_2|^2 \leq \langle x_1 - x_2, J_h x_1 - J_h x_2 \rangle \leq |x_1 - x_2|\, |J_h x_1 - J_h x_2|$$

which implies (3.10) since $1 + \alpha h > 0$.

(3.11) is a direct consequence of (3.10) and (3.8).

(3.12). We have

$$(1 + \alpha h) h A_h = (1 + \alpha h) I - (1 + \alpha h) J_h = (I - C) + \alpha h I,$$

where $C := (1 + \alpha h) J_h$.

By (3.10) $|Cx_1 - Cx_2| \leq |x_1 - x_2|$ hence $\langle (I - C)x_1 - (I - C)x_2, x_1 - x_2 \rangle \geq 0$. Hence

$$\langle A_h x_1 - A_h x_2, x_1 - x_2 \rangle = \frac{1}{h(1 + \alpha h)} \langle (I - C)x_1 - (I - C)x_2, x_1 - x_2 \rangle + \frac{\alpha h}{1 + \alpha h}|x_1 - x_2|^2$$

which implies (3.12). □

3.2 Moreau–Yosida approximation

In this section we define ϕ_h, the Moreau–Yosida approximation of ϕ. In Proposition 3.1 we give some properties of ϕ_h for a fixed h and in Proposition 3.2 the behavior of ϕ_h as $h \to 0$.

Definition 3.2. Let ϕ be as in Definition 3.1 and φ as in (3.4). Let $h \in I_\alpha$. Then

$$\phi_h(x) := \varphi(J_h x), \quad x \in X. \tag{3.13}$$

Proposition 3.1. *Let ϕ, ϕ_h be as above. Then*

$$\phi_h(x) = \frac{h}{2}|A_h x|^2 + \phi(J_h x), \quad x \in X. \tag{3.14}$$

$\phi_h \in C^{1,1}(X; \mathbb{R})$, $\nabla\phi_h = A_h$ and $\phi_h - \frac{\alpha}{1+\alpha h}\, e$ is convex.

Proof. (3.14) is a direct consequence of the definition of $J_h x$, $A_h x$ and (3.4).

Next we show that $\nabla\phi(x) = A_h x$ for every $x \in X$. Let $x, y \in X$. From (3.9) and the monotonicity of $\partial\psi$ we get

$$\psi(J_h y) - \psi(J_h x) \geq \langle A_h x - \alpha J_h x, J_h y - J_h x \rangle.$$

Using (3.7) and (3.14) we obtain

$$\begin{aligned} \phi_h(y) - \phi_h(x) &= \psi(J_h y) - \psi(J_h x) + \frac{\alpha}{2}|J_h y|^2 - \frac{\alpha}{2}|J_h x|^2 + \frac{h}{2}|A_h y|^2 - \frac{h}{2}|A_h x|^2 \\ &\geq \langle A_h x - \alpha J_h x, J_h y - J_h x \rangle + \frac{\alpha}{2}|J_h y|^2 - \frac{\alpha}{2}|J_h x|^2 + \frac{h}{2}|A_h y|^2 - \frac{h}{2}|A_h x|^2. \end{aligned}$$

By (3.8)

$$\langle A_h x - \alpha J_h x, J_h x - J_h y\rangle = \langle A_h x - \alpha J_h x, x - y\rangle - \langle A_h x - \alpha J_h x, hA_h x - hA_h y\rangle .$$

By rearranging terms we have

$$\phi_h(y) - \phi_h(x) - \langle A_h x, y - x\rangle \geq \frac{h}{2}|A_h x - A_h y|^2 + \frac{\alpha}{2}|J_h x - J_h y|^2. \tag{3.15}$$

Exchanging x and y in (3.15) and adding and subtracting $A_h y$ we get

$$\phi_h(x) - \phi_h(y) - \langle A_h x, x - y\rangle \geq \frac{h}{2}|A_h x - A_h y|^2 + \frac{\alpha}{2}|J_h x - J_h y|^2 + \langle A_h y - A_h x, x - y\rangle .$$

Using (3.10) and (3.11) we arrive at

$$\big|\phi_h(y) - \phi_h(x) - \langle A_h x, y - x\rangle\big| \leq M|y - x|^2 \tag{3.16}$$

for some $M > 0$ independent of x and $y \in X$. Hence $A_h x = \nabla\phi_h(x)$ and since $A_h \in \mathrm{Lip}(X)$ we have $\phi_h \in C^{1,1}(X;\mathbb{R})$.

From (3.12) $A_h - \frac{\alpha}{1+\alpha h} I$ is monotone hence $\phi_h - \frac{\alpha}{1+\alpha h} e$ is convex. □

In what follows we shall consider the behavior of J_h, A_h and ϕ_h as h tends to zero. In order to treat the cases $\alpha \geq 0$ and $\alpha < 0$ at the same time, we introduce

$$h_\alpha := \begin{cases} 1 & \text{if } \alpha \geq 0, \\ \frac{1}{2|\alpha|} & \text{if } \alpha < 0, \end{cases} \tag{3.17}$$

then

$$1 + h\alpha \in [\tfrac{1}{2}, 1 + |\alpha|] \quad \text{for} \quad 0 < h \leq h_a. \tag{3.18}$$

We shall use the following notation. Let $x \in D(\partial\psi)$, with ψ as in (3.7). Observe that the set $\{y \in X : y \in \partial\psi(x)\}$ is a nonempty closed convex subset of X. Hence it has a unique element of minimal norm, which we denote by $(\partial\psi)^\circ x$, i.e.

$$\big|(\partial\psi)^\circ x\big| \leq |y| \quad \text{for any} \quad y \in \partial\psi(x). \tag{3.19}$$

Next we establish some properties of $J_h x$, $A_h x$ and $\phi_h(x)$ as functions of $h \in (0, h_\alpha)$.

Lemma 3.4.

$$\sup_{h\in(0,h_\alpha)} |A_h x| < \infty \qquad \textit{if } x \in D(\partial\psi). \tag{3.20}$$

$$\sup_{h\in(0,h_\alpha)} |J_h x| < \infty \qquad \textit{for every } x \in X. \tag{3.21}$$

$$\inf_{h\in(0,h_\alpha)} \phi(J_h x) > -\infty \qquad \textit{for every } x \in X. \tag{3.22}$$

Proof. (3.20). Let $(x, y) \in \partial\psi$. From (3.9) and the monotonicity of $\partial\psi$ we have

$$\frac{1}{h}\langle y - A_h x + \alpha J_h x, x - J_h x\rangle \geq 0$$

hence by (3.8)

$$|A_h x|^2 \leq \langle y, A_h x\rangle + \alpha\langle J_h x, A_h x\rangle = \langle y, A_h x\rangle + \alpha\langle x, A_h x\rangle - \alpha h|A_h x|^2.$$

Then $(1+h\alpha)|A_h x|^2 \le (|y| + |\alpha|\,|x|)|A_h x|$ which implies by (3.18), (3.19)

$$|A_h x| \le 2\big(|(\partial\psi)^\circ x| + |\alpha|\,|x|\big).$$

(3.21). Let $x \in X$ and $\widehat{x} \in D(\partial\psi)$. Set $C := \sup_{h\in(0,h_\alpha)} |A_h\widehat{x}|$. Using (3.8), (3.10), (3.18) and (3.20) we get

$$|J_h x| \le |J_h x - J_h\widehat{x}| + |J_h\widehat{x}| \le 2|x-\widehat{x}| + |\widehat{x}| + h|A_h\widehat{x}| \le 2|x-\widehat{x}| + |\widehat{x}| + h_\alpha \cdot C.$$

(3.22). Let $x \in X$ and set $M := \sup_{h\in(0,h_\alpha)} |J_h x|$. Then by using (3.14), (3.2), (3.7)

$$\phi(J_h x) = \psi(J_h x) + \frac{\alpha}{2}\,|J_h x|^2 \ge -|b|M + c - \frac{|\alpha|}{2}\,M^2. \qquad \square$$

Lemma 3.5.

$$\lim_{h\to 0} |x - J_h x| = 0 \qquad \textit{iff } x \in \overline{D(\partial\psi)}, \tag{3.23}$$

$$\sup_{h\in(0,h_a)} \phi_h(x) = +\infty \qquad \textit{if } x \notin \overline{D(\partial\psi)}. \tag{3.24}$$

Proof. (3.23). For any $\widehat{x} \in D(\partial\psi)$ we have by (3.10), (3.18), (3.8)

$$|x - J_h x| \le |x-\widehat{x}| + |\widehat{x} - J_h\widehat{x}| + |J_h\widehat{x} - J_h x| \le 3|x-\widehat{x}| + h|A_h\widehat{x}|,$$

which implies the *if* part of (3.23) in view of (3.20). Conversely, if $\lim_{h\to 0} |x - J_h x| = 0$ then $x \in \overline{D(\partial\psi)}$ since $J_h x \in D(\partial\psi)$ by (3.9).

(3.24). Using (3.14) and (3.22) it is sufficient to show $\sup_{h\in(0,h_\alpha)} h|A_h x|^2 = +\infty$ if $x \notin \overline{D(\partial\psi)}$. Note that

$$h|A_h x|^2 = |x - J_h x| \cdot |A_h x| \ge \operatorname{dist}\big(x, \overline{D(\partial\psi)}\big)|A_h x|$$

since $J_h x \in D(\partial\psi)$. Since $d(x, \overline{D(\partial\psi)}) > 0$ by assumption, it remains to show that $\sup_{h\in(0,h_\alpha)} |A_h x| = +\infty$ for $x \notin \overline{D(\partial\psi)}$. If $M := \sup_{h\in(0,h_\alpha)} |A_h x| < \infty$, then $|x - J_h x| \le hM$ by (3.8) and $\lim_{h\to 0} |x - J_h x| = 0$ contradicting (3.23). $\square$

We conclude this section by establishing the convergence of ϕ_h as $h \to 0$.

Proposition 3.2. *Let ϕ, ψ, ϕ_h be as above. Then*

$$\phi_h(x) \uparrow \phi(x) \quad \textit{for every} \quad x \in X \ \textit{as } h \downarrow 0. \tag{3.25}$$

$$D(\partial\psi) \subseteq D(\phi) \subseteq \overline{D(\partial\psi)} = \overline{D(\phi)}. \tag{3.26}$$

Proof. By (3.4) and (3.13) we have $\phi_{h_1}(x) \le \phi_{h_2}(x)$ for $0 < h_2 < h_1 \le h_\alpha$, $x \in X$. Moreover, since for every $x, y \in X$ we have $\phi_h(x) = \varphi(J_h x) \le \varphi(y)$, choosing $y = x$ we obtain $\phi_h(x) \le \varphi(x) = \phi(x)$. Consequently, by (3.24) if $x \notin \overline{D(\partial\psi)}$, $\sup_{h\in(0,h_\alpha)} \phi_h(x) = +\infty$, hence $x \notin D(\phi)$. This implies (3.25) for $x \notin \overline{D(\partial\psi)}$ and the inclusion $D(\phi) \subseteq \overline{D(\partial\psi)}$ in (3.26). If $x \in \overline{D(\partial\psi)}$ and $h_n \in (0, h_\alpha)$, $h_n \downarrow 0$ as n tends to infinity, we have by (3.23) $\lim_{n\to\infty} |x - J_{h_n} x| = 0$ and by the lower semicontinuity of ϕ

$$\phi(x) \le \varliminf_{n\to\infty} \phi(J_{h_n} x) \le \varliminf_{n\to\infty} \phi_{h_n}(x) \le \varlimsup_{n\to\infty} \phi_{h_n}(x) \le \phi(x)$$

which implies (3.25).

Finally, $D(\partial\psi) \subseteq D(\psi) = D(\phi)$, hence also $\overline{D(\partial\psi)} \subseteq \overline{D(\phi)}$. Since $D(\phi) \subseteq \overline{D(\partial\psi)}$, (3.26) follows. $\square$

3.3 A quasi-contractive semigroup associated with ϕ

Let $\phi : X \to (-\infty, +\infty]$ be proper, l.s.c. and such that $\phi - \alpha e$ is convex for some $\alpha \in \mathbb{R}$. Let $h \in (0, h_\alpha]$ and ϕ_h the Moreau–Yosida approximation of ϕ. We consider the problem

$$\frac{du}{dt}(t) + A_h u(t) = 0, \quad t \in \mathbb{R}, \tag{3.27}$$

together with the condition

$$u(0) = x \quad \text{with} \quad x \in X. \tag{3.28}$$

In view of Propositions 1.2 and 3.1 this problem possesses exactly one solution which we denote by $u_{h,x}$ (or simply u_h) and we set

$$S_h(t)x := u_{h,x}(t), \quad t \in \mathbb{R}, \ x \in X. \tag{3.29}$$

Moreover, the family $\{S_h(t)\}_{t\in\mathbb{R}}$ is a C_0-group of operators on X satisfying

$$\big|S_h(t)x - S_h(t)y\big| \le e^{-\frac{\alpha}{1+\alpha h}(t-s)}\big|S_h(s)x - S_h(s)y\big| \tag{3.30}$$

for $s < t$ and $x, y \in X$ since $\nabla\phi_h = A_h$ and $\phi_h - \frac{\alpha}{1+\alpha h}\, e$ is convex.

The aim of this section is to establish the following

Theorem 3.1. *For every $x \in \overline{D(\phi)}$ and $t \ge 0$:*

$$S(t)x := \lim_{h\to 0} S_h(t)x \quad \textit{exists in } (X, |\cdot|), \tag{3.31}$$

$$S(t)x \in \overline{D(\phi)}. \tag{3.32}$$

The family of operators $\{S(t)\}_{t\ge 0} : \overline{D(\phi)} \to \overline{D(\phi)}$ is a C_0-semigroup satisfying

$$[S(t)]_{\mathrm{Lip}} \le e^{-\alpha t}, \quad t \ge 0. \tag{3.33}$$

The idea of the proof is to prove (3.31)–(3.32) for $x \in D(\partial\psi)$ and then to use a simple approximation argument together with the estimate (3.30).

Proof. Let $h \in (0, h_\alpha]$ and $x \in D(\partial\psi)$.

Step 1. By Lemma 3.4, $M_1 := \sup\limits_{h\in(0,h_\alpha)} |A_h(x)| < \infty$. Let $T > 0$. We claim that

$$|A_h u_h(t)| \le M_1 e^{2|\alpha|T} =: M_2(\alpha, T) \quad \text{for} \quad h \in (0, h_0) \text{ and } t \in [0, T]. \tag{3.34}$$

Indeed, from (3.30) with $s = 0$, $y = S_h(h)x$ with $h > 0$ we obtain by dividing by h:

$$\Big|\frac{1}{h}\big(u_h(t) - u_h(t+h)\big)\Big| \le e^{2|\alpha|T}\Big|\frac{1}{h}\big(u_h(0) - u_h(h)\big)\Big|,$$

hence

$$|\dot u_h(t)| \le e^{2|\alpha|T}|\dot u_h(0)| = e^{2|\alpha|T}|A_h(x)| \le e^{2|\alpha|T} M_1.$$

Since $\dot u_h(t) = -A_h u_h(t)$ we are done.

Let $0 < h < \lambda \le h_\alpha$ and $t \in [0, T]$.

Step 2. We have

$$\langle A_h u_h(t) - A_\lambda u_\lambda(t), u_h(t) - u_\lambda(t)\rangle \ge -2|\alpha|\, |u_h(t) - u_\lambda(t)|^2 - \lambda M_3, \tag{3.35}$$

where

$$M_3 := \big(8|\alpha|h_\alpha + 4\big)M_2^2(\alpha, T). \tag{3.36}$$

From the monotonicity of $\partial\psi$ and from (3.9) we get

$$\big\langle \big(A_h u_h(t) - \alpha J_h u_h(t)\big) - \big(A_\lambda u_\lambda(t) - \alpha J_\lambda u_\lambda(t)\big), J_h u_h(t) - J_\lambda u_\lambda(t)\big\rangle \ge 0.$$

This implies

$$\langle A_h u_h(t) - A_\lambda u_\lambda(t), J_h u_h(t) - J_\lambda u_\lambda(t)\rangle \ge \alpha\big|J_h u_h(t) - J_\lambda u_\lambda(t)|^2.$$

From (3.8) and (3.34) we obtain

$$\big|J_h u_h(t) - J_\lambda u_\lambda(t)|^2 \le 2|u_h(t) - u_\lambda(t)|^2 + 8M_2^2 h_\alpha \lambda,$$

and

$$\langle A_h u_h(t) - A_\lambda u_\lambda(t), J_h u_h(t) - J_\lambda u_\lambda(t)\rangle \ge \langle A_h u_h(t) - A_\lambda u_\lambda(t), u_h(t) - u_\lambda(t)\rangle - 4M_2^2\lambda.$$

Then (3.35) follows.

Step 3. From $\dot u_h(t) + A_h u_h(t) = 0$, $\dot u_\lambda(t) + A_\lambda u_\lambda(t) = 0$ and (3.35) we obtain

$$\frac{1}{2}\frac{d}{dt}|u_h(t) - u_\lambda(t)|^2 = \langle \dot u_h(t) - \dot u_\lambda(t), u_h(t) - u_\lambda(t)\rangle \le 2|\alpha|\,|u_h(t) - u_\lambda(t)|^2 + \lambda M_3.$$

Using the fact that $|u_h(0) - u_\lambda(0)|^2 = 0$ we arrive at

$$|u_h(t) - u_\lambda(t)|^2 \le \lambda M_3 \cdot M_4 \quad \text{for some} \quad M_4 = M_4(\alpha, T). \tag{3.37}$$

Step 4 (convergence for $x \in D(\partial\psi)$).

It follows from (3.37) that if $h_n \to 0$ as $n \to \infty$, $\{u_{h_n}(t)\}_{n\ge1}$ is a Cauchy sequence in $(X, |\cdot|)$. Set

$$S(t)x := \lim_{n\to\infty} u_{h_n}(t). \tag{3.38}$$

Clearly $S(t)x := \lim_{h\to0} u_h(t) = \lim_{h\to0} S_h(t)x$. Since $T > 0$ is arbitrary, $S(t)x$ is well defined for every $t > 0$. From (3.37) the convergence is uniform on $[0, T]$ hence $t \mapsto S(t)x \in C([0, T]; X)$, $T > 0$. From (3.8), (3.34) we get

$$\big|S(t)x - J_{h_n} u_{h_n}(t)\big| \le \big|S(t)x - u_{h_n}(t)\big| + h_n M_1.$$

Observe that $J_{h_n} u_{h_n}(t) \in D(\partial\psi)$ by (3.9), hence $S(t) \in \overline{D(\partial\psi)} = \overline{D(\phi)}$ by (3.26).

Step 5 (convergence for $x \in \overline{D(\phi)}$).

Let $x \in \overline{D(\phi)}$, $\varepsilon > 0$ and $T > 0$. Then for $\widehat{x} \in D(\partial\psi)$ we have

$$\begin{aligned}|S_h(t)x - S_\lambda(t)x| &\le |S_h(t)x - S_h(t)\widehat{x}| + |S_h(t)\widehat{x} - S_\lambda(t)\widehat{x}| + |S_\lambda(t)\widehat{x} - S_\lambda(t)x| \\ &\le 2e^{2|\alpha|T}|x - \widehat{x}| + |S_h(t)\widehat{x} - S_\lambda(t)\widehat{x}|, \quad t \in [0, T].\end{aligned}$$

Since $\overline{D(\partial\psi)} = \overline{D(\phi)}$, we can choose $\widehat{x} \in D(\phi)$ such that the first term in the last inequality is less than $\varepsilon/2$ and we can find $\overline{h} \in (0, h_\alpha]$ such that $|S_h(t)\widehat{x} - S_\lambda(t)\widehat{x}| \le \varepsilon/2$

for $t \in [0,T]$ and $0 < h < \lambda \leq \overline{h}$. As in Step 4 we conclude that $\lim_{h\to 0} S_h(t)x$ exists in X and we denote it by $S(t)x$, $t \geq 0$. By uniformity on $[0,T]$, $t \mapsto S(t)x$ is continuous on $[0,T]$. Property (3.33) follows from (3.30) with $s=0$ and (3.31).

Next we prove (3.32). Let $x_n \in D(\partial\psi)$, $n \geq 1$, be such that $\lim_{n\to\infty} |x_n - x| = 0$. Then

$$|S(t)x - S(t)x_n| \leq e^{-\alpha t}|x - x_n| \geq 0 \quad \text{as} \quad n \to \infty.$$

Since $S(t)x_n \in \overline{D(\phi)}$, $n \geq 1$, the same holds for $S(t)x$.

Step 6 (semigroup property).

Let $x \in \overline{D(\phi)}$, $t, s \geq 0$, $h \in (0, h_\alpha]$. We have

$$\begin{aligned} |S(t+s)x - S(t)S(s)x| &\leq |S(t+s)x - S_h(t+s)x| + |S_h(t+s)x - S_h(t)S_h(s)x| \\ &\quad + |S_h(t)S_h(s)x - S_h(s)S(s)x| + |S_h(t)S(s)x - S(t)S(s)x| \\ &\leq |S(t+s)x - S_h(t+s)x| + e^{2|\alpha|t}|S_h(s)x - S(s)x| \\ &\quad + |S_h(t)S_h(s)x - S(t)S(s)x| \to 0 \quad \text{as} \quad h \to 0. \end{aligned}$$

Hence $\{S(t)\}_{t\geq 0}$ is a semigroup of operators on $\overline{D(\phi)}$. □

3.4 "Existence" theorem

Let $\phi : X \to (-\infty, +\infty]$ be as in Section 3.3 and let $\{S(t)\}_{t\geq 0}$ be the C_0-semigroup defined in Theorem 3.1. We have

Theorem 3.2. *For every $u_0 \in \overline{D(\phi)}$ the function $u : [0,\infty) \to X$ defined by $u(t) := S(t)u_0$, $t \geq 0$, is a solution to* (EVI) *with initial value u_0.*

Proof. We already know that $u \in C([0,\infty); X)$ by Theorem 3.1. We have to show that for every $a, b \in \mathbb{R}$ with $0 < a < b$ the following hold:

i) $u \in \mathrm{AC}([a,b]; X)$,

ii) $u(t) \in D(\phi)$, $t \in [a,b]$,

iii) u satisfies for every $z \in D(\phi)$:

$$\frac{1}{2}\frac{d}{dt}|u(t) - z|^2 + \frac{\alpha}{2}|u(t) - z|^2 + \phi(u(t)) \leq \phi(z) \quad \text{a.e. in } (a,b). \tag{3.39}$$

In order to establish i)–iii) we first prove the following estimate. There exists $C = C(\phi, \alpha, u_0, a, b) > 0$ such that

$$|A_h u_h(t)| \leq C, \quad h \in (0, h_\alpha),\ t \in [a,b], \tag{3.40}$$

where

$$u_h(t) := S_h(t)u_0, \quad t \in \mathbb{R},\ h \in (0, h_a). \tag{3.41}$$

We recall that $u_h \in C^1(\mathbb{R}; x)$ and satisfies (3.27), (3.28). From (3.30) with $x = S_h(h)u_0$, $y = u_0$, $h > 0$, we obtain as in Step 1 of the proof of Theorem 1.1:

$$t \mapsto e^{\frac{\alpha}{1+\alpha h}t}|\dot{u}_h(t)| \quad \text{is nonincreasing,} \quad t \geq 0. \tag{3.42}$$

Taking the innerproduct of (3.27) (where $u = u_h$) with $te^{\frac{2\alpha}{1+\alpha h}t}|\dot{u}_h(t)|$ and integrating on $[0, a]$ we obtain

$$\int_0^a te^{\frac{2\alpha}{1+\alpha h}t}|\dot{u}_h(t)|^2\,dt + \int_0^a te^{\frac{2\alpha}{1+\alpha h}t}\,\langle A_h u_h(t), \dot{u}_h(t)\rangle\,dt = 0.$$

Since $A_h u_h(t) = \nabla\phi_h(u_h(t))$ by Proposition 3.1 we have

$$\langle A_h u_h(t), \dot{u}_h(t))\rangle = \frac{d}{dt}\,\phi_h(u_h(t)).$$

Using (3.42) we obtain

$$\begin{aligned}\frac{a^2}{2}\,e^{\frac{2\alpha}{1+\alpha h}a}|\dot{u}_h(a)|^2 \le \int_0^a te^{\frac{2\alpha}{1+\alpha h}t}|\dot{u}_h(t)|^2\,dt &= -\int_0^a te^{\frac{2\alpha}{1+\alpha h}t}\frac{d}{dt}\,\phi_h(u_h(t))\,dt\\ &= -ae^{\frac{2\alpha}{1+\alpha h}a}\phi_h(u_h(a)) + \int_0^a \frac{d}{dt}\big(te^{\frac{2\alpha}{1+\alpha h}t}\big)\phi_h(u_h(t))\,dt.\end{aligned}$$

By (3.14), (3.7),

$$\phi_h(u_h(t)) \ge \phi(J_h u_h(t)) \ge \psi(J_h u_h(t)) - \frac{|\alpha|}{2}\,|J_h u_h(t)|^2, \quad t \ge 0.$$

By Lemma 3.1 there exist $a_1, b_1 \in \mathbb{R}$ depending only on ψ such that

$$\psi(J_h u_h(t)) \ge a_1|J_h u_h(t)| + b_1.$$

From Step 2 of the proof of Theorem 3.1 we obtain for $0 < h \le \lambda \le h_\alpha$:

$$|J_h u_h(t) - J_\lambda u_\lambda(t)|^2 \le 2\lambda M_3 M_4 + 8M_2^2 h_\alpha \lambda,$$

where $M_4 = M_4(\alpha, T)$, $T = b$. This implies that there exists a constant $C_1 = C_1(\phi, \alpha, u_0, a, b) > 0$ such that

$$|J_h u_h(t)| \le C_1, \quad t \in [a, b]. \tag{3.43}$$

Hence there exists $C_2 = C_2(\phi, \alpha, u_0, a, b) > 0$ such that

$$\phi_h(u_h(t)) \ge -C_2, \quad t \in [a, b],\ h \in (0, h_\alpha). \tag{3.44}$$

It follows that

$$\frac{a^2}{2}\,e^{\frac{2\alpha}{1+\alpha h}a}|\dot{u}_h(a)|^2 + ae^{\frac{2\alpha}{1+\alpha h}a}\big(\phi_h(u_h(a)) + C_2\big) \le ae^{\frac{2\alpha}{1+\alpha h}a}C_2 + C_3\bigg|\int_0^a \phi_h(u_h(t))\,dt + C_2\bigg|,$$

where $C_3 = C_3(\alpha, a) > 0$.

Using (3.44) again we obtain

$$e^{\frac{2\alpha}{1+\alpha h}a}|\dot{u}_h(a)|^2 \le C_4\int_0^a \phi_h(u_h(t))\,dt + C_5, \tag{3.45}$$

where $C_4, C_5 > 0$ depend only on ϕ, α, u_0, a, b.

It remains to estimate $\int_0^a \phi_h(u_h(t))\,dt$. By Proposition 1.2 we have

$$(3.46)\quad \frac{1}{2}\frac{d}{dt}|u_h(t)-z|^2+\frac{\alpha}{2}|u_h(t)-z|^2+\phi_h(u_h(t))\le\phi_h(z),$$
$$h\in(0,h_a),\ t\in\mathbb{R},\ z\in X.$$

Integrating on $[0,a]$ we obtain

$$(3.47)\quad \frac{1}{2}|u_h(a)-z|^2+\frac{\alpha}{2}\int_0^a|u_h(t)-z|^2\,dt+\int_0^a\phi_h(u_h(t))\,dt\le a\phi_h(z)+\frac{1}{2}|u_0-z|^2.$$

Choosing $z\in D(\phi)$ we obtain $\phi_h(z)\le\phi(z)$ and noting that $\sup_{h\in(0,h_a)}\max_{t\in[0,a]}|u_h(t)|<\infty$ we find $C_6=C_6(\phi,\alpha,u_0,a,b)>0$ such that

$$(3.48)\qquad \int_0^a\phi_h(u_h(t))\,dt\le C_6,\quad h\in(0,h_\alpha).$$

Finally (3.42), (3.45) and (3.48) imply (3.40).

Next we prove i). Since $|\dot u_h(t)|\le C$, $t\in[a,b]$, $h\in(0,h_\alpha)$ we get $|u(t)-u(s)|\le C|t-s|$, $a\le s,t\le b$, hence $u\in\mathrm{Lip}([a,b];X)\subset\mathrm{AC}([a,b];X)$.

ii). From (3.15) we have for $z\in D(\phi)$, $t\in[a,b]$:

$$\phi_h(u_h(t))\le\phi(z)+|A_hu_h(t)|(|u_h(t)|+|z|)+|\alpha|\big(|J_hu_h(t)|^2+|J_hz|^2\big).$$

Using (3.40), Theorem 3.1, (3.43) and (3.21), we find $\widehat{C}=\widehat{C}(\phi,\alpha,u_0,a,b)>0$ such that

$$(3.49)\qquad \phi_h(u_h(t))\le\widehat{C},\quad t\in[a,b],\ h\in(0,h_\alpha).$$

Take $h_n\in(0,h_\alpha)\to0$ as $n\to\infty$. Since $\phi(J_{h_n}u_{h_n}(t))\le\phi_{h_n}(u_{h_n}(t))\le C$ and $J_{h_n}u_{h_n}(t)\to u(t)$ as $n\to\infty$, $t\in[a,b]$, we obtain by the lower semicontinuity of ϕ that

$$(3.50)\qquad \phi(u(t))\le\widehat{C},\quad t\in[a,b].$$

Finally we prove iii). Observe that $t\mapsto\phi(u(t))$ is l.s.c. on $[a,b]$, hence bounded below, which together with (3.50) implies that $\phi(u)\in L^\infty(a,b)$. Moreover, we obtain as $h_n\to0$, $a\le s<t\le b$:

$$\phi(u(t))\le\varliminf_{n\to\infty}\phi(J_{h_n}u_{h_n}(t))\le\varliminf_{n\to\infty}\phi_{h_n}(u_{h_n}(t))\le\widehat{C},$$

where the second inequality follows from (3.14). Using (3.44) we obtain

$$(3.51)\qquad \int_s^t\phi(u(r))\,dr\le\int_s^t\varliminf_{n\to\infty}\phi_{h_n}(u_{h_n}(r))\,dr\le\varliminf_{n\to\infty}\int_s^t\phi_{h_n}(u_{h_n}(r))\,dr.$$

Integrating (3.46) on $[s,t]\subset[a,b]$, taking $z\in D(\phi)$, using Theorem 3.1 and (3.51) we obtain as $h_n\to0$:

$$\frac{1}{2}|u(t)-z|^2-\frac{1}{2}|u(s)-z|^2+\frac{\alpha}{2}\int_s^t|u(r)-z|^2\,dr+\int_s^t\phi(u(r))\,dr\le(t-s)\phi(z).$$

Dividing by $(t-s)$, using the absolute continuity of $t\mapsto|u(t)-z|^2$, $t\mapsto\int_s^t\phi(u(r))\,dr$ we finally get

$$(3.52)\qquad \frac{1}{2}\frac{d}{dt}|u(t)-z|^2+\frac{\alpha}{2}|u(t)-z|^2+\phi(u(t))\le\phi(z)\quad\text{a.e. in }(a,b).$$

This completes the proof of Theorem 3.2. □

Remark. Observe that the set of measure 0 for which (3.52) does not hold depends only on u and $\phi(u)$. So it can be taken independent of z. It follows that (3.52) is equivalent to $u(t) \in D(\partial\psi)$ a.e. in (a,b) and $-\dot{u}(t) - \alpha u(t) \in \partial\psi(u(t))$ a.e. in (a,b).

4 The main existence and approximation theorem

Let (X,d) be a complete metric space, let $\phi : X \to (-\infty,+\infty]$ be proper and lower semicontinuous and let $\alpha \in \mathbb{R}$. Consider the Evolution Variational Inequality (EVI) introduced in Definition 2.1. The goal of this section is to establish the existence of a solution to (EVI) with arbitrary initial value $u_0 \in \overline{D(\phi)}$ under additional assumptions on ϕ which strictly extend the α-convexity condition of the Hilbert space case. It is quite remarkable that this can be done! Before stating these assumptions we reformulate the condition of α-convexity in a way which is more appropriate for the metric space case. We recall that a function $\phi : H \to (-\infty,+\infty]$ is called α-convex if the function $\phi - \alpha e$ is convex where $e(x) := \frac{1}{2}\langle x, x\rangle$, $x \in H$. Clearly the function e is α-convex, for all $\alpha \leq 1$.

Observing that

$$e\big((1-t)y_0 + ty_1\big) = (1-t)e(y_0) + te(y_1) - t(1-t)e(y_0 - y_1) \tag{4.1}$$

for every $y_0, y_1 \in H$ and $t \in \mathbb{R}$ we easily deduce that $\phi : H \to (-\infty,+\infty]$ is α-convex iff it satisfies

$$\phi\big((1-t)y_0 + ty_1\big) \leq (1-t)\phi(y_0) + t\phi(y_1) - \alpha t(1-t)e(y_0 - y_1) \tag{4.2}$$

for every $y_0, y_1 \in D(\phi)$ and every $t \in [0,1]$.

Since $e(y_0 - y_1) = \frac{1}{2}d^2(y_0,y_1)$, where $d^2(y_0,y_1) = \langle y_0 - y_1, y_0 - y_1\rangle$, we see that condition (4.2) can be expressed in term of the distance function d in the Hilbert space $(H, \langle\ ,\ \rangle)$. In Lemma 3.2 we introduced the function φ in (3.4) which can be rewritten as

$$\varphi(y) := \begin{cases} \frac{1}{2h}d^2(x,y) + \phi(y), & y \in D(\phi), \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.3}$$

for $h > 0$ and for $x \in X$.

It follows from (4.1) that ϕ is α-convex iff φ is $(\frac{1}{h} + \alpha)$-convex.

In Section 3 the $(\frac{1}{h}+\alpha)$-convexity of φ for every $h \in I_\alpha$ (defined in (3.1)), equivalently for $\frac{1}{h} + \alpha > 0$, plays an essential role. The notion of the $(\frac{1}{h}+\alpha)$-convexity of φ relates the values of φ on a segment of the form $[0,1] \ni t \mapsto (1-t)y_0 + ty_1$, to the values of $\varphi(y_0)$ and $\varphi(y_1)$. In a general metric space the segment between y_0 and y_1 will be replaced by a map $\gamma : [0,1] \to D(\phi)$ satisfying $\gamma(0) = y_0$ and $\gamma(1) = y_1$.

We are now in a position to formulate the first additional assumption on ϕ:

(H_1) There exists $\alpha \in \mathbb{R}$ such that for every $x, y_0, y_1 \in D(\phi)$ there exists a map $\gamma : [0,1] \to D(\phi)$ satisfying $\gamma(0) = y_0$, $\gamma(1) = y_1$ for which the following inequality holds:

$$\begin{aligned} \frac{1}{2h}\,d^2(x,\gamma(t)) + \phi(\gamma(t)) \leq (1-t)\Big[\frac{1}{2h}\,d^2(x,y_0) + \phi(y_0)\Big] \\ + t\Big[\frac{1}{2h}\,d^2(x,y_1) + \phi(y_1)\Big] - \Big(\frac{1}{h} + \alpha\Big)\frac{1}{2}t(1-t)d^2(y_0,y_1) \end{aligned} \tag{4.4}$$

for every $t \in [0,1]$ and for every $h \in I_\alpha$.

In the Hilbert space case it follows from Lemma 3.1 that if $\phi : H \to (-\infty, +\infty]$ is l.s.c. and α-convex for some $\alpha \in \mathbb{R}$ then ϕ is bounded from below on every closed ball, i.e.

$$\text{(4.5)}\quad \text{for every } x \in X \text{ and } r > 0 \text{ there exists } m \in \mathbb{R} \text{ such that } \phi(y) \geq m \\ \text{for every } y \in X \text{ satisfying } d(x,y) \leq r.$$

In the metric space case condition (H_1) together with lower semicontinuity does not imply (4.5). However, the boundedness from below of ϕ on some closed ball together with (H_1) will do it, as we shall see below. Therefore we assume

(H_2) There exist $x_* \in D(\phi)$, $r_* > 0$ and $m_* \in \mathbb{R}$ such that $\phi(y) \geq m_*$ for every $y \in X$ satisfying $d(x_*, y) \leq r_*$.

The next lemma plays the role of Lemma 3.1 in Section 3.

Lemma 4.1. *Let $\phi : X \to (-\infty, +\infty]$ be proper and satisfy (H_1) and (H_2). Let α be as in (H_1) and x_*, r_*, m_* be as in (H_2). Then for every $y \in X$*

$$\text{(4.6)}\qquad \begin{cases} \phi(y) \geq m_* & \text{if } d(x_*, y) \leq r_*, \\ \phi(y) \geq c - bd(x_*, y) + \frac{1}{2}\alpha d^2(x_*, y) & \text{if } d(x_*, y) > r_*, \end{cases}$$

where $c := \phi(x_*)$ and $b := \frac{1}{r_*}(\phi(x_*) - m_*) - \frac{1}{2}\alpha_+ r_*$ with $\alpha_+ := \max(\alpha, 0)$.

Proof. The first part of (4.6) is simply (H_2). We prove the second part. Assume $y \in D(\phi)$ with $d(x_*, y) > r_*$. From (H_1) with $x := x_*$, $y_0 := x_*$, $y_1 := y$ and $t := \frac{r_*}{d(x_*,y)} \in (0,1)$ we find $y_* := \gamma(t) \in D(\phi)$ independent of $h \in I_\alpha$ such that

$$\text{(4.7)}\quad \frac{1}{2h}\, d^2(x_*, y_*) + \phi(y_*) \leq (1-t)\Big[\frac{1}{2h}\, d^2(x_*, x_*) + \phi(x_*)\Big] \\ + t\Big[\frac{1}{2h}\, d^2(x_*, y) + \phi(y)\Big] - \Big(\frac{1}{h} + \alpha\Big)\frac{1}{2} t(1-t) d^2(x_*, y)$$

for every $h \in I_\alpha$.

Multiplying by h (> 0) and letting h tend to zero in (4.7) we get

$$\frac{1}{2}\, d^2(x_*, y_*) \leq \frac{t^2}{2}\, d^2(x_*, y) = \frac{1}{2}\, r_*^2,$$

hence by (H_2)

$$\text{(4.8)}\qquad \phi(y_*) \geq m_*.$$

Using (4.8), the nonnegativity of the first term in (4.7) and $d(x_*, x_*) = 0$ we obtain

$$\phi(y) \geq \phi(x_*) - \frac{1}{t}(\phi(x_*) - m_*) - \Big(\frac{1}{h} + \alpha\Big)\frac{t}{2}\, d^2(x_*, y) + \frac{\alpha}{2}\, d^2(x_*, y).$$

In case $\alpha \geq 0$ we let h tend to $+\infty$ and in case $\alpha < 0$ we let h tend to $\frac{1}{|\alpha|}$.

Using the definition of t we obtain (4.6). □

In what follows it will be convenient to explicit the dependence in x and h of the function φ. We set

$$\Phi(h, x; y) := \frac{1}{2h} d^2(x, y) + \phi(y), \quad h > 0,\ x, y \in X. \tag{4.9}$$

As a simple consequence of Lemma 4.1 we obtain

Corollary 4.1. *Let $\phi : X \to (-\infty, +\infty]$ be as in Lemma 4.1, and $\alpha \in \mathbb{R}$ be as in (H_1). Then for every $h > 0$ satisfying $\frac{1}{h} + \alpha > 0$, for every $\overline{x} \in X$, $M > 0$ there exist $\beta > 0$ and $\gamma \in \mathbb{R}$ such that*

$$\Phi(h, x; y) \geq \beta d^2(\overline{x}, y) + \gamma \quad \text{for every } x \in X \text{ such that } d(x, \overline{x}) \leq M \text{ and for every } y \in X. \tag{4.10}$$

Proof (sketch). Use

$$d^2(x, y) \geq (1 - \varepsilon^2) d^2(\overline{x}, y) - M^2(1/\varepsilon^2 - 1)$$

and

$$d^2(x_*, y) \leq (1 + \eta^2) d^2(\overline{x}, y) + (1 + 1/\eta^2) d^2(x_*, \overline{x})$$

for $0 < \varepsilon, \eta < 1$. □

Under the assumptions of Corollary 4.1 the function $y \mapsto \Phi(h, x; y)$ is bounded from below. We define $\phi_h(x)$ as its infimum on X.

Definition 4.1. Let ϕ be as in Lemma 4.1, $h + \frac{1}{\alpha} > 0$ with $h > 0$ and α as in (H_1).

$$\phi_h(x) := \inf_{y \in X} \Phi(h, x; y). \tag{4.11}$$

Remark 4.1.

1) ϕ_h is a map from X into $\mathbb{R}$.

2) The notation ϕ_h is consistent with the notation of Section 3. Indeed, in Definition 3.2 $\phi_h(x) := \Phi(h, x; J_h x)$ where $J_h x$ is the unique minimizer of $y \mapsto \Phi(h, x; y)$. In this section the existence and uniqueness of such a minimizer will be obtained only for $x \in \overline{D(\phi)}$.

Lemma 4.2. *Let $\phi : X \to (-\infty, +\infty]$ be proper, l.s.c. and satisfy (H_1), (H_2). Then for every $h \in I_\alpha$ the function $\phi_h : X \to \mathbb{R}$ is continuous and for every $x \in \overline{D(\phi)}$ the function $X \ni y \mapsto \Phi(h, x; y)$ possesses a unique global minimizer element of $D(\phi)$ which we denote by $J_h x$.*

Proof. 1. *Continuity of ϕ_h.*

Let $x_n, \overline{x} \in X$, $n \geq 1$, be such that $\lim_{n \to \infty} d(x_n, \overline{x}) = 0$. Let $y \in D(\phi)$, then $\phi_h(x_n) \leq \Phi(h, x_n; y)$, $n \geq 1$, hence

$$\overline{\lim_{n \to \infty}}\, \phi_h(x_n) \leq \overline{\lim_{n \to \infty}}\, \Phi(h, x_n; y) = \Phi(h, \overline{x}; y).$$

Taking the infimum over $y \in D(\phi)$ we get

$$\overline{\lim_{n \to \infty}}\, \phi(x_n) \leq \phi_h(\overline{x}) < \infty. \tag{4.12}$$

Let $y_n \in D(\phi)$, $n \geq 1$, be such that

$$\Phi(h, x_n; y_n) \leq \phi_h(x_n) + \frac{1}{n}, \quad n \geq 1.$$

In view of Corollary 4.1 there exists $C > 0$ such that $d(\overline{x}, y_n) \leq C$, $n \geq 1$.

We have $\phi_h(\overline{x}) \leq \Phi(h, \overline{x}; y_n)$, $n \geq 1$, hence

$$\begin{aligned}
\phi_h(\overline{x}) &\leq \varliminf_{n\to\infty} \Phi(h, \overline{x}; y_n) = \varliminf_{n\to\infty} \Big\{ \frac{1}{2h} d^2(\overline{x}, y_n) - \frac{1}{h} d(x_n, \overline{x}) d(\overline{x}, y_n) + \phi(y_n) \Big\} \\
&\qquad \text{(since } d(\overline{x}, y_n) \text{ is bounded)} \\
&= \varliminf_{n\to\infty} \Big\{ \frac{1}{2h} \big(d(\overline{x}, y_n) - d(\overline{x}, x_n) \big)^2 + \phi(y_n) \Big\} \\
&\leq \varliminf_{n\to\infty} \Big\{ \frac{1}{2h} d^2(x_n, y_n) + \phi(y_n) \Big\} \leq \varliminf_{n\to\infty} \phi_h(x_n).
\end{aligned}$$

2) *Global minimizer.*

Let $\overline{x} \in \overline{D(\phi)}$ and let $\{y_n\}_{n\geq 1} \subset D(\phi)$ be a minimizing sequence, i.e. $\lim_{n\to\infty} \Phi(h, \overline{x}; y_n) = \phi_h(\overline{x})$. As in the proof of Lemma 3.2, in view of the lower semicontinuity of $\Phi(h, \overline{x}; \cdot)$ and the completeness of (X, d), it is sufficient to prove that $(y_n)_{n\geq 1}$ is a Cauchy sequence. If $\overline{y}$ denotes the limit, note that $\Phi(h, \overline{x}; \overline{y}) < \infty$ hence $\overline{y} \in D(\phi)$. In order to show that (y_n) is a Cauchy sequence we use assumption (H_1) with $x := x_n$, $y_0 := y_n$, $y_1 := y_m$, $t = \frac{1}{2}$, where $(x_n)_{n\geq 1} \subset D(\phi)$ such that $\lim_{n\to\infty} d(x_n, \overline{x}) = 0$. Let $C_1 > 0$ be such that $d(x_n, \overline{x}) \leq C_1$, $n \geq 1$. From (H_1) we obtain the existence of $y_{n,m} \in D(\phi)$ satisfying

$$\Phi(h, x_n; y_{n,m}) \leq \frac{1}{2} \Phi(h, x_n; y_n) + \frac{1}{2} \Phi(h, x_n; y_m) - \frac{1}{8} \Big(\frac{1}{h} + \alpha \Big) d^2(y_n, y_m).$$

Since $\Phi(h, x_n; y_{n,m}) \geq \phi_h(x_n)$, we get

$$d^2(y_n, y_m) \leq 4 \Big(\frac{1}{h} + \alpha \Big)^{-1} \Big[\big(\Phi(h, x_n; y_n) - \phi_h(x_n) \big) + \Phi(h, x_n; y_m) - \phi_h(x_n) \Big], \tag{4.13}$$

for $m, n \geq 1$.

Next we show that the right-hand side of (4.13) tends to zero as $m, n \to \infty$. By Corollary 4.1 we see that any minimizing sequence is bounded, in particular there exists $C_2 > 0$ such that $d(\overline{x}, y_n) \leq C_2$, $n \geq 1$. It follows that

$$\begin{aligned}
|\Phi(h, x_n; y_n) - \Phi(h, \overline{x}; y_n)| &= \frac{1}{2h} |d^2(x_n, y_n) - d^2(\overline{x}, y_n)| \\
&\leq \frac{1}{2h} d(x_n, \overline{x}) \big(d(x_n, y_n) + d(\overline{x}, y_n) \big) \leq \frac{1}{2h} (C_1 + 2C_2) d(x_n, \overline{x}) \to 0
\end{aligned}$$

as $n \to \infty$. In view of the continuity of ϕ_h, we get

$$\begin{aligned}
|\Phi(h, x_n; y_n) - \phi_h(x_n)| &\leq |\Phi(h, x_n; y_n) - \Phi(h, \overline{x}; y_n)| + |\Phi(h, \overline{x}; y_n) - \phi_h(\overline{x})| \\
&\qquad + |\phi_h(\overline{x}) - \phi_h(x_n)| \to 0.
\end{aligned}$$

Finally

$$\begin{aligned}
|\Phi(h, x_m; y_m) - \Phi(h, x_n; y_m)| &= \frac{1}{2h} |d^2(x_m, y_m) - d^2(x_n, y_m)| \\
&\leq \frac{1}{2h} d(x_m, x_n) \cdot 2(C_1 + C_2) \to 0 \quad \text{as} \quad m, n \to \infty.
\end{aligned}$$

Since $|\phi_h(x_n) - \phi_h(x_m)| \to 0$, it follows that the right-hand side of (4.13) tends to zero.

Finally we prove the uniqueness of the minimizer. Since every minimizing sequence is a Cauchy sequence it is easy to see that the minimizer is unique (construct a new minimizing sequence from two minimizing sequences (u_n) and (v_n) converging respectively to $\overline{u}$ and $\overline{v}$. Then the new minimizing sequence converges to $\overline{w} = \overline{u} = \overline{v}$). □

For the formulation of Theorem 4.1 we need to introduce the notion of local slope of the functional ϕ.

Definition 4.2. Let (Y, d_Y) be a metric space and $\phi : Y \to (-\infty, +\infty]$ be proper. Let $x \in D(\phi)$. Then

$$\begin{aligned} &|\partial\phi|(x) := \overline{\lim_{y \overset{\to}{\neq} x}} \frac{\big(\phi(x) - \phi(y)\big)^+}{d(x,y)} && \text{if } x \text{ is not isolated in } D(\phi), \\ &|\partial\phi|(x) := 0 && \text{otherwise.} \end{aligned} \tag{4.14}$$

Set $D(|\partial\phi|) := \{x \in D(\phi) : |\partial\phi|(x) < \infty\}$. $|\partial\phi|(x)$ is called the *local slope of* ϕ *at* x.

Remark 4.2. If X is a Hilbert space and $\phi : X \to (-\infty, +\infty]$ is proper, l.s.c. and convex, then $x \in D(\phi)$ belongs to $D(|\partial\phi|)$ iff $x \in D(\partial\phi)$. In this case $|\partial\phi|(x) = |(\partial\phi)^\circ x|$ (see [AGS], prop. 1.4.4).

The next proposition is the analogue of Proposition 3.2.

Proposition 4.1 ([AGS], Lemma 3.1.3, p. 61, and Lemma 3.1.2, p. 60)**.** *Let* $\phi : X \to (-\infty, +\infty]$ *be proper, l.s.c. and satisfy* (H_1) *and* (H_2)*. Then*

i) *if* $h > 0$, $1 + h\alpha > 0$ (α *from* (H_1)), $x \in \overline{D(\phi)}$, *then* $J_h x \in D(|\partial\phi|)$ *and*

$$|\partial\phi|(J_h x) \leq \frac{1}{h} d(x, J_h x). \tag{4.15}$$

ii) *if* $h > 0$, $1 + h\alpha > 0$, $x \in \overline{D(\phi)}$ *then*

$$\phi(J_h x) \leq \phi_h(x) \leq \phi(x), \tag{4.16}$$

if $h_1 > h_0 > 0$, $1 + h_i\alpha > 0$, $i = 0, 1$, *then*

$$\phi_{h_1}(x) \leq \phi_{h_0}(x), \quad x \in X, \tag{4.17}$$

$$d(J_{h_0}x, x) \leq d(J_{h_1}x, x), \quad x \in \overline{D(\phi)}, \tag{4.18}$$

$$\phi(J_{h_1}x) \leq \phi(J_{h_0}x), \quad x \in \overline{D(\phi)}, \tag{4.19}$$

iii) *if* $x \in \overline{D(\phi)}$, *then*

$$d(x, J_h x) \downarrow 0 \quad \textit{as } h \downarrow 0, \tag{4.20}$$

$$\phi(J_h x) \uparrow \phi(x) \quad \textit{as } h \downarrow 0, \tag{4.21}$$

$$\phi_h(x) \uparrow \phi(x) \quad \textit{as } h \downarrow 0. \tag{4.22}$$

iv)

$$\overline{D(|\partial\phi|)} = \overline{D(\phi)}. \tag{4.23}$$

Proof. i) By definition (see Lemma 4.2) $J_h x$ satisfies

$$\phi(J_h x) - \phi(y) \le \tfrac{1}{2h}\, d^2(x,y) - \tfrac{1}{2h}\, d^2(x, J_h x) \le \tfrac{1}{2h}\, d(y, J_h x)\big(d(x,y) + d(x, J_h x)\big) \tag{4.24}$$

for every $y \in D(\phi)$. If $J_h x$ is isolated in $D(\phi)$, then $|\partial\phi|(J_h x) = 0$ and (4.15) holds. Otherwise there exists a sequence $(y_n) \subset D(\phi)$ such that $y_n \ne J_h x$, $n \ge 1$, and $\lim_{n\to\infty} d(y_n, J_h x) = 0$. From (4.24) we obtain

$$\varlimsup_{n\to\infty} \frac{\big(\phi(J_h x) - \phi(y_n)\big)^+}{d(J_h x, y_n)} \le \frac{1}{h}\, d(x, J_h x)$$

hence

$$|\partial\phi|(J_h x) = \varlimsup_{y \underset{\ne}{\to} J_h x} \frac{\big(\phi(J_h x) - \phi(y)\big)^+}{d(J_h x, y)} \le \frac{1}{h}\, d(x, J_h x).$$

ii) For any $x \in \overline{D(\phi)}$ we have

$$\phi(J_h x) \le \phi(J_h x) + \frac{1}{2h}\, d^2(x, J_h x) = \phi_h(x) \le \Phi(h, x; x) = \phi(x).$$

Let $0 < h_0 < h_1$ with $1 + \alpha h_i > 0$, $i = 0, 1$. (4.17) is a trivial consequence of the definition of ϕ_h. Concerning (4.18) we have

$$\begin{aligned}
\frac{1}{2h_0}\, d^2(x, J_{h_0}x) + \phi(J_{h_0}x) &\le \frac{1}{2h_0}\, d^2(x, J_{h_1}x) + \phi(J_{h_1}x)\\
&\le \Big(\frac{1}{2h_0} - \frac{1}{2h_1}\Big) d^2(x, J_{h_1}x) + \Phi(h_1, x; J_{h_1}x)\\
&\le \Big(\frac{1}{2h_0} - \frac{1}{2h_1}\Big) d^2(x, J_{h_1}x) + \frac{1}{2h_1}\, d^2(x, J_{h_0}x) + \phi(J_{h_0}x).
\end{aligned}$$

Hence

$$\Big(\frac{1}{2h_0} - \frac{1}{2h_1}\Big) d^2(x, J_{h_0}x) \le \Big(\frac{1}{2h_0} - \frac{1}{2h_1}\Big) d^2(x, J_{h_1}x),$$

and (4.18) follows.

Finally from $\Phi(h_1, x; J_{h_1}x) \le \Phi(h_1, x; J_{h_0}x)$ we obtain

$$\phi(J_{h_1}x) \le \frac{1}{2h_1}\left(d^2(x, J_{h_0}x) - d^2(x, J_{h_1}x)\right) + \phi(J_{h_0}x) \le \phi(J_{h_0}x)$$

in view of (4.18).

iii) We have

$$d^2(x, J_h x) \le -2h\phi(J_h x) + d^2(x,y) + 2h\phi(y)$$

for every $y \in D(\phi)$. Since $-\phi(J_h x) \le -\phi(J_{h_0}x)$, $0 < h < h_0$, we obtain

$$\varlimsup_{h\to 0} d^2(x, J_h x) \le d^2(x,y)$$

for every $y \in D(\phi)$, hence $\lim_{h\to 0} d^2(x, J_h x) = 0$ since $x \in \overline{D(\phi)}$. Hence (4.20) follows from (4.18) and part i).

(4.21) follows from (4.16), (4.19), (4.20) and the lower semicontinuity of ϕ. Finally, (4.22) is a consequence of (4.16), (4.17) and (4.21).

iv) Part i) and (4.20) imply $\overline{D(\phi)} \subseteq \overline{D(|\partial\phi|)}$ which implies (4.23). □

Next we give a characterization of the local slope of ϕ from which one can prove the lower semicontinuity of the functional $|\partial\phi|$.

Proposition 4.2 ([AGS], Theorem 2.4.9, p. 53, Cor. 2.4.10, p. 54)**.** *Under the assumptions of Proposition* 4.1 *we have*

(i) *For every $x \in D(\phi)$, x not isolated in $D(\phi)$:*

$$|\partial\phi|(x) = \sup_{\substack{y\in D(\phi)\\ y\neq x}} \left(\frac{\phi(x)-\phi(y)}{d(x,y)} + \frac{\alpha}{2}\, d(x,y)\right)^{+} \tag{4.25}$$

where α is as in (H_1).

(ii) *The functional $|\partial\phi| : D(\phi) \to [0,\infty]$ is l.s.c.*

Proof. i) Let $x \in D(\phi)$ not isolated in $D(\phi)$. For any $\rho \in \mathbb{R}$ we have

$$|\partial\phi|(x) = \varlimsup_{\substack{z\to x\\ z\in D(\phi)}} \left(\frac{\phi(x)-\phi(z)}{d(x,z)} + \frac{1}{2}\,\rho d(x,z)\right)^{+} \le \sup_{\substack{z\neq x\\ z\in D(\phi)}} \left(\frac{\phi(x)-\phi(z)}{d(x,z)} + \frac{1}{2}\,\rho d(x,z)\right)^{+},$$

in particular for $\rho = \alpha$. If the right-hand side of (4.25) is equal to zero, we are done. Otherwise we can restrict the set on which the supremum is taken to the elements $z \in D(\phi)$, $z \neq x$ for which we have

$$\phi(x) - \phi(z) + \frac{1}{2}\,\alpha d^2(x,z) > 0. \tag{4.26}$$

Next we use assumption (H_1) with x, $y_0 := x$ and $y_1 := z$, where z satisfies (4.26). Multiplying (4.4) by h and letting h tend to zero, we obtain

$$d^2(x,\gamma(t)) \le t^2 d^2(x,z), \quad t \in [0,1]. \tag{4.27}$$

Using assumption (H_1) again with the same x, y_0, y_1 and $\{\gamma(t)\}_{t\in[0,1]}$ we fix $h > 0$ (with $1+\alpha h > 0$) and obtain (deleting the first term in (4.4))

$$\phi(x) - \phi(\gamma(t)) \ge \left(\frac{\phi(x)-\phi(z)}{d(x,z)} + \frac{1}{2h}\big(\alpha h(1-t) - t\big)d(x,z)\right) t d(x,z), \tag{4.28}$$

for any $t \in [0,1]$.

Since $h > 0$ is fixed in (4.28) and z satisfies (4.26), there exists $t_0 \in (0,1]$ such that the RHS of (4.28) is positive for $t \in (0,t_0)$. Hence $\gamma(t) \neq x$ for $t \in (0,t_0)$, otherwise the LHS would be zero. For $t \in (0,t_0)$ we can divide (4.28) by $d(x,\gamma(t))$ and using the sign of the RHS together with (4.27) we obtain

$$\frac{\phi(x)-\phi(\gamma(t))}{d(x,\gamma(t))} \ge \left(\frac{\phi(x)-\phi(z)}{d(x,z)} + \frac{1}{2h}\big(\alpha h(1-t) - t\big)d(x,z)\right).$$

Hence

$$|\partial\phi|(x) \ge \varlimsup_{t\downarrow 0} \frac{\phi(x)-\phi(\gamma(t))}{d(x,\gamma(t))} \ge \frac{\phi(x)-\phi(z)}{d(x,z)} + \frac{1}{2}\,\alpha d(x,z) > 0,$$

and

$$|\partial\phi|(x) \geq \sup_{\substack{z\neq x\\ z\in D(\phi)}} \left(\frac{\phi(x)-\phi(z)}{d(x,z)} + \frac{1}{2}\,\alpha d(x,z)\right)^+ .$$

ii) Let $x \in D(\phi)$ and $y \neq x$, $y \in D(\phi)$. Let $(x_n) \subset D(\phi)$ be such that $\lim_{n\to\infty} d(x_n, x) = 0$. There exists $n_0 \geq 1$ such that $x_n \neq y$, $n \geq n_0$. We have

$$\varliminf_{n\to\infty} \sup_{\substack{z\neq x_n\\ z\in D(\phi)}} \left(\frac{\phi(x_n)-\phi(z)}{d(x_n,z)} + \frac{\alpha}{2}\,d(x_n,z)\right)^+$$
$$\geq \varliminf_{n\to\infty} \left(\frac{\phi(x_n)-\phi(y)}{d(x_n,y)} + \frac{\alpha}{2}\,d(x_n,y)\right)^+ \geq \left(\frac{\phi(x)-\phi(y)}{d(x,y)} + \frac{\alpha}{2}\,d(x,y)\right)^+,$$

where we used the lower semicontinuity of ϕ in the last inequality. Taking the supremum over $y \in D(\phi)$, $y \neq x$, we obtain

$$|\partial\phi|(x) \leq \varliminf_{n\to\infty} |\partial\phi|(x_n),$$

in view of (4.25). □

The following estimates will be useful for the proof of Theorems 4.1 and 4.2.

Proposition 4.3 ([AGS], Theorem 3.1.6, p. 64)**.** *Under the assumptions of Proposition* 4.1, *and if* $h > 0$, $1 + h\alpha > 0$, *we have*

i) *for* $x \in D(\phi)$,

$$d^2(x, J_h x) \leq 2(1+h\alpha)^{-1} h\big(\phi(x) - \phi_h(x)\big), \tag{4.29}$$

ii) *for* $x \in D(|\partial\phi|)$,

$$\phi(x) - \phi_h(x) \leq \tfrac{1}{2}(1+h\alpha)^{-1} h|\partial\phi|^2(x), \tag{4.30}$$
$$|\partial\phi|(J_h x) \leq (1+h\alpha)^{-1}|\partial\phi|(x), \tag{4.31}$$
$$\phi(x) - \phi(J_h x) \leq \tfrac{1}{2} h(1+h\alpha)^{-2}(2+h\alpha)|\partial\phi|^2(x), \tag{4.32}$$

iii) *for* $x \in \overline{D(\phi)}$

$$x \in D(|\partial\phi|) \quad \textit{iff} \quad \sup_{\substack{h>0\\ 1+h\alpha\geq 1/2}} |\partial\phi|(J_h x) < \infty,$$
$$\textit{iff} \quad \sup_{\substack{h>0\\ 1+h\alpha\geq 1/2}} \frac{d(x, J_h x)}{h} < \infty,$$

iv) *for* $x \in D(\phi)$

$$x \in D(|\partial\phi|) \quad \textit{iff} \quad \sup_{\substack{h>0\\ 1+h\alpha\geq 1/2}} \frac{\phi(x)-\phi_h(x)}{h} < \infty,$$

v) *for* $x \in D(|\partial\phi|)$

$$|\partial\phi|(x) = \lim_{h\to 0} |\partial\phi|(J_h x) = \lim_{h\to 0} \frac{d(x, J_h x)}{h} = \lim_{h\to 0}\Big(2\,\frac{\phi(x)-\phi_h(x)}{h}\Big)^{1/2},$$

vi) *for* $x \in D(|\partial\phi|)$

$$|\partial\phi|(x) = 0 \text{ iff } \exists h_0 > 0 \text{ with } 1 + h_0\alpha > 0$$
$$\text{such that } x = J_{h_0}x \text{ iff for all } h > 0 \text{ with } 1 + \alpha h > 0\text{: } x = J_h x.$$

Proof. i) Let $x \in D(\phi)$. Use (4.42) with $z = x$.
ii)

$$\frac{\phi(x) - \phi_h(x)}{h} = \frac{\phi(x) - \phi(J_h x)}{h} - \frac{d^2(x, J_h x)}{2h^2} \leq |\partial\phi|(x)\,\frac{d(x, J_h x)}{h} - (1 + h\alpha)\frac{d^2(x, J_h x)}{2h^2}$$

by using (4.25). Then (4.30) follows from

$$|\partial\phi|(x)\,\frac{d(x, J_h(x)}{h} \leq \frac{1}{2}|\partial\phi|^2(x)(1 + h\alpha)^{-1} + \frac{1}{2h^2}d^2(x, J_h x)(1 + h\alpha).$$

(4.31) is a combination of (4.15), (4.29) and (4.30).
(4.32) follows from the definition of ϕ_h, (4.30) and (4.29).
iii), iv), v) are easy consequences of the previous inequalities and the l.s.c. of $|\partial\phi|$.
vi) follows from (4.30), (4.29), (4.18) and v). □

Definition 4.3. We denote by J_h the operator from $\overline{D(\phi)}$ into $D(\phi)$ defined by $x \mapsto J_h x$ where $J_h x$ is defined in Lemma 4.2.

We can now state the first main result of this section.

Theorem 4.1. *Let (X, d) be a complete metric space and let $\phi : X \to (-\infty, +\infty]$ be proper, l.s.c. and satisfy assumptions (H_1) with $\alpha \in \mathbb{R}$ and (H_2). Then, for every $x \in D(|\partial\phi|)$ (see Definition 4.2), (EVI) with α of assumption (H_1) possesses one and only one solution u with initial value $u(0) = x$. Moreover the following holds:*

(4.33) $\quad \lim_{n\to\infty} (J_{t/n})^n x = u(t)$ *for every* $t > 0$,

(4.34) $\quad u(t) \in D(|\partial\phi|)$ *for every* $t > 0$,

(4.35) $\quad u|_{[0,T]} \in \mathrm{Lip}([0, T]; X)$ *for every* $T > 0$,

(4.36) $\quad [0, \infty) \ni t \mapsto \phi(u(t))$ *is nonincreasing,*

(4.37) $\quad [0, \infty) \ni t \mapsto e^{\alpha t}|\partial\phi|(u(t))$ *is nonincreasing and right-continuous,*

(4.38) $\quad \phi(u(t)) = \lim_{n\to\infty} \phi\big((J_{t/n})^n x\big)$ *for every* $t > 0$,

(4.39) $\quad \frac{1}{2}\int_0^t |\dot{u}|^2(s)\,ds + \frac{1}{2}\int_0^t |\partial\phi|^2(u(s))\,ds + \phi(u(t)) \leq \phi(x)$

for every $t \geq 0$, where $|\dot{u}|(s)$ is defined in Theorem 1.2.
Finally, if we set

(4.40) $$S(t)x := u(t), \quad t \geq 0,$$

where u is the unique solution to (EVI) *with initial value $u(0) = x$, then $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup of operators on $D(|\partial\phi|)$ satisfying*

(4.41) $$[S(t)]_{\mathrm{Lip}} \leq e^{\alpha t}, \quad t \geq 0.$$

Proof. *Step* 1 (variational inequality for $J_h x$). Let $x \in D(\phi)$ and let $h > 0$ with $1+\alpha h > 0$. Let $J_h x$ be as in Lemma 4.2. Then

$$\frac{1}{2h}\big[d^2(J_h x, z) - d^2(x, z)\big] + \frac{\alpha}{2}\, d^2(J_h x, z) + \phi_h(x) \le \phi(z) \tag{4.42}$$

for every $z \in D(\phi)$. Indeed, by Lemma 4.2 for $\hat{z} \in D(\phi)$

$$\frac{1}{2h}\, d^2(x, J_h x) + \phi(J_h x) \le \frac{1}{2h}\, d^2(x, \hat{z}) + \phi(\hat{z}). \tag{4.43}$$

Let $z \in D(\phi)$. Using (H_1) with $x := x$, $y_0 := z$ and $y_1 := J_h x$ and substituting $\hat{z} = \gamma(t)$, $t \in (0,1)$ in (4.43), we get

$$\frac{1}{2h}\, d^2(x, J_h x) + \phi(J_h x) \le (1-t)\Big[\frac{1}{2h}\, d^2(x, z) + \phi(z)\Big] + t\Big[\frac{1}{2h}\, d^2(x, J_h x) + \phi(J_h x)\Big] - \frac{1}{2}\, t(1-t)\Big(\frac{1}{h} + \alpha\Big) d^2(J_h x, z).$$

Hence

$$(1-t)\Big[\frac{1}{2h}\, d^2(x, J_h x) + \phi(J_h x)\Big] \le (1-t)\Big[\frac{1}{2h}\, d^2(x, z) + \phi(z)\Big] - \frac{1}{2}\, t(1-t)\Big(\frac{1}{h} + \alpha\Big) d^2(J_h x, z).$$

Dividing by $(1-t)$ and letting t tend to 1 we obtain

$$\frac{1}{2h}\, d^2(x, J_h x) + \phi(J_h x) \le \frac{1}{2h}\, d^2(x, z) + \phi(z) - \frac{1}{2}\Big(\frac{1}{h} + \alpha\Big) d^2(J_h x, z),$$

which is (4.42).

Step 2 (estimate for $d^2(J_\gamma^m x, J_\delta^n x)$). Let $x \in D(|\partial\phi|)$, $\gamma, \delta > 0$ such that $1 + \alpha\gamma > 0$, $1 + \alpha\delta > 0$ and let m, n be nonnegative integers. We want to estimate $d^2((J_\gamma)^m x, (J_\delta)^n x)$ where we use the notation $(J_\gamma)^0 x = (J_\delta)^0 x := x$. The idea is to find an estimate first in the case $m = 0$ or $n = 0$ and then to find a recursive inequality which enables us to find an estimate for all $m, n \ge 1$. A basic tool will be Lemma A2 of Appendix 2. We restrict ourselves to the case $\alpha \le 0$.

Case $n = 0$ or $m = 0$; $\alpha \le 0$. We have for $x \in D(|\partial\phi|)$, $\gamma > 0$, $1 + \alpha\gamma > 0$, $m \ge 1$,

$$d^2(J_\gamma^m x, x) \le (m\gamma)^2 (1 + \alpha\gamma)^{-2m} |\partial\phi|^2(x). \tag{4.44}$$

Indeed, setting $z = x$ in (4.42) and multiplying by $2h$, replacing h by γ, we obtain

$$d^2(J_\gamma x, x) \le (1 + \alpha\gamma)^{-1} 2\gamma[\phi(x) - \phi_\gamma(x)]. \tag{4.45}$$

$$d^2(J_\gamma^m x, x) \le \Big(\sum_{k=1}^{m} d(J_\gamma^k x, J_\gamma^{k-1} x)\Big)^2 \le m \sum_{k=1}^{m} d^2(J_\gamma^k x, J_\gamma^{k-1} x)$$

follows from the triangle inequality and the Cauchy–Schwarz inequality. Using (4.45) we have

$$d^2(J_\gamma^m x, x) \le 2m\gamma(1 + \alpha\gamma)^{-1} \sum_{k=1}^{m} \big[\phi(J_\gamma^{k-1} x) - \phi_\gamma(J_\gamma^{k-1} x)\big].$$

Next we use (4.30) to obtain

$$d^2(J_\gamma^m x, x) \le m\gamma^2 (1 + \alpha\gamma)^{-2} \sum_{k=1}^{m} |\partial\phi|^2(J_\gamma^{k-1} x).$$

Finally, using (4.31) we get

$$d^2(J_\gamma^m x, x) \le m\gamma^2(1+\alpha\gamma)^{-2}\left(\sum_{k=1}^{m}(1+\alpha\gamma)^{-2(k-1)}\right)|\partial\phi|^2(x).$$

Since $\alpha \le 0$ we arrive at

$$d^2(J_\gamma^m x, x) \le m^2\gamma^2(1+\alpha\gamma)^{-2m}|\partial\phi|^2(x)$$

which is (4.44).

Similarly we have for $m = 0$, $n \ge 1$, $\alpha \le 0$, $\delta > 0$, $1 + \alpha\delta > 0$:

$$d^2(J_\delta^n x, x) \le (n\delta)^2(1+\alpha\delta)^{-2n}|\partial\phi|^2(x). \tag{4.46}$$

Case $n \ge 1$, $m \ge 1$, $\alpha \le 0$. We have for $x \in D(|\partial\phi|)$, $\gamma, \delta, 1+\alpha\gamma, 1+\alpha\delta > 0$, $\alpha \le 0$, $m, n \ge 1$:

$$\begin{aligned} d^2(J_\gamma^m x, J_\delta^n x) \le |\partial\phi|^2(x) \cdot \max\bigl\{(1+\alpha\gamma)^{-2(m+1)}, (1+\alpha\delta)^{-2(n+1)}\bigr\} \\ \cdot \bigl\{\bigl[(m\gamma - n\delta) + (m-n)\alpha\gamma\delta\bigr]^2 + (\gamma+\delta)\cdot\min(m\gamma, n\delta)\bigr\}. \end{aligned} \tag{4.47}$$

Indeed, let $1 \le i \le m$, $1 \le j \le n$, $x_0 = y_0 := x$, $x_i := J_\gamma x_{i-1}$, $y_j := J_\delta y_{j-1}$. Using (4.16) and (4.42) we obtain for $z, \hat{z} \in D(\phi)$

$$\frac{1}{2\gamma}\bigl[d^2(x_i, z) - d^2(x_{i-1}, z)\bigr] + \frac{\alpha}{2}\, d^2(x_i, z) + \phi(x_i) \le \phi(z), \tag{4.48}$$

$$\frac{1}{2\delta}\bigl[d^2(y_j, \hat{z}) - d^2(y_{j-1}, \hat{z})\bigr] + \frac{\alpha}{2}\, d^2(y_j, \hat{z}) + \phi(y_j) \le \phi(\hat{z}). \tag{4.49}$$

Setting $z := y_j$ in (4.48), $\hat{z} := x_i$ in (4.49), adding (4.48) and (4.49) and multiplying by $2\gamma\delta$ we have

$$d^2(x_i, y_j) \le \frac{\gamma}{(\gamma+\delta) + 2\alpha\gamma\delta}\, d^2(x_i, y_{j-1}) + \frac{\delta}{(\gamma+\delta) + 2\alpha\gamma\delta}\, d^2(x_{i-1}, y_j). \tag{4.50}$$

Multiplying (4.50) by $(1+\alpha\gamma)^i(1+\alpha\delta)^j$ and defining, also for $i = 0$, $j = 0$,

$$a_{i,j} := (1+\alpha\gamma)^i(1+\alpha\delta)^j d^2(x_i, y_j), \tag{4.51}$$

we obtain

$$a_{i,j} \le \frac{\gamma(1+\alpha\delta)}{\gamma+\delta+2\alpha\gamma\delta}\, a_{i,j-1} + \frac{\delta(1+\alpha\gamma)}{\gamma+\delta+2\alpha\gamma\delta}\, a_{i-1,j}\,.$$

Setting

$$\hat{\gamma} := \gamma(1+\alpha\delta), \quad \hat{\delta} := \delta(1+\alpha\gamma), \tag{4.52}$$

we arrive at

$$a_{i,j} \le \frac{\hat{\gamma}}{\hat{\gamma}+\hat{\delta}}\, a_{i,j-1} + \frac{\hat{\delta}}{\hat{\gamma}+\hat{\delta}}\, a_{i-1,j}\,. \tag{4.53}$$

From (4.44), (4.52) and using $\alpha \le 0$ (hence $(1+\alpha\gamma)^{-1}, (1+\alpha\delta)^{-1} \ge 1$), we get

$$a_{i,0} \le |\partial\phi|^2(x)\cdot(1+\alpha\gamma)^{-m}(1+\alpha\delta)^{-2}(i\hat{\gamma})^2, \tag{4.54}$$

similarly from (4.46)

$$a_{0,j} \le |\partial\phi|^2(x)\cdot(1+\alpha\delta)^{-n}(1+\alpha\gamma)^{-2}(j\hat{\delta})^2. \tag{4.55}$$

Now, since

$$|\partial\phi|^2(x)\max\bigl\{(1+\alpha\gamma)^{-m}(1+\alpha\delta)^{-2},(1+\alpha\delta)^{-n}(1+\alpha\gamma)^{-2}\bigr\} \\ \le |\partial\phi|^2(x)\max\bigl\{(1+\alpha\gamma)^{-(m+2)},(1+\alpha\delta)^{-(n+2)}\bigr\},$$

we can use Lemma A2 with

$$K=|\partial\phi|^2(x)\max\bigl\{(1+\alpha\gamma)^{-(m+2)},(1+\alpha\delta)^{-(n+2)}\bigr\},\quad \hat{\gamma}=\gamma,\quad \hat{\delta}=\delta,\quad r=2.$$

Then (4.47) follows from Lemma A2, (4.51), (4.52), $\hat{\gamma}\le\gamma$, $\hat{\delta}\le\delta$ (4.52, $\alpha\le 0$).

Step 3 (convergence of $\{(J_{t/n})^n x\}$). Let $x\in D(|\partial\phi|)$, $t>0$, $\alpha\le 0$ and $n_0\in\mathbb{N}$ be such that

$$1+\alpha\,\frac{t}{n_0}>0. \tag{4.56}$$

Let $m,n\ge n_0$, then $(J_{t/m})^m x$ and $(J_{t/n})^n x$ are well defined (by Lemma 4.2) and in view of (4.47) with $\gamma:=\frac{t}{m}$, $\delta:=\frac{t}{n}$ we obtain

$$d(J^m_{t/m}x,J^n_{t/n}x)\le|\partial\phi|(x)\cdot t\cdot\max\bigl\{(1+\alpha t/m)^{-(m+1)},(1+\alpha t/n)^{-(n+1)}\bigr\} \\ \cdot\Bigl(\frac{1}{m}+\frac{1}{n}+(\alpha t)^2\Bigl(\frac{1}{m}-\frac{1}{n}\Bigr)^2\Bigr)^{1/2}. \tag{4.57}$$

Since $\lim\limits_{m\to\infty}(1+\alpha t/m)^{-(m+1)}=e^{-\alpha t}$, the sequence $\{(J_{t/n})^n x\}_{n\ge n_0}$ is a Cauchy sequence in (X,d), which is complete. We set

$$u(t):=\lim_{n\to\infty}(J_{t/n})^n x,\quad t>0, \tag{4.58}$$

and we have the estimate

$$d\bigl(u(t),(J_{t/n})^n x\bigr)\le|\partial\phi|(x)\cdot t\Bigl(\frac{1}{n}+\Bigl(\frac{\alpha t}{n}\Bigr)^2\Bigr)^{1/2} \\ \cdot\max\Bigl(e^{-\alpha t},\Bigl(1+\alpha\,\frac{t}{n}\Bigr)^{-(n+1)}\Bigr),\quad t>0,\quad n\ge n_0. \tag{4.59}$$

Next we show that $u(t)\in D(|\partial\phi|)$. By (4.31) we have $|\partial\phi|(J_{t/n}x)\le\bigl(1+\alpha\frac{t}{n}\bigr)^{-1}|\partial\phi|(x)$, and by induction $|\partial\phi|((J_{t/n}x)^n)\le\bigl(1+\alpha\frac{t}{n}\bigr)^{-n}|\partial\phi|(x)$. In view of the lower semicontinuity of $|\partial\phi|(\cdot)$ (Proposition 4.2(ii)), we get

$$|\partial\phi|(u(t))\le e^{-\alpha t}|\partial\phi|(x),\quad t>0. \tag{4.60}$$

Step 4 (local Lipschitz continuity of u). Let $x\in D(|\partial\phi|)$ and set

$$u(0):=x, \tag{4.61}$$

and for $t>0$, $\alpha\le 0$, $u(t)$ is defined by (4.58).

From (4.46) with $\delta = \frac{t}{n}$, $n \geq n_0$, (4.56), we get

$$d\big((J_{t/n})^n x, x\big) \leq t\Big(1 + \alpha\,\frac{t}{n}\Big)^n |\partial\phi|(x).$$

By taking the limit as $n \to \infty$, we obtain

$$d(u(t), u(0)) \leq t e^{-\alpha t} |\partial\phi|(x), \quad t > 0, \tag{4.62}$$

which implies the continuity of u at 0.

Now we take $0 < s < t$, $m = n \geq n_0$ and $\gamma := \frac{t}{n}$, $\delta := \frac{s}{n}$. Applying (4.47) we obtain

$$d^2(J^n_{t/n}x, J^n_{s/n}x) \leq |\partial\phi|^2(x)\Big(1 + \alpha\,\frac{t}{n}\Big)^{-2(n+1)}\Big\{(t-s)^2 + \frac{t+s}{n}\cdot s\Big\}.$$

Hence by taking the limit, we have

$$d(u(t), u(s)) \leq |\partial\phi|(x) e^{-\alpha t}|t - s|, \quad 0 < s < t.$$

Taking the limit as $s \to 0$, we arrive at

$$d(u(t), u(s)) \leq |\partial\phi|(x) e^{-\alpha t}|t - s|, \quad 0 \leq s < t.$$

If $\alpha > 0$ then u is also a solution to (EVI) with $\alpha = 0$, hence we obtain for any $\alpha \in \mathbb{R}$

$$d(u(t), u(s)) \leq |\partial\phi|(x) e^{\alpha^- t}|t - s|, \quad 0 \leq s < t, \tag{4.63}$$

where $\alpha^- := \max(-\alpha, 0)$.

Step 5 (u is a solution to (EVI)). Let $x \in D(|\partial\phi|)$ and $\alpha \in \mathbb{R}$ as in assumption (H_1). If $\alpha \leq 0$, then for $h > 0$, $1 + \alpha h > 0$, $J_h x$ is well defined by Lemma 4.2 and satisfies (4.42). Moreover, the estimates (4.44), (4.46), (4.47), (4.57), (4.59), (4.60), (4.63) hold. We defined $u : [0, \infty) \to X$ in (4.58) and (4.62). We shall prove in this section that u is a solution to (EVI) with initial value $u(0) = x$ where $\alpha \leq 0$ is as above. If $\alpha > 0$, then for every $h > 0$, $J_h x$ is well defined by Lemma 4.2 and satisfies the "variational inequality" (4.42) where $\alpha > 0$, hence *also* for $\alpha = 0$. Therefore it follows from the proofs of Steps 2, 3 and 4 that $J_h x$ satisfies all estimates mentioned above with $\alpha = 0$. As a consequence, $\lim_{n\to\infty} (J_{t/n})^n x$ exists for every $t > 0$ and we can define $u(t)$ as in (4.58) for $t > 0$ and $u(0) = x$. Then u satisfies (4.34) and (4.35). In this case we also want to prove that u is a solution to (EVI) with $\alpha > 0$! For proving this we start from the "variational inequality" (4.42) with $\alpha > 0$. From now on we take $\alpha \in \mathbb{R}$ and distinguish the cases $\alpha \leq 0$ and $\alpha > 0$ if necessary.

In view of (4.35) and of Proposition 2.1 it is sufficient to prove that u is an "integral solution" to (EVI), that is for every $0 < a < b$, $\phi \circ u \in L^1(a, b)$ and $\phi \circ u$ satisfies (2.2). It follows from the continuity of u that if $\phi \circ u \in L^1(a, b)$, $\phi \circ u$ satisfies (2.2) with $0 < a < b$, a, b rationals, then u is an "integral solution" to (EVI). Let $0 < a < b$ with a and b rational numbers. There exist $s > 0$ rational, $p > q > 0$ integers such that $a = qs$ and $b = ps$. Let $k_0 \in \mathbb{N}$ be such that

$$1 + \alpha\,\frac{s}{k_0} > 0 \tag{4.64}$$

and let $k \geq k_0$. Then

$$(J_{s/k})^{qk} x = \big(J_{\frac{qs}{qk}}\big)^{qk} x = \big(J_{\frac{a}{qk}}\big)^{qk} x \overset{k\to\infty}{\longrightarrow} u(a), \tag{4.65}$$

and

$$(J_{s/k})^{pk}x = \big(J_{\frac{ps}{pk}}\big)^{pk}x = \big(J_{\frac{b}{pk}}\big)^{pk}x \overset{k\to\infty}{\longrightarrow} u(b). \tag{4.66}$$

Next we set $h := \frac{s}{k}$. In view of (4.64) $x_m := J_h^m x$, $m \geq 1$, is well defined by Lemma 4.2. We set $x_0 := x$. For any $z \in D(\phi)$, $m \geq 1$, we have by (4.42) and (4.16)

$$\frac{1}{2}\big(d^2(x_m, z) - d^2(x_{m-1}, z)\big) + \frac{\alpha h}{2}\, d^2(x_m, z) + h\phi(x_m) \leq h\phi(z). \tag{4.67}$$

Adding (4.67) from $m := qk+1$ to $m := pk$ we obtain

$$\frac{1}{2}\big(d^2(x_{pk}, z) - d^2(x_{qk}, z)\big) + \frac{\alpha}{2}\frac{s}{k}\sum_{l=qk+1}^{pk} d^2(x_l, z) + \frac{s}{k}\sum_{l=qk+1}^{pk}\phi(x_l) \leq \frac{s}{k}\sum_{l=qk+1}^{pk}\phi(z) = (b-a)\phi(z). \tag{4.68}$$

Now we want to take the limit of (4.68) as $k \to \infty$. Observe that in view of (4.65) and (4.66), $\lim\limits_{k\to\infty} x_{pk} = u(b)$ and $\lim\limits_{k\to\infty} x_{qk} = u(a)$.

The next lemma will take care of the limit of the third and fourth terms in (4.68).

Lemma 4.3. *Let x, u, s, a, b, p, q be as above and let $k \geq k_0$ where k_0 satisfies* (4.64). *We have*

(i) *if $\varphi : X \to \mathbb{R}$ is Lipschitz continuous on bounded subsets of X, then*

$$\lim_{k\to\infty}\frac{s}{k}\sum_{l=qk+1}^{pk}\varphi\big((J_{s/k})^l x\big) = \int_a^b \varphi(u(t))\,dt. \tag{4.69}$$

(ii) *if $\phi : X \to (-\infty, +\infty]$ is as in Theorem* 4.1, *then $\phi \circ u$ is l.s.c.* (*hence Lebesgue measurable*), *$\phi\circ u|_{[a,b]}$ is bounded below. Moreover, if $C \geq 0$ is such that $\phi(u(t))+C \geq 0$, $t \in [a,b]$, then*

$$\int_a^b \big(\phi(u(t)) + C\big)\,dt \leq \varliminf_{k\to\infty}\frac{s}{k}\sum_{l=qk+1}^{pk}\phi\big((J_{s/k})^l x\big) + C(b-a). \tag{4.70}$$

In particular, if the right-hand side of (4.70) *is finite, then $\phi \circ u|_{[a,b]} \in L^1(a,b)$.*

Before proving Lemma 4.3 we apply it in order to prove that $\phi \circ u|_{[a,b]} \in L^1(a,b)$ and satisfies (2.2). Since the function $y \mapsto d^2(y,z)$ is Lipschitz continuous on bounded subsets of X,

$$\big(d^2(y,z) - d^2(\hat{y},z)\big) \leq d(y,\hat{y})\big(d(y,z) + d(\hat{y},z)\big),$$

we can use Lemma 4.3(i) in order to prove that the third term in (4.68) converges to $\frac{\alpha}{2}\int_a^b d^2(u(t),z)\,dt$ as $k \to \infty$. It follows that

$$\varliminf_{k\to\infty}\frac{s}{k}\sum_{l=qk+1}^{pk}\phi(x_l) \leq (b-a)\phi(z) - \frac{1}{2}\, d^2(u(b),z) + \frac{1}{2}\, d^2(u(a),z) - \frac{\alpha}{2}\int_a^b d^2(u(t),z)\,dt < \infty. \tag{4.71}$$

It follows from Lemma 4.3(ii) that $\phi \circ u|_{[a,b]} \in L^1(a,b)$ and from (4.70) that u satisfies (2.2). Hence u is a solution to (EVI).

It remains to prove Lemma 4.3.

Proof of Lemma 4.3. (i) Since $u|_{[a,b]} \in C[a,b]$, we have $\varphi \circ u|_{[a,b]} \in C[a,b]$ and

$$\int_a^b \varphi(u(t))\,dt = \lim_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \varphi\Big(u\Big(l\,\frac{s}{k}\Big)\Big). \tag{4.72}$$

Note that $\big\{u\big(l\frac{s}{k}\big) : k \geq k_0,\ qk+1 \leq l \leq pk\big\} \subset u([a,b])$ is bounded in X. By (4.59) we have

$$\begin{aligned} d\Big(u\Big(l\,\frac{s}{k}\Big), (J_{s/k})^l x\Big) &= d\Big(u\Big(l\,\frac{s}{k}\Big), (J_{\frac{sl}{kl}})^l x\Big) \\ &\leq |\partial\phi|(x)\cdot\Big(\frac{ls}{k}\Big)\cdot\Big(\frac{1}{l} + \Big(\alpha\,\frac{ls}{k}\Big)^2 \frac{1}{l^2}\Big)^{1/2}\cdot C(|\alpha|,b) \end{aligned} \tag{4.73}$$

for some constant $C = C(\alpha,b) > 0$, since $e^{-\alpha ls/k} \leq e^{|\alpha| b}$ and $\lim_{k\to\infty}\big(1+\frac{1}{\alpha}\frac{t}{n}\big)^{-(n+1)} = e^{-\alpha t} \leq e^{|\alpha| b}$. Since $0 < \frac{ls}{k} \leq b$, it follows that $\big\{(J_{s/k})^l x : k \geq k_0,\ qk+1 \leq l \leq pk\big\}$ is bounded in X.

Let $k \geq k_0$ and $qk+1 \leq l \leq pk$. Then there exists $M > 0$ such that

$$\Big|\varphi\Big(u\Big(l\,\frac{s}{k}\Big)\Big) - \varphi\big((J_{s/k})^l x\big)\Big| \leq M d\Big(u\Big(l\,\frac{s}{k}\Big), (J_{s/k})^l x\Big),$$

in view of the Lipschitz continuity of φ on bounded subsets of X. By using (4.73) we get

$$\Big|\varphi\Big(u\Big(l\,\frac{s}{k}\Big)\Big) - \varphi\big((J_{s/k})^l x\big)\Big| \leq M|\partial\phi|(x) C(|\alpha|,b)\big(1+(\alpha s)^2\big)^{1/2}\cdot s\cdot \frac{l^{1/2}}{k}.$$

Hence

$$\frac{s}{k}\Big|\sum_{l=qk+1}^{pk}\Big(\varphi\Big(u\Big(l\,\frac{s}{k}\Big)\Big) - \varphi\big(u(J_{s/k})^l x\big)\Big)\Big| \leq M|\partial\phi|(x) C'(|\alpha|,b,s)\frac{1}{k^2}\sum_{l=qk+1}^{pk} l^{1/2} = O\Big(\frac{1}{\sqrt{k}}\Big),$$

as $k \to \infty$. Therefore in view of (4.72) we obtain (4.69).

(ii) Since $u \in C([a,b];X)$ and ϕ is l.s.c., $\phi \circ u$ is l.s.c. and since $[a,b]$ is compact, $\phi \circ u|_{[a,b]}$ is bounded below. Let $\overline{C} \geq 0$ be such that $\phi(u(t)) + \overline{C} \geq 0$, $t \in [a,b]$. Then $\int_a^b \big(\phi(u(t)) + \overline{C}\big)\,dt$ is well defined, possibly equal to $+\infty$. Next we show that ϕ is bounded below on the set $B := \big\{(J_{s/k})^l x : k \geq k_0,\ qk \leq l \leq pk\big\}$, where x is as in Theorem 4.1 and k_0 satisfies (4.64) and q, p are defined as above. Note that $B \subset D(\phi)$. Suppose for contradiction that ϕ is not bounded below on B. For $k \geq k_0$ let $l_k \in \mathbb{N}$ be such that $qk \leq l_k \leq pk$ and

$$\phi_k := \phi\big((J_{s/k})^{l_k} x\big) = \min\big\{\phi(J_{s/k})^l x : qk \leq l \leq pk\big\}.$$

There exists a subsequence $\phi_{j(k)}$ tending to $-\infty$ as $k \to +\infty$. Let $t_k := l_k \cdot \frac{s}{k}$, $k \geq k_0$. Since $t_k \in [a,b]$, there exist a subsequence of $t_{j(k)}$ *still* denoted by $t_{j(k)}$, and $\bar{t} \in [a,b]$ such that $\lim_{k\to\infty} t_{j(k)} = \bar{t}$. We claim that $\lim_{k\to\infty} d(u(\bar{t}), J_{s/j(k)}^{l_{j(k)}} x) = 0$.

Clearly $\lim_{k\to\infty} d(u(\bar{t}), u(t_{j(k)})) = 0$. Set $m_k := l_{j(k)}$. In view of (4.59), using the same constant $C(|\alpha|, b)$ as in (4.73),

$$d\Big(u\Big(m_k \cdot \frac{s}{j(k)}\Big), \big(J_{\frac{s}{j(k)}}\big)^{m_k} x\Big) = d\Big(u\Big(m_k \cdot \frac{s}{j(k)}\Big), \big(J_{\frac{m_k s}{j(k)}/m_k}\big)^{m_k} x\Big)$$
$$\leq |\partial\phi|(x) C(|\alpha|, b) \cdot b \cdot \Big(\frac{q}{j(k)} + (\alpha s)^2 \frac{1}{(j(k))^2}\Big)^{1/2} \to 0 \quad \text{as } k \to \infty,$$

which proves the claim.

Since ϕ is l.s.c., we have

$$\phi(u(\bar{t})) \leq \varliminf_{k\to\infty} \phi\big(\big(J_{\frac{s}{j(k)}}\big)^{m_k} x\big) = \varliminf_{k\to\infty} \phi_{j(k)} = -\infty,$$

a contradiction. Therefore ϕ is bounded below on the set B, and there exists $C \geq \overline{C} \geq 0$ such that $\phi(y) + C \geq 0$ whenever $y = u(t)$ for some $t \in [a, b]$ or $y \in B$.

Let $\widetilde{\phi}(y) := \max(\phi(y), -C)$, $y \in X$. Then $\widetilde{\phi} : X \to (-\infty, +\infty]$ is proper, l.s.c. and satisfies $\widetilde{\phi} \geq -C$, $\widetilde{\phi}(u(t)) = \phi(u(t))$, $t \in [a, b]$, and $\widetilde{\phi}(y) = \phi(y)$, $y \in B$. Next we approximate $\widetilde{F}$ by Lipschitz continuous functions φ_n. Let $\varphi_n(y) := \inf\big\{\widetilde{\phi}(z) + nd(y, z) : z \in X\big\}$ where $n \geq 1$, $y \in X$. Then one verifies that $\varphi_n \geq -C$, $\varphi_n \leq \varphi_{n+1}$, $\varphi_n \uparrow \widetilde{\phi}$ as $n \to \infty$ and $\varphi_n \in \mathrm{Lip}(X; \mathbb{R})$. For each $n \in \mathbb{N}$ we can apply part (i) of Lemma 4.3 and we get

$$\int_a^b \varphi_n(u(t))\, dt = \lim_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \varphi_n\big((J_{s/k})^l x\big)$$
$$\leq \varliminf_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \widetilde{\phi}\big((J_{s/k})^l\big) = \varliminf_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \phi\big((J_{s/k})^l\big) =: J.$$

Suppose $J < \infty$, otherwise there is nothing to prove. We have $\varphi_n + C \geq 0$ and

$$\int_a^b \big(\varphi_n(u(t)) + C\big)\, dt \leq J + C(b - a), \quad n \geq 1.$$

By the monotone convergence theorem, we get

$$\int_a^b \big(\widetilde{\phi}(u(t)) + C\big)\, dt \leq J + C.$$

Since $\widetilde{\phi}(u(t)) = \phi(u(t))$, $t \in [a, b]$, we get $\phi \circ u + C \in L^1(a, b)$, hence $\phi \circ u|_{[a,b]} \in L^1(a, b)$ and $\int_a^b \phi(u(t))\, dt \leq J$. □

Step 6 (proof of (4.33)–(4.38) and (4.40)–(4.41)). The function u defined above is the unique solution to (EVI) with initial value $u(0) = x$ by the A priori estimate 2.1. Claim (4.33) is clear by definition (4.58), (4.34) is a consequence of (4.60), (4.35) follows from (4.63). Let $\{S(t)\}_{t\geq 0}$ be the family of operators defined as in (4.40) by $S(t)x := u(t)$, $t \geq 0$. Then $S(t)$ maps $D(|\partial\phi|)$ into itself by (4.34), clearly $S(0)$ is the identity map on $D(|\partial\phi|)$. If $h > 0$ and $v(t) := u(t + h)$, $t \geq 0$, then v is a solution to (EVI) with initial value $v(0) = u(h)$. By uniqueness we have $S(t + h)x = S(t)u(h) = S(t)S(h)x$, hence

$S(t+h) = S(t)S(h)$, which is the semigroup property of $\{S(t)\}_{t\geq 0}$. Then (4.41) follows from A priori estimate 2.1. Next we prove (4.36). As a consequence of (4.16) we have

$$\phi\big((J_{t/n})^n x\big) \leq \phi(J_{t/n}x) \leq \phi(x), \quad n \geq n_0,\ t > 0,\ x \in D(|\partial\phi|),$$

where $1 + \alpha\frac{t}{n_0} > 0$.

Since ϕ is l.s.c., we obtain from (4.33) $\phi(S(t)x) \leq \phi(x)$. If $h > 0$, then

$$\phi(u(t+h)) = \phi(S(t+h)x) = \phi(S(t)S(h)x) \leq \phi(S(h)x) = \phi(u(h)),$$

which proves (4.36). Similarly we have from (4.31) for $x \in D(|\partial\phi|)$, $t > 0$ and $n \geq n_0$

$$|\partial\phi|(J_{t/n}x) \leq \Big(1 + \alpha\frac{t}{n}\Big)^{-1}|\partial\phi|(x),$$

hence

$$|\partial\phi|\big((J_{t/n})^n x\big) \leq \Big(1 + \alpha\frac{t}{n}\Big)^{-n}|\partial\phi|(x),$$

and by l.s.c. $|\partial\phi|(u(t)) \leq e^{-\alpha t}|\partial\phi|(x)$. Now let $h > 0$,

$$\begin{aligned} e^{\alpha(t+h)}|\partial\phi|(u(t+h)) &= e^{\alpha(t+h)}|\partial\phi|(S(t+h)x) \\ &= e^{\alpha(t+h)}|\partial\phi|(S(t)S(h)x) \leq e^{\alpha(t+h)}e^{-\alpha t}|\partial\phi|(S(h)x) = e^{\alpha h}|\partial\phi|(u(h)), \end{aligned}$$

which proves the first assertion in (4.37). The right-continuity follows from lower semicontinuity and nonincreasingness.

It remains to prove (4.38). Since $(J_{t/n})^n x \to u(t)$ and ϕ is l.s.c., we have

$$\phi(u(t)) \leq \varliminf_{n\to\infty} \phi\big((J_{t/n})^n x\big), \quad t > 0. \tag{4.74}$$

In view of (4.25) we have for $y \in D(\phi)$ and $z \in D(|\partial\phi|)$:

$$\phi(y) \geq \phi(z) - |\partial\phi|(z)\cdot d(y,z) + \frac{\alpha}{2}\,d^2(y,z). \tag{4.75}$$

Substituting $y = u(t)$ and $z = (J_{t/n})^n x$ in (4.74) we obtain

$$\phi\big((J_{t/n})^n x\big) \leq \phi(u(t)) + |\partial\phi|\big((J_{t/n})^n x\big)\cdot d\big((J_{t/n})^n x, u(t)\big) - \frac{\alpha}{2}\,d^2\big((J_{t/n})^n x, u(t)\big). \tag{4.76}$$

Using (4.31) we have $|\partial\phi|\big((J_{t/n})^n x\big) \leq (1+\alpha\frac{t}{n})^{-n}|\partial\phi|(x)$, hence from (4.33) and (4.76) we arrive at

$$\varlimsup_{n\to\infty} \phi\big((J_{t/n})^n x\big) \leq \phi(u(t)), \quad t > 0, \tag{4.77}$$

which together with (4.74) implies (4.38).

Step 7 (proof of (4.39)). We need the following

Lemma 4.4 ([AGS], Theorem 3.1.4, p. 62)**.** *Let $\phi : X \to (-\infty, +\infty]$ be as in Theorem 4.1. Let $h > 0$ be such that $1 + \alpha h > 0$ where α is as in (H_1). Then for any $y \in D(\phi)$ we have*

$$\phi(y) - \phi_h(y) = \frac{1}{2}\int_0^h \frac{d^2(y, J_s y)}{s^2}\,ds. \tag{4.78}$$

Proof of Lemma 4.4. In view of the assumptions on h, $J_s y$ is well defined for $0 < s \le h$ and $s \mapsto d^2(y, J_s y)$ is nondecreasing by (4.20), hence Borel measurable as well as $d^2(y, J_s y)/s^2$. Moreover, let $N(y) \subset (0, h)$ denote the at most countable set of points of discontinuity of $s \mapsto d^2(y, J_s y)$. Since $\lim_{\tilde{h} \to 0} \phi_{\tilde{h}}(y) = \phi(y)$ by (4.22), it is sufficient to prove

$$\phi_{\bar{h}_0}(y) - \phi_{\bar{h}_1}(y) = \frac{1}{2} \int_{\bar{h}_0}^{\bar{h}_1} \frac{d^2(y, J_s y)}{s^2}\, ds \tag{4.79}$$

for $0 < \bar{h}_0 < \bar{h}_1$ such that $1 + \alpha \bar{h}_1 > 0$.

Next we claim that $y \mapsto \phi_h(y) \in \mathrm{Lip}[\bar{h}_0, \bar{h}_1]$. Let $h_0, h_1 \in [\bar{h}_0, \bar{h}_1]$. Then we have

$$\phi_{h_0}(y) - \phi_{h_1}(y) \le \Phi(h_0, y; J_{h_1} y) - \Phi(h_1, y; J_{h_1} y) = \frac{1}{2h_0}\, d^2(y, J_{h_1} y) - \frac{1}{2h_1}\, d^2(y, J_{h_1} y),$$

hence

$$\phi_{h_0}(y) - \phi_{h_1}(y) \le \frac{1}{2}\, \frac{h_1 - h_0}{h_0 h_1}\, d^2(y, J_{h_1} y). \tag{4.80}$$

Choosing $h_0 < h_1$, we get in view of (4.22), (4.20):

$$\big|\phi_{h_0}(y) - \phi_{h_1}(y)\big| \le (h_1 - h_0)\, \frac{1}{2}\, \frac{1}{(\bar{h}_0)^2}\, d^2(y, J_{\bar{h}_1} y)$$

which proves the claim. It follows that the derivative of $h \mapsto \phi_h(y)$ exists a.e. in $(\bar{h}_0, \bar{h}_1)$ and that

$$\phi_{\bar{h}_0}(y) - \phi_{\bar{h}_1}(y) = \int_{\bar{h}_1}^{\bar{h}_0} \Big(\frac{d}{dh}\, \phi_h(y) \Big)\, dh.$$

We claim that for $h \in (\bar{h}_0, \bar{h}_1) \setminus N(y)$

$$\frac{d}{dy}\, \phi_h(y) = -\frac{1}{2}\, \frac{d^2(y, J_h y)}{h^2} \tag{4.81}$$

which implies (4.79). Interchanging h_0 and h_1 in (4.80) we obtain

$$\phi_{h_0}(y) - \phi_{h_1}(y) \ge -\frac{1}{2}\, \frac{h_0 - h_1}{h_0 h_1}\, d^2(y, J_{h_0} y) = \frac{1}{2}\, \frac{h_1 - h_0}{h_0 h_1}\, d^2(y, J_{h_0} y). \tag{4.82}$$

Assuming $h_0 < h_1$ in (4.80), (4.82) we get

$$\frac{1}{2}\, \frac{1}{h_0 h_1}\, d^2(y, J_{h_0} y) \le \frac{\phi_{h_0}(y) - \phi_{h_1}(y)}{h_0 - h_1} \le \frac{1}{2}\, \frac{1}{h_0 h_1}\, d^2(y, J_{h_1} y)$$

and recalling that $\lim_{h \to \tilde{h}} d^2(y, J_h y) = d^2(y, J_{\tilde{h}} y)$ for $\tilde{h} \notin N(y)$, we obtain (4.81) for every $h \in (\bar{h}_0, \bar{h}_1) \setminus N(y)$. □

In order to prove (4.39) we introduce uniform (dyadic) partitions of the interval $[0, t]$: for $k \ge 1$ we set

$$h_k := t \cdot 2^{-k}, \quad t_i^k := i h_k, \quad 0 \le i \le 2^k, \tag{4.83}$$

and we choose $k \geq k_0 \geq 1$ where k_0 satisfies

$$1 + \alpha h_{k_0} > 0, \tag{4.84}$$

ensuring that $J_{h_k} x$ is well defined.

Using the notation $(J_{h_k})^0 x = x$, we define the following functions associated with the above partitions, where $1 \leq i \leq 2^k$:

$$\overline{u}_k(s) := \begin{cases} x, & s = 0, \\ (J_{h_k})^i x, & s \in (t_{i-1}^k, t_i^k], \end{cases} \tag{4.85}$$

$$\widetilde{u}_k(s) := \begin{cases} x, & s = 0, \\ J_{(s-t_{i-1}^k)}(J_{h_k})^{i-1} x, & s \in (t_{i-1}^k, t_i^k], \end{cases} \tag{4.86}$$

$$v_k(s) := \begin{cases} 0, & s = 0, \\ \dfrac{d\big((J_{h_k})^i x, (J_{h_k})^{i-1} x\big)}{h_k}, & s \in (t_{i-1}^k, t_i^k], \end{cases} \tag{4.87}$$

and

$$w_k(s) := \begin{cases} 0, & s = 0, \\ \dfrac{d\big(\widetilde{u}_k(s), (J_{h_k})^{i-1} x\big)}{s - t_{i-1}^k}, & s \in (t_{i-1}^k, t_i^k]. \end{cases} \tag{4.88}$$

Clearly v_k, w_k are nonnegative real-valued functions on $[0, t]$ which are Borel measurable (for w_k see the proof of Lemma 4.4).

We have

$$\begin{aligned} \frac{1}{2} \int_{t_{i-1}^k}^{t_i^k} w_k^2(s)\, ds &= \frac{1}{2} \int_0^{h_k} w_k^2(s + t_{i-1}^k)\, ds \\ &= \frac{1}{2} \int_0^{h_k} \frac{d^2\big((J_{n_k})^{i-1} x, J_s (J_{h_k})^{i-1} x\big)}{s^2} = \phi\big((J_{h_k})^{i-1} x\big) - \phi_{h_k}\big((J_{h_k})^{i-1} x\big), \end{aligned}$$

where we used (4.88) and (4.78). Using the definition of ϕ_{h_k} we obtain

$$\phi_{h_k}\big((J_{h_k})^{i-1} x\big) = \frac{1}{2h_k} d^2\big((J_{h_k})^{i-1} x, (J_{h_k})^i x\big) + \phi\big((J_{h_k})^i x\big).$$

Next, observing that

$$\frac{1}{2h_k} d^2\big((J_{h_k})^{i-1} x, (J_{h_k})^i x\big) = \frac{1}{2} \int_{t_{i-1}^k}^{t_i^k} v_k^2(s)\, ds,$$

we arrive at

$$\frac{1}{2} \int_{t_{i-1}^k}^{t_i^k} v_k^2(s)\, ds + \frac{1}{2} \int_{t_{i-1}^k}^{t_i^k} w_k^2(s)\, ds = \phi\big((J_{h_k})^{i-1} x\big) - \phi\big((J_{h_k})^i x\big).$$

Summing over i from 1 to z^k we obtain

$$\frac{1}{2} \int_0^t v_k^2(s)\, ds + \frac{1}{2} \int_0^t w_k^2(s)\, ds = \phi(x) - \phi\big((J_{t/2^k})^{2^k} x\big).$$

Using (4.38) we get

$$\varliminf_{k\to\infty} \frac{1}{2}\int_0^t v_k^2(s)\,ds + \varliminf_{k\to\infty} \frac{1}{2}\int_0^t w_k^2(s)\,ds \leq \varliminf_{k\to\infty} \left(\frac{1}{2}\int_0^t v_k^2(s)\,ds + \frac{1}{2}\int_0^t w_k^2(s)\,ds\right) = \phi(x) - \phi(u(t)). \tag{4.89}$$

Next we prove that

$$\int_0^t |\partial\phi|^2(u(s))\,ds \leq \varliminf_{k\to\infty} \int_0^t w_k^2(s)\,ds, \tag{4.90}$$

and for some subsequence $j(k)$

$$\int_0^t |\dot{u}|^2(s)\,ds \leq \varliminf_{k\to\infty} \int_0^t v_{j(k)}^2(s)\,ds. \tag{4.91}$$

Notice that (4.89), (4.90) and (4.91) imply (4.39).

In order to establish (4.90) and (4.91) we first prove that for every $s \in [0, T]$

$$\lim_{k\to\infty} d\big(u(s), \overline{u}_k(s)\big) = \lim_{k\to\infty} d\big(u(s), \widetilde{u}_k(s)\big) = 0. \tag{4.92}$$

Clearly $d(u(0), \overline{u}(0)) = d(u(0), \widetilde{u}(0)) = 0$. Let $s \in (0, t]$ and $\varepsilon > 0$. For every $k \geq k_0$ there exists a unique $i \in \{1, \dots, 2^k\}$, depending on k, such that $s \in (t_{i-1}^k, t_i^k]$. Since u is Lipschitz continuous on $[0, t]$, there exists $k_1 \geq k_\varepsilon$ such that $d(u(s), u(t_{i-1}^k)) \leq \varepsilon/2$ for $k \geq k_1$. On the other hand, by (4.59), there exists $C_1 = C_1(t, k_0)$ such that

$$d\big(u(t_{i-1}^k), \overline{u}_k(t_{i-1}^k)\big) = d\big(u(t_{i-1}^k), \big(J_{\frac{(i-1)h_k}{(i-1)}}\big)^{i-1}x\big) \leq |\partial\phi|(x) C_1 \cdot \frac{1}{\sqrt{i-1}}.$$

Since $\lim_{k\to\infty} (i-1)2^{-k} = s > 0$, we have $\lim_{k\to\infty} i(k) = \infty$, hence by the triangle inequality, for k large enough, $d\big(u(s), \overline{u}(s)\big) \leq \varepsilon$, which proves the first part of (4.92). Next we estimate $d\big(\overline{u}_k(s), \widetilde{u}_k(s)\big)$, $s \in (0, t]$. We have $\widetilde{u}_k(s) = J_{\delta_k}(J_{h_k})^{i-1}x$, where $i = i(k)$ is as above and $\delta_k := s - t_{i-1}^k$. So

$$d\big(\overline{u}_k(s), \widetilde{u}_k(s)\big) \leq d\big(J_{\delta_k}(J_{h_k})^{i-1}x, (J_{h_k})^{i-1}x\big) + d\big(J_{h_k}(J_{h_k})^{i-1}x, (J_{h_k})^{i-1}x\big) \leq \big(\delta_k(1+\delta_k\alpha)^{-1} + h_k(1+h_k\alpha)^{-1}\big)|\partial\phi|\big((J_{h_k})^{i-1}x\big)$$

by using (4.29), (4.30).

By (4.31), for $x \in D(|\partial\phi|)$, $|\partial\phi|((J_{h_k})^{i-1}x)$ is bounded. Since $0 < \delta_k \leq h_k \to 0$, we have

$$\lim_{k\to\infty} d\big(\widetilde{u}_k(s), \overline{u}_k(s)\big) = 0,$$

which implies the second part of (4.92).

Now for $s \in (t_{i-1}^k, t_i^k)$ by (4.88), (4.15) and (4.86) we obtain

$$w_k(s) = \frac{d\big(J_{\delta_k}(J_{h_k})^{i-1}x, (J_{h_k})^{i-1}x\big)}{\delta_k} \geq |\partial\phi|\big(J_{\delta_k}(J_{h_k})^{i-1}x\big) = |\partial\phi|(\widetilde{u}_k(s)).$$

Since $|\partial\phi|$ is l.s.c. we get by (4.92)

$$\varliminf_{k\to\infty} w_k(s) \geq \varliminf_{k\to\infty} |\partial\phi|\big(\widetilde{u}_k(s)\big) \geq |\partial\phi|(u(s)).$$

Therefore by Fatou's lemma

$$\int_0^t |\partial\phi|^2(u(s))\,ds \le \int_0^t \varliminf_{k\to\infty} w_k^2(s)\,ds \le \varliminf_{k\to\infty} \int_0^t w_k^2(s)\,ds$$

which proves (4.90).

Finally we establish (4.91). By (4.89) there exist $M > 0$ and a subsequence $j(k)$ such that

$$\int_0^t (v_{j(k)})^2(s)\,ds \le M. \tag{4.93}$$

Therefore there exist a subsequence, still denoted by $j(k)$, and $\overline{v} \in L^2(0,t)$, $\overline{v} \ge 0$ a.e., such that $v_{j(k)}$ converges weakly to $\overline{v}$ in $L^2(0,t)$ and

$$\int_0^t \overline{v}^2(s)\,ds \le \varliminf_{k\to\infty} \int_0^t v_{j(k)}^2(s)\,ds. \tag{4.94}$$

Since $d\big(\overline{u}_k(t_{i-1}^k), \overline{u}_k(t_i^k)\big) = \int_{t_{i-1}^k}^{t_i^k} v_k(s)\,ds$, given $0 \le s_1 < s_2 \le t$, we can find sequences $(s_{1,k})$, $(s_{2,k})$ converging to s_1, s_2, such that

$$d\big(\overline{u}_k(s), \overline{u}_k(t)\big) \le \int_{s_{1,k}}^{s_{2,k}} v_k(s)\,ds.$$

In view of (4.92), (4.93) we obtain

$$d\big(u(s_1), u(s_2)\big) \le \int_{s_1}^{s_2} \overline{v}(s)\,ds.$$

Hence the metric derivative of u, $|\dot{u}|(s)$ satisfies $|\dot{u}|(s) \le \overline{v}(s)$ a.e. on $(0,t)$. By (4.94),

$$\int_0^t |\dot{u}|^2(s)\,ds \le \int_0^t \overline{v}^2(s)\,ds \le \varliminf_{k\to\infty} \int_0^t v_{j(k)}^2(s)\,ds,$$

which is (4.91). This completes the proof of Theorem 4.1. □

We arrive at the main result of this section.

Theorem 4.2 (Ambrosio–Gigli–Savaré, see [AGS]). *Let (X,d) be a complete metric space and let $\phi : X \to (-\infty, +\infty]$ be proper, lower semicontinuous. Let assumptions (H_1) with $\alpha \in \mathbb{R}$ and (H_2) be satisfied. Then there exists a C_0-semigroup $\{S(t)\}_{t\ge 0}$ on $\overline{D(\phi)}$ satisfying $[S(t)]_{\mathrm{Lip}} \le e^{-\alpha t}$, $t \ge 0$, such that for every $x \in \overline{D(\phi)}$ the function $u : [0,\infty) \to X$ defined by $u(t) := S(t)x$, $t \ge 0$, is the unique solution to* (EVI) *with initial value $u(0) = x$. Moreover, the following properties of the function u hold:*

i) *$\phi \circ u(t) \le \phi_{c(t)}(x)$ for every $t > 0$ such that $1 + \alpha c(t) > 0$ where*

$$c(t) := \int_0^t e^{\alpha s}\,ds;$$

ii) *the map $[0,\infty) \ni t \mapsto \phi \circ u(t)$ is nonincreasing and right-continuous;*

iii) *the map $[0,\infty) \ni t \mapsto e^{-2\alpha^- t}\phi \circ u(t)$ is convex, where $\alpha^- := \max(-\alpha, 0)$;*

iv) *$u(t) \in D(|\partial\phi|)$ for every $t > 0$ and*

$$\frac{t}{2}|\partial\phi|^2(u(t)) \leq e^{2\alpha^- t}(\phi(x) - \phi_t(x))$$

for every $t > 0$ such that $1 + \alpha t > 0$;

v) *the map $(0, \infty) \ni t \mapsto e^{\alpha t}|\partial\phi|(u(t))$ is nonincreasing and right-continuous;*

vi)

$$\frac{d^+}{dt}(\phi \circ u)(t) = -|\partial\phi|^2(u(t)) = -|\dot{u}_+|^2(t)$$

for every $t > 0$ where $|\dot{u}_+|(t) := \lim_{s \downarrow t} \frac{d(u(t),u(s))}{s-t}$ is the right metric derivative of u at t;

vii)

$$\phi \circ u(s) - \phi \circ u(t) = \int_s^t \left\{ \frac{1}{2}|\partial\phi|^2(u(r)) + \frac{1}{2}|\dot{u}|^2(r) \right\} dr$$

for every $0 \leq s < t$;

viii) *for every $0 < a < b$, $u_{[a,b]} \in \mathrm{Lip}([a, b]; X)$ and*

$$[u|_{[a,b]}]_{\mathrm{Lip}} \leq |\partial\phi|(u(a))e^{\alpha^-(b-a)};$$

ix) *$u(t) = \lim_{n\to\infty} (J_{t/n})^n x$ for every $t > 0$;*

x) *$\phi(u(t)) = \lim_{n\to\infty} \phi\big((J_{t/n})^n x\big)$ for every $t > 0$;*

xi) *if $\alpha > 0$ then ϕ has a unique minimizer $\overline{x} \in D(\phi)$ and $d(u(t), \overline{x}) \leq e^{-\alpha t} d(x, \overline{x})$ for every $t \geq 0$;*

xii) *if $\alpha = 0$ then*

$$d^2(u(t), (J_{t/n})^n x) \leq \frac{t}{n}\big(\phi(x) - \phi_{t/n}(x)\big) \leq \frac{t^2}{2n^2}|\partial\phi|^2(x),$$

for every $t > 0$.

Remark 4.3. The proof of the convergence of $\{J^n_{t/n}x\}$ used in Theorem 4.1 gives a weaker estimate than the estimate in xii). However, it is simpler than the one given in [AGS].

Proof of Theorem 4.2.

Step 1. "Extension of $\{S(t)\}_{t\geq 0}$".

Let $x \in \overline{D(\phi)} = \overline{D(|\partial\phi|)}$ and let $t \geq 0$. Let $\{S(t)\}_{t\geq 0}$ be the semigroup defined in Theorem 4.1. Since $S(t) : D(|\partial\phi|) \to D(|\partial\phi|)$ is Lipschitz continuous and $\overline{D(|\partial\phi|)}$ is complete, there exists a unique continuous extension of $S(t)$ to $\overline{D(\phi)}$ still denoted by $S(t)$. Clearly $S(t) : \overline{D(\phi)} \to \overline{D(\phi)}$ is also Lipschitz continuous and satisfies $[S(t)]_{\mathrm{Lip}} \leq e^{\alpha t}$. Let $(x_n) \subset D(|\partial\phi|)$ be such that $x_n \to x$. Then, for $t, s \geq 0$, $S(t + s)x = \lim S(t + s)x_n = \lim S(t)S(s)x_n = S(t)S(s)x$. Since $S(0) = I$, $\{S(t)\}_{t\geq 0}$ satisfies the semigroup property. Let $t_n \geq 0$ be such that $t_n \to t$, and let $y \in D(|\partial\phi|)$. Then $d(S(t)x, S(t_n)x) \leq d(S(t)x, S(t)y) + d(S(t)y, S(t_n)y) + d(S(t_n)y, S(t_n)x) \leq (e^{\alpha t} + e^{\alpha t_n})d(x, y) + d(S(t_n)y, S(t)y)$. Hence $\overline{\lim}\, d(S(t)x, S(t_n)x) \leq 2e^{\alpha t}d(x, y)$. Since $D(|\partial\phi|)$ is dense in $\overline{D(\phi)}$, we have $\overline{\lim}\, d(S(t)x, S(t_n)x) = 0$, hence $\{S(t)\}_{t\geq 0} : \overline{D(\phi)} \to \overline{D(\phi)}$ is a C_0 α-contractive semigroup on $\overline{D(\phi)}$.

Step 2. "$u(t) := S(t)x$ is an integral solution to (EVI)".

Let (x_n) be as in Step 1 and let $u_n(t) := S(t)x_n$, $n \geq 1$, $u(t) := S(t)x$, $t \geq 0$. Since $d(u_n(t), u(t)) \leq e^{\alpha t} d(x, x_n)$, the sequence $\{u_n(\cdot)\}$ converges uniformly to $u(\cdot)$ on intervals $[0, T]$, $T > 0$. Let $0 < a < b$ and $z \in D(\phi)$. Since ϕ is l.s.c. we have $\phi(u(b)) \leq \underline{\lim}\, \phi(u_n(b))$. Hence there exists $C_1 \in \mathbb{R}$ such that $\phi(u_n(b)) \geq \phi(u(b)) - C_1 =: C$. Since $\phi \circ u_n$ is nonincreasing in $[a, b]$, $\phi \circ u_n(t) \geq C$ for $t \in [a, b]$, $n \geq 1$. We have

$$\int_a^b \phi(u_n(t))\, dt \leq \frac{1}{2}\, d^2(u_n(a), z) - \frac{1}{2}\, d^2(u_n(b), z) - \frac{\alpha}{2} \int_a^b d^2(u_n(t), z)\, dt + (b - a)\phi(z).$$

In view of what preceeds we can apply Fatou's lemma, and noticing that $\phi \circ u$ is l.s.c., hence Borel measurable, we obtain

$$\int_a^b (\phi \circ u(t) + c)\, dt \leq \frac{1}{2}\, d^2(u(a), z) - \frac{1}{2}\, d^2(u(b), z) - \frac{\alpha}{2} \int_a^b d^2(u(t), z)\, dt + (b - a)(\phi(z) + c).$$

Therefore $\phi \circ u \in L^1(a, b)$ and u satisfies (2.2).

Step 3. "$u(t) := S(t)x$ is a solution to (EVI) and proof of i), ii) and iv)."

In order to prove that u is a solution to (EVI), in view of Proposition 2.1 and Step 2, it is sufficient to show that $u \in \mathrm{Lip}([a, b]; X)$ for every $0 < a < b$. By (4.63) (recalling that (4.63) is proved under the assumption $\alpha \leq 0$), and by the semigroup property, we have

$$d(u_n(t), u_n(s)) \leq |\partial\phi|(u_n(a)) e^{|\alpha|(b-a)}(t - s)$$

for $0 < a \leq s < t \leq b$, $n \geq 1$, where u_n is defined in Step 2. Thus if we can find $a_0 > 0$ such that for every $a \in (0, a_0)$, $|\partial\phi|(u_n(a))$ is bounded, then u will be a solution to (EVI). We set

$$c(t) := \int_0^t e^{\alpha s}\, ds$$

for $t > 0$ and choose $a_0 > 0$ such that $1 + \alpha a_0 > 0$ and $1 + \alpha c(a_0) > 0$. Clearly if $0 < a < a_0$, then $1 + \alpha a > 0$ and $1 + \alpha c(a) > 0$ too.

Let $a \in (0, a_0)$. We first establish a bound for $\phi\big(u_n\big(\frac{a}{2}\big)\big)$ and prove i). Since u_n satisfies (EVI), we obtain by multiplication by $e^{\alpha s}$ and integration on $[0, t]$, for some $t > 0$:

$$\frac{1}{2}\, e^{\alpha t} d^2(u_n(t), z) - \frac{1}{2}\, d^2(u_n(0), z) + \int_0^t e^{\alpha s} \phi(u_n(s))\, ds \leq c(t)\phi(z), \quad z \in D(\phi).$$

Using the fact that $\phi \circ u_n$ is nonincreasing we get

$$\phi \circ u_n(t) \leq \frac{1}{c(t)} \int_0^t e^{\alpha s} \phi(u_n(s))\, ds \leq \phi(z) + \frac{1}{2c(t)}\, d^2(u_n(0), z).$$

Hence, assuming $1 + \alpha c(t) > 0$ and taking the infimum over $z \in D(\phi)$, we obtain (see Definition 4.1)

$$(\phi \circ u_n)(t) \leq \phi_{c(t)}(u_n(0)). \tag{4.95}$$

Since $\phi_{c(t)}$ is continuous (see Lemma 4.2) and $u_n(0) = x_n \to x$, there exists $C_1(t) > 0$ independent of n such that $(\phi \circ u_n)(t) \leq C_1(t)$, $n \geq 1$, in particular,

$$\phi\Big(u_n\Big(\frac{a}{2}\Big)\Big) \leq C_1\Big(\frac{a}{2}\Big), \quad n \geq 1. \tag{4.96}$$

This equation will be used later.

Notice also that since $\phi_{c(t)}$ is continuous and ϕ is l.s.c., then for $t > 0$ such that $1 + \alpha c(t) > 0$ we have

$$\phi \circ u(t) \leq \phi_{c(t)}(x) \tag{4.97}$$

which establishes i). Next we shall find a bound for $|\partial\phi|(u_n(a))$ and to this end we first prove iv) in the special case $x \in D(|\partial\phi|)$. For the sake of clarity we denote x by y in this case and set $v(t) := S(t)y$, $t \geq 0$. Let $t > 0$ be such that $1 + \alpha t > 0$. From Theorem 4.1 (4.39) we get

$$\frac{1}{2}\int_0^t |\partial\phi|^2(v(s))\,ds \leq \phi(y) - \left[\phi(v(t)) + \frac{1}{2}\int_0^t |\dot{v}|^2(s)\,ds\right].$$

Since $v \in \mathrm{Lip}([0,t];X)$ we have

$$d(v(0), v(t)) \leq \int_0^t |\dot{v}|(s)\,ds$$

and by Jensen's inequality

$$\frac{1}{t}\,d^2(v(0), v(t)) \leq \int_0^t |\dot{v}|^2(s)\,ds.$$

It follows that

$$\frac{1}{2}\int_0^t |\partial\phi|^2(v(s))\,ds \leq \phi(y) - \left[\phi(v(t)) + \frac{1}{2t}\,d^2(y, v(t))\right] \leq \phi(y) - \phi_t(y).$$

Next we use (4.37) which implies $[0,\infty) \ni s \mapsto e^{-2\alpha^- s}|\partial\phi|^2(v(s))$ nonincreasing, where $\alpha^- := \max(-\alpha, 0)$. Therefore

$$\begin{aligned}\frac{t}{2}\,e^{-2\alpha^- t}|\partial\phi|^2(v(t)) &\leq \left(\frac{1}{2}\int_0^t e^{2\alpha^- s}\,ds\right)e^{-2\alpha^- t}|\partial\phi|^2(v(t))\\ &\leq \frac{1}{2}\int_0^t e^{2\alpha^- s}e^{-2\alpha^- s}|\partial\phi|^2(v(s))\,ds \leq \phi(y) - \phi_t(y).\end{aligned}$$

This establishes iv) in the case $y = x \in D(|\partial\phi|)$.

Now we are in a position to prove the bound for $|\partial\phi|(u_n(a))$. Indeed by choosing $y = u_n\big(\frac{a}{2}\big)$, we have $u_n(a) = S\big(\frac{a}{2}\big)y = v\big(\frac{a}{2}\big)$, hence

$$\frac{a}{4}e^{-2\alpha^-(a/2)}|\partial\phi|^2(u_n(a)) \leq \phi\Big(u_n\Big(\frac{a}{2}\Big)\Big) - \phi_{a/2}\Big(u_n\Big(\frac{a}{2}\Big)\Big).$$

Since $\phi\big(u_n\big(\frac{a}{2}\big)\big)$ is bounded by $C_1\big(\frac{a}{2}\big)$ and $\phi_{a/2}\big(u_n\big(\frac{a}{2}\big)\big)$ is bounded ($\phi_{a/2}$ is continuous and $u_n\big(\frac{a}{2}\big) \to u\big(\frac{a}{2}\big)$), there exists $C_2 > 0$ independent of $n \geq 1$ such that $|\partial\phi|(u_n(a)) \leq C_2$, $n \geq 1$. Therefore u is a solution to (EVI).

Next we prove that $u(t) \in D(|\partial\phi|)$ for every $t > 0$. Observing that $u_n(a) = S\big(\frac{a}{2}\big)u_n\big(\frac{a}{2}\big)$ we have as above

$$\frac{a}{4}\,e^{-\alpha^- a}|\partial\phi|^2(u_n(a)) \leq \phi\Big(u_n\Big(\frac{a}{2}\Big)\Big) - \phi_{a/2}\Big(u_n\Big(\frac{a}{2}\Big)\Big) \leq \phi_{c(a/2)}(x_n) - \phi_{a/2}\Big(u_n\Big(\frac{a}{2}\Big)\Big), \quad n \geq 1.$$

Hence, since $|\partial\phi|(\cdot)$ is l.s.c., we obtain

$$|\partial\phi|^2(u(a)) \le \frac{4}{a}\, e^{\alpha^- a}\big[\phi_{c(a/2)}(x) - \phi_{a/2}\big(u\big(\tfrac{a}{2}\big)\big)\big] < \infty.$$

Hence $S(a)x \in D(|\partial\phi|)$ for every $x \in \overline{D(\phi)}$ and $a > 0$ such that $1 + \alpha a > 0$ and $1 + \alpha c(\alpha) > 0$. It easily follows that $S(t)x \in D(|\partial\phi|)$ for every $x \in \overline{D(\phi)}$ and $t > 0$.

Next we prove ii). Let $t > 0$ be such that $1 + \alpha c(t) > 0$ and let $x \in \overline{D(\phi)}$. Then $\phi(S(t)x) \le \phi_{c(t)}(x) \le \phi(x)$, the last inequality being a consequence of (4.16). Clearly $\phi(S(nt)x) \le \phi(S(n-1)t)x) \le \phi(x)$ for every $n \ge 1$ and $x \in \overline{D(\phi)}$. Hence $\phi(S(t)x) \le \phi(x)$ for every $x \in \overline{D(\phi)}$ and $t > 0$. Using the semigroup property we obtain $\phi(S(t+h)x) = \phi(S(t)S(h)x) \le \phi(S(h)x)$, $t, h > 0$, which proves ii).

Finally, we prove the inequality in iv). Let $t > 0$ be such that $1 + \alpha t > 0$. There exists $h_0 > 0$ such that $1 + \alpha(t+h) > 0$ for $0 < h \le h_0$. Let $x \in \overline{D(\phi)}$. Since $S(h)x \in D(|\partial\phi|)$ we have by what precedes

$$\frac{t}{2}|\partial\phi|^2(S(t)S(h)x) \le e^{2\alpha^- t}\big(\phi(S(h)x) - \phi_t(S(h)x)\big) \le e^{2\alpha^- t}\big(\phi(x) - \phi_t(S(h)x)\big).$$

Choosing a sequence $h_n \downarrow 0$ we obtain

$$\frac{t}{2}\,|\partial\phi|^2(S(t)x) \le \underline{\lim}\, e^{2\alpha^- t}\big(\phi(x) - \phi_t(S(h_n)x)\big) = e^{2\alpha^- t}(\phi(x) - \phi_t(x)).$$

Step 4. "Proof of v) and viii)".

v) Let $h > 0$. Then $S(h)x \in D(|\partial\phi|)$ by iv). Hence

$$[0,\infty) \ni t \mapsto e^{\alpha t}|\partial\phi|(u(t+h)) = e^{\alpha t}|\partial\phi|(S(t)S(h)x)$$

is nonincreasing by Theorem 4.1 (4.37) and right-continuous since $t \mapsto e^{\alpha t}|\partial\phi|(u(t+h))$ is l.s.c. This completes the proof of v).

viii) Let $0 < a < b$. Set $v(s) := u(s+a)$, $s \ge 0$. Then $v(0) \in D(||\partial\phi|)$ and since $v(s) = S(s)S(a)x = S(s)v(0)$ we have $s \mapsto v(s) \in \mathrm{Lip}([0,T];X)$ for every $T > 0$ by Theorem 4.1 (4.35). Moreover, by (4.63) we have

$$d(v(s_1), v(s_2)) \le |\partial\phi|(v(0))e^{\alpha^- s_2}|s_2 - s_1|, \quad 0 \le s_1 < s_2.$$

Setting $s_1 := s - a$, $s_2 := t - b$ we arrive at viii).

Step 5. "Proof of xi)".

Existence and uniqueness of a minimizer $\overline{x}$. Let $\alpha > 0$. In view of (H_1) we have for every $x, y, z \in D(\phi)$, $h > 0$, $\gamma(0) = y$, $\gamma(1) = z$:

$$\frac{1}{2h}\, d^2\big(x, \gamma\big(\tfrac12\big)\big) + \phi\big(\gamma\big(\tfrac12\big)\big)$$
$$\le \frac12\Big[\frac{1}{2h}\, d^2(x,y) + \phi(y)\Big] + \frac12\Big[\frac{1}{2h}\, d^2(x,z) + \phi(z)\Big] - \frac18\Big(\frac1h + \alpha\Big)d^2(y,z).$$

Letting $h \to \infty$ we obtain

$$\frac18\,\alpha d^2(y,z) \le \frac12\Big[\phi(y) - \phi\big(\gamma\big(\tfrac12\big)\big)\Big] + \frac12\Big[\phi(z) - \phi\big(\gamma\big(\tfrac12\big)\big)\Big]. \tag{4.98}$$

Since $\alpha > 0$, ϕ is bounded below by Lemma 4.1. Set $\rho := \inf \phi$, and let $\{x_n\}_{n\geq 1}$ be a minimizing sequence. We get, by (4.98),

$$d^2(y_m, y_n) \leq \frac{4}{\alpha}\left[(\phi(y_m) - \rho) + (\phi(y_n) - \rho)\right], \quad m, n \geq 1.$$

Hence $\{y_n\}_{n\geq 1}$ is a Cauchy sequence with limit $\overline{x}$. Therefore

$$\rho \leq \phi(\overline{x}) \leq \underline{\lim}\, \phi(y_n) = \rho$$

and $\overline{x}$ is a minimizer of ϕ.

Uniqueness follows from (4.98). Since $\overline{x}$ is also a minimizer of $\Phi(h, \overline{x}; \cdot)$ we have $J_h\overline{x} = \overline{x}$ for each $h > 0$. Hence from (4.42) with $x := \overline{x}$ and (4.16) we obtain

$$\phi(\overline{x}) = \frac{1}{2h}\left[d^2(J_h\overline{x}, z) - d^2(\overline{x}, z)\right] + \phi(\overline{x})$$
$$= \frac{1}{2h}\left[d^2(J_h\overline{x}, z) - d^2(\overline{x}, z)\right] + \phi_h(\overline{x}) \leq \phi(z) - \frac{\alpha}{2}\, d^2(\overline{x}, z)$$

for every $h > 0$ and $z \in D(\phi)$. It follows that $v(t) := \overline{x}$ for $t \geq 0$ is a solution to (EVI) with initial value $\overline{x}$. Now xi) is a consequence of the A priori estimate 2.1.

Step 6. "Proof of iii), vi) and vii)".

Suppose at first that $x \in D(|\partial\phi|)$. Then by (4.39), (A4.1), (4.36), (4.35), (4.37) we obtain

$$\frac{1}{2}\int_0^t |\dot{u}|^2(r)\, dr + \frac{1}{2}\int_0^t |\partial\phi|^2(u(r))\, dr \leq \phi(x) - \phi(u(t))$$
$$\leq \int_0^t |\partial\phi|(u(r))|\dot{u}|(r)\, dr \leq \frac{1}{2}\int_0^t |\dot{u}|^2(r)\, dr + \frac{1}{2}\int_0^t |\partial\phi|^2(u(r))\, dr$$

for $t > 0$. This implies vii) with $s = 0$. Using the semigroup property we obtain vii) for every $0 < s < t$. Using the semigroup property again and $u(s) \in D(|\partial\phi|)$ for any $s > 0$ we establish vii) for any $0 < s < t$ where $x \in \overline{D(\phi)}$. Finally, observing that $\phi(x) = \sup_{s>0} \phi \circ u(s)$ (by ii) and the lower semicontinuity of ϕ), we obtain vii) with $s = 0$. This completes the proof of vii).

Notice that we also proved that

$$\int_s^t \big(|\dot{u}|(r) - |\partial\phi|(u(r))\big)^2\, dr = 0, \quad 0 < s < t.$$

This implies $|\dot{u}|(r) = |\partial\phi|(u(r))$ a.e. in $(0, \infty)$. It follows that

$$\phi \circ u(s) - \phi \circ u(t) = \int_s^t |\partial\phi|^2(u(r))\, dr = \int_s^t |\dot{u}|^2(r)\, dr < \infty, \quad 0 < s < t.$$

Using the right-continuity of $|\partial\phi|(u(\cdot))$ (which follows from v)), we have

$$\frac{d^+}{dt}(\phi \circ u)(t) = -|\partial\phi|^2(u(t)) \quad \text{for every } t > 0.$$

Next we establish the second equality of vi). We have for $t, h > 0$

$$d(u(t+h), u(t)) \leq \int_t^{t+h} |\dot{u}|(r)\, dr = \int_t^{t+h} |\partial\phi|(u(r))\, dr,$$

hence

$$\overline{\lim_{h\downarrow 0}}\, \frac{d(u(t+h),u(t))}{h} \le |\partial\phi|(u(t)) \tag{4.99}$$

by using the right-continuity of $|\partial\phi|(u(\cdot))$. If $|\partial\phi|(u(t)) = 0$, then we have $|\dot{u}_+|(t) = 0 = |\partial\phi|(u(t))$. We suppose now that $|\partial\phi|(u(t)) > 0$. By Proposition 4.2 we have

$$\begin{aligned}|\partial\phi|(u(t)) &\ge \left(\frac{\phi(u(t)) - \phi(u(t+h))}{d(u(t),u(t+h))} + \frac{1}{2}\,\alpha d(u(t),u(t+h))\right)^+ \\ &\ge \frac{\phi(u(t) - \phi(u(t+h))}{h}\,\frac{h}{d(u(t),u(t+h))} + \frac{1}{2}\,\alpha d(u(t),u(t+h))\end{aligned}$$

for $t,h > 0$, $h \le 1$. Hence

$$\frac{d(u(t),u(t+h))}{h}\,|\partial\phi|(u(t)) \ge \frac{\phi(u(t) - \phi(u(t+h))}{h} - \frac{1}{2}\,|\alpha|[u]_{\mathrm{Lip},[t,t+1]}\cdot d(u(t),u(t+h)).$$

It follows that

$$|\partial\phi|(u(t))\,\underline{\lim_{h\downarrow 0}}\,\frac{d(u(t),u(t+h))}{h} \ge -(\phi\circ u)'(t) = |\partial\phi|^2(u(t)),$$

hence

$$\underline{\lim_{h\downarrow 0}}\,\frac{d(u(t),u(t+h))}{h} \ge |\partial\phi|(u(t)),$$

which together with (4.99) completes the proof of vi).

Finally we prove iii). In view of the right-continuity of $t \mapsto e^{-2\alpha^- t}\phi\circ u(t)$ by ii), it suffices to prove that the function is convex on $(0,\infty)$. Let $0 < a < b$, $\rho := \min_{[a,b]} \phi\circ u = \phi\circ u(b)$. We have

$$\frac{d^+}{dt}\, e^{-2\alpha^- t}(\phi(u(t)) - \rho) = -e^{-2\alpha^- t}|\partial\phi|^2(u(t)) - 2\alpha^- e^{-2\alpha^- t}(\phi(u(t)) - \rho)$$

which is nondecreasing. Therefore (by absolute continuity) $t \mapsto e^{-2\alpha^- t}\phi\circ u(t) - \rho e^{-2\alpha^- t}$ is convex as well as $t \mapsto e^{-2\alpha^- t}\phi\circ u(t)$. This completes the proof of Step 6.

Step 7. "Proof of ix)".

Let $y \in D(|\partial\phi|)$, $n \ge 1$ and $t > 0$. We have

$$d\big(S(t)x,(J_{t/n})^n x\big) \le d(S(t)x,S(t)y) + d\big(S(t)y,(J_{t/n})^n y\big) + d\big((J_{t/n})^n y,(J_{t/n})^n x\big).$$

In view of Theorem 4.1 and the quasi-contractivity of $S(\cdot)$, we have

$$\overline{\lim_{n\to\infty}}\, d\big(S(t)x,(J_{t/n})^n x\big) \le e^{-\alpha t}d(x,y) + \overline{\lim_{n\to\infty}}\, d\big((J_{t/n})^n y,(J_{t/n})^n x\big).$$

Then the claim follows from the density of $D(|\partial\phi|)$ in $\overline{D(\phi)}$ and the estimate

$$\overline{\lim_{n\to\infty}}\, d\big((J_{t/n})^n y,(J_{t/n})^n x\big) \le e^{3\alpha^- t}d(x,y).$$

In order to prove this estimate we set $h := t/n$, $x_k := (J_h)^k x$, $x_0 := x$, $y_k := (J_h)^k y$, $y_0 := y$, $1 \le k \le n$. We recall that for $1 \le k \le n$

$$\begin{aligned}&\frac{\alpha}{2}\,d^2(y_k,z) + \frac{1}{2h}\big(d^2(y_k,z) - d^2(y_{k-1},z)\big) \le \phi(z) - \phi(y_k), \quad z \in D(\phi) \\ &\frac{\alpha}{2}\,d^2(x_k,\widehat{z}) + \frac{1}{2h}\big(d^2(x_k,\widehat{z}) - d^2(x_{k-1},\widehat{z})\big) \le \phi(\widehat{z}) - \phi(x_k), \quad \widehat{z} \in D(\phi).\end{aligned} \tag{4.100}$$

Choosing $z := x_k$, $\widehat{z} := y_{k-1}$, adding and discarding the first term in (4.100) when $\alpha \geq 0$, we get

$$\tag{4.101} \frac{1}{2h}\big[d^2(x_k,y_k) - d^2(x_{k-1},y_{k-1})\big] \leq \phi(y_{k-1}) - \phi(y_k) + \frac{\alpha^-}{2}\big[d^2(x_k,y_k) + d^2(x_k,y_{k-1})\big].$$

When $\alpha \geq 0$ we have telescopic sums and we arrive at

$$\frac{1}{2h}\big[d^2(x_n,y_n) - d^2(x_0,y_0)\big] \leq \phi(y_0) - \phi(y_n).$$

Since $\lim_{n\to\infty} \phi(y_n) = \phi(S(t)y)$ by (4.38), we obtain $\overline{\lim}_{n\to\infty} d^2(x_n,y_n) \leq d^2(x_0,y_0)$ which completes the proof of ix) in the case $\alpha \geq 0$.

Next we consider the case $\alpha < 0$. Majorizing in (4.101) the term $d^2(x_k,y_{k-1})$ by $2d^2(x_k,y_k) + 2d^2(y_k,y_{k-1})$ we obtain

$$(1-3|\alpha|h)d^2(x_k,y_k) \leq d^2(x_{k-1},y_{k-1}) + 2h\big[\phi(y_{k-1}) - \phi(y_k)\big] + 2|\alpha|hd^2(y_k,y_{k-1}).$$

Setting $z := y_{k-1}$ in (4.100) we get

$$d^2(y_k,y_{k-1}) \leq 2h(1-|\alpha|h)\big[\phi(y_{k-1}) - \phi(y_k)\big].$$

Hence, when $3|\alpha|h < 1$, we obtain

$$(1-3|\alpha|h)d^2(x_k,y_k) \leq d^2(x_{k-1},y_{k-1}) + 2h\big[1+2|\alpha|h\big].$$

Multiplying by $(1-3|\alpha|h)^{k-1}$, we have

$$(1-3|\alpha|h)^k d^2(x_k,y_k) \leq (1-3|\alpha|h)^{k-1} d^2(x_{k-1},y_{k-1}) + 2h(1+2|\alpha|h)\big(\phi(y_{k-1}) - \phi(y_k)\big).$$

By adding we arrive at

$$(1-3|\alpha|h)^n d^2(x_n,y_n) \leq d^2(x,y) + 2h(1+2|\alpha|h)\big(\phi(y) - \phi(y_n)\big).$$

As in the case $\alpha \geq 0$ we obtain

$$\overline{\lim_{n\to\infty}}\,(1-3|\alpha|h)^n d^2(x_n,y_n) \leq d^2(x_0,y_0),$$

hence

$$\overline{\lim_{n\to\infty}}\, d^2(x_n,y_n) \leq e^{3|\alpha|t} d^2(x_0,y_0).$$

This completes the proof of the case $\alpha < 0$.

Step 8. "Proof of x)".

Without loss of generality we may assume $\alpha \leq 0$. By lower semicontinuity of ϕ it is sufficient to show: $\overline{\lim}_{n\to\infty} \phi(J^n_{t/n}x) \leq \phi(u(t))$, $t > 0$. Moreover, by Theorem 4.2 iii), $e^{-2\alpha^- t}\phi \circ u(t)$ is convex hence continuous on $(0,\infty)$. Consequently it is sufficient to prove that

$$\tag{4.102} \overline{\lim_{n\to\infty}}\, \phi(J^n_{t/n}x) \leq \phi(u(t-\varepsilon)) \quad \text{for all} \quad \varepsilon \in (0,t).$$

Fix $\varepsilon \in (0,t)$. Assume that (4.102) does not hold. Let $\delta > 0$ be such that

$$\phi(J^n_{t/n}x) > \phi(u(t-\varepsilon)) + \delta \tag{4.103}$$

for infinitely many n. Let n be such that (4.103) holds and $1 + \frac{\alpha t}{n} > 0$. Set $x^n_k := J^k_{t/n}x$, $1 \le k \le n$. Then we have by (4.16)

$$\phi(x^n_k) > \phi(u(t-\varepsilon)) + \delta, \quad 1 \le k \le n. \tag{4.104}$$

Then for all $z \in D(\phi)$, by (4.42), (4.16),

$$d^2(x^n_k, z) - d^2(x^n_{k-1}, z) + \alpha\,\frac{t}{n}\,d^2(x^n_k, z) \le 2\,\frac{t}{n}\big[\phi(z) - \phi(x^n_k)\big].$$

Hence

$$\Big(1 + \alpha\,\frac{t}{n}\Big)^k d^2(x^n_k, z) - \Big(1 + \alpha\,\frac{t}{n}\Big)^{k-1} d^2(x^n_{k-1}, z) \le 2\,\frac{t}{n}\Big(1 + \alpha\,\frac{t}{n}\Big)^{k-1}\big[\phi(z) - \phi(x^n_k)\big].$$

If $0 < m < n$ we have

$$\begin{aligned}\Big(1 + \alpha\,\frac{t}{n}\Big)^n d^2(x^n_n, z) - \Big(1 + \alpha\,\frac{t}{n}\Big)^m d^2(x^n_m, z)& \\ \le 2\,\frac{t}{n}\sum_{k=m+1}^{n}\Big(1 + \alpha\,\frac{t}{n}\Big)^{k-1}\big[\phi(z) - \phi(x^n_k)\big]&.\end{aligned} \tag{4.105}$$

Now we choose m depending on n such that $\frac{t\cdot m}{n} \to t - \varepsilon$ as $n \to \infty$ (i.e., $m \sim n\frac{t-\varepsilon}{t}$). Then $x^n_m \overset{n\to\infty}{\longrightarrow} u(t-\varepsilon)$ by Theorem 4.2 ix). Let $z := u(t-\varepsilon)$. Choose n large enough so that $d^2(x^n_m, z) < \varepsilon\delta e^{\alpha\varepsilon}$. We obtain by (4.104) and (4.105)

$$\begin{aligned}0 \le d^2(x^n_n, z)\Big(1 + \alpha\,\frac{t}{n}\Big)^n &\le \Big(1 + \alpha\,\frac{t}{n}\Big)^m d^2(x^n_m, z) - 2\,\frac{t}{n}\sum_{k=m+1}^{n}\Big(1 + \alpha\,\frac{t}{n}\Big)^{k-1}\delta \\ &\le \Big(1 + \alpha\,\frac{t}{n}\Big)^m \varepsilon\delta e^{\alpha\varepsilon} - 2\,\frac{t}{n}\sum_{k=m+1}^{n}\Big(1 + \alpha\,\frac{t}{n}\Big)^{k-1}\delta.\end{aligned}$$

At the limit for $n \to \infty$ we have

$$0 \le e^{\alpha(t-\varepsilon)}\varepsilon\delta e^{\alpha\varepsilon} - 2\int_{t-\varepsilon}^{t} e^{\alpha s}\,ds\,\delta \le -\delta\varepsilon e^{\alpha t} < 0. \qquad \square$$

5 Gradient flows in probability spaces

The aim of this section is to present some applications of the theory developed in Sections 2 and 4. The metric space (X,d) will be a metric space of probability measures on $\mathbb{R}^N$ equipped with the Kantorovich–Rubinstein–Wasserstein distance.

In Section 5.1 we introduce some notation concerning Borel probability measures on metric spaces, in Section 5.2 the space $\big(\mathcal{P}_2(\mathbb{R}^N), W_2(\cdot,\cdot)\big)$ is defined and in Section 5.3 a basic convexity property of the distance function W_2 is established. In Sections 5.4 and 5.5 we consider examples of functionals ϕ on $\mathcal{P}_2(\mathbb{R}^N)$ which satisfy the assumptions of Theorem 4.1.

5.1 Preliminaries

Let (Y, d) be a metric space. We denote by $\mathcal{B}(Y)$ the Borel σ-algebra generated by the open sets of Y and by $\mathcal{P}(Y)$ the collection of all Borel probability measures on Y. Let (Y, d_Y) and (Z, d_Z) be metric spaces and $f : Y \to Z$ be a Borel map (i.e. $f^{-1}(A) \in \mathcal{B}(Y)$ for all $A \in \mathcal{B}(Z)$) and let $\mu \in \mathcal{P}(Y)$. We denote by $f_\#\mu$ the image measure (of μ under f) defined by

$$(f_\#\mu)(A) := \mu(f^{-1}(A)), \qquad A \in \mathcal{B}(Z).$$

We have $f_\#\mu \in \mathcal{P}(Z)$. Finally, if $(Y_i, d_i)_{i=1}^M$ are metric spaces, we introduce the metric

$$d\big((y_1, \ldots, y_M), (\widehat{y}_1, \ldots, \widehat{y}_M)\big) := \Big(\sum_{k=1}^{M} d_k^2(y_k, \widehat{y}_k)\Big)^{1/2},$$

with $y_i, \widehat{y}_i \in Y_i$, $i = 1, \ldots, M$, on the product space

$$\prod_{k=1}^{M} Y_k = Y_1 \times \ldots \times Y_M.$$

We denote by π^i, $i = 1, \ldots, M$, the projection maps $\pi^i\big((y_1, \ldots, y_M)\big) := y_i$ and by $\pi^{i,j}$, $i, j = 1, \ldots, M$, $i \neq j$, the maps

$$\pi^{i,j}(y_1, \ldots, y_M) = (y_i, y_j).$$

We recall that the maps π^i, $\pi^{i,j}$ are continuous, hence Borel.

If $\mu^i \in \mathcal{P}(X_i)$, $1 \le i \le M$, we set

$$\Gamma(\mu^1, \ldots, \mu^M) := \{\mu \in \mathcal{P}(Y_1 \times \ldots \times Y_M) : \pi^i_\#\mu = \mu^i,\ 1 \le i \le M\}. \tag{5.1}$$

Notice that $\Gamma(\mu^1, \ldots, \mu^M) \ni \mu^1 \otimes \ldots \otimes \mu^M$, the product measure of $\mu^1, \ldots, \mu^M$.

We conclude this subsection by recalling a proposition which plays an important role in the sequel.

Proposition 5.1. *Let (X_i, d_i), $i = 1, 2, 3$, be complete separable metric spaces and let $\mu^{1,2} \in \mathcal{P}(X_1 \times X_2)$, $\mu^{1,3} \in \mathcal{P}(X_1 \times X_3)$ be such that $\pi^1_\#\mu^{1,2} = \pi^1_\#\mu^{1,3}$. Then there exists $\mu \in \mathcal{P}(X_1 \times X_2 \times X_3)$ such that $\pi^{1,2}_\#\mu = \mu^{1,2}$ and $\pi^{1,3}_\#\mu = \mu^{1,3}$.*

Similarly, if $\mu^{1,2} \in \mathcal{P}(X_1 \times X_2)$, $\mu^{2,3} \in \mathcal{P}(X_2 \times X_3)$ be such that $\pi^2_\#\mu^{1,2} = \pi^1_\#\mu^{2,3}$, then there exists $\mu \in \mathcal{P}(X_1 \times X_2 \times X_3)$ such that $\pi^{1,2}_\#\mu = \mu^{1,2}$ and $\pi^{2,3}_\#\mu = \mu^{2,3}$.

5.2 The space $\big(\mathcal{P}_2(\mathbb{R}^N), W_2(\cdot, \cdot)\big)$

From now on we shall consider Borel probability measures on $\mathbb{R}^N$ equipped with the euclidean metric. We shall identify $\mathbb{R}^N \times \mathbb{R}^M$ with $\mathbb{R}^{N+M}$. We recall that on $\mathcal{P}(\mathbb{R}^N)$ the "narrow" convergence of measures can be metrized. We shall use the β-distance ("dual bounded Lipschitz" distance). For $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^N)$

$$\beta(\mu^1, \mu^2) := \sup\Big\{\Big|\int_{\mathbb{R}^N} f\, d\mu^1 - \int_{\mathbb{R}^N} f\, d\mu^2\Big| : f \in BL(\mathbb{R}^N),\ \sup_{x \in \mathbb{R}^N} |f(x)| + [f]_{\mathrm{Lip}} \le 1\Big\} \tag{5.2}$$

where $BL(\mathbb{R}^N)$ is the set of bounded and Lipschitz continuous functions $f : \mathbb{R}^N \to \mathbb{R}$.

Moreover we have: $\mu^n, \mu \in \mathcal{P}(\mathbb{R}^N)$, $n \geq 1$, satisfy

$$(5.3)\qquad \beta(\mu^n, \mu) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{iff} \quad \int_{\mathbb{R}^N} f\, d\mu^n \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} f\, d\mu$$

for every $f : \mathbb{R}^N \to \mathbb{R}$ bounded and continuous.

We recall that the metric space $(\mathcal{P}(\mathbb{R}^N), \beta)$ is complete and separable. We now introduce the space

$$(5.4)\qquad \mathcal{P}_2(\mathbb{R}^N) := \Big\{\mu \in \mathcal{P}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^2\, d\mu(x) < \infty\Big\}$$

and the Wasserstein metric defined by

$$(5.5)\qquad W_2(\mu^1, \mu^2) := \Big[\inf\Big\{\int_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^2\, d\mu(x,y) : \mu \in \Gamma(\mu^1,\mu^2)\Big\}\Big]^{1/2},$$

where $\mu^1, \mu^2 \in \mathcal{P}_2(\mathbb{R}^N)$. We have $\beta(\mu^1, \mu^2) \leq W_2(\mu^1, \mu^2)$, see Exercise 5.1.

The infimum in (5.5) is actually a *minimum* and we use the notation: for $\mu^1, \mu^2 \in \mathcal{P}_2(\mathbb{R}^N)$,

$$(5.6)\qquad \Gamma_0(\mu^1, \mu^2) := \Big\{\mu \in \Gamma(\mu^1,\mu^2) : W_2^2(\mu^1,\mu^2) = \int_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^2\, d\mu(x,y)\Big\}.$$

So we have

$$(5.7)\qquad \Gamma_0(\mu^1,\mu^2) \neq \varnothing \quad \text{for every} \quad \mu^1,\mu^2 \in \mathcal{P}(\mathbb{R}^N).$$

We also have $\mu^n, \mu \in \mathcal{P}_2(\mathbb{R}^N)$, $n \geq 1$, satisfy

$$(5.8)\qquad W_2(\mu^n, \mu) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{iff} \quad \int_{\mathbb{R}^N} f\, d\mu^n \to \int_{\mathbb{R}^N} f\, d\mu$$

for every $f : \mathbb{R}^N \to \mathbb{R}$ continuous for which there exist $C_1, C_2 > 0$ (depending on f) such that

$$|f(x)| \leq C_1 + C_2|x|^2 \quad \text{for all} \quad x \in \mathbb{R}^N$$

(function with quadratic growth).

The space $\big(\mathcal{P}_2(\mathbb{R}^N), W_2(\cdot,\cdot)\big)$ is a *complete* metric space. The *support* $\operatorname{supp}\mu \subset \mathbb{R}^N$ of $\mu \in \mathcal{P}(\mathbb{R}^N)$ is the *closed* set defined by

$$(5.9)\qquad \operatorname{supp}\mu := \{x \in \mathbb{R}^N : \mu(U) > 0 \text{ for each open set } U \text{ of } \mathbb{R}^N \text{ containing } x\}.$$

For $\mu \in \mathcal{P}(\mathbb{R}^N)$, $\operatorname{supp}\mu = \{x\}$ for some $x \in \mathbb{R}^N$ iff $\mu = \delta_x$, the point (Dirac) measure at x. Notice that $\Gamma(\delta_x, \mu) = \{\delta_x \otimes \mu\}$ for any $\mu \in \mathcal{P}(\mathbb{R}^N)$, hence

$$(5.10)\qquad W_2^2(\delta_x, \mu) = \int_{\mathbb{R}^N} |x-y|^2\, d\mu(y), \quad \mu \in \mathcal{P}_2(\mathbb{R}^N).$$

In particular,

$$W_2^2(\delta_x, \delta_y) = d^2(x,y), \quad x, y \in \mathbb{R}^N.$$

Moreover, for $\mu \in \mathcal{P}(\mathbb{R}^N)$, $\operatorname{supp}\mu$ is a finite set $\{x_1, \dots, x_n\} \subset \mathbb{R}^N$ iff there exist $\alpha_1, \dots, \alpha_n \geq 0$ with $\sum\limits_{i=1}^n \alpha_i = 1$ such that $\mu = \sum\limits_{i=1}^n \alpha_i \delta_{x_i}$. The collection of finitely supported measures is dense in $\big(\mathcal{P}_2(\mathbb{R}^N), W_2(\cdot,\cdot)\big)$. It follows that $\big(\mathcal{P}_2(\mathbb{R}^N), W_2(\cdot,\cdot)\big)$ is *separable.*

5.3 "Convexity" of the function $W_2(\mu, \cdot)$

We want to apply Theorem 4.1 when $X = \mathcal{P}_2(\mathbb{R}^N)$ and $d = W_2$. We begin by considering a constant functional ϕ, i.e. $\phi(x) = \phi_0 \in \mathbb{R}$ for every $x \in X$. For any $u_0 \in X$ the constant function $u(t) = u_0$, $t \geq 0$, is a solution to (EVI) (by direct verification) for every $\alpha \leq 0$, in particular for $\alpha = 0$. Moreover, in view of the A priori estimate 2.1 this is the only solution to (EVI) with u_0 as initial value. If we want to apply Theorem 4.1 in order to obtain existence we need to verify condition (H_1) for $\alpha = 0$. This obviously implies the following "convexity" condition on $W^2(\cdot,\cdot)$:

(5.11) For every $\mu^1, \mu^2, \mu^3 \in \mathcal{P}_2(\mathbb{R}^N)$ there exists a map $\gamma : [0,1] \to \mathcal{P}_2(\mathbb{R}^N)$ such that $\gamma(0) = \mu^2,\ \gamma(1) = \mu^3,$ satisfying

$$W_2^2(\mu^1, \gamma(t)) \leq (1-t)W_2^2(\mu^1, \mu^2) + tW_2^2(\mu^1, \mu^3) - t(1-t)W_2^2(\mu^2, \mu^3)$$

for every $t \in [0,1]$.

Assuming that (5.11) holds we could consider functionals $\phi : \mathcal{P}_2(\mathbb{R}^N) \to (-\infty, +\infty]$ which satisfy

$$\phi(\gamma(t)) \leq (1-t)\phi(\mu^2) + t\phi(\mu^3) - \frac{\alpha}{2} t(1-t)W_2^2(\mu^2, \mu^3), \quad t \in [0,1],$$

on the same curve γ as in (5.11). For such ϕ, (H_1) would be satisfied.

See Exercise 5.2.

Proceeding as in the Hilbert space case we could think of the curve γ defined by

$$\gamma(t) := (1-t)\mu^2 + t\mu^3, \quad t \in [0,1],\ \mu^2 \neq \mu^3.$$

However, choosing $\mu^1 = \delta_0$ we get in view of (5.10)

$$\begin{aligned} W_2^2(\delta_0, (1-t)\mu^2 + t\mu^3) &= \int_{\mathbb{R}^N} |y|^2 \, d((1-t)\mu^2 + t\mu^3)(y) \\ &= (1-t)W_2^2(\delta_0, \mu^2) + tW_2^2(\delta_0, \mu^3), \quad t \in [0,1]. \end{aligned}$$

Choosing $t = \frac{1}{2}$ in (5.11) we obtain $W_2^2(\mu^2, \mu^3) = 0$ which is impossible. Therefore another choice of curve is needed.

In case $N = 1$ condition (5.11) is satisfied even with equality sign for the curve $\gamma : [0,1] \to \mathcal{P}_2(\mathbb{R}^N)$ defined as follows: given any $\mu \in \Gamma_0(\mu^2, \mu^3)$,

$$\gamma(t) := ((1-t)\pi^1 + t\pi^2)_{\#}\mu, \quad t \in [0,1].$$

See [AGS], p. 204.

However, if $N \geq 2$ the opposite inequality holds, i.e.

$$W_2^2(\mu^1, \gamma(t)) \geq (1-t)W_2^2(\mu^1, \mu^2) + tW_2^2(\mu^1, \mu^3) - t(1-t)W_2^2(\mu^2, \mu^3), \quad t \in [0,1].$$

See [AGS], p. 162.

Fortunately there is another curve for which (5.11) holds. Let $\mu^1, \mu^2, \mu^3 \in \mathcal{P}(\mathbb{R}^N)$. In view of (5.7) there exist $\mu^{1,i} \in \Gamma_0(\mu^1, \mu^i)$, $i = 2, 3$. By Proposition 5.1 there exists

(5.12) $$\mu \in \Gamma(\mu^1, \mu^2, \mu^3) \quad \text{satisfying} \quad \pi^{1,i}_{\#}\mu = \mu^{1,i}, \quad i = 2, 3.$$

Set

$$(5.13) \qquad \gamma(t) := ((1-t)\pi^2 + t\pi^3)_{\#}\mu, \quad t \in [0,1],$$

where μ satisfies (5.12). We have for $t \in [0,1]$: $\gamma(t) \in \mathcal{P}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |x|^2 \, d\gamma(x) = \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} \big|(1-t)y + tz\big|^2 \, d\mu(x,y,z)$$
$$\leq \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} \big(|y|^2 + |z|^2\big) \, d\mu(x,y,z) = \int_{\mathbb{R}^N} |y|^2 \, d\mu^2(y) + \int_{\mathbb{R}^N} |z|^2 \, d\mu^3(z) < \infty,$$

hence $\gamma(t) \in \mathcal{P}_2(\mathbb{R}^N)$.

Furthermore we associate with μ satisfying (5.12):

$$(5.14) \qquad W_\mu^2(\mu^2,\mu^3) := \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} |y-z|^2 \, d\mu(x,y,z)$$

which is (proceeding as above) finite.

Proposition 5.2. *Let $\mu^1,\mu^2,\mu^3 \in \mathcal{P}_2(\mathbb{R}^N)$. Let $\mu \in \Gamma(\mu^1,\mu^2,\mu^3)$ satisfy* (5.12) *and let $\gamma : [0,1] \to \mathcal{P}_2(\mathbb{R}^N)$ be as in* (5.13). *Then we have*

$$(5.15) \quad W_2^2(\mu^1,\gamma(t)) \leq (1-t)W_2^2(\mu^1,\mu^2) + tW_2^2(\mu^1,\mu^3) - t(1-t)W_\mu^2(\mu^2,\mu^3), \quad t\in[0,1],$$

where $W_\mu^2(\mu^2,\mu^3)$ is defined as in (5.14).

Furthermore we have

$$(5.16) \qquad W_\mu^2(\mu^2,\mu^3) \geq W_2^2(\mu^2,\mu^3),$$

hence γ satisfies (5.11).

Proof. Set $\delta_t := ((1-t)\pi^{1,2} + t\pi^{1,3})_{\#}\mu$, $t\in[0,1]$. Then $\delta_t \in \mathcal{P}(\mathbb{R}^N\times\mathbb{R}^N)$ for $t\in[0,1]$. Moreover, since

$$((1-t)\pi^{1,2} + t\pi^{1,3})(x_1,x_2,x_3) = (x_1,(1-t)x_2 + tx_3),$$

we have $\pi^1_{\#}\delta_t = \mu^1$ and $\pi^2_{\#}\delta_t = \gamma(t)$, hence $\delta_t \in \Gamma(\mu^1,\gamma(t))$. It follows that

$$\begin{aligned} W_2^2(\mu^1,\gamma(t)) &\leq \int_{\mathbb{R}^N\times\mathbb{R}^N} |u-v|^2 \, d\delta_t(u,v) \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} \big|x_1 - (1-t)x_2 - tx_3\big|^2 \, d\mu(x_1,x_2,x_3) \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} \big|(1-t)(x_1-x_2) + t(x_1-x_3)\big|^2 \, d\mu(x_1,x_2,x_3) \\ &= (1-t)W_2^2(\mu^1,\mu^2) + tW_2^2(\mu^1,\mu^3) - t(1-t)W_\mu^2(\mu^2,\mu^3) \end{aligned}$$

by (5.12) and the Hilbertian identity (4.1).

Moreover

$$W_\mu^2(\mu^2,\mu^3) = \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} |y-z|^2 \, d\mu(x,y,z)$$
$$= \int_{\mathbb{R}^N\times\mathbb{R}^N} |y-z|^2 \, d(\pi^{2,3}_{\#}\mu)(y,z) \geq W_2^2(\mu^2,\mu^3)$$

since $\pi^{2,3}_{\#}\mu \in \Gamma(\mu^2,\mu^3)$. □

The previous result motivates the following definitions.

Definition 5.1. A curve $\gamma_\mu : [0,1] \to \mathcal{P}_2(\mathbb{R}^N)$ as in (5.13) where $\mu \in \Gamma(\mu^1, \mu^2, \mu^3)$ satisfies (5.12) is called a *generalized geodesic* between μ^2 and μ^3 with base μ^1.

Definition 5.2. A proper functional $\phi : \mathcal{P}_2(\mathbb{R}^N) \to (-\infty, +\infty]$ is called *λ-convex along generalized geodesics* if there exists $\lambda \in \mathbb{R}$ such that for every $\mu^1, \mu^2, \mu^3 \in D(\phi)$ there exists a generalized geodesic γ_μ for which the following holds:

$$\phi(\gamma_\mu(t)) \le (1-t)\phi(\mu^2) + t\phi(\mu^3) - \frac{\lambda}{2} t(1-t) W_\mu^2(\mu^2, \mu^3) \quad \text{for every } t \in [0,1], \tag{5.17}$$

where W_μ^2 is defined in (5.14).

Corollary 5.1. *Let $\phi : \mathcal{P}_2(\mathbb{R}^N) \to (-\infty, +\infty]$ be proper and λ-convex along generalized geodesics. Then the map*

$$\Phi(h, \mu^1; \mu) := \begin{cases} \frac{1}{2h} W_2^2(\mu^1, \mu) + \phi(\mu), & \mu \in D(\phi), \\ +\infty & \textit{otherwise}, \end{cases}$$

$h > 0$, $\mu^1 \in \mathcal{P}_2(\mathbb{R}^N)$, satisfies assumption (H_1) with $\alpha = \lambda$.

5.4 The "potential energy" functional

Let $V : \mathbb{R}^N \to (-\infty, +\infty]$ be proper, l.s.c. and λ-convex (i.e., $V - \lambda e$ is convex) for some $\lambda \in \mathbb{R}$. In view of Theorem 4.1 with $X = \mathbb{R}^N$ equipped with the euclidean metric we can associate with V a C_0-semigroup $\{T(t)\}_{t\ge 0}$, $T(t) : \overline{D(V)} \to \overline{D(V)}$ satisfying $[T(t)]_{\mathrm{Lip}} \le e^{-\lambda t}$, $t \ge 0$, such that the function

$$t \mapsto u(t) := T(t)x, \quad x \in \overline{D(V)},\ t \ge 0, \tag{5.18}$$

is the unique solution to (EVI) with initial value x.

Given $\mu_0 \in \mathcal{P}(\overline{D(V)})$ we can define

$$S(t)\mu_0 := T(t)_{\#}\mu_0, \quad t \ge 0. \tag{5.19}$$

Clearly $S(t)$ maps $\mathcal{P}(\overline{D(V)})$ into itself and satisfies the semigroup property. Indeed, $S(0)\mu_0 = I|_{\overline{D(V)}\#}\mu_0 = \mu_0$ and for $t, s \ge 0$

$$\begin{aligned} S(t+s)\mu_0 = T(t+s)_{\#}\mu_0 &= (T(t)T(s))_{\#}\mu_0 \\ &= T(t)_{\#}(T(s)_{\#}\mu_0) = S(t)(S(s)\mu_0) = (S(t)S(s))\mu_0 \end{aligned}$$

for every $\mu_0 \in \mathcal{P}(\overline{D(V)})$.

Moreover, if $\mu_0 \in \mathcal{P}(\overline{D(V)})$, $f \in BC(\overline{D(V)})$, $t_n, t \ge 0$ are such that $t_n \to t$, then

$$\begin{aligned} \int_{\overline{D(V)}} f(x)\, d(S(t_n)\mu_0) &= \int_{\overline{D(V)}} f(T(t_n)x)\, d\mu_0 \\ &\to \int_{\overline{D(V)}} f(T(t)x)\, d\mu_0 = \int_{\overline{D(V)}} f(x)\, d(S(t)\mu_0) \end{aligned}$$

where we used the Lebesgue dominated convergence theorem. It follows that $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup on $(\mathcal{P}(\overline{D(V)}), \beta)$ where β is the metric defined in (5.2) and $\mathbb{R}^N$ is replaced by $\overline{D(V)}$. We can also identify $\mathcal{P}(\overline{D(V)})$ with $\{\mu \in \mathcal{P}(\mathbb{R}^N) : \operatorname{supp}\mu \subseteq \overline{D(V)}\}$ by setting

$$\mu(\mathbb{R}^N \setminus \overline{D(V)}) = 0 \quad \text{if} \quad \mu \in \mathcal{P}(\overline{D(V)}). \tag{5.20}$$

In this way we can view $\mathcal{P}(\overline{D(V)})$ as a subset of $\mathcal{P}(\mathbb{R}^N)$.

The fact that $\mathcal{P}(\overline{D(V)})$ is a *closed* subset of $(\mathcal{P}(\mathbb{R}^N), \beta)$ is a consequence of the following

Lemma 5.1 (Prop. 5.1.8 [AGS]). *Let (Y, d) be a separable metric space. Let $\mu^n, \mu \in \mathcal{P}(Y)$, $n \geq 1$, be such that $\beta(\mu^n, \mu) \to 0$ as $n \to \infty$. Then for every $x \in \operatorname{supp}\mu$ there exists a subsequence $(n_k)_{k=1}^{\infty}$ such that*

$$x_{n_k} \in \operatorname{supp}\mu_{n_k},\ k \geq 1 \quad \textit{and} \quad d(x_{n_k}, x) \to 0 \ \textit{as}\ k \to \infty. \tag{5.21}$$

Similarly we can define

$$\mathcal{P}_2(\overline{D(V)}) := \{\mu \in \mathcal{P}_2(\mathbb{R}^N) : \operatorname{supp}\mu \subseteq \overline{D(V)}\}. \tag{5.22}$$

Since $\beta(\mu^1, \mu^2) \leq W_2(\mu^1, \mu^2)$ for every $\mu^1, \mu^2 \in \mathcal{P}_2(\mathbb{R}^N)$, $\mathcal{P}_2(\overline{D(V)})$ is a closed subset of $\big(\mathcal{P}_2(\mathbb{R}^N), W_2(\cdot,\cdot)\big)$. Now we claim that $\mathcal{P}_2(\overline{D(V)})$ is invariant under the semigroup $S(t)$, i.e.,

$$S(t)\mathcal{P}_2(\overline{D(V)}) \subseteq \mathcal{P}_2(\overline{D(V)}), \quad t \geq 0. \tag{5.23}$$

Indeed, for $t \geq 0$, $\overline{x} \in \overline{D(V)}$ and $\mu_0 \in \mathcal{P}_2(\overline{D(V)})$ we have

$$\begin{aligned}\int_{\overline{D(V)}} d^2(T(t)\overline{x}, y)\, d(S(t)\mu_0)(y) &= \int_{\overline{D(V)}} d^2(T(t)\overline{x}, T(t)y)\, d\mu_0(y) \\ &\leq e^{-2\lambda t} \int_{\overline{D(V)}} d^2(\overline{x}, y)\, d\mu_0(y) < \infty.\end{aligned}$$

Moreover, we have for $\mu^1, \mu^2 \in \mathcal{P}_2(\overline{D(V)})$:

$$W_2(S(t)\mu^1, S(t)\mu^2) \leq e^{-\lambda t} W_2(\mu^1, \mu^2), \quad t \geq 0. \tag{5.24}$$

Indeed, if $\mu \in \Gamma_0(\mu^1, \mu^2)$ we get

$$\begin{aligned}W_2^2(S(t)\mu^1, S(t)\mu^2) &\leq \int_{\overline{D(V)}\times\overline{D(V)}} |x-y|^2\, d\big((S(t)\times S(t))_{\#}\mu\big)(x,y) \\ &= \int_{\overline{D(V)}\times\overline{D(V)}} |T(t)x - T(t)y|^2\, d\mu(x,y) \\ &\leq e^{2\lambda t} \int_{\overline{D(V)}\times\overline{D(V)}} |x-y|^2\, d\mu(x,y) = e^{-2\lambda t} W_2^2(\mu^1, \mu^2).\end{aligned}$$

Hence the semigroup $\{S(t)\}_{t\geq 0}$ is a λ-contractive semigroup on $\mathcal{P}_2(\overline{D(V)})$. Next we show that it is also a C_0-semigroup with respect to the metric $W_2(\cdot,\cdot)$. Let $f \in C(\mathbb{R}^n;\mathbb{R})$

be such that there exist $C_1, C_2 \geq 0$ such that $|f(x)| \leq C_1 + C_2|x|^2$, $x \in \mathbb{R}^N$. Let $\mu_0 \in \mathcal{P}_2(\overline{D(V)})$, $t_n, t \geq 0$ be such that $t_n \to t$. We have to show

$$\lim_{n\to\infty} \int_{\overline{D(V)}} f(x)\, d(S(t_n)\mu_0)(x) = \int_{\overline{D(V)}} f(x)\, d(S(t)\mu_0)(x),$$

equivalently

$$\lim_{n\to\infty} \int_{\overline{D(V)}} f(T(t_n)x)\, d\mu_0(x) = \int_{\overline{D(V)}} f(T(t)x)\, d\mu_0(x). \tag{5.25}$$

Since $f(T(t_n)x) \to f(T(t)x)$ as $n \to \infty$ and

$$\begin{aligned}\big|f(T(t_n)x)\big| &\leq C_1 + C_2|T(t_n)x|^2 \leq C_1 + 2C_2\big|T(t_n)x - T(t_n)\overline{x}\big|^2 + 2C_2\big|T(t_n)\overline{x}\big|^2 \\ &\leq C_1 + 2C_2e^{-2\lambda t_n}|x-\overline{x}|^2 + 2C_2|T(t_n)\overline{x}|^2 \leq C_3 + C_4|x-\overline{x}|^2\end{aligned}$$

for some $C_3, C_4 \geq 0$ and $\overline{x} \in \overline{D(V)}$, (5.25) holds as a consequence of $\mu_0 \in \mathcal{P}_2(\overline{D(V)})$ and the Lebesgue dominated convergence theorem.

It is remarkable that such a contractive C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{P}_2(\overline{D(V)})$ is the semigroup associated with a proper, l.s.c. functional $\phi_V : \mathcal{P}_2(\mathbb{R}^N) \to (-\infty, +\infty]$ satisfying assumption (H_1) with $\alpha = \lambda$ and assumption (H_2).

This functional is the so-called *potential energy functional* defined by

$$\phi_V(\mu) := \int_{\mathbb{R}^N} V(x)\, d\mu(x), \quad \mu \in \mathcal{P}_2(\mathbb{R}^N). \tag{5.26}$$

The right-hand side of (5.26) is well-defined (possibly $+\infty$) since V^- the negative part of V ($V = V^+ - V^-$) has "quadratic growth" as a consequence of Lemma 3.1 and λ-convexity, hence

$$\int_{\overline{D(V)}} V^-(x)\, d\mu(x) < +\infty \quad \text{for} \quad \mu \in \mathcal{P}_2(\mathbb{R}^N).$$

Theorem 5.1. *Let $V : \mathbb{R}^N \to (-\infty, +\infty]$ be proper, l.s.c. and λ-convex for some $\lambda \in \mathbb{R}$ and let $\phi_V : \mathcal{P}_2(\mathbb{R}^N) \to (-\infty, +\infty]$ be the functional defined by* (5.26)*. Then ϕ_V is proper, l.s.c. and satisfies assumption (H_1) with $\alpha = \lambda$ and assumption (H_2). Moreover*

$$\overline{D(\phi_V)} = \{\mu \in \mathcal{P}_2(\mathbb{R}^N) : \operatorname{supp}\mu \subseteq \overline{D(V)}\}$$

and the semigroup associated with ϕ_V by Theorem 4.1 *is equal to the semigroup $\{S(t)\}_{t\geq 0}$ defined by* (5.19) *with $\mu_0 \in \mathcal{P}_2(\mathbb{R}^N)$ with* $\operatorname{supp}\mu_0 \subseteq \overline{D(V)}$.

Proof. 1) ϕ_V is *proper*. Since V is proper there exists $x_0 \in D(V)$ hence $\phi_V(\delta_{x_0}) = V(x_0) < \infty$ and $\delta_{x_0} \in D(\phi_V)$.

2) ϕ_V is *l.s.c.* Let $\mu^n, \mu \in \mathcal{P}(\mathbb{R}^N)$ be such that $W_2(\mu^n, \mu) \to 0$ as $n \to \infty$. For $h > 0$ such that $\frac{1}{h} + \lambda > 0$ we denote by $V_h \in C^{1,1}(\mathbb{R}^N; \mathbb{R})$ the Yosida–Moreau approximation of V as in Section 3. We have:

$$\exists k_0 > 0 \quad \text{such that } \forall k \geq k_0 :$$

$$V_{1/k}(x) \leq V_{1/(k+1)}(x),\ x \in \mathbb{R}^N \quad \text{and} \quad \sup_{k\geq k_0} V_{1/k}(x) = V(x).$$

Since $V_{1/k}$ has quadratic growth we have for every $k \geq k_0$

$$\int_{\mathbb{R}^N} V_{1/k}\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}^N} V_{1/k}\, d\mu^n \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^N} V_{1/k}\, d\mu^n \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^N} V\, d\mu^n.$$

By taking the supremum over k we obtain by the monotone convergence theorem

$$\int_{\mathbb{R}^N} V\, d\mu \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^N} V\, d\mu^n.$$

3) ϕ_V is *λ-convex along generalized geodesics*. Let $\mu^1, \mu^2, \mu^3 \in D(\phi_V)$, $\mu \in \Gamma(\mu^1, \mu^2, \mu^3)$ satisfy (5.12), γ_μ be as in (5.13) and $W_\mu^2(\cdot,\cdot)$ as in (5.16). In view of the λ-convexity of V, we have for $t \in [0,1]$:

$$\begin{aligned}
\phi_V(\gamma_\mu(t)) &= \int_{\mathbb{R}^N} V(x)\, d\gamma_\mu(t)(x) \\
&= \int_{\mathbb{R}^N} V(x)\, d\big(((1-t)\pi^2 + t\pi^3)_\#\mu\big)(x) = \int_{(\mathbb{R}^N)^3} V((1-t)y + tz)\, d\mu(x,y,z) \\
&\leq (1-t)\int_{(\mathbb{R}^N)^3} V(y)\, d\mu(x,y,z) + t\int_{(\mathbb{R}^N)^3} V(z)\, d\mu(x,y,z) - \frac{\lambda}{2} t(1-t) W_\mu^2(\mu^2,\mu^3) \\
&= (1-t)\phi_V(\mu^2) + t\phi_V(\mu^3) - \frac{\lambda}{2} t(1-t) W_\mu^2(\mu^2,\mu^3).
\end{aligned}$$

4) ϕ_V *satisfies assumption* (H_2). Let $x_0 \in D(V)$, $r_x > 0$ and consider the closed ball in $\mathcal{P}_2(\mathbb{R}^N)$:

$$\overline{B}(\delta_{x_0}, r_x) := \{\mu \in \mathcal{P}_2(\mathbb{R}^N) : W_2^2(\delta_{x_0}, \mu) \leq r_x^2\}.$$

Since V^- has quadratic growth and since $|y|^2 \leq 2|y - x_0|^2 + 2|x_0|^2$ we have for $\mu \in \overline{B}(\delta_{x_0}, r_x)$:

$$\phi_V(\mu) \geq -\int_{\mathbb{R}^N} V^-\, d\mu \geq -C_1 - C_2 \int_{\mathbb{R}} |y|^2\, d\mu(y) > -\infty \quad \text{for some } C_1, C_2 > 0.$$

5) $\overline{D(\phi_V)} \subseteq \mathcal{P}_2(\overline{D(V)})$. Since $\mathcal{P}_2(\overline{D(V)})$ is closed in $(\mathcal{P}_2(\mathbb{R}^N), W_2(\cdot,\cdot))$ it is sufficient to show that $D(\phi_V) \subseteq \mathcal{P}_2(\overline{D(V)})$ or, equivalently, $\mathcal{P}_2(\overline{D(V)})^c \subseteq D(\phi_V)^c$. Suppose $\mu \in \mathcal{P}_2(\mathbb{R}^N)$ with $\operatorname{supp}\mu \not\subseteq \overline{D(V)}$. Then there exist x and U open such that $x \in U \subset \overline{D(V)}^c$, with $\mu(U) > 0$. Since $V(x) = +\infty$ for all $x \in U$, $\int_{\mathbb{R}^N} V(x)\, d\mu = +\infty$.

6) $\mathcal{P}_2(\overline{D(V)}) \subseteq \overline{D(\phi_V)}$. Let $\mu \in \mathcal{P}_2(\mathbb{R}^N)$ with $\operatorname{supp}\mu \subseteq \overline{D(V)}$. Then there exists a sequence of convex combinations of Dirac measures

$$\sum_{k=1}^{M_n} \alpha_{k,n}\delta_{x_{k,n}} \quad \text{with} \quad \alpha_{k,n} \geq 0,\ \sum_{k=1}^{M_n} \alpha_{k,n} = 1,$$

$x_{k,n} \in \overline{D(V)}$ satisfying

$$W_2\Big(\sum_{k=1}^{M_n} \alpha_{k,n}\delta_{x_{k,n}}, \mu\Big) \overset{n\to\infty}{\longrightarrow} 0.$$

There exists a sequence $y_{k,n} \in D(V)$ such that $|x_{k,n} - y_{k,n}| \leq \frac{1}{n}$. Hence

$$W_2\Big(\sum_{k=1}^{M_n} \alpha_{k,n}\delta_{y_{k,n}}, \mu\Big) \to 0 \quad \text{and} \quad \sum_{k=1}^{M_n} \alpha_{k,n}\delta_{y_{k,n}} \in D(\phi_V).$$

7) In view of 1)–6) and Corollary 5.1 the functional ϕ_V satisfies all assumptions of Theorem 4.1. We denote by $\{\widetilde{S}(t)\}_{t\ge 0}$ the semigroup introduced in Theorem 4.1. It remains to show that $S(t)=\widetilde{S}(t)$, $t\ge 0$.

To this end we consider the (resolvent) operator $J_h:\overline{D(\phi_V)}\to\overline{D(\phi_V)}$, $h>0$, $\frac{1}{h}+\lambda>0$, introduced in Lemma 4.2. Similarly we denote by $j_h:\overline{D(V)}\to\overline{D(V)}$ the (resolvent) operator associated with V in $\mathbb{R}^N$, i.e., for $x\in\overline{D(V)}$, $j_h x$ is the unique minimizer of the functional $\widehat{x}\mapsto\frac{1}{2h}|x-\widehat{x}|^2+V(\widehat{x})$. We claim that $J_h\mu=(j_h)_{\#}\mu$ for $\mu\in\overline{D(\phi_V)}$. Indeed, for any $\widehat{\mu}\in D(\phi_V)$, $\mu\in\overline{D(\phi_V)}$ and $\gamma\in\Gamma_0(\mu,\widehat{\mu})$ we have

$$\begin{aligned}
&\frac{1}{2h}\,W_2^2(\mu,\widehat{\mu})+\phi(\widehat{\mu})=\frac{1}{2h}\int_{\mathbb{R}^N\times\mathbb{R}^N}|x_1-x_2|^2\,d\gamma(x_1,x_2)+\int_{\mathbb{R}^N\times\mathbb{R}^N}V(x_2)\,d\gamma(x_1,x_2)\\
&=\int_{\mathbb{R}^N\times\mathbb{R}^N}\Big(\frac{1}{2h}\,|x_1-x_2|^2+V(x_2)\Big)\,d\gamma(x_1,x_2)\\
&\ge\int_{\mathbb{R}^N\times\mathbb{R}^N}\Big(\frac{1}{2h}\,|x_1-j_hx_1|^2+V(j_hx_1)\Big)\,d\gamma(x_1,x_2)\\
&=\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{2h}\,|x_1-x_2|^2\,d\big((\mathrm{id}\times j_h)_{\#}\mu\big)(x_1,x_2)+\int_{\mathbb{R}^N}V(x)\,d\big((j_h)_{\#}\mu\big)(x)\\
&\ge\frac{1}{2h}\,W_2^2\big(\mu,(j_h)_{\#}\mu\big)+\phi((j_h)_{\#}\mu),
\end{aligned}$$

since $(\mathrm{id}\times j_h)_{\#}\mu\in\Gamma(\mu,(j_h)_{\#}\mu)$.

In view of the uniqueness of the minimizer we obtain $J_h=(j_h)_{\#}\mu$, as claimed. Then it is easy to verify that $(J_{t/n})^n\mu=(j_{t/n})^n{}_{\#}\mu$, for n large enough and $t>0$, $\mu\in\overline{D(\phi_V)}$.

In view of Theorem 4.1, $W_2\big((J_{t/n})^n\mu,\widehat{S}(t)\mu\big)\overset{n\to\infty}{\longrightarrow}0$ hence $\beta\big((J_{t/n})^n\mu,\widehat{S}(t)\mu\big)\overset{n\to\infty}{\longrightarrow}0$. It is easy to verify that

$$\beta\big(((j_{t/n})^n)_{\#}\mu,(T(t))_{\#}\mu\big)\overset{n\to\infty}{\longrightarrow}0$$

since $(j_{t/n})^nx\to T(t)x$, $x\in\overline{D(V)}$ (in view of Theorem 4.1 applied to $X=\mathbb{R}^N$, $d(x,y):=|x-y|$ and $\phi:=V$). Hence $\widehat{S}(t)\mu=S(t)\mu$, $t>0$ and $\mu\in\overline{D(\phi_V)}$. □

5.5 The negative of the Gibbs–Boltzmann entropy functional

In this subsection we give another important functional which satisfies assumption (H_1) with $\alpha=0$ and assumption (H_2) of Section 4.

Let $\mu\in\mathcal{P}_2(\mathbb{R}^N)$. If μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$ and ρ denotes its density we set

$$\begin{aligned}
&\phi_E(\mu):=\int_{\mathbb{R}^N}\rho\log\rho\,dx \quad \text{and}\\
&\phi_E(\mu):=+\infty \qquad\qquad \text{otherwise.}
\end{aligned}\tag{5.27}$$

The functional $-\int_{\mathbb{R}^N}\rho\log\rho\,dx$ is known as the Gibbs–Boltzmann entropy of the density ρ on $\mathbb{R}^N$.

Let us show that the right-hand side of (5.27) is well-defined, more precisely, that the negative part of $\rho\mapsto\rho\log\rho$ is integrable with respect to the Lebesgue measure whenever $\mu\in\mathcal{P}_2(\mathbb{R}^N)$. Set

$$h(s):=\begin{cases}0 & \text{if } s=0,\\ s\log s & \text{if } s>0\end{cases}\quad\text{and } h^-(s):=-\min(h(s),0).$$

Let $C > 0$ be such that $h(s) \le C\sqrt{s}$, $s \in [0,1]$. For any $\Omega \subset \mathbb{R}^N$ Borel measurable consider the sets $\Omega_0 := \Omega \cap \{\rho(x) \le \exp(-|x|)\}$ and $\Omega_1 := \Omega \cap \{\exp(-|x|) < \rho(x) \le 1\}$. Then

$$\int_\Omega h^-(\rho(x))\,dx = \int_{\Omega_0} h^-(\rho(x))\,dx + \int_{\Omega_1} h^-(\rho(x))\,dx \le C\int_\Omega e^{-|x|/2}\,dx + \int_\Omega |x|\rho(x)\,dx.$$

Let $R > 0$ and $B_R := \{x \in \mathbb{R}^N : |x| < R\}$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} h^-(\rho(x))\,dx &= \int_{B_R} h^-(\rho(x))\,dx + \int_{\mathbb{R}^N\setminus B_R} h^-(\rho(x))\,dx \\ &\le \int_{B_R} h^-(\rho(x))\,dx + C\int_{\mathbb{R}^N\setminus B_R} e^{-|x|/2}\,dx + \int_{\mathbb{R}^N\setminus B_R} |x|\rho(x)\,dx.\end{aligned}$$

Observe that the last term is bounded by

$$\varepsilon\int_{\mathbb{R}^N\setminus B_R} |x|^2\rho(x)\,dx + \frac{1}{4\varepsilon}\int_{\mathbb{R}^N\setminus B_R} \rho(x)\,dx$$

which is finite for every $\varepsilon > 0$ (see [JKO], p. 9).

Theorem 5.2. *The functional $\phi_E : \mathcal{P}_2(\mathbb{R}^N) \to (-\infty, +\infty]$ defined in (5.27) is proper, l.s.c., α-convex along generalized geodesics with $\alpha = 0$ and satisfies assumption (H_2).*

Proof. See [AGS]. □

Exercise 5.1. Let $\mu^1, \mu^2 \in \mathcal{P}_2(\mathbb{R}^N)$. Prove $\beta(\mu^1,\mu^2) \le W_2(\mu^1,\mu^2)$. (*Hint*: use $\gamma \in \Gamma_0(\mu^1,\mu^2)$ to rewrite $\int_{\mathbb{R}^N} f\,d\mu^1 - \int_{\mathbb{R}^N} f\,d\mu^2$.)

Exercise 5.2 (M. Wortel). Show that in any metric space (X,d) for every $x_0, x_1, x_2 \in X$ and $t \in [0,1]$ we have:

$$d^2(x_0,\gamma(t)) \le (1-t)d^2(x_0,x_1) + td^2(x_0,x_2) - t(1-t)d^2(x_1,x_2)$$

where

$$\gamma(t) = \begin{cases} x_1 & t = 0, \\ x_0 & t \in (0,1), \\ x_2 & t = 1. \end{cases}$$

Which proper functionals $\phi : X \to (-\infty, +\infty]$ satisfy assumption (H_1) with γ as above and $\alpha = 0$?

Exercise 5.3. Suppose that in Theorem 5.1 the function V satisfies the additional assumption $V \in C^1(\mathbb{R}^N;\mathbb{R})$. Let $\{T(t)\}_{t\ge0}$ be defined as in (5.18) and $\{S(t)\}_{t\ge0}$ as in (5.19). Let $\mu_t := S(t)\mu_0$ with $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$, $t \ge 0$. Show that $\{\mu_t\}_{t\ge0}$ satisfies the PDE

$$\int_{(0,\infty)}\int_{\mathbb{R}^N} \Big(\partial_t\varphi(t,x) + \langle -\nabla_x V(x), \nabla_x\varphi(t,x)\rangle\Big)\,d\mu_t(x)\,dt = 0$$

for every $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R}^N)$.

Appendix 1

The aim of this appendix is to recall, mostly without proofs, some results concerning functions of bounded variation.

Let (X,d) be a (not necessarily complete) metric space. Let $a,b\in\mathbb{R}$ with $a<b$ and let $u:[a,b]\to X$. Given a partition π, $a=t_0<t_1<\ldots<t_n=b$, let

$$V(\pi;u) := \sum_{i=1}^{n} d\big(u(t_{i-1}),u(t_i)\big).$$

Then u is said to be of *bounded variation* (with respect to the metric d) if $\sup_{\pi} V(\pi;u)<\infty$. We denote by $\mathrm{BV}([a,b];X)$ the collection of all X-valued functions which are of bounded variation. We use the notation

$$\text{(A1.1)}\qquad V(u;[a,b]) := \sup_{\pi} V(\pi;u) \quad \text{over all partitions } \pi \text{ of } [a,b].$$

Clearly if $u\in\mathrm{Lip}([a,b];X)$ then $u\in\mathrm{BV}([a,b];X)$ and $V(u;[a,b])\le[u]_{\mathrm{Lip}}(b-a)$. As in the case $X=\mathbb{R}$ one shows that if $u\in\mathrm{BV}([a,b];X)$ and $c\in(a,b)$ then $u|_{[a,c]}\in\mathrm{BV}([a,c];X)$, $u|_{[c,b]}\in\mathrm{BV}([c,b];X)$ and

$$\text{(A1.2)}\qquad V(u;[a,b]) = V\big(u|_{[a,c]};[a,c]\big)+V\big(u|_{[c,b]};[c,b]\big).$$

We shall denote by $V_u(t)$ the real-valued function defined by

$$\text{(A1.3)}\qquad V_u(t) := V(u;[a,t]),\quad t\in[a,b].$$

We have for $a\le s<t\le b$

$$\text{(A1.4)}\qquad d\big(u(s),u(t)\big)\le V_u(t)-V_u(s)=V(u;[s,t]).$$

The function $V_u(\cdot)$ is nondecreasing and satisfies $V_u(a)=0$.

Let $v:[a,b]\to X$. If there exists a nondecreasing function $M:[a,b]\to\mathbb{R}$ such that

$$d\big(v(s),v(t)\big)\le M(t)-M(s)$$

holds for all $a\le s<t\le b$, then $v\in\mathrm{BV}([a,b];X)$ and $V_v(t)\le M(t)-M(a)$, $t\in[a,b]$.

It follows from (A1.4) that if $u\in\mathrm{BV}([a,b];X)$, then the set where u is not continuous is at most countable. Also if $V_u(\cdot)$ is continuous then clearly u is continuous. On the other hand, it can be shown as in the case $X=\mathbb{R}$ that if u is right (resp. left) continuous at $t\in[a,b]$ then $V_u(\cdot)$ is also right (resp. left) continuous at t.

The next lemma is useful.

Lemma A1 ([Br73], Appendix)**.** *Let $u\in\mathrm{BV}([a,b];X)$. Then we have for all h in $(0,b-a)$*

$$\text{(A1.5)}\qquad \int_a^{b-h} d\big(u(t),u(t+h)\big)\,dt\le hV(u;[a,b]).$$

Proof. Since the set of discontinuity of u is at most countable, the same holds for the bounded functions $t \mapsto d(u(t), u(t+h))$, $t \mapsto V_u(t)$ and $t \mapsto V_u(t+h)$ on $[a, b-h]$. Hence these functions are integrable. Using (A1.4) we have

$$\int_a^{b-h} d\big(u(t), u(t+h)\big)\, dt \le \int_a^{b-h} V_u(t+h) - V_u(t)\, dt$$
$$= \int_{a+h}^{b} V_u(t)\, dt - \int_a^{b-h} V_u(t)\, dt \le \int_{b-h}^{b} V_u(t)\, dt \le hV_u(b) = hV(u; [a,b]). \quad \square$$

A function $u \in C([a,b]; X)$ is not necessarily of bounded variation but if u is absolutely continuous (see Definition 1.1), then it is of bounded variation and $V_u(\cdot) \in \mathrm{AC}[a,b]$ as in the case $X = \mathbb{R}$. Conversely, if $u \in \mathrm{BV}([a,b]; X)$ and $V_u(\cdot) \in \mathrm{AC}[a,b]$ then $u \in \mathrm{AC}([a,b]; X)$.

Let $v : [a,b] \to X$ be such that there exists a function $M : [a,b] \to X$ nondecreasing and absolutely continuous. Then by what precedes we have $v \in \mathrm{BV}([a,b]; X)$ and $V_v(t) \le M(t) - M(a)$, $t \in [a,b]$. It is easy to verify that $V_v(\cdot) \in \mathrm{AC}[a,b]$ hence $v \in \mathrm{AC}([a,b]; X)$. Notice that M is absolutely continuous iff there exists $m \in L^1(a,b)$ nonnegative such that $M(t) - M(s) = \int_s^t m(r)\, dr$, $a \le s < t \le b$. It follows that for $v : [a,b] \to X$ we have $v \in \mathrm{AC}([a,b]; X)$ iff there exists $m \in L^1(a,b)$ nonnegative such that

$$\text{(A1.6)} \qquad d\big(v(s), v(t)\big) \le \int_s^t m(r)\, dr, \quad a \le s < t \le b.$$

In this case (A1.6) implies $V_v(t) - V_v(s) \le \int_s^t m(r)\, dr$, hence

$$\int_s^t \frac{d}{dr} V_v(r)\, dr \le \int_s^t m(r)\, dr, \quad a \le s < t \le b.$$

It follows that $\frac{d}{dt} V_v(r) \le m(r)$ a.e. in (a,b).

We conclude this Appendix by showing that if $u \in \mathrm{AC}([a,b]; X)$, then the metric derivative $|\dot{u}|(t)$ (see Theorem 1.1) exists for almost all $t \in (a,b)$, $|\dot{u}| \in L^1(a,b)$ and

$$|\dot{u}|(t) = \frac{d}{dt} V_u(t) \quad \text{a.e. in } (a,b).$$

Proof ([AGS], Theorem 1.1.2). Let $u \in \mathrm{AC}([a,b]; X)$ and let N_u be a subset of (a,b) with measure zero such that $\frac{d}{dt} V_u(t)$ exists for every $t \in (a,b) \setminus N_u$. Since $u([a,b])$ is compact in X, it is separable. There exists a countable subset E of $u([a,b])$ which is dense in $u([a,b])$. For every $e \in E$ the functions $d(e, u(\cdot)) \in \mathrm{AC}[a,b]$ and let N_e be a subset of (a,b) with measure zero such that $\frac{d}{dt} d(e, u(t))$ exists for every $t \in (a,b) \setminus N_e$.

Set $N := N_u \cup \bigcup_{e \in E} N_e$. For $t \in (a,b) \setminus N$ set

$$\ell(t) := \sup_{e \in E} \Big| \frac{d}{dt}\, d(e, u(t)) \Big| \quad \text{and} \quad \ell(t) = 0,\ t \in N.$$

Then ℓ is nonnegative and measurable. We have

$$d\big(u(s), u(t)\big) = \sup_{e \in E} \big| d(e, u(s)) - d(e, u(t)) \big| \le \int_s^t \ell(r)\, dr, \quad a \le s < t \le b.$$

Let $t \in (a,b) \setminus N$. Then

$$\ell(t) = \sup_{e \in E} \varliminf_{s \to t} \frac{\big|d(e,u(t)) - d(e,u(s))\big|}{|t-s|} \le \varliminf_{s \to t} \frac{\big|d(u(t),u(s))\big|}{|t-s|}$$
$$\le \varliminf_{s \to t} \frac{|V_u(t) - V_u(s)|}{|t-s|} = \frac{d}{dt} V_u(t).$$

It follows that $\ell \in L^1(a,b)$. Let N_ℓ be a subset of (a,b) of measure zero such that every $t \in (a,b) \setminus N_\ell$ is a Lebesgue point of ℓ. For every $t \in (a,b) \setminus N_\ell$ we have

$$\varlimsup_{s \to t} \frac{d(u(s),u(t))}{|t-s|} \le \ell(t).$$

Hence for every $t \in (a,b) \setminus (N \cup N_\ell)$ we have

$$\varlimsup_{s \to t} \frac{d(u(s),d(u(t))}{|t-s|} \le \varliminf_{s \to t} \frac{d(u(s),d(u(t))}{|t-s|} \le \frac{d}{dt} V_u(t).$$

Therefore on this set the metric derivative $|\dot{u}|(t)$ exists and $|\dot{u}|(t) = \ell(t) \le \frac{d}{dt} V_u(t)$.

On the other hand, since $d(u(s),u(t)) \le \int_s^t \ell(r)\,dr$, $a \le t < s \le b$, we have $\frac{d}{dt} V_u(t) \le \ell(t)$ a.e. in (a,b). It follows that $|\dot{u}|(t) = \frac{d}{dt} V_u(t)$ a.e. in (a,b). ☐

Appendix 2

The purpose of this appendix is to state and prove a lemma which is used in the proof of Theorem 4.1. It is a (symmetric) variant of a lemma due to Crandall–Liggett [CL71].

Lemma A2. *Let r, γ, δ, K be real numbers satisfying*

$$0 < r \le 2, \quad \gamma, \delta, K > 0. \tag{A2.1}$$

Let m, n be positive integers. Let $\{a_{i,j}\}_{\substack{0 \le i \le m \\ 0 \le j \le n}}$ be nonnegative *real numbers satisfying*

$$a_{i,j} \le \frac{\gamma}{\gamma+\delta} a_{i,j-1} + \frac{\delta}{\gamma+\delta} a_{i-1,j}, \quad 1 \le i \le m,\ 1 \le j \le n, \tag{A2.2}$$

$$a_{i,0} \le K(i\gamma)^r, \qquad 1 \le i \le m, \tag{A2.3}$$

$$a_{0,j} \le K(j\delta)^r, \qquad 1 \le j \le n. \tag{A2.4}$$

Then for $1 \le i \le m$ and $1 \le j \le n$

$$a_{i,j} \le K\big[(i\gamma - j\delta)^2 + (\gamma+\delta)\min(i\gamma, j\delta)\big]^{r/2}. \tag{A2.5}$$

Proof. i) *Case $r = 2$.* First observe that if

$$\alpha = (\alpha_1, \dots, \alpha_m), \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \quad \text{and} \quad H = (h_{i,j})_{\substack{1 \le i \le m \\ 1 \le j \le n}},$$

where $\alpha_i, \beta_j, h_{i,j}$ are real numbers, there exists one and only one $m \times n$ matrix $U = (u_{i,j})$ defined recursively by

$$
\begin{aligned}
&\text{(A2.6)} \quad u_{i,j} = \frac{\gamma}{\gamma+\delta}\, u_{i,j-1} + \frac{\delta}{\gamma+\delta}\, u_{i-1,j} + \frac{\gamma\delta}{\gamma+\delta}\, h_{i,j}, \quad 1 \le i \le m,\ 1 \le j \le n,\\
&\text{(A2.7)} \quad u_{i,0} = \alpha_i, \qquad 1 \le i \le m,\\
&\text{(A2.8)} \quad u_{0,j} = \beta_j, \qquad 1 \le j \le n.
\end{aligned}
$$

If we denote this matrix by $U = U(\alpha, \beta, H)$ we have $U = U(\alpha, 0, 0) + U(0, \beta, 0) + U(0, 0, H)$ and the maps $\alpha \mapsto U(\alpha, 0, 0)$, $\beta \mapsto U(0, \beta, 0)$, $H \mapsto U(0, 0, H)$ are linear. Moreover, if α_i, β_j and $h_{i,j}$ are nonnegative, then $u_{i,j}$ are nonnegative. From these considerations it follows that

$$
\text{(A2.9)} \qquad a_{i,j} \le K b_{i,j}, \quad 1 \le i \le m, \quad 1 \le j \le n,
$$

where $b_{i,j}$ satisfies (A2.6) with $H = 0$, (A2.7) with $\alpha_i = (i\gamma)^2$ and (A2.8) with $\beta_j = (j\delta)^2$.

Let $c_{i,j} = (i\gamma - j\delta)^2$, $0 \le i \le m$, $0 \le j \le n$. Then $c_{i,j}$ satisfies (A2.6) with $h_{i,j} = -(\gamma+\delta)$, (A2.7) with $\alpha_i = (i\gamma)^2$ and (A2.8) with $b_j = (j\delta)^2$. Indeed

$$
\begin{aligned}
&\frac{\gamma}{\gamma+\delta}(i\gamma - (j-1)\delta)^2 + \frac{\delta}{\gamma+\delta}((i-1)\gamma - j\delta)^2\\
&\quad = \frac{\gamma}{\gamma+\delta}\, c_{i,j} + \frac{\delta}{\gamma+\delta}\, c_{i,j} + 2\,\frac{\gamma\delta}{\gamma+\delta}\, c_{i,j} - 2\,\frac{\gamma\delta}{\gamma+\delta}\, c_{i,j} + \frac{\gamma\delta^2}{\gamma+\delta} + \frac{\gamma^2\delta}{\gamma+\delta}\\
&\qquad = c_{i,j} + (\gamma+\delta)\,\frac{\gamma\delta}{\gamma+\delta}\,.
\end{aligned}
$$

Setting $d_{i,j} = b_{i,j} - c_{i,j}$ we deduce that $d_{i,j}$ satisfies (A2.6) with $h_{i,j} = \gamma + \delta$, (A2.7) with $\alpha_i = 0$ and (A2.8) with $\beta_j = 0$. Therefore $\widetilde{d}_{i,j} := \frac{1}{\gamma+\delta}\, d_{i,j}$ satisfies (A2.6) with $h_{i,j} = 1$, (A2.7) with $\alpha_i = 0$ and (A2.8) with $\beta_j = 0$.

Finally, we observe that $e_{i,j} := \gamma i$, $0 \le i \le m$, $0 \le j \le n$, satisfies (A2.6) with $h_{i,j} = 1$, (A2.7) with $\alpha_i \ge 0$ and (A2.8) with $\beta_j = 0$. Indeed,

$$
\frac{\gamma}{\gamma+\delta}\, e_{i,j-1} + \frac{\delta}{\gamma+\delta}\, e_{i-1,j} = \frac{\gamma}{\gamma+\delta}(\gamma i) + \frac{\delta}{\gamma+\delta}\,\gamma(i-1) = \gamma i - \frac{\gamma\delta}{\gamma+\delta} = e_{i,j} - \frac{\gamma\delta}{\gamma+\delta}\,.
$$

It follows that $\widetilde{d}_{i,j} \le e_{i,j}$. Similarly if $f_{i,j} := \delta_j$, $0 \le i \le m$, $0 \le j \le n$, then $\widetilde{d}_{i,j} \le f_{i,j}$, hence $\widetilde{d}_{i,j} \le \min(i\gamma, j\delta)$. Consequently

$$
a_{i,j} \le K b_{i,j} = K(c_{i,j} + d_{i,j}) = K\big(c_{i,j} + (\gamma+\delta)\widetilde{d}_{i,j}\big) \le K\big[(i\gamma - j\delta)^2 + \gamma\delta \min(i\gamma, j\delta)\big],
$$

which is (A2.5) with $r = 2$.

Case $0 < r < 2$. Set $b_{i,j} = (a_{i,j})^{2/r}$. Since $2/r > 1$ we have

$$
b_{i,j} = (a_{i,j})^{2/r} \le \Big(\frac{\gamma}{\gamma+\delta}\, a_{i,j-1} + \frac{\delta}{\gamma+\delta}\, a_{i-1,j}\Big)^{2/r} \le \frac{\gamma}{\gamma+\delta}\, b_{i,j-1} + \frac{\delta}{\gamma+\delta}\, b_{i-1,j}
$$

by Jensen's inequality. Moreover

$$
b_{i,0} \le K^{2/r}(i\gamma)^2, \quad 1 \le i \le m, \quad \text{and} \quad b_{0,j} \le K^{2/r}(j\delta)^2, \quad 1 \le j \le n.
$$

Since $a_{i,j} = (b_{i,j})^{r/2}$, the result follows from case i). □

Appendix 3

The aim of this appendix is to state without proofs some results of the theory of "nonlinear semigroups" on Banach and Hilbert spaces.

Notation

Let X be a nonempty set and let $A, B \subset X \times X$.

$$\begin{aligned} D(A) &:= \{x \in X : \exists y \in X \text{ such that } (x,y) \in A\} \\ R(A) &:= \{y \in X : \exists x \in X \text{ such that } (x,y) \in A\} \\ A^{-1} &:= \{(y,x) \in X \times X : (x,y) \in A\} \\ I &:= \{(x,x) \in X \times X : x \in X\} \\ A \circ B &:= \{(x,y) \in X \times X : \exists z \in X \text{ with } (x,z) \in B \text{ and } (z,y) \in A)\} \end{aligned}$$

Let X be a *real vector space*. If $A, B \subset X \times X$, and $\lambda \in \mathbb{R}$, one sets

$$\begin{aligned} A \pm B &:= \{(x, y \pm z) : (x,y) \in A,\ (x,z) \in B\} \\ \lambda A &:= \{(x, \lambda y) : (x,y) \in A\}. \end{aligned}$$

Let $(X, \|\cdot\|)$ be a normed space.

Definition. A nonempty subset B of $X \times X$ is called *accretive* ($-B$ *dissipative*) if, for every $\lambda > 0$,

$$(I + \lambda B)^{-1} : R(I + \lambda B) \to X$$

is single-valued (i.e. $(I+\lambda B)^{-1}x$ is a singleton for every $x \in R(I+\lambda B)$ or, equivalently, $(I+\lambda B)^{-1}$ is the graph of a function from $R(I+\lambda B)$ into X. By abuse of notation we shall also denote the element of this singleton by $(I+\lambda B)^{-1}x$), and we have

$$\|(I+\lambda B)^{-1}x_1 - (I+\lambda B)^{-1}x_2\| \leq \|x_1 - x_2\|$$

for every $x_1, x_2 \in R(I+\lambda B)$.

Remark. Clearly a nonempty set $B \subset X \times X$ is accretive iff

$$\|x_1 - x_2\| \leq \|(x_1 - x_2) + \lambda(y_1 - y_2)\|$$

for every $\lambda > 0$ and every $(x_i, y_i) \in B$, $i = 1, 2$.

Remark. If B is accretive then $\lambda B + \mu I$ is also accretive for $\lambda, \mu > 0$. In particular, if $A \subset X \times X$ is such that $A + \omega I$ is accretive for some $\omega \in \mathbb{R}$, then $(I+\lambda A)^{-1}$ is the graph of a function whenever $\lambda > 0$ satisfies $\omega\lambda < 1$.

Theorem A3 ([CL71])**.** *Let $(X, \|\cdot\|)$ be a real Banach space and let $A \subset X \times X$ be such that there exists $\omega \in \mathbb{R}$ for which $A + \omega I$ is accretive. Suppose that there exists $\lambda_0 > 0$ such that*

$$R(I + \lambda A) \supseteq \overline{D(A)} \tag{A3.1}$$

for all $\lambda \in (0, \lambda_0)$, where $\overline{D(A)}$ denotes the closure of $D(A)$ in $(X, \|\cdot\|)$.

Then

$$\lim_{n\to\infty} \left[\left(I + \tfrac{t}{n}A\right)^{-1}\right]^n x \tag{A3.2}$$

exists for $x \in \overline{D(A)}$ *and* $t > 0$.

Let $S(0)x = x$ *and* $S(t)x$ *be the limit in* (A3.2) *for* $x \in \overline{D(A)}$ *and* $t > 0$. *Then* $\{S(t)\}_{t\geq 0}$ *is a* C_0*-semigroup on* $\overline{D(A)}$ *which satisfies* $[S(t)]_{\mathrm{Lip}} \leq e^{\omega t}$, $t \geq 0$. *Moreover, if* $x \in D(A)$ *and* $u(t) := S(t)x$ *for* $t \geq 0$, *then* $u|_{[0,T]} \in \mathrm{Lip}([0,T];X)$ *for every* $T > 0$.

Next (for simplicity) we suppose in addition that the set $A \subset X \times X$ satisfies instead of (A3.1) the stronger assumption

$$R(I + \lambda A) = X \tag{A3.3}$$

for all $\lambda > 0$ such that $\omega\lambda < 1$. Then the following holds:

i) If u defined above is strongly right-differentiable at some $t \in [0,\infty)$, then

$$u(t) \in D(A) \quad \text{and} \quad -\frac{d^+}{dt}u(t) \in Au(t). \tag{A3.4}$$

ii) If $v \in C([0,T];X)$ for some $T > 0$ satisfies

$$v(0) \in D(A), \tag{A3.5}$$

$$v \in \mathrm{AC}([\varepsilon,T];X) \quad \text{for every} \quad \varepsilon \in (0,T), \tag{A3.6}$$

$$v \text{ is strongly differentiable a.e. in } (0,T), \tag{A3.7}$$

$$v(t) \in D(A) \text{ a.e. in } (0,T), \tag{A3.8}$$

$$-\frac{d}{dt}v(t) \in Av(t) \text{ a.e. in } (0,T), \tag{A3.9}$$

then

$$v(t) = S(t)v(0) \quad \text{for every} \quad t \in (0,T].$$

Hilbert space case (see [Br73] and references)

If $A + \omega I$ is accretive for some $\omega \in \mathbb{R}$, if assumption (A3.3) holds and $x \in D(A)$, then $t \mapsto S(t)x$ is right-differentiable for every $t \geq 0$.

If moreover $A + \omega I$ is the subdifferential of a proper, lower semicontinuous, convex function $\phi : X \to (-\infty,+\infty]$ and $x \in \overline{D(A)}$ then $t \mapsto S(t)x$ is right-differentiable for every $t > 0$.

Appendix 4

In this appendix we state and prove a result used in the proof of Theorem 4.1.

Proposition A4. *Let* (X,d) *be a complete metric space and let* $\phi : X \to (-\infty,+\infty]$ *be proper, l.s.c. If* ϕ *satisfies* (H_1), *then* $|\partial\phi|$ *is a* strong upper gradient, *i.e. for every* $u \in \mathrm{AC}([a,b];X)$, *the Borel function* $|\partial\phi| \circ u$ *satisfies*

$$\big|\phi(u(t)) - \phi(u(s))\big| \leq \int_s^t |\partial\phi|(u(r))|\dot{u}|(r)\,dr \tag{A4.1}$$

for every $a \leq s < t \leq b$. *In particular, if* $\big(|\partial\phi| \circ u\big) \cdot |\dot{u}| \in L^1(a,b)$, *then* $\phi \circ u \in \mathrm{AC}[a,b]$ *and*

$$|(\phi \circ u)'(t)| \leq |\partial\phi|(u(t))|\dot{u}|(t) \quad \text{a.e. in } (a,b). \tag{A4.2}$$

Proof [AGS]. Let $u \in \mathrm{AC}([a,b];X)$. Since $u|_{[s,t]} \in \mathrm{AC}([s,t];X)$ for $a \le s < t \le b$, it is sufficient to show that if $(|\partial\phi| \circ u) \cdot |\dot{u}| \in L^1(a,b)$, then $\phi \circ u \in \mathrm{AC}[a,b]$ and (A4.2) holds. First we show that (A4.2) is a consequence of $\phi \circ u \in \mathrm{AC}[a,b]$. Let

$$A := \big\{t \in (a,b) : \phi \circ u \text{ is differentiable at } t \text{ and } |\dot{u}|(t) \text{ exists}\big\}.$$

We observe that $(a,b) \setminus A$ has measure zero. Let $t \in A$ and without loss of generality we may assume $(\phi \circ u)'(t) \neq 0$. Therefore, when $s \in A \setminus \{t\}$ belongs to a suitable neighborhood of t, we have $d(u(t),u(s)) > 0$, $\phi \circ u(t) - \phi \circ u(s) > 0$ if either $(\phi \circ u)'(t) > 0$ and $t > s$ or $(\phi \circ u)'(t) < 0$ and $t < s$. Consequently, if $(\phi \circ u)'(t) > 0$ then

$$\begin{aligned}\big|(\phi \circ u)'(t)\big| = (\phi \circ u)'(t) &= \lim_{\substack{s \uparrow t \\ s \in A}} \frac{\phi \circ u(t) - \phi \circ u(s)}{d(u(t),u(s))} \, \frac{d(u(t),u(s))}{t-s} \\ &\le \varlimsup_{\substack{s \uparrow t \\ s \in A}} \frac{\phi \circ u(t) - \phi \circ u(s)}{d(u(t),u(s))} \lim \frac{d(u(t),u(s))}{t-s} \le |\partial\phi|(u(t)) \cdot |\dot{u}|(t)\end{aligned}$$

and if $(\phi \circ u)'(t) < 0$ then

$$\big|(\phi \circ u)'(t)\big| = -(\phi \circ u)'(t) = \lim_{\substack{s \downarrow t \\ s \in A}} \frac{\phi \circ u(t) - \phi \circ u(s)}{d(u(t),u(s))} \, \frac{d(u(t),u(s))}{s-t} \le |\partial\phi|(u(t)) \cdot |\dot{u}|(t)$$

This establishes (A4.2) under the assumption $\phi \circ u \in \mathrm{AC}[a,b]$.

Next we assume $|\partial\phi|(u) \cdot |\dot{u}| \in L^1(a,b)$ and prove that $\phi \circ u \in \mathrm{AC}[a,b]$. We recall that if $(\widetilde{X}, \widetilde{d})$ is a metric space and $\widetilde{\phi} : \widetilde{X} \to (-\infty, +\infty]$ is proper, then the global slope of $\widetilde{\phi}$ at $\widetilde{x} \in \widetilde{X}$, denoted by $|\partial\widetilde{\phi}_g|(\widetilde{x})$, is defined by

$$|\partial\widetilde{\phi}_g|(\widetilde{x}) = \begin{cases} 0 & \text{if } \widetilde{X} = \{\widetilde{x}\} \\ \sup\limits_{\widetilde{y} \neq \widetilde{x}} \dfrac{\big(\widetilde{\phi}(\widetilde{x}) - \widetilde{\phi}(\widetilde{y})\big)^+}{\widetilde{d}(\widetilde{x},\widetilde{y})} & \text{otherwise.} \end{cases}$$

Clearly $|\partial\widetilde{\phi}_g|(\widetilde{x}) \ge |\partial\widetilde{\phi}|(\widetilde{x})$ (local slope). We also recall that if $\widetilde{\phi}$ is l.s.c. then $|\partial\widetilde{\phi}_g|$ is also l.s.c. Indeed, if $\widetilde{X}$ is not a singleton, $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ with $\widetilde{x} \neq \widetilde{y}$, $x_n \in \widetilde{X}$, $n \ge 1$, such that $\lim\limits_{n\to\infty} \widetilde{d}(\widetilde{x}_n, \widetilde{x}) = 0$, then $\widetilde{x}_n \neq \widetilde{y}$ for n large enough and therefore

$$\varliminf_{n\to\infty} \big|\partial\widetilde{\phi}_g\big|(\widetilde{x}_n) \ge \varliminf_{n\to\infty} \frac{\big(\widetilde{\phi}(\widetilde{x}_n) - \widetilde{\phi}(\widetilde{y})\big)^+}{d(\widetilde{x}_n,\widetilde{y})} \ge \frac{\big(\widetilde{\phi}(\widetilde{x}) - \widetilde{\phi}(\widetilde{y})\big)^+}{d(\widetilde{x},\widetilde{y})}\,.$$

The lower semicontinuity follows by taking the supremum over all $\widetilde{y} \in \widetilde{X}$.

Next we choose $\widetilde{X} := u([a,b])$, $\widetilde{d} = d$ and observe that $(\widetilde{X}, \widetilde{d})$ is a compact metric space. Moreover, we define $\widetilde{u}(t) := u(t)$, $t \in [a,b]$ and $\widetilde{\phi}(\widetilde{x}) := \phi(\widetilde{x})$, $\widetilde{x} \in \widetilde{X}$. Clearly $\widetilde{\phi}$ is proper, l.s.c., $\widetilde{u} \in \mathrm{AC}([a,b];\widetilde{X})$ and $\phi \circ u \in \mathrm{AC}[a,b]$ iff $\widetilde{\phi} \circ \widetilde{u} \in \mathrm{AC}[a,b]$.

Now, let α be as in assumption (H_1) for ϕ and let $D := \operatorname{diam} \widetilde{X}$. We have for $\widetilde{x} \in \widetilde{X}$,

$$\big|\partial\widetilde{\phi}_g\big|(\widetilde{x}) \le \sup_{\substack{\widetilde{y} \neq \widetilde{x} \\ \widetilde{y} \in \widetilde{X}}} \left(\frac{\widetilde{\phi}(\widetilde{x}) - \widetilde{\phi}(\widetilde{y})}{d(\widetilde{x},\widetilde{y})} + \frac{\alpha}{2}\, d(\widetilde{x},\widetilde{y}) \right)^+ + \frac{|\alpha|}{2}\, D.$$

By Proposition 4.2, we obtain

$$\big|\partial\widetilde{\phi}_g\big|(\widetilde{x}) \le |\partial\widetilde{\phi}|(\widetilde{x}) + \frac{|\alpha|}{2}\,D,$$

hence

$$\big|\partial\widetilde{\phi}_g\big|(\widetilde{x}) \le |\partial\phi|(\widetilde{x}) + \frac{|\alpha|}{2}\,D.$$

Therefore

$$\big|\partial\widetilde{\phi}_g\big|(\widetilde{u}(t)) \le |\partial\phi|(\widetilde{u}(t)) + \frac{|\alpha|}{2}\,D = |\partial\phi|(u(t)) + \frac{|\alpha|}{2}\,D.$$

Since $|\dot{u}|(t) = \big|\dot{\widetilde{u}}\big|(t)$, we have

$$\big|\partial\widetilde{\phi}_g\big|(\widetilde{u}(t))\big|\dot{\widetilde{u}}\big|(t) \le |\partial\phi|(u(t))|\dot{u}|(t) + \frac{|\alpha|}{2}\,|\dot{u}|(t), \quad \text{a.e. in } (a,b).$$

Noticing that $\big|\partial\widetilde{\phi}_g\big| \circ \widetilde{u}$ is l.s.c., we have by using the assumption on u that $\big(|\partial\widetilde{\phi}_g| \circ \widetilde{u}\big)\cdot\big|\dot{\widetilde{u}}\big| \in L^1(a,b)$.

We observe that by using the space $\widetilde{X}$ we can assume without loss of generality that in the assumption on u the local slope can be replaced by the global one. In order to simplify the notation we shall replace $\widetilde{X}, \widetilde{\phi}, \widetilde{u}, \widetilde{d}$ by X, ϕ, u, d. Next we recall that if $u \in \mathrm{AC}([a,b];X)$ and $\sigma(t) := V(u;[a,t])$, $t \in [a,b]$, then $\sigma : [a,b] \to [0,\infty)$ is nondecreasing, absolutely continuous. Setting $L := \sigma(b)$, we define $\tau(s) := \min\{t \in [a,b] : \sigma(t) = s\}$ for $s \in [0,L]$. Then $\tau : [0,L] \to [a,b]$ is nondecreasing, left continuous. Setting $\widehat{u}(s) := u(\tau(s))$, $s \in [0,L]$, we have (see [AGS], Lemma 1.1.4, arc-length parametrization)

$$u = \widehat{u} \circ \sigma, \quad \widehat{u} \in \mathrm{Lip}([0,L];X) \quad \text{and} \quad [\widehat{u}]_{\mathrm{Lip}} \le 1.$$

We have $\phi \circ u = \phi \circ (\widehat{u} \circ \sigma) = (\phi \circ \widehat{u}) \circ \sigma$. Setting $\varphi := \phi \circ \widehat{u}$ we have $\phi \circ u = \varphi \circ \sigma$. Therefore since σ is nondecreasing and absolutely continuous, $\phi \circ u \in \mathrm{AC}[a,b]$ provided $\varphi \in \mathrm{AC}[a,b]$. Since ϕ is l.s.c. and $\widehat{u} \in \mathrm{Lip}([0,L];X)$, φ is l.s.c.

Next we show that φ is absolutely continuous. Set $g(s) := \big|\partial\phi_g\big|(\widehat{u}(s))$, $s \in [0,L]$. Then g is l.s.c. and for $0 \le s_1, s_2 \le L$

$$\big(\varphi(s_1) - \varphi(s_2)\big)^+ \le g(s_1)d(\widehat{u}(s_1), \widehat{u}(s_2)) \le g(s_1)|s_2 - s_1|.$$

It follows that

$$\text{(A4.3)} \qquad |\varphi(s_1) - \varphi(s_2)| \le \max(g(s_1), g(s_2))|s_1 - s_2|, \quad 0 \le s_1, s_2 \le L.$$

Moreover,

$$\int_0^L g(s)\,ds = \int_a^b g(\sigma(t))\,\frac{d\sigma}{dt}(t)\,dt = \int_a^b |\partial\phi_g|(\widehat{u}\circ\sigma)(t)|\dot{u}|(t)\,dt = \int_a^b |\partial\phi_g|(u(t))|\dot{u}|(t)\,dt$$

is finite. Hence $g \in L^1(a,b)$.

One concludes the proof by showing as in [AGS, p. 29] that a function $\varphi : [0,L] \to \mathbb{R}$ which is l.s.c. and satisfies (A4.3) with $g \in L^1(a,b)$ is absolutely continuous. □

References

[AGS] L. Ambrosio, N. Gigli, G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel 2005.

[ABHN] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, vol. 96, Birkhäuser, Basel 2001.

[Br73] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies No. 5, Notas de Matemática (50), North-Holland Publishing Co., Amsterdam 1973.

[CD1] Ph. Clément, W. Desch, *Some remarks on gradient flows in metric spaces*, in preparation.

[CD2] Ph. Clément, W. Desch, *A "Crandall–Liggett" approach to "Gradient flows" in metric spaces*, in preparation.

[CL71] M. G. Crandall, T. M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. 93 (1971), pp. 265–298.

[JKO] R. Jordan, D. Kinderlehrer, F. Otto, *The variational formulation of the Fokker–Planck equation*, SIAM J. Math. Anal. 29 (1998), pp. 1–17 (electronic).

[R] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. 17 (1966), pp. 497–510.